Improving device-independent weak coin flipping protocols

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Abstract

Weak coin flipping is the cryptographic task where Alice and Bob remotely flip a coin but want opposite outcomes. This work studies this task in the device-independent regime where Alice and Bob neither trust each other, nor their quantum devices. The best protocol was devised over a decade ago by Silman, Chailloux, Aharon, Kerenidis, Pironio, and Massar with bias $\varepsilon \leq 0.33664$, where the bias is a commonly adopted security measure for coin flipping protocols. This work presents two techniques to lower the bias of such protocols, namely self-testing and abort-phobic compositions. We apply these techniques to the SCAKPM '11 protocol above and, assuming a continuity conjecture, lower the bias to $\varepsilon \approx 0.29104$. We believe that these techniques could be useful in the design of device-independent protocols for a variety of other tasks.

Independently of weak coin flipping, en route to our results, we show how one can test $n-1$ out of n devices, and estimate the performance of the remaining device, for later use in the protocol. The proof uses linear programming and, due to its generality, may find applications elsewhere.

Contents

1 Introduction

Coin flipping is the two-party cryptographic primitive where two parties, henceforth called Alice and Bob, wish to flip a coin, but where—and this is the non-trivial requirement—they do not trust each other. This primitive was introduced by Blum [Blu83] who also introduced the first (classical) protocol achieving this functionality. In this work, we concentrate on *weak* coin flipping (WCF) protocols where Alice and Bob desire opposite outcomes. Since then, a series of quantum protocols were introduced with successively improved security. Mochon, in his tour de force, finally settled the question about the limits of the security in the quantum regime by proving the *existence* of quantum protocols with security approaching the ideal limit [Moc07]. This was followed by a flurry of results which achieved diverse cryptographic functionality assuming WCF as a black-box, such as strong coin flipping [CK09], bit commitment [CK11], a variant of oblivious transfer [CGS13], leader election [Gan09] and dice rolling [AS], establishing the importance of WCF in the quantum setting. Returning to Mochon, his work was quite technical and based on the notion of *point games*, a concept introduced by Kitaev. Interestingly, his work was never published—only a preprint was available. Subsequently, a sequence of works have continued the study of point games. In particular, the proof of existence was eventually simplified and peer reviewed [ACG$^+$14] and explicit protocols were reported after more than a decade of Mochon's work [ARW19, ARV].[1] Yet, we note that all of this work is in the *device-dependent* setting where *Alice and Bob trust their quantum devices*. Very little is known in the *device-independent* setting where a cheating player is allowed to control an honest player's quantum devices, opening up a plethora of new cheating strategies that were not considered in the previously mentioned references.

We introduce some basic concepts to facilitate further discussion. The prefix *weak* in weak coin flipping refers to the situation where Alice and Bob desire opposite outcomes of the coin. (We have occasion to discuss *strong* coin flipping protocols, where Alice or Bob could try to bias the coin towards either outcome, but it is not the focus of this work.) When designing weak coin flipping protocols, the security goals are as follows.

Correctness for honest parties:	If Alice and Bob are honest, then they share the same outcome of a protocol $c \in \{0, 1\}$, and c is generated uniformly at random by the protocol.
Soundness against cheating Alice:	If Bob is honest, then a dishonest (i.e., cheating) Alice cannot force the outcome $c = 0$.
Soundness against cheating Bob:	If Alice is honest, then a dishonest (i.e., cheating) Bob cannot force the outcome $c = 1$.

The commonly adopted goal of two-party protocol design is to assume perfect correctness and then minimize the effects of a cheating party, i.e., to make it as sound as possible. This way, if no parties cheats, then the protocol at least does what it is meant to still. With this in mind, we need a means to quantify the effects of a cheating party. It is often convenient to have a single measure to determine if one protocol is better than another. For this purpose, we use *cheating probabilities* (denoted p_B^* and p_A^*) and *bias* (denoted ϵ), defined as follows.

- p_A^*: The maximum probability with which a dishonest Alice can force an honest Bob to accept the outcome $c = 0$.
- p_B^*: The maximum probability with which a dishonest Bob can force an honest Alice to accept the outcome $c = 1$.
- ε: The maximum amount with which a dishonest party can bias the probability of the outcome away from uniform. Explicitly, $\varepsilon = \max\{p_A^*, p_B^*\} - 1/2$.

These definitions are not complete in the sense that we have not yet specified what a cheating Alice or a cheating Bob are allowed to do, or of their capabilities. In this work, we study *information theoretic security*—Alice and Bob are only bounded by the laws of quantum mechanics. For example, they are not bounded by polynomial-time quantum computations. In addition to this, we study the security in the *device-independent* regime where we assume Alice and Bob have complete control over the quantum devices when they decide to "cheat".

[1] [ARVW24] is the distilled concise version of these works and [ARVW22] is the comprehensive version subsuming both works. Interestingly, Miller [Mil20] used techniques from Mochon's proof to show that protocols approaching the ideal limit must have an exponentially increasing number of messages.

When studying device-independent (DI) protocols, one should first consider whether or not secure classical protocols are known (since these are not affected by the DI assumption). It was proved that every classical WCF protocol[2] has bias $\varepsilon = 1/2$, which is the worst possible value (see [Kit03, HW11]). Thus, it makes sense to study quantum WCF protocols in the DI setting, especially if one with bias $\varepsilon < 1/2$ can be found. Indeed, Silman, Chailloux, Aharon, Kerenidis, Pironio, and Massar presented a protocol (see Protocol S) in [SCA$^+$11] with $p_A^* = \cos^2(\pi/8) \approx 0.853$ and $p_B^* = 3/4$. We briefly discuss this protocol because we build on this result but defer the details. To this end, denote by *boxes*, the spatially separable constituents of an untrusted *quantum device*, each of which accepts a classical input and produces a classical output. For instance, an untrusted quantum device corresponding to the GHZ game[3] consists of three *boxes*, each accepting and outputting a single bit. Returning to the protocol in [SCA$^+$11], Protocol S starts with Alice possessing *two* boxes and Bob possessing *one* box which are together supposed to contain the GHZ state and measurements.[4] As the protocol proceeds, they, in addition to exchanging classical information, operate these boxes and exchange them.[5] As is, Protocol S has bias $\varepsilon \approx 0.353$ but in [SCA$^+$11], Protocol S is composed many times to lower the bias to $\varepsilon \leq 0.33664$.[6]

1.1 Main Result

In this work, we provide two techniques for lowering the bias of weak coin flipping protocols and apply them to (the natural weak coin flipping variant of) Protocol S, to obtain the following.

Theorem 1. *There exist device-independent weak coin flipping protocols with bias, ε, approaching 0.29014, assuming two continuity conjectures, Conjectures 22 and 21, hold.*

Before discussing the proof, we note that Protocol S was, in fact, a strong coin flipping protocol and we begin by turning it into a weak coin flipping protocol—Protocol W—in the most natural way. Again, we defer the explicit description of the protocol and informally describe the basic idea: since weak coin flipping has the notion of a "winner" (if $c = 0$ Alice wins and if $c = 1$ Bob wins) we have the party who does not win, conduct an additional test.

The proof of our theorem relies on two key techniques. Our first technique is to add a pre-processing step to Protocol W which *self-tests* the boxes shared by Alice and Bob at the start of the protocol. Our second technique is to compose and analyse the resulting protocols in a new way,[7] which we call *abort-phobic* composition.

1.2 First technique: Self-testing

In the original Protocol S and its WCF variant, Protocol W, a cheating party may control what measurement is performed in the boxes of the other party and how the state of the boxes is correlated to its own quantum memory. This is more general than *device-dependent* protocols, where for instance, the measurements are known to the honest player. However, we employ the concept of self-testing to stop Bob (or Alice) from applying such a strategy. Intuitively, self-testing is a powerful property which allows one to, just from certain input-output behaviours of given devices (satisfying minimal assumptions), conclude uniquely which quantum states and measurements constitute the devices (up to relabelling). The GHZ state which was used in Protocols S and W can be self-tested. Clearly, this property has the potential to improve their security.[8]

We define two variants of Protocol W: Protocol P, where Alice self-tests Bob before executing Protocol W, and Protocol Q, where Bob self-tests Alice instead. Skipping the details, the basic construction is almost trivial. Alice and Bob start with n triples of boxes (constituting n untrusted quantum devices). When Alice self-tests, for instance, Alice asks Bob to send all but one randomly selected triple and tests if the GHZ test passes for these. If so, the

[2] also holds for strong coin flipping

[3] A GHZ game is a 3-player non-local game where each player is asked a single bit question and produces a single bit single bit answer; we review this in Section 2.

[4] They specify the best quantum strategy for winning the GHZ game.

[5] Any protocol described using boxes is readily converted into one where Alice and Bob communicate over an insecure quantum channel; see Section A.

[6] While here compositions are done in a specific context, universal composition of weak coin flipping protocols has recently been studied in [WHBT24].

[7] The composition in [SCA$^+$11] may also be seen as "abort-phobic" but their analysis doesn't rely on the "abort" probability; their bound essentially neglects the abort event.

[8] In [SCA$^+$11], it was noted that self-testing doesn't help improve the security of Protocol S. Alternatively stated, Protocol S has the curious property that its device dependent variant has the same security as it (the device dependent variant).

remaining triple is used for the actual protocol. A large enough n forces a dishonest Bob to not tamper with the boxes too much, as suggested above. Indeed, Protocol P (i.e. when Alice self-tests) already allows us to reduce the cheating probabilities.[9]

Proposition 2 (Informal. See Proposition 10 for a formal statement)**.** *For Protocol P, i.e. where Alice self-tests Bob, the cheating probabilities, in the limit of $n \to \infty$, are*

$$p_A^* = \cos^2(\pi/8) \approx 0.85355 \quad \text{and} \quad p_B^* \approx 0.6667, \tag{1}$$

assuming a continuity conjecture.

For comparison, recall that for Protocol S (it turns out, also for Protocol W), $p_A^* = \cos^2(\pi/8)$ and $p_B^* = 3/4$. We prove this lemma in two stages. In the *first* stage (see Section 4), we assume perfect self-testing: the self-testing step results in exactly specifying (up to a relabelling) the state and measurements governing Alice's boxes. This may be seen as taking $n \to \infty$ in the self-testing step. It is known that for device-dependent protocols, where Alice and Bob trust their devices, the cheating probabilities can be cast as values of semi-definite programs (SDPs) [Kit03, Moc07]. Perfect self-testing allows us to, therefore, express Bob's cheating probabilities as an SDP. Its numerical evaluation yields the quoted value. Analysis for Alice's cheating probability is unchanged from Protocol W. In the *second* stage (see Section 5), we analyse the protocol with n finite and prove that it converges to the case above, under a precisely stated continuity conjecture. The analysis consists of the following two key conceptual steps.

Self-testing in a cryptographic setting. Since Alice tests $n-1$ devices, at best she can conclude (with some confidence) that the last device wins the GHZ game with probability $1-\epsilon(n)$ where $\epsilon(n)$ decreases with n. We therefore need a *robust* self-testing result that allows one to conclude, in particular, the following: if the success probability of a device in a GHZ test is close to unity, then the states and measurements constituting the device are close to GHZ states and measurements (up to a relabelling), in say trace distance. Fortunately, such *robust* self-testing results are known for the GHZ game [MS13] and so it only remains to show the first statement, i.e. estimating the success probability of the remaining device, based on testing $n-1$ devices. While this looks straightforward, formalising and proving this statement turns out to be a bit subtle (see Section 5.3). For instance, related statements are known (e.g. [VV14]) when all devices are measured but they do not apply to our setting where one device is left for later use in the protocol. Our statement holds quite generally for any game with perfect completeness,[10] and is proved using linear programming. It may, therefore, be of independent interest. In particular, our statement holds for both Protocols P and Q.

Continuity conjecture. Consider Protocol P where Alice self-tests. Using the result described above, one can conclude that the state and measurements in the remaining pair of boxes held by Alice, are ϵ-close (in trace distance) to the GHZ state and measurements (up to relabelling). If they were exactly the same, the cheating probabilities could be expressed as the value, say η_0, of an SDP. When they are not, one can still write an optimisation problem whose solution, say $\eta(\epsilon)$, bounds the cheating probabilities, but it is unclear if this optimisation problem is an SDP. The main issues are that the adversary can use states of arbitrary dimensions (that could scale with ϵ) and that the optimisation is over both states and measurements. We conjecture that $\lim_{\epsilon\to 0}\eta(\epsilon) = \eta_0$, i.e. the analysis with ϵ-close boxes converges to the analysis done assuming Alice holds GHZ boxes. Since $\eta(0) = \eta_0$, if the conjecture fails, it must mean there is a discontinuity in $\eta(\epsilon)$ near $\epsilon = 0$ and this would be very surprising. The precise continuity statements for Protocols P and Q are stated as Conjectures 21 and 22, respectively.

While we leave the proofs of these conjectures to future work, we remark that we do not see any obvious obstacles for proving Conjecture 21. Conjecture 22 (when Bob self-tests), on the other hand, seems more involved as the box held by Bob is not measured right after the self-test step, but only later—after some interaction has taken place between the two parties. This makes it more difficult to relax the optimisation problem (corresponding to the cheating probabilities) to obtain an SDP in dimensions independent of ϵ.

[9] Protocol Q (i.e. when Bob self-tests) does not result in a lower bias. However, as we show, when protocols are composed using our second technique, Protocol Q helps lower the bias.

[10] i.e. there is a quantum strategy for winning the game with probability 1

1.3 Second technique: abort-phobic composition

It can happen, that for a given WCF protocol, $p_B^* \neq p_A^*$, in which case we say the protocol is *polarised*. As we saw earlier, it is known (e.g. [SCA$^+$11]) that composing a polarised protocol with itself (or other protocols) can effectively reduce the bias. Our second improvement is a modified way of composing protocols when there is a non-zero probability that the honest player catches the cheating player. Let us start by recalling the standard way of composing protocols.

Standard composition. For a protocol with cheating probabilities p_B^* and p_A^*, we say that it has polarity towards Alice when it satisfies $p_A^* > p_B^*$. Similarly, we say that it has polarity towards Bob when $p_B^* > p_A^*$. Given a polarised protocol $\mathcal{R}$, we may switch the roles of Alice and Bob since the definition of coin flipping is symmetric. To make the polarity explicit, we define $\mathcal{R}_A$ to be the version of the protocol with $p_A^* > p_B^*$ and $\mathcal{R}_B$ to be the version with $p_B^* > p_A^*$. With this in mind, we can now define a simple composition.

Protocol 3 (Winner-gets-polarity composition)**.** *Alice and Bob agree on a protocol $\mathcal{R}$.*

1. *Alice and Bob execute protocol $\mathcal{R}$.*
2. *If Alice wins, she polarises the second protocol towards herself, i.e., they now use the protocol $\mathcal{R}_A$ to determine the final outcome.*
3. *If Bob wins, he polarises the second protocol towards himself, i.e., they now use the protocol $\mathcal{R}_B$ to determine the final outcome.*

The standard composition above is a sensible way to balance the cheating probabilities of a protocol. For instance, if $\mathcal{R}$ has cheating probabilities p_A^* and p_B^* with $p_A^* > p_B^*$, then the composition gets to decide "who gets to be Alice" in the second run. We can easily compute Alice's cheating probability in the composition as

$$(p_A^*)^2 + (1 - p_A^*)p_B^* < p_A^* \tag{2}$$

and Bob's as

$$p_B^* p_A^* + (1 - p_B^*)p_B^* < p_A^*. \tag{3}$$

This does indeed reduce the bias since the maximum cheating probability is now smaller.

Abort-phobic composition. The "traditional" way of considering WCF protocols is to view them as only having two outcomes "Alice wins" (when $c = 0$) or "Bob wins" ($c = 1$). This is because Alice can declare herself the winner if she catches Bob cheating. Similarly, Bob can declare himself the winner if he catches Alice cheating.[11] This is completely fine when we consider "one-shot" versions of these protocols, but we lose something when we compose them. For instance, in the simple composition used in Protocol 3, Bob should not really accept to continue onto the second protocol if he catches Alice cheating in the first. That is, if he knows Alice cheated, he can declare himself the winner of the entire protocol. In other words, the cheating probabilities (2) and (3) may get reduced even further. For purposes of this discussion, suppose Bob adopts a cheating strategy which has a probability v_B of him winning ($c = 1$), a probability v_A of him losing ($c = 0$), and a probability $v_\perp$ of Alice catching him cheating. Then his cheating probability in the (abort-phobic) version of the simple composition is now

$$v_B \cdot p_A^* \,+\, v_A \cdot p_B^* \,+\, v_\perp \cdot 0. \tag{4}$$

This quantity may be a strict improvement if $v_\perp > 0$ when $v_B = p_B^*$.

The concept of abort-phobic composition is simple. Alice and Bob keep using WCF protocols and the winner (at that round) gets to choose the polarity of the subsequent protocol. However, if either party *ever aborts*, then it is game over and the cheating player loses *the entire composite protocol*.

One may think it is tricky to analyse abort-phobic compositions, but we may do this one step at time. To this end, we introduce the concept of *cheat vectors*.

Definition 4 ($\mathbb{C}_A, \mathbb{C}_B$; Alice and Bob's cheat vectors)**.** Given a protocol $\mathcal{R}$, we say that $(v_A, v_B, v_\perp)$ is a cheat vector for (dishonest) Bob if there exists a cheating strategy where,

[11] In doing so, we implicitly assume that the protocol has perfect correctness—when both players are honest, the probability of abort is zero.

v_B is the probability with which Alice accepts the outcome $c = 1$,
v_A is the probability with which Alice accepts the outcome $c = 0$,
$v_\perp$ is the probability with which Alice aborts.

We denote the set of cheat vectors for (dishonest) Bob by $\mathbb{C}_B(\mathcal{R})$. Cheat vectors for (dishonest) Alice and $\mathbb{C}_A(\mathcal{R})$ are analogously defined keeping the notation v_A for her winning, v_B for her losing, and $v_\perp$ for Bob aborting.

In this work, we show how to capture cheat vectors as the feasible region of a semi-definite program, from which we can optimize

$$v_B \cdot p_A^* + v_A \cdot p_B^* + v_\perp \cdot 0. \tag{5}$$

For this to work, we assume we have p_A^* and p_B^* for the protocol that comes in the second round. A simplifying observation is that once we solve for the optimal cheating probabilities in the abort-phobic composition in this way, we can then fix those probabilities and compose again. In other words, we are recursively composing the abort-phobic composition, from the *bottom up*.

By using abort-phobic compositions with Protocol P (where Alice self-tests) one obtains protocols which converge onto a bias of $\varepsilon \approx 0.31486$ proving the first part of the main result. For the second part, we place Protocol P at the bottom, and Protocol Q (where Bob self-tests) on higher layers, to obtain protocols whose bias approaches $\varepsilon \approx 0.29104$. These results are also contingent on the assumption that Conjectures 21 and 22 hold.

1.4 Applications

The concept of polarity extends beyond finding WCF protocols and, as such, the "winner-gets-polarity" concept allows for WCF to be used in other compositions. Indeed, we can use it to balance the cheating probabilities in *any* polarised protocol for any symmetric two-party cryptographic task for which such notions can be properly defined.

For instance, many *strong* coin flipping protocols can be thought of as polarised. For an example, Protocol S is indeed a polarised strong coin flipping protocol. Thus, by balancing the cheating probabilities of that protocol using our DI WCF protocol, we get the following corollary.

Corollary 5. *Suppose Conjectures 21 and 22 hold. Then, there exist DI strong coin flipping protocols where no party can cheat with probability greater than* 0.33192.

To contrast, for [SCA+11], the bound on cheating probabilities was 0.336637. There are likely more examples of protocols which can be balanced in a DI way using this idea.

2 New protocols using self-testing | First Technique

We start by recalling the DI strong coin flipping protocol introduced in [SCA+11], Protocol S, and introduce its weak coin flipping variant Protocol W. We then describe the new Protocols P and Q, where Alice and Bob respectively perform the self-testing step. We also give more formal security guarantees associated with these. Their proofs constitute Sections 4 and 5.

Notation We often use single calligraphic symbols $\mathcal{S}, \mathcal{W}, \mathcal{P}$ and $\mathcal{Q}$ to succinctly refer to the aforementioned protocols. We use, for instance, $p_A^*(\mathcal{W})$ (resp. $p_B^*(\mathcal{W})$) to denote the maximum probability with which a dishonest Alice (resp. Bob) can force an honest Bob (resp. Alice) to output "Alice" (resp. "Bob") in an execution of $\mathcal{W}$. When we say, for instance, consider a tripartite device, viz. a triple of boxes $\square^A, \square^B, \square^C$, we mean that there is a tripartite quantum state and local measurements associated with these boxes. The input to the box selects the measurement setting and the output is the measurement outcome as governed by quantum theory (see Definition 24). When we speak of Alice and Bob exchanging boxes, we understand that the these states and the description of the measurement settings are sent over a (possibly insecure) quantum communication channel (see Definitions 25 and 27 in Section A).

We recall the GHZ test before starting our main discussion as this is at the heart of these protocols.

Definition 6. Suppose we are given a tripartite device, viz. a triple of boxes, $\square^A, \square^B$ and $\square^C$, which accept binary inputs $a, b, c \in \{0, 1\}$ and produces binary output $x, y, z \in \{0, 1\}$ respectively. The boxes pass the GHZ test if $a \oplus b \oplus c = xyz \oplus 1$, given the inputs satisfy $x \oplus y \oplus z = 1$.

It is known that no classical triple of boxes can pass the GHZ test with certainty but quantum boxes can.

Claim 7. Quantum boxes pass the GHZ test with certainty (even if they cannot communicate), for the state $|\psi\rangle_{ABC} = \frac{|000\rangle_{ABC}+|111\rangle_{ABC}}{\sqrt{2}}$, and measurement[12] $\frac{\sigma_x+\mathbb{I}}{2}$ for input 0 and $\frac{\sigma_y+\mathbb{I}}{2}$ for input 1 (in the notation introduced earlier, $M^A_{0|0} = |+\rangle\langle+|$, $M^A_{1|0} = |-\rangle\langle-|$ and so on, where $|\pm\rangle = \frac{|0\rangle\pm|1\rangle}{\sqrt{2}}$).

The proof is easier to see in the case where the outcomes are ± 1; it follows from the observations that $\sigma_y \otimes \sigma_y \otimes \sigma_y |\psi\rangle = -|\psi\rangle$, $\sigma_x \otimes \sigma_x \otimes \sigma_x |\psi\rangle = |\psi\rangle$ and the anti-commutation of σ_x and σ_y matrices, i.e. $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$.

In fact a stronger property holds. If a triple of boxes passes the GHZ test with certainty, it can be shown that up to a local isometry, the state and measurements are as in Claim 7 above.

Lemma 8. *Let $a, b, c, x, y, z \in \{0, 1\}$. Consider a triple of quantum boxes, specified by projectors*[13] $\{M^A_{a|x}, M^B_{b|y}, M^C_{c|z}\}$ *acting on finite dimensional Hilbert spaces $\mathcal{H}^A, \mathcal{H}^B$ and $\mathcal{H}^C$, and $|\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C =: \mathcal{H}^{ABC}$. If the triple pass the GHZ test with probability $1 - \epsilon$ (for $0 \le \epsilon < 1$), then there exists a local isometry,*

$$\Phi = \Phi^A \otimes \Phi^B \otimes \Phi^C : \mathcal{H}^{ABC} \to \mathcal{H}^{ABC} \otimes \mathbb{C}^{2\times 3}$$

and a decreasing function of ϵ, $f(\epsilon)$ such that

$$\|\Phi(|\psi\rangle) - |\chi\rangle \otimes |\text{junk}\rangle\| \le f(\epsilon), \tag{6}$$

$$\left\|\Phi\left(M^D_{d|t}|\psi\rangle\right) - \Pi^D_{d|t}|\text{GHZ}\rangle \otimes |\text{junk}\rangle\right\| \le f(\epsilon) \quad \forall D \in \{A, B, C\},\ \text{and } d, t \in \{0, 1\} \tag{7}$$

where $|\text{GHZ}\rangle = \frac{|000\rangle+|111\rangle}{\sqrt{2}} \in \mathbb{C}^{2\times 3}$, $|\text{junk}\rangle \in \mathcal{H}^{ABC}$ is some arbitrary state and $\{\Pi^A_{a|x}, \Pi^B_{b|y}, \Pi^C_{c|z}\}$ are projectors corresponding to σ_x on the first, second and third qubit of $|\text{GHZ}\rangle$ respectively, for $x = 0$ and corresponding to σ_y for $x = 1$, as in Claim 7.

Proof. Proofs of robust self-testing for GHZ can be found in [MS13] and [McK14]. ☐

2.1 Original protocols

Protocol S is defined as follows.

Protocol S A DI-SCF protocol with $p^*_A = \cos^2 \pi/8$ and $p^*_B = 3/4$ ([SCA$^+$11])

Alice has one box and Bob has two boxes. Each box takes one binary input and gives one binary output and are designed to play the optimal GHZ game strategy. (Who creates and distributes the boxes is not important in the DI setting.)

1. Alice chooses a uniformly random input to her box $x \in_R \{0, 1\}$ and obtains the outcome a. She chooses another uniformly random bit $r \in_R \{0, 1\}$ and computes $s = a \oplus (x \cdot r)$. She sends s to Bob.
2. Bob chooses a uniformly random bit $g \in_R \{0, 1\}$ and sends it to Alice. (We may think of g as Bob's "guess" for the value of x.)
3. Alice sends x to Bob. They both compute the output $c = x \oplus g$. (This is the outcome of the protocol if no-one abort.)
4. Bob tests Alice

Test 1: Alice sends a to Bob. Bob sees if $s = a$ or $s = a \oplus x$. If this is not the case, he aborts.

Test 2: Bob chooses $y, z \in_R \{0, 1\}$ uniformly at random such that $x \oplus y \oplus z = 1$ and then performs a GHZ using x, y, z as the inputs and a, b, c as the output from the three boxes. He aborts if this test fails.

5. If Bob does not abort, they both accept the value of c as the outcome of the protocol.

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[12] we added the identity so that the eigenvalues associated become $0, 1$ instead of $-1, 1$.

[13] This is without loss of generality; given POVMs and a state, one can always construct projectors and a state on a larger Hilbert space which preserves the statistics, using Naimark's theorem.

We now discuss the correctness and soundness of Protocol S. From Claim 7, it is clear that when both players follow the protocol using GHZ boxes (Definition 6), Bob never aborts and they win with equal probabilities. As for the security, [SCA+11] proved the following.

Lemma 9 (Security of SCF)**.** *[SCA+11] Let $\mathcal{S}$ denote the protocol corresponding to Protocol S. Then, the success probability of cheating Bob,*[14] $p_B^*(\mathcal{S}) \leq \frac{3}{4}$ *and that of cheating Alice,* $p_A^*(\mathcal{S}) \leq \cos^2(\pi/8)$.

Further, both bounds are saturated by a quantum strategy which uses a GHZ state and the honest player measures along the σ_x/σ_y *basis corresponding to input* $0/1$ *into the box. Cheating Alice measures along* $\sigma_{\hat{n}}$ *for* $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ *while cheating Bob measures his first box along* σ_x *and second along* σ_y.

Note that both players can cheat maximally assuming they share a GHZ state and the honest player measures along the associated basis. This is why it was asserted that even though the cheating player could potentially tamper with the boxes before handing them to the honest player, exploiting this freedom does not offer any advantage to the cheating player.

Clearly, if we take Protocol S as is and treat it like a weak coin flipping protocol, this conclusion would continue to hold. As motivated in the introduction, we consider a minor, yet crucial, modification to Protocol S. Observe that in Protocol S only Bob performs the test round, while in weak coin flipping there is a notion of Alice winning and Bob winning which may be leveraged. More precisely, if $x \oplus g = 0$, i.e. the outcome corresponding to "Alice wins", we can imagine that Bob continues to perform the test to ensure (at least to some extend) that Alice did not cheat. However, if $x \oplus g = 1$, i.e. the outcome corresponding to "Bob wins", we can require Alice to now complete the GHZ test to ensure that Bob did not cheat. Since we analyse this protocol in detail, we state it as Protocol W, somewhat redundantly below. We have emphasised the changes compared to Protocol S in blue.

Protocol W Weak Coin Flipping version of Protocol S (Italics indicate the differences with Protocol S)

Alice has one box and Bob has two boxes. Each box takes one binary input and gives one binary output and are designed to play the optimal GHZ game strategy. (Who creates and distributes the boxes is not important in the DI setting.)

1. Alice chooses a uniformly random input to her box $x \in_R \{0, 1\}$ and obtains the outcome a. She chooses another uniformly random bit $r \in_R \{0, 1\}$ and computes $s = a \oplus (x \cdot r)$. She sends s to Bob.

2. Bob chooses a uniformly random bit $g \in_R \{0, 1\}$ and sends it to Alice. (We may think of g as Bob's "guess" for the value of x.)

3. Alice sends x to Bob. They both compute the output $c = x \oplus g$. This is the outcome of the protocol assuming neither Alice nor Bob aborts.

4. Test rounds:

   (a) *If $x \oplus g = 0$,* Bob tests Alice

   Test 1: Alice sends a to Bob. Bob sees if $s = a$ or $s = a \oplus x$. If this is not the case, he aborts.

   Test 2: Bob chooses $y, z \in_R \{0, 1\}$ uniformly at random such that $x \oplus y \oplus z = 1$ and then performs a GHZ using x, y, z as the inputs and a, b, c as the output from the three boxes. He aborts if this test fails.

   (b) If $x \oplus g = 1$, Alice tests Bob

   Test 3: Alice chooses $y, z \in_R \{0, 1\}$ uniformly at random such that $x \oplus y \oplus z = 1$ and sends them to Bob. Bob inputs y, z into his boxes, obtains and sends b, c to Alice. Alice tests if x, y, z as inputs and a, b, c as outputs, satisfy the GHZ test. She aborts if this test fails.

5. If Alice and Bob do not abort, they both accept the value of c as the outcome of the protocol.

While it is not surprising that $p_A^*(\mathcal{W}) = p_A^*(\mathcal{P}) = \cos^2(\pi/8)$, it turns out that $p_B^*(\mathcal{W}) = p_B^*(\mathcal{P}) = 3/4$, despite the additional test that Alice performs i.e. P_B^* for Protocol W is not lowered. Yet, this is not quite a setback—one can show that the best cheating strategy now deviates from the GHZ state and measurements for the honest

[14] For SCF, P_B^* is max{Pr[Bob can force Alice to output 1], Pr[Bob can force Alice to output 0]}; P_A^* is analogously defined.

player, suggesting that a cheating player *does* benefit from tampering with the boxes. Consequently, adding a self-testing step before initiating Protocol W, may potentially improve its security and as we shall see in the following subsections, it indeed does.

A remark about the limitation of self-testing in this setting. We note that no self-testing scheme can be concocted which simultaneously self-tests Alice and Bob's boxes. More precisely, no such procedure can ensure that Alice and Bob share a GHZ state (Alice one part, Bob the other two, for instance) because this would mean perfect (or near perfect) SCF is possible which,[15] recall, is forbidden even in the device dependent case.[16] In fact, [BAHS23] shows that the impossibility extends to all devices that produce non-product correlations.

2.2 Alice self-tests | Protocol $\mathcal{P}$

We begin by explicitly stating Protocol P (where Alice self-tests her boxes before initiating Protocol W). In the honest implementation, the quantum device—the triple of boxes—used in Protocol P are characterised by the GHZ state and measurements (see Claim 7).

Protocol P Alice self-tests

Alice starts with n boxes, indexed from I_1 to I_n. Bob starts with $2n$ boxes, the first half indexed by II_1 to II_n and the last half indexed by III_1 to III_n. The triple of boxes $(\mathrm{I}_i, \mathrm{II}_i, \mathrm{III}_i)$ is meant to play the optimal GHZ game strategy.

1. Alice selects a uniformly random index $i \in \{1, \dots, n\}$ and asks Bob to send her all the boxes *except* those indexed by II_i and III_i.
2. Alice performs $n-1$ GHZ tests using the $n-1$ triples of boxes she has, making sure there is no communication between any of them, e.g. by shielding the boxes (in the relativistic settings, coin flipping is possible).
3. Alice aborts if *any* of the GHZ tests fail. Otherwise, she announces to Bob that they can use the remaining boxes for Protocol W.

Proposition 10. *Let $\mathcal{P}$ denote Protocol P and suppose Conjecture 21 holds. Then, in the $n \to \infty$ limit (i.e. very large number of devices are used in the self-test step), Alice's cheating probability $p_A^*(\mathcal{P}) \leq \cos^2(\pi/8) \approx 0.852$ and Bob's cheating probability $p_B^*(\mathcal{P}) \leq 0.667$.*

We defer the proof to Section 4.1 and Section 5.1. As remarked in the introduction, the value for $p_B^*(\mathcal{P})$ is lower than $p_B^*(\mathcal{W})$ and was obtained by numerically solving the corresponding SDP while the analysis for cheating Alice is the same as that of the original protocol.

To write the associated security statement that we use later for compositions, we make the following assumption. (We also state the corresponding statement for Protocol Q, i.e. when Bob self-tests.)

Assumption 11. *In protocol $\mathcal{P}$ ($\mathcal{Q}$, resp.), Alice (Bob, resp.) does not perform the box verification step and instead it is assumed that her box is (his boxes are, resp.) taken from a triple of boxes which win the GHZ game with probability $1-\epsilon$. Here ϵ is a decreasing function of n satisfying $\lim_{n\to\infty} \epsilon = 0$.*

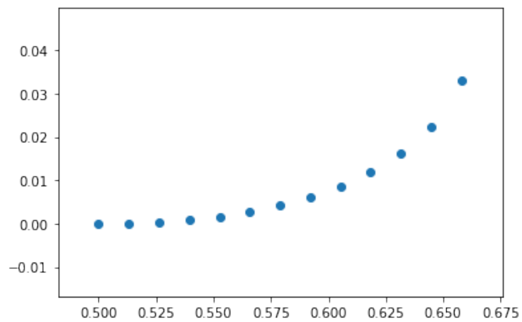
Recalling the definition of cheat vectors from the introduction (see Definition 4), and using Assumption 11 above instead of the continuity conjecture, Conjecture 21, we have the following:

Lemma 12. *Let $\mathcal{P}$ denote Protocol P and suppose Assumption 11 holds. Further, let $c_0, c_1, c_\perp \in \mathbb{R}$, and $\mathbb{C}_B(\mathcal{P})$ be the set of cheat vectors for Bob (see Figure 1a). Then, as $n \to \infty$, the solution to the optimisation problem $\max(c_0\alpha + c_1\beta + c_\perp\gamma)$ over $\mathbb{C}_B(\mathcal{P})$ approaches that of an SDP over variables of constant dimension (wrt n). (In particular, i.e. for $c_0 = c_\perp = 0$ and $c_1 = 1$, $p_B^*(\mathcal{P}) \approx 0.667$.)*

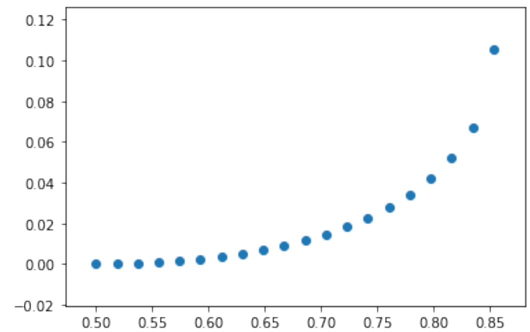
The fact that optimising linear functions in Bob's cheat vectors is an SDP becomes useful in Section 3 when we compose these protocols. Again, the proofs are deferred to Sections 4.1 and 5.1.

[15] More precisely, note that once a GHZ state has been shared, and we are in the device dependent setting, then the protocol would be for each player to make a projective σ_z measurement. What the malicious player does becomes irrelevant for the security.

[16] More precisely, Kitaev [Kit03] showed that for any SCF protocol, $\epsilon \geq \frac{1}{\sqrt{2}} - \frac{1}{2}$. Note that the protocol in this paper does not violate the bound—we propose coin flipping protocols with not-too-small cheating probabilities.



(a) Cheat vectors $\mathbb{C}_B(\mathcal{P})$. The x-axis represents v_B and the y-axis represents the smallest $v_\perp$, given v_B.



(b) Cheat vectors $\mathbb{C}_A(\mathcal{Q})$. The x-axis represents v_B and the y-axis represents the smallest $v_\perp$, given v_B.

Figure 1: Cheat vectors for Protocol P and Protocol Q. Observe how compared to Figure 1a, the abort probabilities in Figure 1b are higher making it more suitable for abort-phobic compositions.

2.3 Bob self-tests | Protocol $\mathcal{Q}$

We analogously define Protocol Q—where Bob self-tests his boxes before initiating Protocol W.

Protocol Q Bob self-tests

Alice starts with n boxes, indexed from I_1 to I_n. Bob starts with $2n$ boxes, the first half indexed by II_1 to II_n and the last half indexed by III_1 to III_n. The triple of boxes $(\text{I}_i, \text{II}_i, \text{III}_i)$ is meant to play the optimal GHZ game strategy.

1. Bob selects a uniformly random index $i \in \{1, \dots, n\}$ and asks Alice to send him all the boxes *except* those indexed by I_i.
2. Bob performs $n-1$ GHZ tests using the $n-1$ triples of boxes he has, making sure there is no communication between any of them.
3. Bob aborts if *any* of the GHZ tests fail. Otherwise, he announces to Alice that they can use the remaining boxes for Protocol W.

Consider Protocol W and Protocol S. Suppose Bob is honest while Alice is malicious, and that at step 3, she sends an x s.t. $x \oplus g = 0$. Under these conditions, observe that Bob's actions are identical in both Protocol W and Protocol S. Since it is already known from Lemma 9 that Alice does not gain anything from tampering with Bob's boxes, the same conclusion holds for Protocol W. Thus, for Protocol Q, we do not expect any improvement in Bob's security, viz. $p_A^*(\mathcal{Q}) = p_A^*(\mathcal{W})$ given that Protocol Q only ensures Alice does not tamper with Bob's boxes. It is also immediate that $p_B^*(\mathcal{Q}) = p_B^*(\mathcal{W})$. This means that we do not see any advantage of self-testing at this stage and therefore the analogue of Proposition 10 is not stated. However, self-testing does help when compositions are considered. More precisely, analogously to Protocol P, optimisation of linear functions of Alice's cheat vectors now becomes an SDP and we reap the benefits of this simplification in the next section.

Lemma 13. *Let $\mathcal{Q}$ denote Protocol Q and suppose Assumption 11 holds. Further, let $c_0, c_1, c_\perp \in \mathbb{R}$, and $\mathbb{C}_A(\mathcal{Q})$ be the set of cheat vectors for Alice (see Figure 1b). Then, as $n \to \infty$, the solution to the optimisation problem $\max(c_0\alpha + c_1\beta + c_\perp\gamma)$ over $(\alpha, \beta, \gamma) \in \mathbb{C}_A(\mathcal{Q})$ approaches that of an SDP of constant dimension (wrt n).*

The proof is again deferred to Section 4.2 and Section 5.2.

3 New protocols using abort-phobic compositions | Second Technique

To convey the key idea, here also we assume Assumption 11 except with $\epsilon = 0$, i.e. the player that does the self-test, in fact, holds boxes corresponding exactly to a GHZ strategy. This simplification allows us to focus on the novelty in our composition technique. In Section 5 we complete the analysis by accounting for all the details skipped here.

Central to the discussion in this section, will be the notion of *polarity* introduced in Section 1.3 and our results will apply to polarised protocols. Note that $\mathcal{W}, \mathcal{P}$, and $\mathcal{Q}$ are all polarised. We begin by recalling that in Section 1.3, again, we had introduced a "standard composition"—the simplest implementation of the "winner gets polarity" idea. Here, we restate this composition with more precision and introduce the notation we use for the more involved cases.

3.1 Composition

The definition below simply formalises the following—execute protocol $\mathcal{X}$ to determine who gets to choose the polarity of protocol $\mathcal{Y}$. We use C with two parameters, as in $C(\mathcal{X}, \mathcal{Y})$, to denote a single composition described above. We use $C(\mathcal{X})$ to denote repeated compositions of $\mathcal{X}$.

Definition 14 ($C(\cdot, \cdot)$ and $C(\cdot)$)**.** Given two polarised WCF protocols, $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X}_A, \mathcal{X}_B$ and $\mathcal{Y}_A, \mathcal{Y}_B$ be their polarisations (see Section 1.3). Define $C(\mathcal{X}, \mathcal{Y})$ as follows:

1. Alice and Bob execute $\mathcal{X}_A$ and obtain outcome $X \in \{A, B, \perp\}$.
2. (a) If $X = A$, execute $\mathcal{Y}_A$ and obtain outcome $Y \in \{A, B, \perp\}$, else
   (b) if $X = B$, execute $\mathcal{Y}_B$ and obtain outcome $Y \in \{A, B, \perp\}$, and finally
   (c) if $X = \perp$, set $Y = \perp$.

   Output Y.

Let $\mathcal{Z}^{i+1} := C(\mathcal{X}, \mathcal{Z}^i)$ for $i \geq 1$, and $\mathcal{Z}^1 := \mathcal{X}$. Then, formally, define $C(\mathcal{X}) := \lim_{i \to \infty} \mathcal{Z}^i$.[17]

The study of such composed protocols is simplified by assuming that in an honest run, neither player outputs $\perp$ (abort), i.e. they either output A or B. We take a moment to explain this.

Consider any protocol $\mathcal{R}$ where, when both players are honest, the probability of abort is zero. The protocols we have described so far, satisfy this property, so long as we assume that honest players can prepare perfect GHZ boxes. Such protocols are readily converted into protocols where an honest player never outputs abort.

For instance, suppose that in the execution of the aforementioned protocol $\mathcal{R}$ (with no-honest-abort), Alice behaves honestly but Bob is malicious. Suppose after interacting with Bob, Alice reaches the conclusion that she must abort. Since she knows that if Bob was honest, the outcome abort could not have arisen, she concludes that Bob is cheating and declares herself the winner, i.e. she outputs A. Similarly, when Bob is honest and after the interaction, reaches the outcome abort, he knows Alice cheated and can therefore declare himself the winner, i.e. output B.

Whenever we modify a protocol so that an honest Alice (Bob) replaces the outcome abort with Alice (Bob) winning, we say Alice (Bob) is *lenient*. This is motivated by the fact that when we compose protocols, if Alice can conclude that Bob is cheating, and she still outputs A instead of aborting, she is giving Bob further opportunity to cheat—she is being lenient.

Definition 15 ($\mathcal{R}$ with lenient players)**.** Suppose $\mathcal{R}$ is a WCF protocol such that when both players are honest, the probability of outcome abort, $\perp$, is zero. Then by "*$\mathcal{R}$ with lenient Alice (Bob)*", which we denote by $\mathcal{R}^{L\perp}$ ($\mathcal{R}^{\perp L}$), we mean that Alice (Bob) follows $\mathcal{R}$ except that the outcome $\perp$ replaced with A (B). Finally, by "*lenient $\mathcal{R}$*", which we denote by $\mathcal{R}^{LL}$, we mean $\mathcal{R}$ with lenient Alice and Bob.

For clarity and conciseness, we define C^{LL} to be compositions with lenient variants of the given protocols. We work out some examples of such protocols and analyse their security in the following section. These can be improved by considering $C^{L\perp}$ and $C^{\perp L}$—compositions where only one player is lenient. We discuss those afterwards.

Definition 16 (C^{LL}, $C^{\perp L}$ and $C^{L\perp}$)**.** Suppose a WCF protocol $\mathcal{X}$ can be transformed into its *lenient* variants (see Definition 15). Then define

$$\begin{aligned} C^{LL}(\mathcal{X}, \mathcal{Y}) &:= C(\mathcal{X}^{LL}, \mathcal{Y}), \\ C^{\perp L}(\mathcal{X}, \mathcal{Y}) &:= C(\mathcal{X}^{\perp L}, \mathcal{Y}), \quad \text{and} \\ C^{L\perp}(\mathcal{X}, \mathcal{Y}) &:= C(\mathcal{X}^{L\perp}, \mathcal{Y}). \end{aligned}$$

[17]This formal notation is defined to let $p_A^*(C(\mathcal{X}))$ mean $\lim_{i\to\infty} p_A^*(\mathcal{Z}^i)$ and similarly for p_B^* and ϵ.

In words, C^{LL} is referred to as a *standard* composition, while $C^{\perp L}$ and $C^{L\perp}$ are referred to as *abort-phobic* compositions. The single argument versions are analogously defined, i.e. $C^{LL}(\mathcal{X}) := C(\mathcal{X}^{LL})$, $C^{L\perp}(\mathcal{X}) := C(\mathcal{X}^{L\perp})$ and $C^{\perp L}(\mathcal{X}^{\perp L})$.

We make two remarks. *First*, the reader might wonder why do we not consider $C^{\perp\perp}$. Nothing prevents us from doing this, however, to obtain an improvement, one must also analyse such a composition. This is, in general, difficult in the device independent setting since one must optimise over both states and measurement settings. However, due to the self-testing step, the (asymptotic) analysis of $C^{\perp L}$ and $C^{L\perp}$ reduces to a semi-definite program which can be solved for protocols with a small number of rounds. *Second*, it is worth clarifying that for our analysis, we need not consider Alice aborting and Bob aborting as separate events. When both players are honest, correctness requires both players produce the same output. When one player is malicious and the other honest, we simply assume that the malicious player can learn the honest player's outcome before producing their outcome. Therefore, it suffices for the malicious player to only care about the output produced by the honest player. This aforementioned assumption only makes the malicious player stronger and therefore can be made in the security analysis.

3.2 Standard Composition | C^{LL}

We begin with the simplest case, standard composition, C^{LL}. Let us take an example. Let $\mathcal{P}$ denote Protocol P and recall (see Lemma 12)

$$p_A^*(\mathcal{P}_A) =: \alpha \approx 0.852,$$
$$p_B^*(\mathcal{P}_A) =: \beta \approx 0.667.$$

Note that therefore $p_A^*(\mathcal{P}_B) = \beta$ and $p_B^*(\mathcal{P}_B) = \alpha$. Further, let $\mathcal{P}' := C^{LL}(\mathcal{P}, \mathcal{P})$, i.e. Alice and Bob (who are both lenient) first execute $\mathcal{P}_A$ and if the outcome is A, they execute $\mathcal{P}_A$, while if the outcome is B, they execute $\mathcal{P}_B$. This is illustrated in Figure 2 where note that the event abort doesn't appear due to the leniency assumption. This allows us to evaluate the cheating probabilities for the resulting protocol as

$$\begin{aligned} p_A^*(\mathcal{P}') &= \alpha\alpha + (1-\alpha)\beta =: \alpha^{(1)}, \quad \text{and} \\ p_B^*(\mathcal{P}') &= \beta\alpha + (1-\beta)\beta =: \beta^{(1)}. \end{aligned} \tag{8}$$

To see this, consider Equation (8). Alice knows that if she wins the first round, her probability of winning is $\alpha > \beta$. She knows that in the first round, she can force the outcome A with probability α. From leniency, she knows that Bob would output B with the remaining probability.[18]

A side remark: one consequence of this simplified analysis is that[19] $\alpha^{(1)} > \beta^{(1)}$. Intuitively, it means that polarity is preserved by the composition procedure. Proceeding similarly, i.e. defining $\mathcal{P}'' := C^{LL}(\mathcal{P}, \mathcal{P}')$, and repeating $k+1$ times overall, one obtains[20]

$$\alpha^{(k+1)} = \alpha\alpha^{(k)} + (1-\alpha)\beta^{(k)}$$
$$\beta^{(k+1)} = \beta\alpha^{(k)} + (1-\beta)\beta^{(k)}.$$

In the limit of $k \to \infty$, one obtains

$$p_A^*(C^{LL}(\mathcal{P})) = p_B^*(C^{LL}(\mathcal{P})) = \lim_{k\to\infty} \alpha^{(k)} = \lim_{k\to\infty} \beta^{(k)} \approx 0.8199.$$

Proceeding similarly, one obtains for $X \in \{A, B\}$ and $\mathcal{X} \in \{\mathcal{W}, \mathcal{Q}\}$,

$$p_X^*(C^{LL}(\mathcal{X})) \approx 0.836\,.$$

We thus have the following.

[18] Without leniency, this probability could have been shared between the outcomes $\perp$ (abort) and B. Consequently, only upper bounds could be obtained on $p_A^*(\mathcal{P}')$ and $p_B^*(\mathcal{P}')$ using only α and β as security guarantees for $\mathcal{P}_A$. Upper bounds, however, would not be enough to determine the polarity of $\mathcal{P}'$ and stymie an unambiguous repetition of the composition procedure (at least as it is defined). One could nevertheless compose by hoping that the upper bounds can be used to accurately represent the polarity. Regardless, this would still yield a protocol and the same calculation would yield correct bounds but the composition itself might be sub-optimal.

[19] $\alpha^{(1)} - \beta^{(1)} = (\alpha-\beta)\alpha - (\alpha-\beta)\beta = (\alpha-\beta)^2 > 0$

[20] Again, note that $\alpha^{(k+1)} - \beta^{(k+1)} = (\alpha^{(k)} - \beta^{(k)})(\alpha - \beta) > 0$.

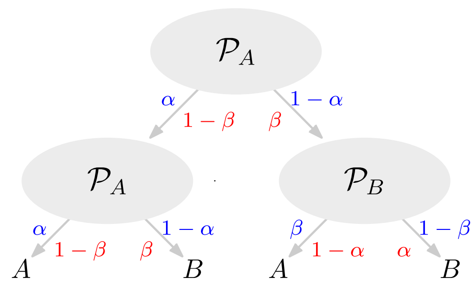


Figure 2: Standard composition of weak coin flipping protocols. Subprotocols only have two outcomes depending on the coin flip. Coloured labels indicate probabilities of outcomes for cheating Alice (blue) and cheating Bob (red).

Theorem 17. *Let $\mathcal{W}$ and $\mathcal{Q}$ denote Protocol W and Protocol Q respectively. Suppose $X \in \{A, B\}$ and $\mathcal{X} \in \{\mathcal{W}, \mathcal{Q}\}$. Then, asymptotically in the security parameter (i.e. in the limit that $\lambda \to \infty$), one has $p_X^*(C^{LL}(\mathcal{X})) \leq 0.836$ and, assuming Conjecture 22 holds, one has $p_X^*(C^{LL}(\mathcal{P})) \leq 0.8199$.*

In Theorem 17, λ is a security parameter that specifies the number of compositions, $k = \lambda$, and the number of devices used in the self-test step, $n = \lambda^3$. The theorem also uses a slightly different definition of C^{LL} from the one described above. Details are deferred to Section 5 but the key idea remains the same.

3.3 Abort Phobic Compositions | $C^{L\perp}, C^{\perp L}$

We now look at the case of abort phobic compositions, $C^{L\perp}$ and $C^{\perp L}$. We work through essentially the same example as above and see what changes in this setting. As usual, let $\mathcal{P}$ denote Protocol P and recall that as before

$$p_A^*(\mathcal{P}_A) =: \alpha \approx 0.852,$$
$$p_B^*(\mathcal{P}_A) =: \beta \approx 0.667.$$

In addition, we know from Lemma 12 that cheat vectors for Bob, $(v_A, v_B, v_\perp) \in \mathbb{C}_B(\mathcal{P}_A)$ admit a nice characterisation courtesy of the self-testing step. Let $\mathcal{P}' := C^{\perp L}(\mathcal{P}, \mathcal{P})$, i.e. Alice and Bob execute $\mathcal{P}_A$ and if the outcome is A, they execute $\mathcal{P}_A$ while if the outcome is B, they execute $\mathcal{P}_B$. Bob is assumed to be lenient so an honest Bob never outputs abort, $\perp$. However, an honest Alice can output abort, $\perp$ so we keep that output in the illustration, Figure 3. Our goal is to find $p_A^*(\mathcal{P}')$ and $p_B^*(\mathcal{P}')$. The former is the same as before because Bob is lenient:

$$p_A^*(\mathcal{P}') = \alpha \cdot \alpha + (1-\alpha) \cdot \beta.$$

Clearly, $p_B^*(\mathcal{P}') \leq \beta\alpha + (1-\beta)\beta$ but this bound may not be tight because $(1-\beta)$ is the combined probability of Alice aborting and Alice outputting A. However, we can use cheat vectors to obtain

$$p_B^*(\mathcal{P}') = \max_{(v_A, v_B, v_\perp) \in \mathbb{C}_B} v_B\alpha + v_A\beta$$

which is an SDP one can solve numerically. Unlike the previous case, the polarity of the resulting protocol, $\mathcal{P}'$, might have flipped (compared to the polarity of $\mathcal{P}$).

Repeating this procedure, one can consider $\mathcal{P}'' := C^{\perp L}(\mathcal{P}, \mathcal{P}')$ and obtain $p_A^*(\mathcal{P}'')$ directly as illustrated above and numerically solve for $p_B^*(\mathcal{P}'')$ using the cheat vectors. After around fifteen such iterations, we found that the cheating probabilities converged to approximately 0.81459. We also observed that the abort probabilities associated with $\mathcal{P}$ were quite small and therefore one could hope that $\mathcal{Q}$ (which denotes Protocol Q) fares better. Proceeding analogously and considering $\mathcal{Q}' := C^{L\perp}(\mathcal{Q}, \mathcal{Q})$, $\mathcal{Q}'' := C^{L\perp}(\mathcal{Q}, \mathcal{Q}')$, etc., we found that the cheating probabilities converged to approximately 0.822655.

Theorem 18. *Let $\mathcal{P}$ and $\mathcal{Q}$ denote Protocol P and Protocol Q respectively. Suppose $X \in \{A, B\}$. Then, asymptotically in the security parameter (i.e. in the limit that $\lambda \to \infty$), one has*

$$p_X^*(C^{\perp L}(\mathcal{P})) \leq 0.81459$$

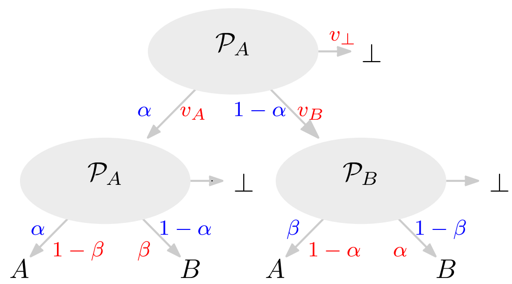


Figure 3: Abort phobic composition for weak coin flipping protocols. Subprotocols have three possible outcomes including abort. Aborting in any subprotocol directly leads to aborting the whole protocol. Coloured labels indicate probabilities of outcomes for cheating Alice (blue) and cheating Bob (red). In the security analysis of cheating Bob, we need to optimise over the cheat vectors $(v_A, v_B, v_\perp) \in \mathbb{C}_B$.

assuming the continuity conjecture, Conjecture 21 (for Protocol P), holds. Further,

$$p_X^*(C^{L\perp}(\mathcal{Q})) \leq 0.822655$$

assuming the continuity conjecture, Conjecture 22 (for Protocol Q), holds.

While by itself $\mathcal{Q}$ doesn't seem to help, one can suppress the bias further, by noting that at the very last step, only the cheating probabilities $p_A^*(\mathcal{Q})$ and $p_B^*(\mathcal{Q})$ played a role (i.e. the fact that the cheating vectors $\mathbb{C}_A$ for $\mathcal{Q}$ had an SDP characterisation was not used). Further, we know that $p_A^*(\mathcal{P}) = p_A^*(\mathcal{Q})$ but $p_B^*(\mathcal{P}) < p_B^*(\mathcal{Q})$, i.e. using $\mathcal{P}$ at the very last step will result in a strictly better protocol.

Theorem 19. *Let $\mathcal{P}$ and $\mathcal{Q}$ denote Protocol P and Protocol Q respectively. Suppose the continuity conjectures, Conjectures 21 and 22, hold. Let $X \in \{A, B\}$,*

$$\begin{aligned} \mathcal{Z}^1 &:= C^{L\perp}(\mathcal{Q}, \mathcal{P}), \quad \text{and} \\ \mathcal{Z}^{k+1} &:= C^{L\perp}(\mathcal{Q}, \mathcal{Z}^k) \quad i > 1. \end{aligned}$$

Then, asymptotically in the security parameter (i.e. in the limit that $\lambda \to \infty$),

$$\lim_{k\to\infty} p_X^*(\mathcal{Z}^k) \leq 0.791044.$$

As was the case for Theorem 17, in the theorems above, λ controls the number of boxes, $n = \lambda^3$, used in the self-test step and the number of compositions $k = \lambda$. Furthermore, the theorems above also use slightly modified definitions for $C^{\perp L}$ and $C^{L\perp}$ as detailed in Section 5.

4 Security proofs | Asymptotic limit

The goal of this section is to analyse the security of Protocol P (resp. Protocol Q) in the simplest setting—assuming that Alice's box (resp. Bob's boxes) indeed correspond to the GHZ state and measurements. To be more precise, in this section, we assume Assumption 11 with $\epsilon = 0$.

Assumption 20 (Restatement: Assumption 11 with $\epsilon = 0$)**.** *In protocol $\mathcal{P}$ (resp. $\mathcal{Q}$), Alice (resp. Bob) does not perform the box verification step and instead it is assumed that her box is (resp. his boxes are) taken from a triple of boxes which win the GHZ game with certainty.*

Lemma 8 asserts that when the winning probability is exactly one (i.e. $\epsilon = 0$ in the lemma), the states and measurements are the same as the GHZ state and σ_x, σ_y measurements, up to local isometries and this allows us to use semi-definite programming. Below, we use the following notation.

- Quantum registers are denoted by capital letters, e.g. A, B, C.

- A pure quantum state in these registers is denoted as a vector $|\psi\rangle \in ABC$ where ABC is interpreted to be the vector space corresponding to the registers A, B, C.
- A mixed quantum state is denoted by $\rho \in \mathsf{Pos}(ABC)$ where $\mathsf{Pos}(ABC)$ is the set of all Hermitian matrices on the vector space ABC with non-negative eigenvalues.

4.1 SDP when Alice self-tests $|\mathcal{P}$

Asymptotic proof of Lemma 12. We prove Lemma 12 under Assumption 20. We begin by making two observations.

First, note that in the protocol, if Alice applies an isometry on her box *after* she has inputted x, obtained the outcome a (and has noted it somewhere), the security of the resulting protocol is unchanged because the rest of the protocol only depends on x and a, and Alice's isometry only amounts to relabelling of the post measurement state. This freedom allows us to simplify the analysis.

Second, in the analysis, we cannot model Alice's random choice, say for x, as a mixed state because Bob can always hold a purification and thus know x. Therefore, we model the randomness using pure states and measure them in the end.

Notation: Other than PQR, all other registers store qubits.

We proceed step by step.

1. We can model (justified below) Alice's act of inputting a random x and obtaining an outcome a from her box through the state
$$|\Psi_0\rangle := \frac{1}{2} \sum_{x,a\in\{0,1\}} |xa\rangle_{XA} |\Phi(x,a)\rangle_{IJ} \tag{9}$$
where X represents the random input and A the output. Here, $|\Phi(x,a)\rangle_{IJ}$ are Bell states (see Equation (11)) and the registers IJ are held by Bob. Alice's act of choosing r at random, computing $s = a \oplus x.r$ is modelled as
$$|\Psi_1\rangle := \frac{1}{2\sqrt{2}} \sum_{x,a,r\in\{0,1\}} |xa\rangle_{XA} |\Phi(x,a)\rangle_{IJ} |r\rangle_R |a \oplus x.r\rangle_S . \tag{10}$$
Finally, Alice's act of sending s is modelled as Alice starting with the state
$$\mathrm{tr}_{IJS}\left[|\Psi_1\rangle\langle\Psi_1|\right] \in \mathsf{Pos}(XAR).$$

Justification for starting with $|\Psi_0\rangle$.
To see why we start with the state $|\Psi_0\rangle$, model Alice's choice of x as $|+\rangle_X$, suppose her measurement result is stored in $|0\rangle_A$, the state of the boxes before measurement is $|\psi\rangle_{PQR}$ and Alice holds P, i.e.
$$|\Psi'_0\rangle := |+\rangle_X |0\rangle_A |\psi\rangle_{PQR} .$$
Let $\{M^P_{a|x}\}$ be the measurement operators corresponding to Alice's box. The measurement process is unitarily modelled as
$$|\Psi'_1\rangle := U_{\text{measure}} |\Psi'_0\rangle = \frac{1}{\sqrt{2}} \sum_{x,a\in\{0,1\}} |x\rangle_X |a\rangle_A M^P_{a|x} |\psi\rangle_{PQR}$$
where
$$U_{\text{measure}} = \sum_{x\in\{0,1\}} |x\rangle\langle x|_X \otimes \left[\mathbb{I}_A \otimes M^P_{0|x} + X_X \otimes M^P_{1|x}\right] \otimes \mathbb{I}_{QR}.$$
Now we harness the freedom of applying an isometry to the post measured state (as observed above). We choose the local isometry in Lemma 8. Without loss of generality, we can assume that Bob had already applied his part of the isometry before sending the boxes (because he can always reverse it when it is his turn). We thus have,
$$\begin{aligned}|\Psi'_2\rangle := \Phi_{PQR} |\Psi'_1\rangle &= \frac{1}{\sqrt{2}} \sum_{x,a\in\{0,1\}} |x\rangle_X |a\rangle_A \Pi^H_{x|a} |\mathrm{GHZ}\rangle_{HIJ} \otimes |\mathrm{junk}\rangle_{PQR} \\ &= \frac{1}{2} \sum_{x,a\in\{0,1\}} |x\rangle_X |a\rangle_A U^H(x,a) |0\rangle_H |\Phi(x,a)\rangle_{IJ} \otimes |\mathrm{junk}\rangle_{PQR}\end{aligned}$$

where

$$|\Phi(x,a)\rangle_{IJ} = \frac{|00\rangle + (-1)^a (i)^x |11\rangle}{\sqrt{2}} \tag{11}$$

and $U^H(x,a)|0\rangle_H$ is $\frac{|0\rangle+(-1)^a(i)^x|1\rangle}{\sqrt{2}}$. Since the state of register H is completely determined by registers X and A, we can drop it from the analysis without loss of generality. Finally, since $|\text{junk}\rangle_{PQR}$ is completely tensored out, we can drop it too without affecting the security. Formally, we can assume that Alice gives Bob the register P at this point.

2. Bob sending g is modelled by introducing $\rho_2 \in \mathsf{Pos}(XARG)$ satisfying $\text{tr}_{IJS}[|\Psi_1\rangle\langle\Psi_1|] = \text{tr}_G(\rho_2)$.

3. At this point, either $x \oplus g$ is zero, in which case Alice's output is fixed or $x \oplus g$ is one, and in that case Bob will already know x because he knows g (he sent it) and Alice will proceed to testing Bob. Formally, therefore, we needn't do anything at this step.

4. Assuming $x \oplus g = 1$, Alice sends y, z to Bob such that $x \oplus y \oplus z = 1$. However, since Bob already knows x, he can deduce z from y. We thus only need to model Alice sending y and Bob responding with $d = b \oplus c$ (because Alice will only use $b \oplus c$ to test the GHZ game, so it suffices for Bob to send d). This amounts to introducing $\rho_3 \in \mathsf{Pos}(XARGYD)$ satisfying $\rho_2 \otimes \frac{\mathbb{I}_Y}{2} = \text{tr}_D(\rho_3)$.

5. Since we postponed the measurements to the end, we add this last step. Alice now measures ρ_3 to determine $x \oplus g$ and if it is one, whether the GHZ test passed. Let

$$\Pi_i := \sum_{x,y\in\{0,1\}:x\oplus g=i} |xg\rangle\langle xg|_{XG} \otimes \mathbb{I}_{ARYD}, \tag{12}$$

$$\Pi^{\text{GHZ}} := \sum_{\substack{x,y\in\{0,1\},\\ a,d\in\{0,1\}:a\oplus d\oplus 1 = xy\cdot(1\oplus x\oplus y)}} |xyad\rangle\langle xyad|_{XYAD} \otimes \mathbb{I}_{RG}. \tag{13}$$

Then, we can write the cheat vector for Alice, i.e. the tuple of probabilities that Alice outputs 0, 1 and abort (see Definition 4), as

$$(\alpha,\beta,\gamma) = (\text{tr}(\Pi_0\rho_3), \text{tr}(\Pi_1\Pi^{\text{GHZ}}\rho_3), \text{tr}(\Pi_1\bar{\Pi}^{\text{GHZ}}\rho_3))$$

where $\bar{\Pi} := \mathbb{I} - \Pi$.

To summarise, the final SDP is as follows: let $|\Psi_1\rangle \in \mathsf{Pos}(XAIJRS)$ be as given in Equation (10), $\rho_2 \in \mathsf{Pos}(XARG)$ and $\rho_3 \in \mathsf{Pos}(XARGYD)$

$$\eta := \max \ \text{tr}([c_0\Pi_0 + \Pi_1(c_1\Pi^{\text{GHZ}} + c_\perp\bar{\Pi}^{\text{GHZ}})]\rho_3) \tag{14}$$

subject to

$$\text{tr}_{IJS}[|\Psi_1\rangle\langle\Psi_1|] = \text{tr}_G(\rho_2)$$
$$\rho_2 \otimes \frac{\mathbb{I}_Y}{2} = \text{tr}_D(\rho_3)$$

where the projectors are defined in Equation (13). □

4.2 SDP when Bob self-tests | $\mathcal{Q}$

Asymptotic proof of Lemma 13. Denote by $\mathcal{S}$ the protocol corresponding to Protocol S.

It is evident that $p_B^*(\mathcal{Q}) \le p_B^*(\mathcal{S})$ because compared to $\mathcal{S}$, in $\mathcal{Q}$ Alice performs an extra test. However, it is not hard to construct a cheating strategy for Bob which lets him saturate the inequality, i.e. $p_B^*(\mathcal{Q}) = p_B^*(\mathcal{S})$.

From Lemma 9, it is also clear that $p_A^*(\mathcal{Q}) = p_A^*(\mathcal{S})$ because the only difference between Bob's actions in $\mathcal{Q}$ and $\mathcal{S}$ is that Bob self-tests to ensure his boxes are indeed GHZ. However, the optimal cheating strategy for $\mathcal{S}$ can be implemented using GHZ boxes.

This establishes the first part of the lemma. For the second part, i.e. establishing that optimising $c_0\alpha+c_1\beta+c_\perp\gamma$ over $(\alpha,\beta,\gamma)\in\mathbb{C}_A$ is an SDP, we proceed as follows. Suppose Assumption 20 holds. Then we can assume that Bob starts with the state

$$\rho_0 := \mathrm{tr}_H(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|_{HIJ}) \tag{15}$$

and the effect of measuring the two boxes can be represented by the application of projectors of pauli operators X and Z.

The justification is similar to that given in the former proof. Suppose Bob holds registers QR of $|\psi\rangle_{PQR}$ which is the combined state of the three boxes. Suppose his measurement operators are $\{M^Q_{b|y}, M^R_{c|z}\}$. Then using the isometry in Lemma 8, Bob can relabel his state (and without loss of generality, we can suppose Alice also relabels according to the aforementioned isometry) to get $\Phi_{PQR}|\psi\rangle_{PQR} = |\mathrm{GHZ}\rangle_{HIJ}\otimes|\mathrm{junk}\rangle_{PQR}$. Further, since $\Phi_{PQR}(M^Q_{b|y}\otimes M^R_{c|z}|\psi\rangle_{PQR}) = \Pi^I_{b|y}\Pi^J_{c|z}|\mathrm{GHZ}\rangle_{HIJ}\otimes|\mathrm{junk}\rangle_{PQR}$ Bob's act of measurement, in the new labelling, corresponds to simply measuring the GHZ state in the appropriate Pauli basis.

1. Bob receiving s from Alice is modelled by introducing $\rho_1\in\mathsf{Pos}(SIJ)$ satisfying $\mathrm{tr}_S(\rho_1)=\rho_0$.
2. Bob sending $g\in_R\{0,1\}$ can be seen as appending a mixed state: $\rho_1\otimes\frac{1}{2}\mathbb{I}_G$.
3. Alice sending x (and a) can be modelled as introducing $\rho_2\in\mathsf{Pos}(AXSIJG)$ satisfying $\mathrm{tr}_A(\rho_2)=\rho_1\otimes\frac{\mathbb{I}_G}{2}$.
4. To model the GHZ test, introduce a register Y in the state $\frac{|0\rangle_Y+|1\rangle_Y}{\sqrt{2}}$. Recall that to perform the GHZ test, we need $x\oplus y\oplus z=1$ i.e. $z=1\oplus y\oplus x$. Further introduce registers B and C to hold the measurement results, define

   $$U := \sum_{y,x\in\{0,1\}}|y\rangle\langle y|_Y\,|x\rangle\langle x|_X\otimes(\mathbb{I}_B\otimes\Pi^I_{0|y}+X_B\otimes\Pi^I_{1|y})\otimes(\mathbb{I}_C\otimes\Pi^J_{0|(1\oplus y\oplus x)}+X_C\otimes\Pi^J_{1|(1\oplus y\oplus x)})\otimes\mathbb{I}_{ASG}. \tag{16}$$

   By construction, $\rho_3 := U\,(|+\rangle\langle+|_Y\otimes|00\rangle\langle 00|_{BC}\otimes\rho_2)\,U^\dagger\in\mathsf{Pos}(YBCAXSIJG)$ models the measurement process.

5. Since we postponed the measurements to the end, we add this step. Define

   $$\Pi_i := \sum_{x,g\in\{0,1\}:x\oplus g=i}|xg\rangle\langle xg|_{XG}\otimes\mathbb{I}_{YABSIJ}$$

   to determine who won. Define

   $$\Pi^{\mathrm{sTest}} := \sum_{s,a,x\in\{0,1\}:s=a\vee s=a\oplus x}|sax\rangle\langle sax|_{SAX}\otimes\mathbb{I}_{GYBCIJ}$$

   to model the first test, i.e. s should either be a or $a\oplus x$. Define

   $$\Pi^{\mathrm{GHZ}} := \sum_{\substack{x,y\in\{0,1\},\\ a,b,c\in\{0,1\}:a\oplus b\oplus c\oplus 1=xy\cdot(1\oplus x\oplus y)}}|xyabc\rangle\langle xyabc|_{XYABC}\otimes\mathbb{I}_{GSIJ}$$

   to model the GHZ test. Let

   $$\Pi^{\mathrm{Test}} := \Pi^{\mathrm{GHZ}}\Pi^{\mathrm{sTest}},\quad \bar{\Pi}^{\mathrm{Test}} := \mathbb{I}-\Pi^{\mathrm{Test}}. \tag{17}$$

   One can then write the cheat vector for Bob, i.e. the tuple of probabilities that Bob outputs $0,1$ and abort (see Definition 4), as

   $$(\alpha,\beta,\gamma) = (\mathrm{tr}(\Pi_0\Pi^{\mathrm{Test}}\rho_3),\mathrm{tr}(\Pi_1\rho_3),\mathrm{tr}(\Pi_0\bar{\Pi}^{\mathrm{Test}}\rho_3)).$$

To summarise, the final SDP is as follows: let $\rho_0\in\mathsf{Pos}(IJ)$ be as defined in Equation (15), $\rho_1\in\mathsf{Pos}(SIJ)$ and $\rho_2\in\mathsf{Pos}(AXSIJG)$. Then,

$$\eta := \max\quad \mathrm{tr}\left([\Pi_0(c_0\Pi^{\mathrm{Test}}+c_\perp\bar{\Pi}^{\mathrm{Test}})+c_1\Pi_1]U\,(|{+}00\rangle\langle{+}00|_{YBC}\otimes\rho_2)\,U^\dagger\right) \tag{18}$$

subject to

$$\mathrm{tr}_S(\rho_1) = \rho_0$$
$$\mathrm{tr}_A(\rho_2) = \frac{1}{2}\rho_1 \otimes \mathbb{I}_G$$

where U is as defined in Equation (16) and the projectors as in Equation (17). □

5 Security Proofs | Finite λ

In this section, we complete the missing steps from the asymptotic analysis.

- We already derived the cheat vector SDP characterisation for Protocol P in Lemma 12 (resp. for Protocol Q in Lemma 13) assuming the boxes win GHZ with certainty, i.e. Assumption 20, instead of assuming the boxes win with probability $1-\epsilon$, i.e. Assumption 11. Section 5.1 (resp. Section 5.2) precisely states the conjecture that using Assumption 11, the security analysis of Protocol P (resp. Protocol Q) converges to that done using Assumption 20.
- Section 5.3, shows how to use the step where $n-1$ devices are tested, to draw the conclusion that the remaining device wins the GHZ game with probability $1-\epsilon$, justifying Assumption 11. The analysis here holds quite generally for any game with perfect completeness and may be of independent interest.
- Finally, Section 5.4, uses this link, Lemma 12 (resp. Lemma 13), and Conjecture 21 (resp. Conjecture 22) to prove the security of Protocol P as stated in Proposition 10. It ends by completing the remaining steps in Section 3 to establish the proofs of security for the composed protocols, as stated in Theorems 17, 18 and 19.

5.1 Continuity conjecture when Alice self-tests | $\mathcal{P}$

Conjecture 21 (Continuity conjecture for $\mathcal{P}$). *Let $c_0, c_1, c_\perp$ be non-negative. Denote by $v_0, v_1, v_\perp$ the probability that Alice outputs $0, 1, \perp$ respectively when Protocol P is executed against some cheating strategy of Bob. Let $0 \le 1-\epsilon'(N) \le 1$ denote the winning probability of the GHZ boxes held by Alice (via Assumption 11). Let $\epsilon := f(\epsilon')$ where f is as in Lemma 8. Consider the objective $\eta_\epsilon = \max c_0 v_0 + c_1 v_1 + c_\perp v_\perp$ where the maximisation is over $v_0, v_1, v_\perp$. Then, $\lim_{\epsilon\to 0} \eta_\epsilon = \eta$ where η is the value of the SDP in Equation (14).*

Conjecture 21 and the asymptotic analysis in Section 4.1 immediately yield Lemma 12.

5.2 Continuity conjecture when Bob self-tests | $\mathcal{Q}$

The analogous statement for the case where Bob self-tests is the following.

Conjecture 22 (Continuity conjecture for $\mathcal{Q}$). *Let $c_0, c_1, c_\perp$ be non-negative. Denote by $v_0, v_1, v_\perp$ the probability that Bob outputs $0, 1, \perp$ respectively when Protocol Q is executed against some cheating strategy of Alice. Let $0 \le 1-\epsilon' \le 1$ denote the winning probability of the GHZ boxes held by Bob (via Assumption 11). Let $\epsilon := f(\epsilon')$ where f is as in Lemma 8. Consider the objective $\eta_\epsilon = \max c_0 v_0 + c_1 v_1 + c_\perp v_\perp$ where the maximisation is over $v_0, v_1, v_\perp$. Then, $\lim_{\epsilon\to 0} \eta_\epsilon = \eta$ where η is the value of the SDP in Equation (18).*

Conjecture 22 and the asymptotic analysis in Section 4.2 immediately yield Lemma 13.

5.3 Estimation of GHZ winning probability

We restate the self-testing procedure using notation tailored to the analysis here. We assume that the $3n$ boxes are described by some joint quantum state and local measurement operators. After playing the GHZ game with $3(n-1)$ of them, and verifying that they all pass, we want to make a statement about the remaining box, whose state $\tilde{\rho}$ is conditioned on the passing of all the other test.

We remark that the expectation value of $E[X_J|J,\Omega]$ accurately describes the expected GHZ value associated to the state of the remaining boxes J, conditioned on having measuring some outcome sequence in the other boxes which passes all the GHZ tests. Note that the conditioning in J is important because otherwise we would get a bound on the GHZ averaged over all boxes, but we are only interested in the remaining box.

Protocol Est-GHZ

1. Pick a box $J \in [n]$ uniformly at random.
2. For $i \in [n]\backslash J$, play the GHZ game with box i, denote outcome of game as $X_i \in \{0,1\}$
3. If

$$\Omega: \quad X_i = 1, \text{ for all } i \in [n]\backslash J \tag{19}$$

4. Then conclude that the remaining box satisfies

$$T: \quad E[X_J|J,\Omega] \geq 1-\epsilon \tag{20}$$

Proposition 23 (Security of Protocol Est-GHZ)**.** *For any implementation of the boxes and choice of $\epsilon > 0$ the joint probability that that the test Ω passes and that the conclusion T is false is small, more precisely,* $\Pr[\Omega \cap \bar{T}] \leq \frac{1}{1-\epsilon+n\epsilon} \leq \frac{1}{n\epsilon}$*, where the first upper-bound is tight.*

This is the correct form of the security statement. It is important to bound the joint distribution of Ω and $\overline{T}$, and not $\Pr[\overline{T}|\Omega]$, conditioning on passing the test Ω. Indeed in the latter case, it would not be possible to conclude anything of value about the remaining box J, as there could be some implementation of the boxes which has a very low expectation value of GHZ, but which passes the test with small but non-zero probability. Consider a hypothetical ideal protocol, which after having chosen J, only passes when T is true. In that case, $\Pr[\Omega \cap \overline{T}] = 0$. Then the actual protocol is equivalent the ideal one, except that it fails with probability $\delta = \frac{1}{1-\epsilon+m\epsilon}$, and so it is δ-close to the ideal algorithm.

Proof. For a given implementation of the boxes, let $p(x_1, \cdots x_n)$ denote the joint probability distribution of passing the GHZ games. Let $S = \{j|\ E[X_j|J=j,\Omega] < 1-\epsilon\} \subset [n]$ be the set of boxes that have an expectation value for GHZ (conditioned on passing in the other boxes) below our target threshold and let $m = |S|$ be the number of such boxes. The value of m is unknown, so we will need to maximise over it in the end.

Let $\alpha = \Pr(\{X_i\}_i = 1)$ and $\beta_j = \Pr(\{X_i\}_{i\neq j} = 1 \cap X_j = 0)$ be respectively the probabilities of the events where all the tests pass, or they all pass except for the jth test. This allows us to rewrite $E[X_j|J=j,\Omega] = \Pr(\{X_i\}_i = 1)/\Pr(\{X_i\}_{i\neq j} = 1) = \alpha/(\alpha+\beta_j)$, and so, by definition of S, we have $\alpha/(\alpha+\beta_j) < (1-\epsilon)$, for $j \in S$, which is equivalent to $\beta_j > \frac{\epsilon}{1-\epsilon}\alpha$.

The aim of the proof is to bound the probability $\Pr[\Omega \cap \overline{T}]$. If we condition and summed over the different values of J, we can rewrite it as

$$\Pr(\Omega \cap \overline{T}) = \sum_j \frac{1}{n}\Pr(\Omega \cap \overline{T}|J=j) = \sum_{j\in S}\frac{1}{n}\Pr(\{X_i\}_{i\neq j} = 1) = \frac{1}{n}\sum_{j\in S}(\alpha+\beta_i)\,, \tag{21}$$

where we have kept the round $j \in S$ ones, conditioned on which T is false. We are thus left with the optimisation problem

$$\max_{\alpha\geq 0,(\beta_i)_i\geq 0} \quad \frac{1}{n}\left(\sum_{j\in S}\alpha+\beta_j\right) \tag{22}$$

$$\text{subject to} \quad \alpha + \sum_{j\in S}\beta_j \leq 1 \tag{23}$$

$$\beta_j \geq \frac{\epsilon}{1-\epsilon}\alpha, \text{ for } j \in S \tag{24}$$

This is a linear problem. Simplifying it by defining $\Sigma = \sum_{j\in S}\beta_j$, gives

$$\begin{aligned}
\max_{\alpha\geq 0,\Sigma\geq 0} \quad & \frac{1}{n}(m\alpha+\Sigma) && (25)\\
\text{subject to} \quad & \alpha+\Sigma\leq 1 && (26)\\
& \Sigma \geq m\frac{\epsilon}{1-\epsilon}\alpha && (27)
\end{aligned}$$

It is easily shown that the maximum is attained for $(\alpha,\Sigma) = \left(\frac{1-\epsilon}{1-\epsilon+m\epsilon}, \frac{m\epsilon}{1-\epsilon+m\epsilon}\right)$ which gives the upper-bound

$$\Pr[\Omega\cap\overline{T}] \leq \frac{1}{n}\max_m \frac{m}{1-\epsilon+m\epsilon} = \frac{1}{1-\epsilon+n\epsilon} \tag{28}$$

We note that the upper-bound is an increasing function of m and so the maximum is attained for $m = n$. This yield the desired upper-bound. From the converse statement, we note that from the present proof we can construct a probability distribution $p(x_1,\cdots x_n)$, which saturates all inequalities, and so the upper-bound $\frac{1}{1-\epsilon+n\epsilon}$ is tight. □

5.4 Putting everything together

Using Lemma 12 (analysis of Protocol P under Assumption 11) and Proposition 23 (in Section 5.3 above), we show how to prove the security of Protocol P, i.e. Proposition 10.

Notation:

- From the previous subsection, recall the definition of event Ω (see Equation (19)) which indicated whether the self-test step cleared, and that of T (see Equation (20)) which indicated whether the last device (hypothetically) clears the GHZ test with probability at least $1-\epsilon$.
- Denote by $\epsilon^{\text{st}} := \Pr(\Omega\bar{T})$ (which, using Proposition 23 we will bound by $1/n\epsilon$).
- Denote by $p_A^{*\epsilon}$ (resp. $p_B^{*\epsilon}$) the probability that Alice (resp. Bob) wins when Protocol P is executed under Assumption 11.

Proof of Proposition 10. We start by considering Protocol P and label it as in Figure 4. It consists of two parts $\mathcal{P}^{\text{st}}$ and $\mathcal{P}^{\epsilon}$.

- $\mathcal{P}^{\text{st}}$ outputs either "abort" or "continue" which are denoted by $\bar{\Omega}$ and Ω respectively.
- $\mathcal{P}^{\epsilon}$ is only executed on event Ω and it outputs either A, B or $\perp$.

Note that, $\lim_{\epsilon\to 0} p_B^{*\epsilon}$ can be calculated using Lemma 12 and $p_A^{*\epsilon} = p_A^*(\mathcal{W})$ since the behaviour of honest Bob is identical for the two protocols. Note also that, essentially by definition, $p^*_{A/B}(\mathcal{P}^\epsilon|T) = p^{*\epsilon}_{A/B}$.

Now, $p_A^*(\mathcal{P}) \leq p_A^{*\epsilon}$ because Bob is honest and so T is always true. Further, a cheating Alice can only decrease her probability of success by having Bob output $\bar{\Omega}$.

As for $p_B^*(\mathcal{P})$, one can write

$$\begin{aligned}
p_B^*(\mathcal{P}) &= p_B^*(\mathcal{P}|\Omega)\Pr(\Omega) + p_B^*(\mathcal{P}|\bar{\Omega})\Pr(\bar{\Omega})\\
&\leq p_B^*(\mathcal{P}|\Omega T) + \Pr(\Omega\bar{T})\\
&\leq p_B^{*\epsilon} + \epsilon^{\text{st}}
\end{aligned}$$

where in the first equality, note that $p_B^*(\mathcal{P}|\bar{\Omega}) = 0$ and from Proposition 23, $\epsilon^{\text{st}} = 1/(n\epsilon)$ bounds the probability that the self-test passed but the conclusion T the device was wrong. Choosing $\epsilon = 1/\lambda$ and $n = \lambda^3$, and taking the limit $\lambda\to\infty$, we obtain the bound asserted in Proposition 10.

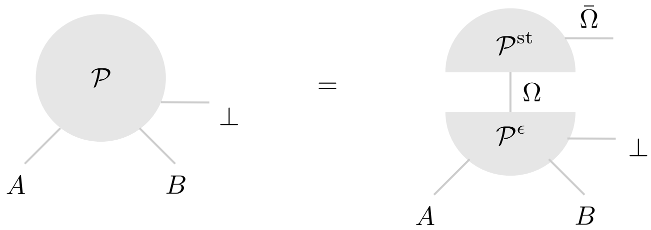


Figure 4: Protocol P broken into two phases: the self-testing phase and the remaining protocol. Above, Ω denotes the outcome that the self-test phase succeeded and $\bar{\Omega}$ that it failed. $\mathcal{P}^{\epsilon}$ denotes continuation of the remaining protocol to output A, B or $\perp$ (abort).

□

We now show how to compose protocols involving the self-test step and conclude the proofs of Theorems 17, 18 and 19.

Proof of Theorems 17, 18 and 19. We start by revisiting lenient compositions. Suppose we are given a protocol $\mathcal{Z}$ with $p_A^*(\mathcal{Z}) = \alpha$ and $p_B^*(\mathcal{Z}) = \beta$, with $\alpha > \beta$ as in Figure 5. Define the lenient composition of $\mathcal{Z}$ with $\mathcal{P}$, $C^{LL}(\mathcal{P}, \mathcal{Z})$ as shown in Figure 6. We have

$$\begin{aligned} p_A^*(C^{LL}(\mathcal{P}, \mathcal{Z}) &\leq p_A^{*\epsilon} \cdot \alpha + (1 - p_A^{*\epsilon}) \cdot \beta \\ p_B^*(C^{LL}(\mathcal{P}, \mathcal{Z}) &\leq (p_B^{*\epsilon} \cdot \alpha + (1 - p_B^{*\epsilon}) \cdot \beta) \cdot \Pr(\Omega T) + \Pr(\Omega \bar{T}) \\ &\leq p_B^{*\epsilon} \cdot \alpha + (1 - p_B^{*\epsilon}) \cdot \beta + \epsilon^{\text{st}} \end{aligned}$$

where the analysis for malicious Alice is the same as before and the analysis for malicious Bob follows by simply writing out the two possibilities that could lead Alice to output B, to wit: event Ω which can be split into two conditioned on T as ΩT and $\Omega \bar{T}$. In both cases, one can bound the probability that Alice outputs B by using Lemma 12 and Proposition 23.

One can similarly analyse the abort phobic composition $C^{\perp L}(\mathcal{P}, \mathcal{Z})$ as shown in Figure 7. Reasoning as above, one can write

$$\begin{aligned} p_A^*(C^{\perp L}(\mathcal{P}, \mathcal{Z})) &\leq p_A^{*\epsilon} \alpha + (1 - p_A^{*\epsilon}) \beta \\ p_B^*(C^{\perp L}(\mathcal{P}, \mathcal{Z})) &\leq \max(v_A \cdot \beta + v_B \cdot \alpha) + \epsilon^{\text{st}} \end{aligned}$$

where the maximisation is over cheat vectors of $\mathcal{P}^{\epsilon}$, i.e. $(v_A, v_B, v_\perp) \in \mathbb{C}_B(\mathcal{P}^{\epsilon})$, and we again used Lemma 12 and Proposition 23.

Clearly, iterating this procedure, in both cases, results in an additional ϵ^{st} term each time to the final cheating probabilities. Thus, for k compositions, one must add $k\epsilon^{\text{st}}$. For $k = \lambda$, $\epsilon = 1/\lambda$ and $n = \lambda^3$, one obtains $\lim_{\lambda \to \infty} k\epsilon^{\text{st}} = 0$ and the compositions converge to the asymptotic case that we already analysed in Section 3. This completes the proof of Theorems 17, 18 and 19.

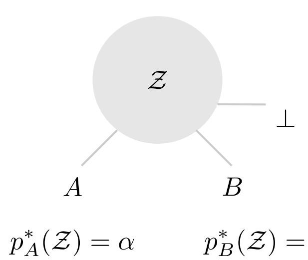


Figure 5: Suppose for protocol $\mathcal{Z}$, one knows $p_A^*(\mathcal{Z}) = \alpha$ and $p_B^*(\mathcal{Z}) = \beta$ where $\alpha > \beta$.

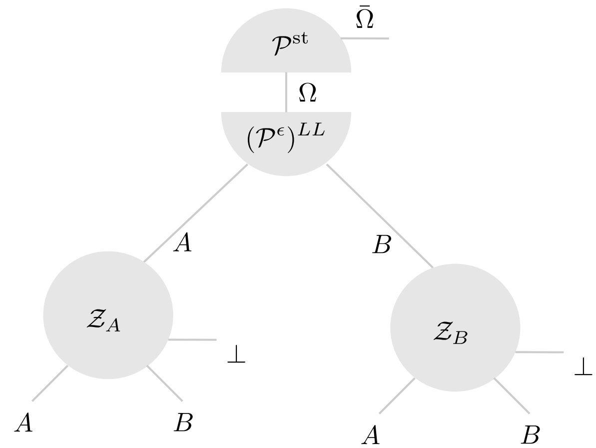


Figure 6: The lenient composition $C^{LL}(\mathcal{P}, \mathcal{Z})$ is shown above. Even though the output $\bar{\Omega}$ is still an abort, the composition is lenient because $(\mathcal{P}^{\epsilon})^{LL}$ means that in $\mathcal{P}^{\epsilon}$, the honest player replaces $\perp$ by declaring themselves the winner.

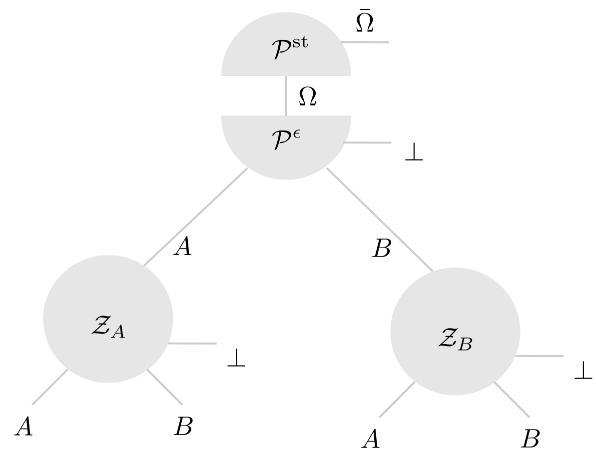


Figure 7: The abort-phobic composition $C^{\perp L}(\mathcal{P}, \mathcal{Z})$ is shown above where $\mathcal{P}^{\epsilon}$ can output abort.

□

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A Device independence and the box paradigm

We describe device independent protocols as classical protocols with one modification: we assume that the two parties can exchange boxes and that the parties can shield their boxes (from the other boxes i.e. the boxes can't communicate with each other once shielded).

Definition 24 (Box)**.** A *box* is a device that takes an input $x \in \mathcal{X}$ and yields an outputs $a \in \mathcal{A}$ where $\mathcal{X}$ and $\mathcal{A}$ are finite sets. Typically, a set of n boxes, taking inputs $x_1, x_2, \dots x_n$ and yielding outputs $a_1, a_2 \dots a_n$ are *characterised* by a joint conditional probability distribution, denoted by

$$p(a_1, a_2 \dots a_n | x_1, x_2 \dots x_n).$$

Further, if $p(a_1, a_2 \dots a_n | x_1, x_2 \dots x_n) = \text{tr}\left[M^1_{a_1|x_1} \otimes M^2_{a_2|x_2} \dots \otimes M^n_{a_n|x_n} \rho\right]$ then we call the set of boxes, *quantum boxes*, where $\{M^i_{a'|x'}\}_{a' \in \mathcal{A}_i}$constitute a POVM for a fixed i and x', ρ is a density matrix and their dimensions are mutually consistent.

Henceforth, we restrict ourselves to quantum boxes.

Definition 25 (Protocol in the box formalism)**.** A generic two-party protocol in the box formalism has the following form:

1. Inputs:
   (a) Alice is given boxes $\square^A_1, \square^A_2 \dots \square^A_p$ and Bob is given boxes $\square^B_1, \square^B_2, \dots \square^B_q$.
   (b) Alice is given a random string r^A and Bob is given a random string r^B (of arbitrary but finite length).
2. Structure: At each round of the protocol, the following is allowed.
   (a) Alice and Bob can locally perform arbitrary but finite time computations on a Turing Machine.
   (b) They can exchange classical strings/messages and boxes.

A protocol in the box formalism is readily expressed as a protocol which uses a (trusted) classical channel (i.e. they trust their classical devices to reliably send/receive messages), untrusted quantum devices and an untrusted quantum channel (i.e. a channel that can carry quantum states but may be controlled by the adversary).

Assumption 26 (Setup of Device Independent Two-Party Protocols)**.** *Alice and Bob*

1. *both have private sources of randomness,*
2. *can send and receive classical messages over a (trusted) classical channel,*
3. *can prevent parts of their untrusted quantum devices from communicating with each other, and*
4. *have access to an untrusted quantum channel.*

We restrict ourselves to a "measure and exchange" class of protocols—protocols where Alice and Bob start with some pre-prepared states and subsequently, only perform classical computation and quantum measurements locally in conjunction with exchanging classical and quantum messages. More precisely, we consider the following (likely restricted) class of device independent protocols.

Definition 27 (Measure and Exchange (Device Independent Two-Party) Protocols)**.** A *measure and exchange (device independent two-party) protocol* has the following form:

1. Inputs:
   (a) Alice is given quantum registers $A_1, A_2, \dots A_p$ together with POVMs[21]

$$\{M^{A_1}_{a|x}\}_a, \{M^{A_2}_{a|x}\}_a, \dots \{M^{A_p}_{a|x}\}_a$$

[21]For concreteness, take the case of binary measurements. By $\{M^{A_1}_{a|x}\}_a$, for instance, we mean $\{M^{A_1}_{0|x}, M^{A_1}_{1|x}\}$ is a POVM for $x \in \{0, 1\}$.

which act on them and Bob is, analogously, given quantum registers $B_1, B_2, \ldots B_q$ together with POVMs

$$\{M^{B_1}_{b|y}\}_b, \{M^{B_2}_{b|y}\}_b, \ldots, \{M^{B_q}_{b|y}\}_b.$$

Alice shields $A_1, A_2, \ldots A_p$ (and the POVMs) from each other and from Bob's lab. Bob similarly shields $B_1, B_2 \ldots B_q$ (and the POVMs) from each other and from Alice's lab.

(b) Alice is given a random string r^A and Bob is given a random string r^B (of arbitrary but finite length).

2. Structure: At each round of the protocol, the following is allowed.

   (a) Alice and Bob can locally perform arbitrary but finite time computations on a Turing Machine.

   (b) They can exchange classical strings/messages.

   (c) Alice (for instance) can

      i. send a register A_l and the encoding of her POVMs $\{M^{A_l}_i\}_i$ to Bob, or

      ii. receive a register B_m and the encoding of the POVMs $\{M^{B_m}_i\}_i$.

   Analogously for Bob.

It is clear that a protocol in the box formalism (Definition 25) which uses only quantum boxes (Definition 24) can be implemented as a measure and exchange protocol (Definition 27).