UPGRADED FREE INDEPENDENCE PHENOMENA FOR RANDOM UNITARIES

DAVID JEKEL AND SRIVATSAV KUNNAWALKAM ELAYAVALLI

Abstract. We study upgraded free independence phenomena for unitary elements u_1, u_2, ... in a matrix ultraproduct constructed from the large-n limit of Haar random unitaries. Using a uniform asymptotic freeness argument and volumetric analysis, we establish freeness of several much larger algebras $\mathcal{A}_j$ containing u_j, which sheds new light on the structural properties of matrix ultraproducts, as well as free products of tracial von Neumann algebras. First, motivated by Houdayer and Ioana's results on free independence of approximate commutants in free products [28], we show that the commutants $\{u_j\}' \cap \prod_{n \to \mathcal{U}} \mathbb{M}_n(\mathbb{C})$ in the matrix ultraproduct are freely independent. We then prove free independence of the entire Pinsker algebras $\mathcal{P}_j$ containing u_j; $\mathcal{P}_j$ by definition is the maximal subalgebra containing u_j with vanishing 1-bounded entropy [23], and $\mathcal{P}_j$ contains for instance any amenable algebra containing u_j as well as the entire sequential commutation orbit of u_j, and it is closed under taking iterated wq-normalizers. Through an embedding argument, we go back and deduce analogous free independence results for $\mathcal{M}^{\mathcal{U}}$ when $\mathcal{M}$ is a free product of Connes embeddable tracial von Neumann algebras $\mathcal{M}_i$, which thus yields a generalization and a new proof of Houdayer–Ioana's results in this case.

1. Introduction

1.1. **Main results.** A fundamental result in random matrix theory is that for independent $n \times n$ Haar random unitaries $U_1^{(n)}$, $U_2^{(n)}$, ..., the trace of any word in the $U_j^{(n)}$'s and their adjoints converges to trace of the corresponding word in the free group [53,56]. Thus, the von Neumann algebra of the free group $L(F_\infty)$ arises from the large n limit of random matrices. Recently, Houdayer and Ioana [28] showed that (for instance) if b_1, b_2, ... are elements in $L(F_\infty)$ such that b_j approximately commutes with the group generator g_j, then b_1, b_2, ... must be approximately freely independent, and thus it is natural to ask whether an analogous statement holds for Haar random unitary matrices. We will give an affirmative answer to this question as well as showing several generalizations.

Theorem A (Asymptotic freeness of approximate commutants). *Let $U_1^{(n)}$, $U_2^{(n)}$, ... be independent $n \times n$ Haar random unitary matrices. Let $B_1^{(n)}$, ..., $B_m^{(n)}$ be random matrices on the same probability space such that $\|B_j^{(n)}\| \leq 1$ and $\lim_{n \to \infty} \|[U_j^{(n)}, B_j^{(n)}]\|_2 = 0$ almost surely for $j = 1, \dots, k$. Then $B_1^{(n)}$, ..., $B_k^{(n)}$ are almost surely asymptotically freely independent.*

Here also each matrix $B_j^{(n)}$ can be replaced by a tuple; see Theorem 2.12. The challenge of Theorem A is that $B_j^{(n)}$ is allowed to depend arbitrarily on the $U_j^{(n)}$. Thus, the proof requires a version of Voiculescu's asymptotic freeness [56, Corollary 2.7] that applies *uniformly* to all the possible values of $B_j^{(n)}$. After considering the diagonalization of the $U_j^{(n)}$'s (see §2.1), we will show a uniform asymptotic freeness result for matrices $B_j^{(n)}$ that are asymptotically supported in ϵn-bands around the diagonal (Lemma 2.10). To guarantee each moment condition, we test it on a δ-dense subset for some small δ by playing off the exponential concentration of measure for the Haar random unitary matrices against the small dimension of the ϵ-bands compared to the ambient matrix space.

Such asymptotic results can be conveniently formulated using ultraproducts of tracial von Neumann algebras (see [8, Appendix A]); intuitively, elements of the ultraproduct $\prod_{n\to\mathcal{U}} \mathcal{M}_n$ capture all possible limiting behaviors of elements x_n from $\mathcal{M}_n$, and thus allow asymptotic or approximate statements to be reformulated as exact statements. Letting $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, Houdayer–Ioana's result in the special case of $L(F_\infty)$ would say that the commutants $\{g_j\}' \cap L(F_\infty)^{\mathcal{U}}$ are freely independent of each other. Meanwhile, the ultraproduct version of Theorem A is the following.

Theorem B (Freeness of relative commutants)**.** *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $\mathcal{Q} = \prod_{n\to\mathcal{U}} M_n(\mathbb{C})$ be the ultraproduct of matrix algebras. Let $U_1^{(n)}$, $U_2^{(n)}$, … be independent $n \times n$ Haar random unitary matrices on a probability space $(\Omega, \mathcal{F}, P)$. For each $\omega \in \Omega$, let $u_j(\omega) = [U_j^{(n)}(\omega)]_{n\in\mathbb{N}} \in \mathcal{Q}$ be the corresponding element of the matrix ultraproduct. Then almost surely*

$$\{u_1(\omega)\}' \cap \mathcal{Q}, \{u_2(\omega)\}' \cap \mathcal{Q}, \ldots$$

are freely independent.

This method of volumetric analysis on the space of matrices goes back at least to von Neumann [57]. Voiculescu formalized the exponential growth rate of volumes of the set of matrix microstate with certain moments through his free entropy χ [54], which he used to show that the free group von Neumann algebra has no Cartan subalgebra [55] (see also [18]). Jung [32] defined the related notion of strong 1-boundedness, a condition of "small microstate dimension" which is independent of the choice of generators of the von Neumann algebra, and which Hayes later captured through the metric-entropy invariant h [23]. Volumetric analysis and high-dimensional concentration of measure in random matrix theory [4,34] form a powerful combination with many applications to the structure of $L(F_\infty)$ [24, 26, 27]. These techniques also relate closely to deep questions about the growth rates of approximate representations [43–45].

Using the toolkit of 1-bounded entropy and a small amount of model theory for von Neumann algebras [13, 14, 29], we show that this upgrading of freeness phenomenon applies not only to the commutants of u_j in $\mathcal{Q}$, but to the much larger Pinsker algebra of u_j. A *Pinsker algebra* [27, Definition after Theorem B] in a von Neumann algebra $\mathcal{M}$ is a maximal von Neumann algebra $\mathcal{P} \subseteq \mathcal{M}$ such that $h(\mathcal{P} : \mathcal{Q}) = 0$ (this terminology is motivated by an analogous construction in ergodic theory). Thanks to [23, Lemma A.12], every diffuse von Neumann subalgebra $\mathcal{A} \subseteq \mathcal{M}$ with $h(\mathcal{A} : \mathcal{M}) = 0$ is contained in a unique Pinsker algebra $\mathcal{P}$. If $\mathcal{A}$ is diffuse amenable, then the Pinsker algebra of $\mathcal{A}$ contains any maximal amenable $\mathcal{B} \supseteq \mathcal{A}$, and the same holds with amenability replaced with property Gamma; moreover, the Pinsker algebra is coarse in $\mathcal{M}$ and in particular closed under taking wq-normalizers [23, Theorem 3.8]; see [27, §1.2].

Theorem C (Freeness of Pinsker algebras)**.** *Let $u_j(\omega)$ be as in Theorem B. Let $\mathcal{P}_j$ be the Pinsker algebra of $u_j(\omega)$. Then almost surely $\mathcal{P}_1$, $\mathcal{P}_2$, … are freely independent.*

This immediately implies the following corollary, for instance.

Corollary D (Freeness of amenable algebras)**.** *Let $u_j(\omega)$ be as in Theorem B. Then almost surely the following statement holds: If $\mathcal{A}_j$ is any amenable subalgebra containing $u_j(\omega)$, then $\mathcal{A}_1$, $\mathcal{A}_2$, … are freely independent.*

Another consequence of Theorem C is freeness of the sequential commutation orbits of u_j studied by [33]; see also [17], [10]. Recall a *Haar unitary* in a tracial von Neumann algebra $\mathcal{M}$ is any unitary element u satisfying $\mathrm{tr}^{\mathcal{M}}(u^m) = 0$ for all $m \in \mathbb{Z} \setminus \{0\}$, or equivalently a unitary u whose spectral measure is the Haar measure on the circle.[1] We denote the set of Haar unitaries by

[1] This sense of Haar unitary is not to be confused with the Haar random unitary matrix $U^{(n)}$ which is a random matrix chosen according to the Haar measure on the $n \times n$ unitary group. For this reason, we will refer to the latter always as a Haar *random* unitary.

$\mathcal{H}(\mathcal{M})$. Following [33], for $u, v \in \mathcal{H}(\mathcal{M})$, we say $u \sim_k v$ if there exist Haar unitaries in $N^{\mathcal{U}}$ $u = u_0, u_1, \dots, u_k = v$ such that $[u_{j-1}, u_j] = 0$ for $j = 1, \dots, k$. We also write $u \sim v$ if $u \sim_k v$ for some k.

The *sequential commutation orbit* of u is its equivalence class under the relation $\sim$. As a consequence of [33, Fact 2.9], the sequential commutation orbit of a Haar unitary u is always contained inside the Pinsker algebra of u. (Actually, in the special case of $L(F_\infty)$, the Pinsker algebra of some $\mathcal{A}$ with $h(\mathcal{A} : L(F_\infty)) = 0$ is equal to the algebra generated by its sequential commutation orbit [33, Theorem 5.4] as a consequence of the recent resolution of the Peterson-Thom conjecture, which occurred through a combination of 1-bounded entropy techniques [24] and strong convergence of tensor product random matrix models [3, 7].)

Corollary E (Freeness of sequential commutation orbits). *Let $u_j(\omega) \in \mathcal{Q} = \prod_{n\to\mathcal{U}} M_n(\mathbb{C})$ be as in the Theorem B. Almost surely, the von Neumann algebras generated by the sequential commutation orbits of $u_1(\omega)$, $u_2(\omega)$, ... respectively are freely independent.*

Although Theorem C focuses on the matrix ultraproduct, we are able to transfer these results to $L(F_\infty)$ and more generally for arbitrary free products of Connes embeddable von Neumann algebras through a natural argument using embeddings and countable saturation (a concept from model theory).

Theorem F (Freeness phenomena in ultrapowers of free products). *Let $(\mathcal{M}_i)_{i\in I}$, ... be diffuse Connes-embeddable tracial von Neumann algebras, and let $\mathcal{M} = *_{i\in I}\mathcal{M}_i$. Let $\mathcal{V}$ be a free ultrafilter on some index set J. Then*

1. *If $\mathcal{A}_i \subseteq \mathcal{M}^{\mathcal{V}}$ with $h(\mathcal{A}_i : \mathcal{M}^{\mathcal{V}}) = 0$ and $\mathcal{A}_i \cap \mathcal{M}_i$ diffuse for each i, then the algebras $(\mathcal{M}_i \vee \mathcal{A}_i)_{i\in I}$ are freely independent.*
2. *Let $\mathcal{C}_i \subseteq \mathcal{M}^{\mathcal{V}}$ be the algebra generated by the sequential commutation orbits of Haar unitaries in $\mathcal{M}_i$. Then $(\mathcal{C}_i)_{i\in I}$ are freely independent. In particular, if $u_i \in \mathcal{M}_i$ are diffuse unitaries, then $\{u_i' \cap \mathcal{M}^{\mathcal{V}}\}_{i\in I}$ are freely independent.*
3. *Let $\mathcal{N}_i \subseteq \mathcal{M}^{\mathcal{V}}$ be the wq-normalizer of $\mathcal{M}_i$. Then $(\mathcal{N}_i)_{i\in I}$ are freely independent.*

In particular, this gives a new proof and generalization of the freeness result for commutants of Houdayer-Ioana [28, Theorem B] in the case of Connes embeddable free products. However, we cannot handle though the case of amalgamated free products through this method. Indeed, it is not even known if free products with amalgamation over a non-amenable algebra preserve Connes embeddability, and if the amalgam is amenable, we expect similar issues to arise as in [27, Remark 5.13].

Theorem F (1) is also closely related to [27, Theorem A], which says that if $\mathcal{A}_i$ is contained in $\mathcal{M}$ with $\mathcal{A}_i \cap \mathcal{M}_i$ diffuse and $h(\mathcal{A}_i : \mathcal{M}) = 0$, then $\mathcal{A}_i$ must be contained in $\mathcal{M}_i$, which in particular means that the $\mathcal{A}_i$'s are freely independent. Now in Theorem F (1), $\mathcal{A}_i$ is not assumed to be in $\mathcal{M}$, only in $\mathcal{M}^{\mathcal{U}}$, but we still obtain free independence of $\mathcal{M}_i \vee \mathcal{A}_i$. We also remark that in this setting one cannot conclude that $\mathcal{A}_i \subseteq \mathcal{M}_i^{\mathcal{U}} \subseteq \mathcal{M}^{\mathcal{U}}$. For instance, suppose that $\mathcal{M}_i^{\mathcal{U}}$ is commutative; then let $\mathcal{A}_i$ be a copy of the hyperfinite II_1 factor that intersects $\mathcal{M}_i$ diffusely, which must exist since all Haar unitaries in $\mathcal{M}^{\mathcal{U}}$ are conjugate. Then $\mathcal{A}_i$ cannot be contained in the commutative algebra $\mathcal{M}_i^{\mathcal{U}}$.

1.2. **Broader context and motivation.** The broader motivation for our work includes a long history of results about the von Neumann algebra $L(F_\infty)$, going back to Murray and von Neumann's first papers in the subject [38, §6]. The challenge of understanding its structure has been addressed by with an astonishing diversity of tools, including not only the probabilistic methods that our paper draws on [18, 23, 24, 32, 55], but also deformation rigidity theory [40, 46, 48, 50]; amenable actions, C^* and boundary theory [11, 12, 39]; closable derivations [41, 42]; free harmonic analysis and non-commutative L_p space theory [37], which was a key ingredient in Ioana and Houdayer's result [28]. We also point out that Popa showed the existence of elements in an ultraproduct freely independent from certain subalgebras through deformation/ridigity techniques in [47, 49].

We were particularly motivated by the question of elementary equivalence of von Neumann algebras, and in particular of $L(F_\infty)$ and matrix ultraproducts. The introduction of ultraproducts in functional analysis naturally inspired the classification question of when $\mathcal{M}^{\mathcal{U}}$ and $\mathcal{N}^{\mathcal{V}}$ are isomorphic for various ultrafilters $\mathcal{U}$ and $\mathcal{V}$. Using mathematical logic, isomorphism of $\mathcal{M}^{\mathcal{U}}$ and $\mathcal{M}^{\mathcal{V}}$ for different ultrapowers depends on the continuum hypothesis [13, Proposition 3.3]. But model theory also provides a powerful tool, known as the Keisler–Shelah theorem, which says that $\mathcal{M}$ and $\mathcal{N}$ admit some isomorphic ultrapowers if and only if they have the same first-order theory (see [15, §2.2]), which here we understand in the sense of model theory for metric structures [5]. In this case, $\mathcal{M}$ and $\mathcal{N}$ are said to be *elementarily equivalent*.

The classification of tracial von Neumann algebras up to elementary equivalence is a challenging problem [20, §4] and in particular very little is known in the setting of II_1 factors without property Gamma. For instance, we do not know if the matrix ultraproducts with different ultrafilters are elementarily equivalent [15, §5] [30, §5.2], we do not know if $L(F_m)$ and $L(F_n)$ are elementarily equivalent for $m \neq n$ [21], nor do we know if $L(F_\infty)$ is elementarily equivalent to a matrix ultraproduct [20, Question 4.6]. One could try to distinguish the theories of these algebras with certain first-order sentences relating to familiar properties such as commutation, free independence, and the like. For instance, the construction in [9] and with minor modifications in [28, Theorems F and G] and [33, §2.4] of non-Gamma II_1 factors that are not elementarily equivalent is motivated by statements such as "for all unitaries u_1 and u_2 with $u_1^2 = u_2^3 = 1$, there (approximately) exist Haar unitaries v_1 and v_2 with $[u_1, v_1] = [v_1, v_2] = [v_2, u_2] = 0$," see [9, Remark 5.9]; these results fit into the formalism of sequential commutation introduced in [33]. Houdayer and Ioana's theorem on freeness of commutants in [28] implies that "there exist unitaries u_1, u_2, ... such that for all b_1, b_2, ... with $[u_j, b_j] = 0$ (approximately), we have that b_1, b_2 ... are (approximately) freely independent" holds in $L(F_\infty)$, while our first result Theorem B shows that a similar statement holds in matrix ultraproducts; see Remark 4.5 for a more precise statement.

Moreover, Theorem C yields further first-order statements that hold in matrix ultraproducts; see Lemma 4.2. Since u_1, u_2, ... arise from the Haar random unitaries, our results also give some information about the values of first-order formulas on independent random unitaries. While Voiculescu's asymptotic freeness theory [53, 56] describes the $*$-moments of random matrices, precious little information is known about formulas that involve quantifiers; see also [30, §5.2], [16, §4].

1.3. **Notation and Organization.** Here *tracial von Neumann algebra* refers to a von Neumann algebra $\mathcal{M}$ with a fixed faithful normal tracial state $\mathrm{tr}^{\mathcal{M}}$. We write $\|x\|_2 = \mathrm{tr}^{\mathcal{M}}(x^*x)^{1/2}$. In particular, when $\mathcal{M} = M_n(\mathbb{C})$, we denote the normalized trace by tr_n and write $\|x\|_2 = \mathrm{tr}_n(x^*x)^{1/2}$. We assume that familiarity with basic theory of tracial von Neumann algebras (see for instance [1, 6, 51, 58]) as well as ultraproducts of tracial von Neumann algebras (see for instance [8, Appendix A], [1, §5.4]).

Concerning the organization of the paper, we first give a self-contained proof of Theorems A and B through concentration and volumetric analysis in §2. Then we prepare ingredients about model theory and 1-bounded entropy in §3, which are subsequently used in the proof of Theorems C and F in §4.

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2. Freeness of commutants

In this section, after recalling some elementary facts about Haar random unitaries and diagonalization, we give our concentration of measure argument for uniform asymptotic freeness, and finally prove Theorems A and B.

2.1. **Useful facts about diagonalization and approximate commutants.**

In order to analyze the commutant of the u_j more easily, it will be convenient to diagonalize the Haar random unitary $U_j^{(n)}$. In fact, up to a small error, we will be able to arrange that $U_j^{(n)}$ has the form $V_j^{(n)} A^{(n)} (V_j^{(n)})^*$ for deterministic $A^{(n)}$ with evenly spaced eigenvalues. Thus, to prove the main results, we can perform our analysis on the model given by conjugates of a diagonal matrix, and then transfer them over to $U_j^{(n)}$'s.

The following are useful elementary observations that can be considered folklore in random matrix theory, but we include the proofs here for completeness. Some of the lemmas in this section will actually be used several times.

Lemma 2.1. *For $k \in \mathbb{N}$ and $j \leq k$, let $I_{k,j}$ be the interval $[2\pi(j-1)/k, 2\pi j/k)$ viewed as a subset of the unit circle S^1. Let*

$$\mathcal{O}_k = \{\mu \in \mathcal{P}(S^1) : 1/k - 1/k^2 < \mu(I_{k,j}^\circ) \leq \mu(\overline{I}_{k,j}) < 1/k + 1/k^2\}$$

Then $\mathcal{O}_k$ is an open neighborhood in $\mathcal{P}(S^1)$ of the Haar measure.

Let $A^{(n)} = \operatorname{diag}(1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1})$. Let B be a diagonal matrix with eigenvalues $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$ where $0 \leq \lambda_1 \leq \cdots \leq \lambda_n < 2\pi$. If the empirical spectral distribution of B is in $\mathcal{O}_k$, then $\|A^{(n)} - B\| < 4\pi/k$.

Proof. To see $\mathcal{O}_k$ that open, first note that $\mu(I_{k,j}^\circ) > 1/k - 1/k^2$ is an open condition in $\mathcal{P}(S^1)$. This follows because it is the disjunction of the conditions $\int f \, d\mu > 1/k - 1/k^2$ for $f \in C(S^1)$ with $0 \leq f \leq \mathbf{1}_{I_{j,k}^\circ}$, since the indicator function of the open set is a supremum of a sequence of continuous functions. Similarly, by taking complements $\mu(\overline{I}_{k,j}) < 1/k + 1/k^2$ is an open condition.

For the second claim, let μ be the empirical spectral distribution of B, and suppose $\mu \in \mathcal{O}_k$. Then

$$(j-1)/k \leq j/k - j/k^2 \leq \mu(I_1 \cup \cdots \cup I_j) \leq j/k + j/k^2 \leq (j+1)/k.$$

Therefore, if $t \in [n]$ with $t/n > (j+1)/k$, then $\lambda_t^{(n)} \geq 2\pi j/k$, and if $t \in [n]$ with $t/n < (j-1)/k$, then $\lambda_t^{(n)} \leq 2\pi j/k$. Overall, this implies that

$$|\lambda_t^{(n)} - 2\pi t/n| \leq \frac{2}{k}, \text{ hence } |\zeta_t^{(n)} - e^{2\pi i \lambda_t^{(n)}}| \leq \frac{2(2\pi)}{k}$$

since the complex exponential function is 1-Lipschitz. Therefore, $\|B_j^{(n)} - A^{(n)}\| \leq 4\pi/k$ as desired. $\square$

Proposition 2.2. *Let $A^{(n)} = \operatorname{diag}(1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1})$, where $\zeta_n = e^{2\pi i/n}$. Then there exists a family of independent Haar random unitaries $U_1^{(n)}$, ..., $U_m^{(n)}$ and another family of independent Haar random unitaries $V_1^{(n)}$, ..., $V_m^{(n)}$ on the same probability space $(\Omega, \mathcal{F}, P)$ such that*

$$\lim_{n\to\infty} \sum_{j=1}^m \|U_j^{(n)} - V_j^{(n)} A^{(n)} (V_j^{(n)})^*\| = 0 \text{ almost surely.}$$

Remark 2.3. The conclusion in the proposition shows that the operator norm goes to zero. For this paper, we only need the weaker statement that the 2-norm goes to zero.

Proof of Proposition 2.2. First, we remark the following: If X is an $n \times n$ Haar random unitary and Y is an $n \times n$ random unitary independent of X, then XY and YXY^* are Haar random unitaries. In the case where Y is deterministic, YX is a Haar unitary since the Haar measure is left-invariant, and then YXY^* is a Haar unitary also since the Haar measure is right-invariant. Now consider Y that is random and independent of X. Since (X, Y) are independent, the joint distribution (X, Y) has a disintegration given by conditioning on the value of Y. Moreover, the conditional distributions of XY and YXY^* given that Y is some fixed value y are the Haar measure on the unitary group, which follows from the case of deterministic Y handled above. Now the distribution of YX and YXY^* respectively are obtained by integrating the conditional distributions given $Y = y$ with respect to the marginal distribution of Y. Hence, they are also equal to the Haar measure on the unitary group.

Now let $X_1^{(n)}$, ..., $X_m^{(n)}$, $Y_1^{(n)}$, ..., $Y_m^{(n)}$ be independent Haar random unitary matrices. Let $U_j^{(n)} = Y_j^{(n)} X_j^{(n)} (Y_j^{(n)})^*$. By the foregoing argument, $U_1^{(n)}, \dots, U_m^{(n)}$ are independent Haar random unitaries. By the spectral theorem, we may write $X_j^{(n)} = W_j^{(n)} B^{(n)} (W_j^{(n)})^*$, where $B^{(n)}$ is diagonal and $W_j^{(n)}$ is unitary. For ζ on the unit circle, let $\arg(\zeta)$ be the value of the argument that is in $[0, 2\pi)$. If $X_j^{(n)}$ has distinct eigenvalues, then there is a unique choice of $B_j^{(n)}$ where the arguments of the diagonal entries are in increasing order. Moreover, in the case when $X_j^{(n)}$ has distinct eigenvalues (which is almost surely), the choice of $W_j^{(n)}$ such that $X_j^{(n)} = W_j^{(n)} B^{(n)} (W_j^{(n)})^*$ is unique as well. It is also straightforward to check that it depends on $X_j^{(n)}$ in a Borel-measurable manner. Since the $X_j^{(n)}$'s are independent of the $Y_j^{(n)}$'s, we also have that $W_j^{(n)}$'s are independent of the $Y_j^{(n)}$'s, and therefore $V_j^{(n)} = Y_j^{(n)} W_j^{(n)}$ is a Haar random unitary, and of course $V_1^{(n)}, \dots, V_m^{(n)}$ are independent since $V_j^{(n)}$ only depends on $X_j^{(n)}$ and $Y_j^{(n)}$ for each j. Overall, we have

$$U_j^{(n)} = V_j^{(n)} B_j^{(n)} (V_j^{(n)})^*,$$

where $U_1^{(n)}, \dots, U_m^{(n)}$ are independent Haar random unitaries, $V_1^{(n)}, \dots, V_m^{(n)}$ are independent Haar random unitaries (though not independent of $U_j^{(n)}$), and $B_j^{(n)}$ is diagonal with eigenvalues listed in order of increasing argument in $[0, 2\pi)$.

Note that

$$\|U_j^{(n)} - V_j^{(n)} A_j^{(n)} (V_j^{(n)})^*\|_2 = \|V_j^{(n)} B_j^{(n)} (V_j^{(n)})^* - V_j^{(n)} A_j^{(n)} (V_j^{(n)})^*\|_2 = \|B_j^{(n)} - A_j^{(n)}\|_2.$$

Therefore, to complete the proof, it suffices to show that $\lim_{n\to\infty} \|B_j^{(n)} - A_j^{(n)}\|_2 = 0$ almost surely.

Let $\mu_j^{(n)}$ be the empirical spectral distribution of $U^{(n)}$, that is, the (random) probability measure on the circle that has a point mass of $1/n$ at each eigenvalue of $U_j^{(n)}$. Note that empirical spectral distributions of $U_j^{(n)}$ and $B_j^{(n)}$ are the same. By standard results about random unitary matrices (see for instance [35, Theorem 4.13]), we have that almost surely $\mu_j^{(n)}$ converges weak-$*$ to the Haar measure on the unit circle, which we denote here by μ. Hence, almost surely, $\mu_j^{(n)}$ is eventually in the neighborhood $\mathcal{O}_k$ in Lemma 2.1. It follows that $\|B_j^{(n)} - A^{(n)}\|$ is eventually less than $4\pi/k$, and since k is arbitrary, this completes the proof. □

This result on diagonalization enables us to reduce the study of approximate commutants of $U_j^{(n)}$ to the case of $A^{(n)}$. In this setting, the approximate commutant is described by ϵ-diagonal or band matrices.

Notation 2.4. For $i, j \in [n]$, let $d_n(i,j)$ denote the distance of i and j modulo n. Let

$$\mathcal{D}_\epsilon^{(n)} := \{B \in M_n(\mathbb{C}) : B_{i,j} = 0 \text{ when } d_n(i,j) > \epsilon n\}.$$

Lemma 2.5. *Let $A^{(n)} = \operatorname{diag}(1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1})$ where $\zeta_n = e^{2\pi i/n}$. Let $B \in M_n(\mathbb{C})$. Then for every $\epsilon > 0$, there exists $B_\epsilon \in \mathcal{D}_\epsilon^{(n)}$ with*

$$\|B - B_\epsilon\|_2 \le \frac{8\sqrt{\pi}}{\epsilon}\|[A^{(n)}, B]\|_2, \qquad \|B_\epsilon\| \le 3\|B\|.$$

Proof. Fix ϵ and B. In the case where $\epsilon < 2/n$, we take B_ϵ to be the projection of B onto diagonal matrices. Note that

$$\|B - B_\epsilon\|_2^2 = \frac{1}{n}\sum_{j \neq k} |B_{j,k}|^2,$$

while

$$\|[A^{(n)}, B]\|_2^2 = \frac{1}{n}\sum_{j \neq k} |\zeta_n^j - \zeta_n^k|^2 |B_{j,k}|^2. \tag{2.1}$$

Note

$$|\zeta_n^j - \zeta_n^k|^2 = |1 - \zeta_n^{j-k}|^2 = 2 - 2\cos(2\pi d_n(j,k)/n) \ge 2 - 2\cos(2\pi/n)$$

By concavity of the sine function on $[0,\pi]$, we have $|x|/\pi \le |\sin x|$ on $[-\pi,\pi]$, and thus by integrating, we get $1 - \cos x \ge x^2/2\pi$. Hence,

$$|\zeta_n^j - \zeta_n^k|^2 \ge \frac{4\pi}{n^2} \ge \pi\epsilon^2.$$

Overall,

$$\|B - B_\epsilon\|_2 \le \frac{1}{\sqrt{\pi}\epsilon}\|[A^{(n)}, B]\|_2$$

Now suppose that $\epsilon \ge 2/n$. Let $m = \lfloor n\epsilon/2 \rfloor \ge 2$. Write $n = qm + r$ for some $r \in \{0, \ldots, m\}$. For $j = 1, \ldots, q$, let P_j be the projection onto the basis vectors $e_{(j-1)m+1}, \ldots, e_{jm}$; and let P_{q+1} be the projection onto the last r basis vectors. Let

$$B_\epsilon = \sum_{d_{q+1}(j,k) \le 1} P_j B P_k.$$

Note that all the indices (j', k') where $(B_\epsilon)_{j',k'} \neq 0$ occur when $d_n(j',k') \le 2m \le n\epsilon$, and thus $B_\epsilon \in \mathcal{D}_\epsilon^{(n)}$. Moreover, by writing

$$B_\epsilon = \sum_{j=1}^{q+1} P_j B P_j + \sum_{j=1}^{q+1} P_j B P_{j+1} + \sum_{j=1}^{q+1} P_j B P_{j-1}$$

(indices considered modulo $q+1$), we see that $\|B_\epsilon\| \leq 3\|B\|$. Next, note that the entries of B_ϵ that have not been zeroed out include all the entries (j',k') with

$$d_n(j',k') \leq m \geq n\epsilon/2 - 1 \geq n\epsilon/4.$$

Therefore,

$$\|B - B_\epsilon\|_2^2 = \frac{1}{n} \sum_{d_n(j,k) > n\epsilon/4} |B_{j,k}|^2 \leq \frac{1}{2 - 2\cos(8\pi/\epsilon)} \|[A^{(n)}, B]\|_2^2 \leq \frac{64\pi}{\epsilon^2} \|[A^{(n)}, B]\|_2^2.$$

□

Corollary 2.6. *Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and write $\mathcal{Q} = \prod_{n\to\mathcal{U}} M_n(\mathbb{C})$. Let $A^{(n)} = \operatorname{diag}(1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1})$, and let $a = [A^{(n)}]_{n\in\mathbb{N}} \in \mathcal{Q}$. Let $V_j^{(n)}$ for $j \in \mathbb{N}$ be independent Haar random unitaries. Fix an outcome ω and let $v_j(\omega) = [U_j^{(n)}(\omega)] \in \mathcal{Q}$. Let*

$$\mathcal{D}_\epsilon = \{[X^{(n)}]_{n\in\mathbb{N}} \in \mathcal{Q} : X^{(n)} \in \mathcal{D}_\epsilon^{(n)} \text{ for } n \in \mathbb{N}\}.$$

Then

$$\{v_j(\omega) a v_j(\omega)^*\}' \cap \mathcal{Q} = \bigcap_{\epsilon>0} v_j(\omega)\mathcal{D}_\epsilon v_j(\omega)^*$$

Proof. Since commutants respect conjugation, it suffices to show that $\{a\}' \cap \mathcal{Q} = \bigcap_{\epsilon>0} \mathcal{D}_\epsilon$. Let $b = [B^{(n)}]_{n\in\mathbb{N}}$ be an element commuting with a, and fix $\epsilon > 0$. Let $B_\epsilon^{(n)}$ be as in Lemma 2.5. Since $\|[A^{(n)}, B^{(n)}]\|_2 \to 0$ along the ultrafilter, we obtain that $\|B^{(n)} - B_\epsilon^{(n)}\|_2 \to 0$ as well. Hence, $b = [B_\epsilon^{(n)}]_{n\in\mathbb{N}}$ is in $\mathcal{D}_\epsilon$, as desired. □

2.2. **Uniform asymptotic freeness via concentration of measure.** For Theorems B and A, we will proceed as follows. By Proposition 2.2, we can replace $U_j^{(n)}$ with $V_j^{(n)} A^{(n)} (V_j^{(n)})^*$. In this section, we will argue that $V_j^{(n)} \mathcal{D}_\epsilon^{(n)} (V_j^{(n)})^*$ are asymptotically free as $n \to \infty$ up to some error tolerance depending on ϵ. Then since matrices that approximately commute with $V_j^{(n)} A^{(n)} (V_j^{(n)})^*$ will, for any ϵ, be approximately in $V_j^{(n)} \mathcal{D}_\epsilon^{(n)} (V_j^{(n)})^*$, we will obtain the desired results.

Hence, our present goal is to obtain a *uniform* approximate asymptotic freeness result for $V_j^{(n)} \mathcal{D}_\epsilon^{(n)} (V_j^{(n)})^*$. This is based on Voiculescu's famous asymptotic freeness theorem.

Theorem 2.7 (Voiculescu's asymptotic freeness [56]). *Let $U_1^{(n)}$, …, $U_m^{(n)}$ be $n \times n$ Haar random unitaries. Let $i_1 \neq i_2 \neq \dots i_k$. Then*

$$\lim_{n\to\infty} \sup_{X_1,\dots,X_k \in B_1^{M_n(\mathbb{C})}} |\mathbb{E} \operatorname{tr}_n \left[U_{i_1}^{(n)} (X_1 - \operatorname{tr}_n(X_1)) (U_{i_1}^{(n)})^* \dots U_{i_k}^{(n)} (X_k - \operatorname{tr}_n(X_k)) (U_{i_k}^{(n)})^* \right] | = 0.$$

Proof. Let $B_r^{M_n(\mathbb{C})}$ denote the r-ball with respect to operator norm. For each n, fix $X_1^{(n)}$, …, $X_k^{(n)}$ in $B_1^{M_n(\mathbb{C})}$ which maximize

$$|\mathbb{E} \operatorname{tr}_n \left[U_{i_1}^{(n)} (X_1^{(n)} - \operatorname{tr}_n(X_1^{(n)})) (U_{i_1}^{(n)})^* \dots U_{i_k}^{(n)} (X_k^{(n)} - \operatorname{tr}_n(X_k^{(n)})) (U_{i_k}^{(n)})^* \right] |. \tag{2.2}$$

A maximizer exists because the quantity depends continuously on the X_j's. It follows from [56, Corollary 2.5] that (2.2) converges to 0 as $n \to \infty$. □

In order to apply this result uniformly to all the matrices $X_j \in \mathcal{D}_\epsilon^{(n)}$, we rely on the high-dimensional concentration of measure phenomenon for Haar unitaries, which has been a staple of random matrix theory since [4]. We recall that the unitary group $\mathbb{U}_n$ (equipped with the Riemannian metric associated to the inner product $\langle \cdot, \cdot \rangle_{\mathrm{Tr}_n}$) satisfies the log-Sobolev inequality with constant $6/n$ [36, Theorem 15]. One can easily deduce the log-Sobolev inequality for the product of several copies of $\mathbb{U}_n$; see e.g. [34, Corollary 5.7], [35, Theorem 5.9]. This in turn implies that it satisfies the Herbst concentration estimate; see e.g. [2, Lemma 2.3.3], [35, Theorem 5.5]. After renormalizing the metric to $\langle \cdot, \cdot \rangle_{\mathrm{tr}_n}$, one obtains the following concentration bound. See [27, §5.3] for further explanation.

Lemma 2.8 (Concentration)**.** *Let $U_1^{(n)}$, ..., $U_m^{(n)}$ be $n \times n$ Haar random unitaries, and let $f : \mathbb{U}_n^m \to \mathbb{C}$ be Lipschitz with respect to $\|\cdot\|_2$. Then*

$$P(|f(U_1^{(n)}, \dots, U_m^{(n)}) - \mathbb{E}[f(U_1^{(n)}, \dots, U_m^{(n)})]| \geq \delta) \leq 4e^{-n^2\delta^2/12\|f\|_{\mathrm{Lip}}^2}$$

Concentration allows us to deduce the following uniform asymptotic freeness result, which is our main technical tool. Here we replace $\mathcal{D}_\epsilon^{(n)}$ with a more general set $S_j^{(n)}$ that has relatively small covering numbers; this added generality will be used in the proof of Lemma 4.1.

Notation 2.9. Let S be a subset of a metric space X. Then $K_\epsilon(S)$ is defined as the smallest cardinality of a set $\Omega \subseteq X$ such that the ϵ-neighborhood of Ω covers S.

Lemma 2.10 (Uniform asymptotic freeness)**.** *For each n, let $S_j^{(n)} \subseteq B_1^{M_n(\mathbb{C})}$ and suppose that*

$$\lim_{n \to \mathcal{U}} \frac{1}{n^2} \log K_\epsilon(S_j^{(n)}) < \delta. \tag{2.3}$$

Let $i_1 \neq i_2 \neq \dots i_k$. Then almost surely

$$\lim_{n \to \mathcal{U}} \sup_{X_1 \in S_1^{(n)}} \dots \sup_{X_k \in S_k^{(n)}} \left| \mathrm{tr}_n \left[U_{i_1}^{(n)} (X_1 - \mathrm{tr}_n(X_1)) (U_{i_1}^{(n)})^* \dots U_{i_k}^{(n)} (X_k - \mathrm{tr}_n(X_k)) (U_{i_k}^{(n)})^* \right] \right| \leq 4k\epsilon + 2k\sqrt{12k\delta}. \tag{2.4}$$

The same statement also holds when $\lim_{n\to\mathcal{U}}$ is replaced by $\limsup_{n\to\infty}$ in both the hypothesis and the conclusion.

Proof. First, fix a set $\Omega_j^{(n)}$ such that the ϵ-neighborhood of $\Omega_j^{(n)}$ covers $S_j^{(n)}$, and such that $|\Omega_j^{(n)}| = K_\epsilon^{(n)}(S_j^{(n)})$. Although $\Omega_j^{(n)}$ is initially not assumed to be a subset of $S_j^{(n)}$, we can replace each element of $\Omega_j^{(n)}$ with an element from $S_j^{(n)}$ in its ϵ-ball. Hence, WLOG assume that $\Omega_j^{(n)} \subseteq S_j^{(n)}$ such that $S_j^{(n)}$ is in the 2ϵ-neighborhood of $\Omega_j^{(n)}$.

Let $m = \max(i_1, \dots, i_k)$. Let

$$f(X_1, \dots, X_k, U_1^{(n)}, \dots, U_m^{(n)}) = \mathrm{tr}_n \left[U_{i_1}^{(n)} (X_1 - \mathrm{tr}_n(X_1)) (U_{i_1}^{(n)})^* \dots U_{i_k}^{(n)} (X_k - \mathrm{tr}_n(X_k)) (U_{i_k}^{(n)})^* \right]$$

Since $\|X_j - \mathrm{tr}_n(X_j)\| \leq \|X_j\| \leq 1$, we see that f is a $2k$-Lipschitz function of $(U_1^{(n)}, \dots, U_m^{(n)})$ with respect to $\|\cdot\|_2$. In particular,

$$\sup_{X_1 \in S_1^{(n)}} \dots \sup_{X_k \in S_k^{(n)}} |f(X_1, \dots, X_k, U_1^{(n)}, \dots, U_m^{(n)})| \leq 4k\epsilon + \sup_{X_1 \in \Omega_1^{(n)}} \dots \sup_{X_k \in \Omega_k^{(n)}} |f(X_1, \dots, X_k, U_1^{(n)}, \dots, U_m^{(n)})|. \tag{2.5}$$

Now by Lemma 2.8, we have for each $(X_1, \ldots, X_k)$ that

$$P(|f(X_1, \ldots, X_k, U_1^{(n)}, \ldots, U_m^{(n)}) - \mathbb{E}[f(X_1, \ldots, X_k, U_1^{(n)}, \ldots, U_m^{(n)})]| \geq 2k\sqrt{12k\delta}) \\ \leq 4e^{-n^2 12k\delta(2k)^2/12(2k)^2} = 4e^{n^2k\delta} \tag{2.6}$$

Hence, by a union bound,

$$P\left(\sup_{X_1 \in \Omega_1^{(n)}} \cdots \sup_{X_k \in \Omega_k^{(n)}} |f(X_1, \ldots, X_k, U_1^{(n)}, \ldots, U_m^{(n)}) - \mathbb{E}[f(X_1, \ldots, X_k, U_1^{(n)}, \ldots, U_m^{(n)})]| \geq \eta \right) \\ \leq 4e^{-n^2k\delta} \prod_{j=1}^k K_\epsilon(S_j^{(n)}).$$

By (2.3), there exists $\delta' < \delta$ and some $A \in \mathcal{U}$ such that for all $n \in A$, for $j = 1, \ldots, k$, we have $K_\epsilon(S_j^{(n)}) \leq e^{n^2\delta'}$. In particular, the previous equation is bounded by

$$4e^{-n^2k\delta}e^{n^2k\delta'} = e^{-n^2k(\delta-\delta')}.$$

Since $\delta' < \delta$, this quantity is summable over $n \in \mathbb{N}$, and in particular, it is summable over $n \in A$. Hence, by the Borel-Cantelli lemma, almost surely

$$\lim_{\substack{n\to\infty \\ n\in A}} \sup_{X_1 \in \Omega_1^{(n)}} \cdots \sup_{X_k \in \Omega_k^{(n)}} |f(X_1, \ldots, X_k, U_1^{(n)}, \ldots, U_m^{(n)}) - \mathbb{E}[f(X_1, \ldots, X_k, U_1^{(n)}, \ldots, U_m^{(n)})]| \\ \leq 2k\sqrt{12k\delta}. \tag{2.7}$$

In particular, this holds for the limit as $n \to \mathcal{U}$. Moreover, by Theorem 2.7, we have that

$$\lim_{n\to\infty} \sup_{X_1 \in \Omega_1^{(n)}} \cdots \sup_{X_k \in \Omega_k^{(n)}} |\mathbb{E}[f(X_1, \ldots, X_k, U_1^{(n)}, \ldots, U_m^{(n)})]| = 0. \tag{2.8}$$

Combining (2.5), (2.7), and (2.8) together with the triangle inequality implies (2.4). The argument with limits as $n \to \infty$ is the same (except that now it is unnecessary to consider choosing a set $A \in \mathcal{U}$). $\square$

Finally, we record the following standard estimate on the covering number of $\mathcal{D}_\epsilon^{(n)}$.

Lemma 2.11. *For $0 < \epsilon < R$, we have*

$$\frac{1}{n^2} \log K_\epsilon(\{X \in \mathcal{D}_\epsilon^{(n)} : \|X\|_2 \leq R\}) \leq 2\epsilon \log \frac{3R}{\epsilon}.$$

Proof. Note that $\dim_{\mathbb{R}} \mathcal{D}_\epsilon^{(n)} \leq 2\epsilon n^2$. Let Ω be a maximal set of ϵ-separated points in $S = \{X \in \mathcal{D}_\epsilon^{(n)} : \|X\|_2 \leq R\}$. Then the ϵ-neighborhood of Ω covers S by maximality. The $\epsilon/2$ balls with centers in Ω are disjoint, and they are contained in the $R + \epsilon/2 \leq 3R/2$-ball centered at zero. Therefore, $|\Omega|$ is at most

$$\left(\frac{3R/2}{\epsilon/2}\right)^{2\epsilon n^2} = \left(\frac{3R}{\epsilon}\right)^{2\epsilon n^2}. \qquad \square$$

2.3. **Proof of freeness of commutants.** All the pieces are now in place to prove Theorems A and B. We start with Theorem B first since it involves fewer approximation arguments.

Proof of Theorem B. As in Proposition 2.2, let $U_j^{(n)}$ be independent Haar random unitaries. Let $A^{(n)} = \operatorname{diag}(1, \zeta_n, \dots, \zeta_n^{n-1}$, and let $V_j^{(n)}$ be independent Haar random unitaries, such that $\|U_j^{(n)} - V_j^{(n)} A^{(n)} (V_j^{(n)})^* \|_2 \to 0$ almost surely as $n \to \infty$. Let $u_j(\omega) = [U_j^{(n)}(\omega)]_{n \in \mathbb{N}} \in \mathcal{Q}$, and let $v_j(\omega) = [V_j^{(n)}(\omega)]_{n \in \mathbb{N}} \in \mathcal{Q}$, and let $a = [A^{(n)}]_{n \in \mathbb{N}} \in \mathcal{Q}$. Thus, almost surely $u_j(\omega) = v_j(\omega) a v_j(\omega)^*$, and so

$$\{u_j(\omega)\}' \cap \mathcal{Q} = \{v_j(\omega) a v_j(\omega)^*\}' \cap \mathcal{Q} = \bigcap_{\epsilon > 0} v_j(\omega) \mathcal{D}_\epsilon v_j(\omega)^*,$$

where the second equality follows from Corollary 2.1 with $\mathcal{D}_\epsilon = \{[X^{(n)}]_{n \in \mathbb{N}} \in \mathcal{Q} : X^{(n)} \in \mathcal{D}_\epsilon^{(n)}\}$.

For each word $i_1 \neq \dots \neq i_k$, and for each $m \in \mathbb{N}$, taking $\epsilon = 1/m$ in Lemma 2.11 and $\delta = 2\epsilon \log(3R/\epsilon)$ in Lemma 2.10, we have that almost surely

$$\limsup_{n \to \infty} \sup_{X_1 \in \mathcal{D}_{1/m}^{(n)}} \dots \sup_{X_k \in \mathcal{D}_{1/m}^{(n)}} \left| \operatorname{tr}_n \left[V_{i_1}^{(n)} (X_1 - \operatorname{tr}_n(X_1)) (V_{i_1}^{(n)})^* \dots V_{i_k}^{(n)} (X_k - \operatorname{tr}_n(X_k)) (V_{i_k}^{(n)})^* \right] \right| \tag{2.9}$$

$$\leq 4k/m + 2k \sqrt{24k \log(3Rm)/m}. \tag{2.10}$$

There are only countably many words and values of m, so almost surely this holds for all i_1, …, i_k and m, and also $\|U_j^{(n)} - V_j^{(n)} A^{(n)} (V_j^{(n)})^* \|_2 \to 0$. In the rest of the proof, we fix an outcome ω in this almost sure event.

To show free independence of the commutants, fix a word $i_1 \neq i_2 \neq \dots \neq i_k$, and fix $x_j \in \{u_j(\omega)\}$ with $\operatorname{tr}^{\mathcal{Q}}(x_j) = 0$. Then for each m, we can write $x_j = [X_j^{(n)}]_{n \in \mathbb{N}}$ for some sequence $X_j^{(n)} \in \mathcal{D}_{1/m}^{(n)}$ by Corollary 2.6. Then (2.9) implies that

$$\left| \operatorname{tr}^{\mathcal{Q}} [v_{i_1}(\omega) x_1 v_{i_1}(\omega)^* \dots v_{i_k}(\omega) x_k v_{i_k}(\omega))^*] \right| \leq 4k/m + 2k \sqrt{24k \log(3Rm)/m}.$$

Since m was arbitrary, the trace is zero. Hence, we obtain free independence as desired. □

For the case of Theorem A, we present here a slightly more general version with each $B_j^{(n)}$ replaced by a tuple.

Theorem 2.12 (Asymptotic freeness of approximate commutants)**.** *Let $U_1^{(n)}$, $U_2^{(n)}$, … be independent $n \times n$ Haar random unitary matrices. For each j, let $\mathbf{B}_j^{(n)} = (B_{j,\ell}^{(n)})_{\ell \in \mathbb{N}}$ be random matrices such that almost surely we have*

$$\limsup_{n \to \infty} \|B_{j,\ell}^{(n)}\| < \infty$$

and

$$\lim_{n \to \infty} \|[B_{j,\ell}^{(n)}, U_j^{(n)}]\|_2 = 0.$$

Then almost surely the tuples $\mathbf{B}_j^{(n)}$ are asymptotically freely independent, that is, for each alternating word $i_1 \neq j_2 \neq \dots \neq i_k$ and any non-commutative polynomials p_1, …, p_k, we have almost surely

$$\lim_{n \to \infty} \operatorname{tr}_n [(p_1(\mathbf{B}_{i_1}^{(n)}) - \operatorname{tr}_n[p_1(\mathbf{B}_{i_1}^{(n)})]) \dots (p_k(\mathbf{B}_{i_k}^{(n)}) - \operatorname{tr}_n[p_k(\mathbf{B}_{i_k}^{(n)})])] = 0. \tag{2.11}$$

Proof. Again, let $A^{(n)}$, $U_j^{(n)}$, and $V_j^{(n)}$ be as in Proposition 2.2.

Fix the word $i_1, \dots, i_k$. Let $C_j^{(n)} = p_j(\mathbf{B}_{i_j}^{(n)})$. Observe that almost surely $\limsup_{n\to\infty} \|C_j^{(n)}\| < \infty$. Moreover, since $\limsup_{n\to\infty} \|B_{j,k}^{(n)}\| < \infty$, we see that

$$\limsup_{n\to\infty} \|[C_j^{(n)}, U_{i_j}^{(n)}]\|_2 = \limsup_{n\to\infty} \|[p_j(\mathbf{B}_{i_j}^{(n)}), U_{i_j}^{(n)}]\|_2 = 0,$$

which follows from checking the case when p_j is a monomial, which in turn follows from the triangle inequality and non-commutative Hölder inequality.

Let $D_j^{(n)} = (V_{i_j}^{(n)})^* C_j^{(n)} V_{i_j}^{(n)}$, and note that almost surely $\|[D_j^{(n)}, A^{(n)}]\|_2 \to 0$. Fix $m \in \mathbb{N}$. By Lemma 2.5, there is a matrix $D_{j,\epsilon}^{(n)} \in \mathcal{D}_{1/m}^{(n)}$ with

$$\|D_{j,\epsilon}^{(n)}\| \leq 3\|D_j^{(n)}\|, \qquad \|D_{j,\epsilon}^{(n)} - D_j^{(n)}\|_2 \leq 8m\sqrt{\pi}\|[D_j^{(n)}, A^{(n)}]\|_2,$$

and it is obvious from the proof of that lemma that $D_{j,\epsilon}^{(n)}$ is a measurable function on the probability space. Recall that (2.9) holds almost surely by Lemma 2.10 and 2.11. Hence, almost surely

$$\limsup_{n\to\infty} \left| \operatorname{tr}_n \left[V_{i_1}^{(n)}(D_{1,\epsilon}^{(n)} - \operatorname{tr}_n(D_{1,\epsilon}^{(n)}))(V_{i_1}^{(n)})^* \dots V_{i_k}^{(n)}(D_{k,\epsilon}^{(n)} - \operatorname{tr}_n(D_{k,\epsilon}^{(n)}))(V_{i_k}^{(n)})^* \right] \right| \\ \leq 4k/m + 2k\sqrt{24k\log(3Rm)/m}.$$

Since $\|D_{j,\epsilon}^{(n)} - D_j^{(n)}\|_2 \to 0$ almost surely and also $\limsup_{n\to\infty} \|D_{j,\epsilon}^{(n)}\| < \infty$ almost surely, we obtain also that

$$\limsup_{n\to\infty} \left| \operatorname{tr}_n \left[V_{i_1}^{(n)}(D_1^{(n)} - \operatorname{tr}_n(D_1^{(n)}))(V_{i_1}^{(n)})^* \dots V_{i_k}^{(n)}(D_k^{(n)} - \operatorname{tr}_n(D_k^{(n)}))(V_{i_k}^{(n)})^* \right] \right| \\ \leq 4k/m + 2k\sqrt{24k\log(3Rm)/m}.$$

Then since m was arbitrary and since $p_j(\mathbf{B}_{i_j}^{(n)}) = C_j^{(n)} = V_{i_j}^{(n)} D_j^{(n)} (V_{i_j}^{(n)})^*$, we obtain (2.11). □

3. Tools for the general approach

Toward the proof of Theorems C and F, we recall some results about model theory of operator algebras, as well as the version of Jung-Hayes 1-bounded entropy [23,32] for types developed in [29]. The reason that we use full types in the proof of our main theorem and not just existential types (which would correspond to the entropy in the presence of Hayes) is explained in Remark 4.3.

3.1. **Model theory background.**

In the proof of Theorem C and Theorem F, we will use several concepts from model theory of tracial von Neumann algebras, in particular *formulas*, *definable predicates*, *types*, *elementary submodels*, and *countable saturation*. We explain below the minimal background for these concepts in tracial von Neumann algebras. For more general background on model theory for metric structures and tracial von Neumann algebras in particular, see [5], [22], [14], [20], [19], [29, §2-3], [31, §2].

Model theory of metric structures: In model theory of metric structures [5], a certain category of objects is formalized through *language* $\mathcal{L}$ which describes the operations (functions from $\mathcal{M}^n$ to $\mathcal{M}$ such as addition or multiplication) and predicates (functions from $\mathcal{M}$ to $\mathbb{R}$ such as the trace or the distance), which can then be used to state axioms for the structures of interest. To state such axioms, one first defines *formulas* as expressions in formal variables $x_1, x_2, \dots$ built from the operations and predicates in the language together with *connectives* and *quantifiers*, as explained in more detail below. Formulas with no free variables are called *sentences*, and a collection of sentences is called a *theory*.

Farah, Hart, and Sherman [15] described the language $\mathcal{L}_{\mathrm{tr}}$ for tracial von Neumann algebras, and a certain theory T_{tr} that encodes the axioms of a tracial von Neumann algebra. In general, $\mathcal{L}$-*structures* are metric spaces equipped with functions corresponding to the operations and predicates in $\mathcal{L}$ (but which do not *a priori* satisfy any particular list of axioms, or theory). Thus, $\mathcal{L}_{\mathrm{tr}}$-structures have formal operations of addition, multiplication, $*$, and trace, but do not necessarily satisfy the $*$-algebra axioms. The $\mathcal{L}_{\mathrm{tr}}$-structures that are actually tracial von Neumann algebras are precisely those which satisfy T_{tr}.

Formulas: In continuous logic, the connectives are given by continuous functions. The quantifiers are supremum and infimum over appropriate sets called *sorts* or *domains*; the domains for $\mathcal{L}_{\mathrm{tr}}$ are the operator norm balls $B_r^{\mathcal{M}}$. $\mathcal{L}_{\mathrm{tr}}$-formulas in variables $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$ are formal expressions built up recursively as follows:

- *Basic formulas:* Let t be an expression formed through addition, multiplication, and $*$-operations (in practice, when evaluated on a von Neumann algebra, t reduces to a $*$-polynomial [2]). Then $\operatorname{Re}\operatorname{tr}(t(\mathbf{x}))$ is a formula.
- *Connectives:* If $\phi_1(\mathbf{x})$, ..., $\phi_k(\mathbf{x})$ are formulas, and $F : \mathbb{R}^k \to \mathbb{R}$ is a continuous function, then $F(\phi_1(\mathbf{x}), \dots, \phi_k(\mathbf{x}))$ is a formula.
- *Quantifiers:* Let $\phi(\mathbf{x}, y)$ be a formula in variables $\mathbf{x}$ and another variable y. Fix $r > 0$ and recall B_r denotes the operator-norm ball of radius r. Then
$$\sup_{y\in B_r} \phi(\mathbf{x}, y) \text{ and } \inf_{y\in B_r} \phi(\mathbf{x}, y)$$
are formulas.

Given a formula ϕ, a tracial von Neumann algebra $\mathcal{M}$, and $x_1, x_2, \dots \in \mathcal{M}$, we can evaluate $\phi^{\mathcal{M}}(\mathbf{x})$ by substituting the actual elements x_j instead of the formal variables, and evaluating each $\sup_{y\in B_r}$ or $\inf_{y\in B_r}$ symbol in the formula as the supremum or infimum over the ball $B_r^{\mathcal{M}}$ in $\mathcal{M}$. The mapping $\phi^{\mathcal{M}} : \mathcal{M}^{\mathbb{N}} \to \mathbb{R}$ is called the *interpretation* of the formula.

Definable predicates: The set of formulas $\mathcal{F}$ forms an algebra over $\mathbb{R}$ because the formulas can be added and multiplied (the addition and multiplication functions $\mathbb{R}^2 \to \mathbb{R}$ count as connectives). From a functional-analytic point of view, it is natural to complete the space of formulas into a Banach algebra (or something similar). In the case of $\mathcal{L}_{\mathrm{tr}}$, we can obtain a "Fréchet algebra" by taking the completion with respect to uniform convergence on each operator norm ball as in [29, §2]. More precisely, for a formula ϕ in variables $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$, for $\mathbf{r} = (r_j)_{j\in\mathbb{N}}$, let
$$\|\phi\|_{\mathbf{r}} = \sup\left\{ |\phi^{\mathcal{M}}(\mathbf{x})| : \mathcal{M} \text{ tracial von Neumann algebra, } \mathbf{x} \in \prod_{j\in\mathbb{N}} B_{r_j}^{\mathcal{M}} \right\}.$$
This defines a collection of seminorms, and the elements of the completion are called *definable predicates relative to* T_{tr} (the notation here mentions T_{tr} because we took the supremum only over $\mathcal{M}$ which satisfy T_{tr} rather than all $\mathcal{L}_{\mathrm{tr}}$-structures).

Types: Naturally, we want to identify the algebra of definable predicates with the algebra of continuous functions on its Gelfand spectrum; see [22, §7.2]. This Gelfand spectrum is precisely the space of *types*. For each tuple $\mathbf{x}$, the *type* of $\mathbf{x}$ is the mapping $\operatorname{tp}^{\mathcal{M}}(\mathbf{x}) : \phi \mapsto \phi^{\mathcal{M}}(\mathbf{x})$ from the algebra of formulas (and more generally definable predicates) to the real numbers. We denote the set of types of countable tuples in tracial von Neumann algebras by $\mathbb{S}(\mathrm{T}_{\mathrm{tr}})$. For $\mathbf{r} \in (0,\infty)^{\mathbb{N}}$, the set of types of tuples $\mathbf{x}$ with $x_j \in B_{r_j}$ will be denoted $\mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$. We equip $\mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$ with the weak-$*$

[2] The definition of formula necessarily precedes writing axioms for von Neumann algebras. Thus, in this definition we don't assume the $*$-algebra axioms, and hence, for instance, $x_1(x_2 + x_3)$ and $x_1x_2 + x_1x_3$ are distinct expressions.

topology (also known as the logic topology), and we equip $\mathbb{S}(\mathrm{T})$ with the inductive limit topology obtained by viewing it as the union of the spaces $\mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$ (see [29, §3.1]). With these definitions in hand, the algebra of definable predicates is isomorphic to $C(\mathbb{S}(\mathrm{T}_{\mathrm{tr}}))$ and the norm $\|\cdot\|_{\mathbf{r}}$ given above coincides with the $C(\mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}}))$ norm.

The space $\mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$ is compact Hausdorff and metrizable [29, Observation 3.14]. In particular, for any point $\mu \in \mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$, there is a nonnegative continuous function f that vanishes only at μ. By [31, Lemma 2.16], f automatically extends to continuous function on $\mathbb{S}(\mathrm{T}_{\mathrm{tr}})$, that is, a definable predicate. We record this result for later use.

Lemma 3.1. *Let $\mu \in \mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$ be a type. Then there exists a nonnegative definable predicate ϕ such that whenever $\mathcal{M}$ is a tracial von Neumann algebra and $\mathbf{x} \in \prod_{j \in B_{r_j}^{\mathcal{M}}}$, we have*

$$\phi^{\mathcal{M}}(\mathbf{x}) = 0 \iff \mathrm{tp}^{\mathcal{M}}(\mathbf{x}) = \mu.$$

In the foregoing discussion, we have described the space of types and the algebra of definable predicates for countably many variables indexed by $\mathbb{N}$. Of course, the same notations and facts make sense for a fixed finite number of free variables rather than countably many variables. Thus, for instance, if $m \in \mathbb{N}$ and $\mathbf{r} \in (0,\infty)^m$, then we denote by $\mathbf{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$ the space of types of m-tuples $\mathbf{x}$ with $\|x_j\| \leq r_j$.

In the proof of Theorem C, we will be concerned with the limiting type of a countable tuple of independent Haar random unitary matrices. In particular, we rely on the fact that this limit exists almost surely. The proof is the same as [16, Lemma 4.2], so we only give a sketch below. In the following, we denote the dual pairing of a type μ with a formula ϕ by $\mu[\phi]$.

Lemma 3.2. *Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $U_1^{(n)}$, $U_2^{(n)}$, ... be independent $n \times n$ Haar random unitaries. For each $\mathcal{L}_{\mathrm{tr}}$-formula ϕ in countably many variables, let*

$$\mu_{\mathrm{Haar}}[\phi] = \lim_{n \to \mathcal{U}} \mathbb{E}\phi^{M_n(\mathbb{C})}(U_1^{(n)}, U_2^{(n)}, \dots).$$

Then almost surely $\lim_{n \to \mathcal{U}} \phi^{M_n(\mathbb{C})}(U_1^{(n)}, U_2^{(n)}, \dots) = \mu_{\mathrm{Haar}}[\phi]$. In other words, $\mathrm{tp}^{M_n(\mathbb{C})}(U_1^{(n)}, U_2^{(n)}, \dots) \to \mu$ weak-$*$ as $n \to \mathcal{U}$ almost surely.

Sketch of proof. Let $\mathbf{r} = (1, 1, \dots)$. Note that every $\mathcal{L}_{\mathrm{tr}}$-formula ϕ can be approximated in $\|\cdot\|_{\mathbf{r}}$ by one which is uniformly Lipschitz (for all tracial von Neumann algebras and all inputs); this follows by taking the connectives $F : \mathbb{R}^k \to \mathbb{R}$ to be Lipschitz; see [16, proof of Lemma 4.2]. By a "3ϵ argument" it suffices to check the claim when ϕ is Lipschitz. In this case, using concentration of measure (Lemma 2.8) and the Borel-Cantelli lemma, we see that almost surely

$$\lim_{n \to \infty} |\phi^{M_n(\mathbb{C})}(U_1^{(n)}, U_2^{(n)}, \dots) - \mathbb{E}\phi^{M_n(\mathbb{C})}(U_1^{(n)}, U_2^{(n)}, \dots)| = 0,$$

which implies the claim of the lemma. □

Remark 3.3. Note that μ_{Haar} may depend on the choice of ultrafilter $\mathcal{U}$, since we do not even know whether the matrix ultraproducts for different ultrafilters are elementarily equivalent. Very little is known at this point about the large n behavior of formulas containing quantifiers on Haar unitaries. See [30, §5.2] and [16, §4] for related discussion and results.

Elementary substructures: If $\mathcal{L}$ is a metric language, and $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, we say that $\mathcal{M}$ is an *elementary submodel of* $\mathcal{N}$ if for every formula ϕ and for every tuple $\mathbf{x}$ in $\mathcal{M}$, we have that $\phi^{\mathcal{M}}(\mathbf{x}) = \phi^{\mathcal{N}}(\mathbf{x})$. Equivalently, $\mathrm{tp}^{\mathcal{M}}(\mathbf{x}) = \mathrm{tp}^{\mathcal{N}}(\mathbf{x})$. We will use the following fact, known as the Downward Löwenheim-Skolem theorem; see [5, Proposition 7.3]. We state it for convenience in the particular case of tracial von Neumann algebras.

Proposition 3.4 (Downward Löwenheim-Skolem theorem)**.** *Let $\mathcal{M}$ be a tracial von Neumann algebra, and let $\mathcal{A} \subseteq \mathcal{M}$ be a separable von Neumann subalgebra. Then there exists a separable elementary substructure $\widehat{\mathcal{M}} \preceq \mathcal{M}$ that contains $\mathcal{A}$.*

Countable saturation: Countable saturation of an $\mathcal{L}$-structure means essentially that sets of formulas in countably many variables that admit approximate solutions must admit exact solutions.

Let Φ be a set of $\mathcal{L}_{\mathrm{tr}}$-formulas in countably many variables $(x_j)_{j\in\mathbb{N}}$ and parameters or constants $(a_j)_{j\in\mathbb{N}}$ from some tracial von Neumann algebra $\mathcal{M}$. Fix some $\mathbf{r} \in (0,\infty)^{\mathbb{N}}$.

- We say that Φ is *satisfiable in* $\prod_{j\in\mathbb{N}} B_{r_j}^{\mathcal{M}}$ if there exists $\mathbf{x} \in \prod_{j\in\mathbb{N}} B_{r_j}^{\mathcal{M}}$ such that $\phi^{\mathcal{M}}(\mathbf{x},\mathbf{a}) = 0$ for all $\phi \in \Phi$.
- We say that Φ is *finitely approximately satisfiable in* $\prod_{j\in\mathbb{N}} B_{r_j}^{\mathcal{M}}$ if for every $\phi_1, \dots, \phi_k$ in Φ and every ϵ, there exists some $\mathbf{x} \in \prod_{j\in\mathbb{N}} B_{r_j}^{\mathcal{M}}$ satisfying $|\phi_j(\mathbf{x},\mathbf{a})| < \epsilon$.
- We say that $\mathcal{M}$ is *countably saturated* if every $\mathbf{r}$, and for every set of formulas Φ in countably many variables and countably many parameters $\mathbf{a}$, if Φ is finitely approximately satisfiable in $\prod_{j\in\mathbb{N}} B_{r_j}^{\mathcal{M}}$, then Φ is satisfiable in $\prod_{j\in\mathbb{N}} B_{r_j}^{\mathcal{M}}$.

Most ultraproducts are countably saturated. Recall that an ultrafilter $\mathcal{U}$ on a set I is said to be *countably incomplete* if there is a countable family of sets in $\mathcal{U}$ with empty intersection. We recall the following fact:

Lemma 3.5 (See [14, Proposition 4.11], [5, Proposition 7.6])**.** *Let $\mathcal{M} = \prod_{i\to\mathcal{U}} \mathcal{M}_i$ for some tracial von Neumann algebras $(\mathcal{M}_i)_{i\in I}$ where I is an infinite index set and $\mathcal{U}$ is a countably incomplete ultrafilter on I. Then $\mathcal{M}$ is countably saturated.*

3.2. **1-bounded entropy for types.** In free entropy theory, 1-bounded entropy is a notion of metric entropy for matricial approximations defined by Hayes [23] and inspired by the work of Jung [32]. Here we describe the version for full types from [29].

If $\mathcal{O}$ is a subset of the type space $\mathbb{S}(\mathrm{T}_{\mathrm{tr}})$ and $\mathbf{r} \in (0,\infty)^{\mathbb{N}}$, we define

$$\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O}) = \left\{ \mathbf{X} \in \prod_{j\in\mathbb{N}} D_{r_j}^{M_n(\mathbb{C})} : \mathrm{tp}^{M_n(\mathbb{C})}(\mathbf{X}) \in \mathcal{O} \right\}.$$

We view this as a microstate space as in Voiculescu's free entropy theory [54]. Entropy of types is defined by the exponential growth rates of covering numbers of these spaces $\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O})$ up to unitary conjugation.

Definition 3.6 (Orbital covering numbers)**.** Given $\Omega \subseteq M_n(\mathbb{C})^{\mathbb{N}}$ and a finite $F \subseteq \mathbb{N}$ and $\epsilon > 0$, we define $N_{F,\epsilon}^{\mathrm{orb}}(\Omega)$ to be the set of $\mathbf{Y} \in M_n(\mathbb{C})^{\mathbb{N}}$ such that there exists a unitary U in $M_n(\mathbb{C})$ and $\mathbf{X} \in \Omega$ such that $\|Y_i - UX_iU^*\|_2 < \epsilon$ for all $i \in F$. If $\Omega \subseteq N_{F,\epsilon}^{\mathrm{orb}}(\Omega')$, we say that Ω' *orbitally* (F,ϵ)*-covers* Ω. We denote by $K_{F,\epsilon}^{\mathrm{orb}}(\Omega)$ the minimum cardinality of a set Ω' that orbitally (F,ϵ)-covers Ω.

Definition 3.7 (1-bounded entropy for types [29])**.** Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For $\mu \in \mathbb{S}(\mathrm{T}_{\mathrm{tr}})$ and $F \subseteq I$ finite and $\epsilon > 0$, we define

$$\mathrm{Ent}_{\mathbf{r},F,\epsilon}^{\mathcal{U}}(\mu) = \inf_{\text{open } \mathcal{O}\ni\mu} \lim_{n\to\mathcal{U}} \frac{1}{n^2} \log K_{F,\epsilon}^{\mathrm{orb}}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O})).$$

Then let

$$\mathrm{Ent}_{\mathbf{r}}^{\mathcal{U}}(\mu) := \sup_{\substack{\text{finite } F\subseteq\mathbb{N} \\ \epsilon>0}} \mathrm{Ent}_{\mathbf{r},F,\epsilon}^{\mathcal{U}}(\mu).$$

and

$$\operatorname{Ent}^{\mathcal{U}}(\mu) := \sup_{\mathbf{r} \in (0,\infty)^{\mathbb{N}}} \operatorname{Ent}^{\mathcal{U}}_{\mathbf{r}}(\mu).$$

We also remark that the same definitions make sense for types of finite tuples $(x_1, \dots, x_m)$ instead of countable tuples. In that setting, one does not need to consider the finite subset F since F can be taken to be $\{1, \dots, m\}$.

By [29, Corollary 4.10], if $\mathbf{x}$ and $\mathbf{y}$ are tuples in a tracial von Neumann algebra $\mathcal{M}$ and $\mathrm{W}^*(\mathbf{x}) = \mathrm{W}^*(\mathbf{y})$, then $\operatorname{tp}^{\mathcal{M}}(\mathbf{x})$ and $\operatorname{tp}^{\mathcal{M}}(\mathbf{y})$ have the same entropy. Hence, for separable $\mathcal{A} \subseteq \mathcal{M}$, one can define $\operatorname{Ent}^{\mathcal{U}}(\mathcal{A} : \mathcal{M})$ as the entropy of any generating tuple for $\mathcal{A}$. More generally, if $\mathcal{A}$ is not necessarily separable, one defines

$$\operatorname{Ent}^{\mathcal{U}}(\mathcal{A} : \mathcal{M}) := \sup\{\operatorname{Ent}^{\mathcal{U}}(\operatorname{tp}^{\mathcal{M}}(\mathbf{x})) : \mathbf{x} \in \mathcal{A}^{\mathbb{N}}\},$$

and by [29, Observation 4.12] this agrees with the entropy of any countable (or finite) generating tuple in the case when $\mathcal{A}$ is separable. Now if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$, then

$$\operatorname{Ent}^{\mathcal{U}}(\mathcal{A} : \mathcal{M}) \leq \operatorname{Ent}^{\mathcal{U}}(\mathcal{B} : \mathcal{M}). \tag{3.1}$$

Entropy for types versus entropy in the presence: The relationship between the entropy for types described above and Hayes' 1-bounded entropy is as follows (see [29, §5] for details). Hayes' 1-bounded entropy $h^{\mathcal{U}}(\mathcal{A} : \mathcal{M})$ of $\mathcal{A}$ in the presence of $\mathcal{M}$ (with respect to the ultrafilter $\mathcal{U}$) arises by looking at microstate spaces for the *existential type* of tuples $\mathbf{x}$ rather than the full type. The existential type describes the evaluation on $\mathbf{x}$ of formulas of the form $\phi(\mathbf{x}) = \inf_{y \in B_r} \psi(\mathbf{x}, y)$, where ψ is a quantifier-free formula (i.e. a formula with no supremum or infimum in it). The space of existential types is equipped with a topology that is non-Hausdorff because a basic neighborhood is defined by one-sided upper bounds on a finite family of inf-formulas. The microstate spaces defined by a neighborhood of the existential type are a special case of microstate spaces defined by neighborhoods of the full type, and hence

$$\operatorname{Ent}^{\mathcal{U}}(\mathcal{A} : \mathcal{M}) \leq h^{\mathcal{U}}(\mathcal{A} : \mathcal{M}); \tag{3.2}$$

see e.g. [29, Lemma 5.13].

Orbital versus relative entropy: There are actually two approaches to defining metric entropy– one based on covering numbers up to unitary conjugation, and one based on covering number of microstate spaces relative to a fixed microstate sequence $A^{(n)}$ for a normal element a with diffuse spectrum. These were shown to be equivalent in [23, Lemma A.5] for the setting of a self-adjoint and for the 1-bounded entropy in the presence h. We need the analogous result for the entropy for types, and with using a Haar unitary instead of a self-adjoint element for the fixed microstate. We include a self-contained proof for convenience; the approach here is slightly different since we fix a very specific $A^{(n)}$ and use Lemma 2.5 rather than using Szarek's covering estimates for Grassmannians [52].

Lemma 3.8. *Let $\mu \in \mathbb{S}_{\mathbf{r}}(\mathrm{T}_{\mathrm{tr}})$ be the type of some infinite tuple $(a, x_1, x_2, \dots)$ such that a is a Haar unitary. Let $A^{(n)} = \operatorname{diag}(1, \zeta_n, \dots, \zeta_n^{n-1})$. For a neighborhood $\mathcal{O}$ of μ, let*

$$\Gamma^{(n)}_{\mathbf{r}}(\mathcal{O}; A^{(n)} \rightsquigarrow a) = \{\mathbf{X}^{(n)} \in \prod_{j \in \mathbb{N}} D^{M_n(\mathbb{C})}_{r_j} : \operatorname{tp}^{M_n(\mathbb{C})}(A^{(n)}, \mathbf{X}^{(n)}) \in \mathcal{O}\}.$$

Then

$$\operatorname{Ent}^{\mathcal{U}}_{\mathbf{r}}(\mu) = \sup_{(F,\epsilon)} \inf_{\mathcal{O} \ni \mu} \lim_{n \to \mathcal{U}} \frac{1}{n^2} \log K_\epsilon(\Gamma^{(n)}_{\mathbf{r}}(\mathcal{O}; A^{(n)} \rightsquigarrow a)).$$

Proof. First, let us show $\leq$. Given (F,ϵ) and given a neighborhood $\mathcal{O}$ of μ, we claim that there is a neighborhood $\mathcal{O}'$ such that $\Gamma_{(1,\mathbf{r})}(\mathcal{O}')$ is contained in the orbital $(F,\epsilon/2)$-neighborhood of $\Gamma_{\mathbf{r}}(\mathcal{O}; A^{(n)} \rightsquigarrow a)$. By construction of the logic topology, we can assume without loss of generality that $\mathcal{O}$ is defined by $|\phi_j - c_j| < \eta$ for some finitely many definable predicates $\phi_1, \dots, \phi_k$. Let k be such that $4\pi/k < \epsilon/2$, and let $\mathcal{O}_k$ be the neighborhood of the Haar measure on the unit circle described in Lemma 2.1; then since the weak-$*$ topology on $\mathcal{P}(S^1)$ is topology given by testing against trigonometric polynomials, there exist $*$-polynomials $p_1, \dots, p_m$ such that for any unitary u, if $\operatorname{Re}\operatorname{tr}(p_j(u)) < 0$, for all j, then the spectral distribution of u is in $\mathcal{O}_k$. Let $\psi(a', \mathbf{x}')$ be the definable predicate

$$\psi^{\mathcal{M}}(a', \mathbf{x}') = \inf_{b \text{ unitary}} \max_j \max(|\phi_j(b, \mathbf{x}') - c_j| - \eta, \operatorname{Re}\operatorname{tr}(p_j(b))).$$

This is indeed a definable predicate since unitaries form a definable set. Let $\mathcal{O}'$ be the set of types satisfying $\psi < 0$. Of course, $\mu \in \mathcal{O}'$ since

$$\psi^{\mathcal{Q}}(a, \mathbf{x}) \leq \max_j \max(|\phi_j(a, \mathbf{x}) - c_j| - \eta, \operatorname{Re}\operatorname{tr}(p_j(a))) < 0.$$

Moreover, if $(A, \mathbf{X})$ is an $n \times n$ matrix tuple whose type is in $\mathcal{O}'$, this implies that there is some unitary B such that $\operatorname{tp}^{M_n(\mathbb{C})}(B, \mathbf{X}) \in \mathcal{O}$ and also the spectral measure of B is in $\mathcal{O}_k$. Then by Lemma 2.1, there is a unitary V such that $\|V^*BV - A^{(n)}\|_2 < 4\pi/k < \epsilon/2$. In particular, $(B, \mathbf{X})$ is in the $(F, \epsilon/2)$ neighborhood of $(VA^{(n)}V^*, \mathbf{X})$, so it is in the orbital $(F, \epsilon/2)$ neighborhood of $\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O} \mid A^{(n)} \rightsquigarrow a)$. Thus, $\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O})$ is in the orbital $(F, \epsilon/2)$ neighborhood of $\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O} \mid A^{(n)} \rightsquigarrow a)$. Hence,

$$K_\epsilon^{\mathrm{orb}}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O}') \leq K_{\epsilon/2}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O}; A^{(n)} \rightsquigarrow a)).$$

From this we obtain

$$\operatorname{Ent}_{\mathbf{r},F,\epsilon}^{\mathcal{U}}(\mu) \leq \inf_{\mathcal{O} \ni \mu} \lim_{n \to \mathcal{U}} \frac{1}{n^2} \log K_{\epsilon/2}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O}; A^{(n)} \rightsquigarrow a)),$$

and then taking the supremum over (F, ϵ) finishes the inequality $\geq$.

For the other direction, again fix (F, ϵ) and $\mathcal{O}$. We want to bound the (F, ϵ) covering number of $\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O} \mid A^{(n)} \rightsquigarrow a)$ in terms of the orbital (F, δ) covering number of $\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O})$ where $\delta \in (0,1)$ is to be chosen later. Let Ω be a set that orbitally (F, δ)-covers $\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O})$. In particular, $\{A^{(n)}\} \times \Gamma_{\mathbf{r}}^{(n)}(\mathcal{O} \mid A^{(n)} \rightsquigarrow a)$ is covered by the $N_{(F,\delta)}(B, \mathbf{X})$ for $(B, \mathbf{X}) \in \Omega$. Thus, our goal is to estimate the plain covering number of the sets

$$S_{(B,\mathbf{X})} = \{A^{(n)}\} \times \Gamma_{\mathbf{r}}^{(n)}(\mathcal{O} \mid A^{(n)} \rightsquigarrow a) \cap N_{(F,\delta)}(B, \mathbf{X}).$$

The idea is that unitaries that we conjugate by must approximately fix $A^{(n)}$ and hence are approximately band matrices. More precisely, if $(A^{(n)}, X_1)$ and $(A^{(n)}, X_2)$ are both in $S_{(B,\mathbf{X})}$, then there is a unitary V such that

$$(A^{(n)}, X_1) \in N_{(F,\delta)}(VA^{(n)}V^*, VX_2V^*).$$

In particular, $\|[V, A^{(n)}]\|_2 < \delta$, and so by Lemma 2.5,

$$d(V, \mathcal{D}_{\sqrt{\delta}}^{(n)} \cap B_3^{M_n(\mathbb{C})}) \leq \frac{8\sqrt{\pi}}{\sqrt{\delta}} 2\delta = 16\sqrt{\pi\delta}.$$

Thus, by Lemma 2.11,

$$K_{\sqrt{\delta}+16\sqrt{\pi\delta}}(\{V \in \mathbb{U}_n : \|[V, A^{(n)}]\|_2 < \delta\}) \leq K_{\sqrt{\delta}}(\mathcal{D}_{\sqrt{\delta}}^{(n)} \cap B_3^{M_n(\mathbb{C}))}) \leq \exp(n^2 \cdot 2\sqrt{\delta} \log(3R/\sqrt{\delta})).$$

Hence, fixing some $(A^{(n)}, X_0) \in S_{(B,\mathbf{X})}$, we have that

$$S_{(B,\mathbf{X})} \subseteq N_{(F,2\delta)}(\{(A^{(n)}, VX_0V^*) : V \in \mathbb{U}_n : \|[V, A^{(n)}]\|_2 < \delta\}).$$

Then since $\|(X_0)_j\| \leq r_j$, we have that $V \mapsto V(X_0)_j V^*$ is $2r_j$-Lipschitz. Let $r_F = \max_{j \in F} r_j$. Then any $(F, \sqrt{\delta} + 16\sqrt{\pi\delta})$-covering of $\{V \in \mathbb{U}_n : \|[V, A^{(n)}]\|_2 < \delta\}$ yields a $(F, r_F(\sqrt{\delta} + 16\sqrt{\pi\delta}))$-covering of the conjugation orbit of $(A^{(n)}, X_0)$ and so a $(F, 2\delta + r_F(\sqrt{\delta} + 16\sqrt{\pi\delta}))$-covering of $S_{(B,\mathbf{X})}$. Hence, choose δ small enough that $2\delta + r_F(\sqrt{\delta} + 16\sqrt{\pi\delta}) < \epsilon$. Then we obtain

$$K_{F,\epsilon}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O} \mid A^{(n)} \rightsquigarrow a)) \leq \sum_{(B,\mathbf{X}) \in \Omega} K_{F,\epsilon}(S_{(B,\mathbf{X})}) \leq K_{F,\delta}^{\mathrm{orb}}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O})) \exp(n^2 \cdot 2\sqrt{\delta} \log(3R/\sqrt{\delta})).$$

Hence,

$$\frac{1}{n^2} \log K_{F,\epsilon}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O} \mid A^{(n)} \rightsquigarrow a)) \leq \frac{1}{n^2} \log K_{F,\delta}^{\mathrm{orb}}(\Gamma_{\mathbf{r}}^{(n)}(\mathcal{O})) + 2\sqrt{\delta} \log(3R/\sqrt{\delta}).$$

Since δ can be chosen arbitarily small, we get the desired inequality. $\square$

4. General freeness results

We are now ready to prove Theorem C and F.

4.1. Free independence phenomena in matrix ultraproducts. In this section, we fix a countably incomplete ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and write $\mathcal{Q} = \prod_{n \to \mathcal{U}} M_n(\mathbb{C})$.

Lemma 4.1. *Let $U_1^{(n)}$, $U_2^{(n)}$, … be independent Haar unitaries. Let $k \in \mathbb{N}$, and let $i_1 \neq i_2 \neq \ldots \neq i_k$. For each $j = 1, \ldots, k$, let $\nu_j \in \mathbb{S}_{(1,1)}(\mathrm{T}_{\mathrm{tr}})$ be the type of some pair (u, x) in $\mathcal{Q}$ where u is a Haar unitary and x has trace zero, such that $\mathrm{Ent}^{\mathcal{U}}(\nu_j) = 0$. Then almost surely, for all x_j with $\mathrm{tp}^{\mathcal{Q}}(u_{i_j}(\omega), x_j) = \nu_j$, we have $\mathrm{tr}^{\mathcal{Q}}(x_1 \ldots x_k) = 0$.*

Proof. By Proposition 2.2, let $U_j^{(n)}$ be an independent family of Haar random unitaries and $V_j^{(n)}$ another independent family of Haar random unitaries such that $\|U_j^{(n)} - V_j^{(n)} A^{(n)} (V_j^{(n)}\|)^* \to 0$ almost surely, where $A^{(n)} = \mathrm{diag}(1, \zeta_n, \ldots, \zeta_n^{n-1})$. For each outcome ω, let $v_j(\omega)$ be the corresponding element of the ultraproduct $\mathcal{Q}$, so that $u_j(\omega) = v_j(\omega) a v_j(\omega)^*$.

Fix $m \in \mathbb{N}$. Since $\mathrm{Ent}^{\mathcal{U}}(\nu_j) = 0$, using Lemma 3.8 for the case of a 1-tuple rather than a countable tuple, there exists some neighborhood $\mathcal{O}_{j,m}$ of ν_j such that

$$\lim_{n \to \mathcal{U}} \frac{1}{n^2} \log K_{1/m}(\Gamma_1^{(n)}(\mathcal{O}_{j,m} \mid A^{(n)} \rightsquigarrow a)) < \frac{1}{m}.$$

(Here we take $\epsilon = 1/m$ and $F = \{1\}$.) Then applying Lemma 2.10 with $\delta = \epsilon = 1/m$, we see that almost surely

$$\lim_{n \to \mathcal{U}} \sup_{Y_1 \in \Gamma_1^{(n)}(\mathcal{O}_{1,m} | A^{(n)} \rightsquigarrow a)} \cdots \sup_{Y_k \in \Gamma_1^{(n)}(\mathcal{O}_{k,m} | A^{(n)} \rightsquigarrow a)} \left| \mathrm{tr}_n \left[V_{i_1}^{(n)} (Y_1 - \mathrm{tr}_n(Y_1)) (V_{i_1}^{(n)})^* \ldots V_{i_k}^{(n)} (Y_k - \mathrm{tr}_n(Y_k)) (V_{i_k}^{(n)})^* \right] \right| \leq 4k/m + 2k\sqrt{12k/m}.$$

By taking the countable intersection, almost surely this holds for *all* $m \in \mathbb{N}$.

Now fix an outcome ω in the almost sure event where the above inequality holds for all m. Suppose that $\mathrm{tp}^{\mathcal{Q}}(u_{i_j}(\omega), x_j) = \nu_j$. Write $x_j = v_{i_j}(\omega) y_j v_{i_j}(\omega)^*$, and represent $Y_j = [Y_j^{(n)}]_{n \in \mathbb{N}}$ with $\|Y_j^{(n)}\| \leq 1$. Recall by Łos's theorem that

$$\lim_{n \to \mathcal{U}} \mathrm{tp}^{M_n(\mathbb{C})}(A^{(n)}, Y_j^{(n)}) = \mathrm{tp}^{\mathcal{Q}}(a, y_j) = \mathrm{tp}^{\mathcal{Q}}(u_{i_j}(\omega), x_j) = \nu_j.$$

Thus, there is some $A \in \mathcal{U}$ such that $\mathrm{tp}^{M_n(\mathbb{C})}(A^{(n)}, Y_j^{(n)}) \in \mathcal{O}_{m,j}$ for each $j = 1, \ldots, k$ for each $n \in A$. Hence,

$$\begin{aligned}|\operatorname{tr}^{\mathcal{Q}}(x_1 \ldots x_k)| &= |\operatorname{tr}^{\mathcal{Q}}(v_{i_1}(\omega) y_1 v_{i_1}(\omega)^* \ldots v_{i_k}(\omega) y_k v_{i_k}(\omega)^*)| \\ &= \lim_{n \to \mathcal{U}} \left| \operatorname{tr}_n \left[V_{i_1}^{(n)}(\omega)(X_1 - \operatorname{tr}_n(X_1))(V_{i_1}^{(n)}(\omega))^* \ldots V_{i_k}^{(n)}(\omega)(X_k - \operatorname{tr}_n(X_k))(V_{i_k}^{(n)}(\omega))^* \right] \right| \\ &\leq 4k/m + 2k\sqrt{12k/m}.\end{aligned}$$

Then since m was arbitrary, we get $\operatorname{tr}^{\mathcal{Q}}(x_1 \ldots x_k) = 0$ as desired. □

In the proof of Theorem 4.4 below, we will want to apply the conclusion of Lemma 4.1 simultaneously to *all* ν_j's satisfying the hypotheses (not just to a countable family of such ν_j's). So we will not be able to do this by taking intersections of almost sure events naïvely. Rather, we proceed by arguing that the conclusion of the lemma is a property of the *type* of $(u_1(\omega), u_2(\omega), \ldots)$. We already know that the type of the Haar unitaries converges almost surely to μ_{Haar} (there we only had to test countably many formulas). We will show that for each $\nu_1, \ldots, \nu_k$ as in Lemma 4.1, the conclusion of the lemma can be expressed in terms of formulas in $(u_1, u_2, \ldots)$, and so it depends only the type. Therefore, since the type μ_{Haar} can be described only by testing countably many formulas, one can indeed obtain the conclusion of Lemma 4.1 for uncountably many choices of $(\nu_1, \ldots, \nu_k)$.

The next lemma follows from Lemma 4.1 from purely model-theoretic considerations, specifically the countable saturation of the matrix ultraproduct $\mathcal{Q}$. Compare for instance [5, proof of Corollary 9.10]. In the following, a *modulus of continuity* refers to a continuous increasing function $w : [0, \infty) \to [0, \infty)$ such that $w(0) = 0$.

Lemma 4.2. *Let $\mu_{\mathrm{Haar}} \in \mathbb{S}_{(1,1,\ldots)}(\mathrm{Th}(\mathcal{Q}))$ be the almost sure limit of the type of Haar random unitary matrices as $n \to \mathcal{U}$. Let $k \in \mathbb{N}$, and let $i_1 \neq i_2 \neq \ldots \neq i_k$. For each $j = 1, \ldots, k$, let $\nu_j \in \mathbb{S}_{(1,1)}(\mathrm{T}_{\mathrm{tr}})$ be the type of some pair (u, x) where u is a Haar unitary and x has trace zero, such that $\mathrm{Ent}^{\mathcal{U}}(\nu_j) = 0$. Then there exist 2-variable nonnegative definable predicates $\phi_1, \ldots, \phi_k$ with $\nu_j[\phi_j] = 0$ for each j and there exists a modulus of continuity $w : [0, \infty) \to [0, \infty)$, such that μ_{Haar} annihilates the formula ψ given by*

$$\psi^{\mathcal{Q}}(u_1, u_2, \ldots) := \sup_{x_1, \ldots, x_k \in B_1^{\mathcal{Q}}} \left(|\operatorname{tr}^{\mathcal{Q}}(x_1 \ldots x_k)| \dot{-} w(\max_j \phi_j^{\mathcal{Q}}(u_{i_j}, x_j)) \right), \tag{4.1}$$

where $a \dot{-} b = \max(a - b, 0)$.

Proof. From Lemma 4.1, we know that if $(u_1, u_2, \ldots)$ realizes the type μ_{Haar} and if $\mathrm{tp}^{\mathcal{Q}}(u_{i_j}, x_j) = \nu_j$ for $j = 1, \ldots, k$, then $\operatorname{tr}^{\mathcal{Q}}(x_1 \ldots x_k) = 0$.

By Lemma 3.1, fix a 2-variable definable predicate $\phi_j \geq 0$ such that for $u, x \in B_1^{\mathcal{Q}}$, we have $\phi_j^{\mathcal{Q}}(u, x) = 0$ if and only if $\mathrm{tp}^{\mathcal{Q}}(u, x) = \nu_j$. We claim that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\max_j \phi_j^{\mathcal{Q}}(u_{i_j}, x_j) < \delta$, then $|\operatorname{tr}^{\mathcal{Q}}(x_1 \ldots x_k)| < \epsilon$. Suppose for contradiction that this fails. Then there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exist $x_1, \ldots, x_k$ with $\phi_j^{\mathcal{Q}}(u_{i_j}, x_j) < \delta$ for each j but $|\operatorname{tr}^{\mathcal{Q}}(x_1 \ldots x_k)| \geq \epsilon$. Consider the definable predicates $\{\phi_j(u_{i_j}, x_j)\}_{j=1}^k \cup \{\epsilon \dot{-} |\operatorname{tr}(x_1 \ldots x_k)|\}$ in variables $(x_1, \ldots, x_k)$ and constants $(u_1, u_2, \ldots)$. Then this set of formulas is approximately satisfiable, and so by countable saturation of $\mathcal{Q}$ (Lemma 3.5), it is satisfiable. That is, there exist $(x_1, \ldots, x_k)$ with $\phi_j(u_{i_j}, x_j) = 0$ and $|\operatorname{tr}^{\mathcal{Q}}(x_1 \ldots x_k)| \geq \epsilon$. This contradicts the conclusion of Lemma 4.1 that we stated at the beginning of the proof.

Knowing that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\max_j \phi_j^{\mathcal{Q}}(u_{i_j}, x_j) < \delta$, then $|\operatorname{tr}^{\mathcal{Q}}(x_1 \dots x_k)| < \epsilon$, one can construct a continuous increasing function $w : [0,\infty) \to [0,\infty)$ with $w(0) = 0$ such that $|\operatorname{tr}^{\mathcal{Q}}(x_1 \dots x_k)| \leq w(\max_j \phi_j^{\mathcal{Q}}(u_{i_j}, x_j))$ for all $x_1, \dots, x_k \in B_1^{\mathcal{Q}}$. $\square$

Remark 4.3. Lemma 4.2 is where we rely the fact on working with the full types rather than the existential types. Since the space of existential types is not Hausdorff, one cannot use continuous functions to separate points in that space. Hence, it is necessary to reason about a larger space which is compact and Hausdorff.

Theorem 4.4. *Let $(u_1, u_2, \dots)$ be a tuple in $\mathcal{Q}$ with type equal to μ_{Haar} of Lemma 3.2. For each $i \in \mathbb{N}$, let $\mathcal{A}_i \ni u_i$ with $\operatorname{Ent}^{\mathcal{U}}(\mathcal{A}_i) = 0$. Then $\mathcal{A}_1$, $\mathcal{A}_2$, ... are freely independent.*

Proof. Let $i_1 \neq i_2 \neq \dots \neq i_k$. Let $x_j \in \mathcal{A}_{i_j}$ with $\operatorname{tr}^{\mathcal{Q}}(x_j) = 0$, and we will show that $\operatorname{tr}^{\mathcal{Q}}(x_1 \dots x_k) = 0$. Let ν_j be the type of (u_j, x_j). Note that by (3.1),

$$\operatorname{Ent}^{\mathcal{U}}(\nu_j) = \operatorname{Ent}^{\mathcal{U}}(\mathrm{W}^*(u_j, x_j)) \leq \operatorname{Ent}^{\mathcal{U}}(\mathcal{A}_j) = 0.$$

Therefore, by Lemma 4.2, there exists some modulus of continuity w such that μ_{Haar} vanishes on the formula ψ of (4.1). In particular, we have

$$|\operatorname{tr}^{\mathcal{Q}}(x_1 \dots x_k)| \dot{-} w(\max_j \phi_j^{\mathcal{Q}}(u_{i_j}, x_j)) = 0.$$

Since $\phi_j^{\mathcal{Q}}(u_{i_j}, x_j) = \nu_j[\phi_j] = 0$, we obtain that $\operatorname{tr}^{\mathcal{Q}}(x_1 \dots x_k) = 0$ as desired. Since this holds for every alternating string $i_1 \neq \dots \neq i_k$ and all trace-zero $x_j \in \mathcal{A}_{i_j}$, we have that $\mathcal{A}_1$, $\mathcal{A}_2$, ... are freely independent as desired. $\square$

Proof of Theorem C. Fix an outcome ω in the almost sure event where the type $(U_1^{(n)}, U_2^{(n)}, \dots)$ converges to μ_{Haar}, or in other words the type of $(u_1(\omega), u_2(\omega), \dots)$ is μ_{Haar}. Let $\mathcal{P}_j$ be the Pinsker algebra containing u_j. Since $\operatorname{Ent}^{\mathcal{U}}(\mathcal{P}_j) \leq h(\mathcal{P}_j : \mathcal{Q}) = 0$ by (3.2), the $\mathcal{P}_j$'s are freely independent by Theorem 4.4. $\square$

Remark 4.5 (On the theory of matrix ultraproducts). Fix 2-variable types ν_j with $\operatorname{Ent}^{\mathcal{U}}(\nu_j) = 0$, and let ψ in Lemma 4.2, note that ψ only depends on the finitely many of u_i's, and since the unitaries form a definable set, the following is a definable predicate with no free variables:

$$\gamma = \inf_{u_1, u_2, \dots \text{ unitary}} \psi(u_1, u_2, \dots).$$

Then the matrix ultraproduct $\mathcal{Q}$ satisfies the sentence $\gamma^{\mathcal{Q}} = 0$ since $\psi^{\mathcal{Q}}$ vanishes on the unitaries with type μ_{Haar}. By similar reasoning as in Lemma 4.2, the free independence of the commutants of the u_j's from Theorem B leads to the following statement: Fix a word $i_1 \neq i_2 \neq \dots \neq i_k$. There is some modulus of continuity w such that

$$\inf_{u_1, u_2, \dots \in \mathcal{Q} \text{ unitary}} \sup_{x_1, \dots, x_k \in B_1^{\mathcal{Q}}} \left(|\operatorname{tr}^{\mathcal{Q}}(x_1 \dots x_k) \dot{-} w(\max_j \|[u_{i_j}, x_j]\|_2) \right) = 0.$$

Note that (for an appropriate choice of w) this sentence is also satisfied by $L(F_\infty)$ on account of [28, Theorem B] (or alternatively Theorem F which we prove in the next section). Hence, these freeness results give some new information about the $\exists\forall$ theory of $\mathcal{Q}$ and that of $L(F_\infty)$, and in particular shows agreement of the two on certain $\exists\forall$ formulas.

4.2. **Free independence phenomena in ultraproducts of free products.**

Proof of Theorem F. First, to prove claims (1), (2), (3), it suffices to consider the case where I is countable. Indeed, for (1), we have to prove that for every $i_1 \neq i_2 \neq \ldots \neq i_k$ if $x_j \in \mathcal{A}_{i_j}$ with trace zero, then $\operatorname{tr}^{\mathcal{M}^{\mathcal{V}}}(x_1 \ldots x_k) = 0$. Let I_0 be a countable subset of I containing the indices $i_1, \ldots, i_k$. Then view $\mathcal{M}$ as a free product of countably many algebras $\mathcal{M}_0 * (*_{i \in I_0} \mathcal{M}_i)$, where $\mathcal{M}_0 = *_{i \in I \setminus I_0} \mathcal{M}_i$. By applying the conclusions of the theorem for this free product decomposition, we obtain $\operatorname{tr}^{\mathcal{M}^{\mathcal{V}}}(x_1 \ldots x_k) = 0$. Hence, in the remainder of the proof assume without loss of generality that $I = \mathbb{N}$.

Now because $\mathcal{M}_i$ is assumed to be diffuse and Connes embeddable, there exists an ultrafilter $\mathcal{V}'$ on some infinite index set[3] such that $\mathcal{M}_i$ embeds into $\mathcal{R}^{\mathcal{V}'}$ for each $i \in I$, where $\mathcal{R}$ is the unique separable hyperfinite II_1 factor. Since we assumed that $\mathcal{A}_i \cap \mathcal{M}_i$ is diffuse, let a_i be a Haar unitary in $\mathcal{M}_i$. We may assume without loss of generality that inclusion $\mathcal{M}_i \to \mathcal{R}^{\mathcal{V}'}$ sends a_i into the diagonal copy of $\mathcal{R}$ in $\mathcal{R}^{\mathcal{V}'}$; this is because all Haar unitaries in $\mathcal{R}^{\mathcal{V}'}$ are conjugate to each other.

Let $\mathcal{S} = *_{i \in I} \mathcal{R} \cong L(F_\infty)$. For convenience of notation, we will denote by $\mathcal{R}_i$ the ith copy of $\mathcal{R}$ in $\mathcal{S}$. The inclusions $\mathcal{R}_i \to \mathcal{S}$ yields an inclusion $\mathcal{R}_i^{\mathcal{V}'}$ into $\mathcal{S}^{\mathcal{V}'}$, and the $\mathcal{R}_i^{\mathcal{V}'}$'s are freely independent in $\mathcal{S}^{\mathcal{V}'}$ by a straightforward limiting argument. Hence, we have mappings

$$*_{i \in I} \mathcal{M}_i \to *_{i \in I} \mathcal{R}_i^{\mathcal{V}'} \to (*_{i \in I} \mathcal{R}_i)^{\mathcal{V}'}.$$

Thus also we have inclusions

$$\mathcal{M}^{\mathcal{V}} = (*_{i \in I} \mathcal{M}_i)^{\mathcal{V}} \to (*_{i \in I} \mathcal{R}_i^{\mathcal{V}'})^{\mathcal{V}} \to ((*_{i \in I} \mathcal{R}_i)^{\mathcal{V}'})^{\mathcal{V}} \cong (*_{i \in I} \mathcal{R}_i)^{\mathcal{W}} = \mathcal{S}^{\mathcal{W}},$$

where $\mathcal{W}$ is a certain ultrafilter on the product of the index sets for $\mathcal{V}'$ and $\mathcal{V}$. We have also arranged that the Haar unitary a_i in $\mathcal{M}_i \cap \mathcal{A}_i \subseteq (*_{i \in I} \mathcal{M}_i)^{\mathcal{V}}$ is contained in $\mathcal{R}_i \subseteq \mathcal{R}_i^{\mathcal{W}} \subseteq \mathcal{S}^{\mathcal{W}}$.

Note that $h(\mathcal{M}_i : \mathcal{S}^{\mathcal{W}}) \leq h(\mathcal{R}_i^{\mathcal{V}'} : \mathcal{S}^{\mathcal{W}}) = 0$ since $\mathcal{R}^{\mathcal{V}'}$ has property Gamma. Let $\mathcal{P}_i$ be the Pinsker algebra of $\mathcal{M}_i$ in $\mathcal{S}^{\mathcal{W}}$, that is, the unique maximal subalgebra with $h(\mathcal{P}_i : \mathcal{S}^{\mathcal{W}}) = 0$. Of course, we have that $\mathcal{R}_i \subseteq \mathcal{P}_i$ since $\mathcal{R}_i \cap \mathcal{M}_i$ is diffuse.

We claim that the $\mathcal{P}_i$'s are freely independent (the three claims in the theorem statement will follow from this, as we explain at the end of the proof). It suffices to show free independence of any separable subalgebras $\mathcal{B}_i$ inside $\mathcal{P}_i$ such that $\mathcal{R}_i \subseteq \mathcal{B}_i$. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and let $\mathcal{Q} = \prod_{n \to \mathcal{U}} M_n(\mathbb{C})$. Let $A^{(n)} = \operatorname{diag}(1, \zeta_n, \ldots, \zeta_n^{n-1})$, and let $a = [A^{(n)}]_{n \in \mathbb{N}} \in \mathcal{Q}$. Fix an embedding $\pi_i : \mathcal{R}_i \to \mathcal{Q}$. Since all Haar unitaries in $\mathcal{Q}$ are conjugate, assume without loss of generality that $\pi_i(a_i) = a$. Let $(V_i^{(n)})_{i \in \mathbb{N}}$ be independent $n \times n$ Haar random unitaries. By Proposition 2.2 and Lemma 3.2), we have that almost surely $\operatorname{tp}^{M_n(\mathbb{C})}(V_1^{(n)} A^{(n)} (V_1^{(n)})^*, V_2^{(n)} A^{(n)} (V_2^{(n)})^*, \ldots)$ converges to the type μ_{Haar} of Lemma 3.2. Thus, fix an outcome ω where this occurs, let $v_i = [V_i^{(n)}(\omega)]_{n \in \mathbb{N}} \in \mathcal{Q}$, and let $u_i = v_i a v_i^*$, so that the type of $(u_1, u_2, \ldots)$ is μ_{Haar}. Note that $v_i \pi_i(\mathcal{R}_i) v_i^*$ contains u_i and is amenable, hence also $\operatorname{Ent}^{\mathcal{U}}(v_i \pi_i(\mathcal{R}_i) v_i^* : \mathcal{Q}) = 0$. Thus, by Theorem 4.4, the algebras $v_i \pi_i(\mathcal{R}_i) v_i^*$ are freely independent. In particular, there is an embedding $\pi : \mathcal{S} = *_{i \in I} \mathcal{R}_i \to \mathcal{Q}$ such that $\pi|_{\mathcal{R}_i} = \operatorname{ad}_{v_j} \circ \pi_i$.

By the downward Löwenheim-Skolem theorem (Proposition 3.4), choose some separable elementary substructure $\widehat{\mathcal{S}}$ of $\mathcal{S}^{\mathcal{W}}$ containing all the $\mathcal{B}_i$'s. We claim that π extends to an embedding $\widehat{\pi} : \widehat{\mathcal{S}} \to \mathcal{Q}$. In more detail, let $\mathbf{x}$ and $\mathbf{y}$ be countable tuple generating $\mathcal{S}$ and $\widehat{\mathcal{S}}$ respectively, and assume they are elements in the unit ball. Since $\mathcal{S} \subseteq \widehat{\mathcal{S}} \subseteq \mathcal{S}^{\mathcal{W}}$, then for every $\epsilon > 0$ and for any finitely many

[3] Since $\mathcal{M}_i$ is not necessarily separable, the index set for $\mathcal{V}'$ may be uncountable.

polynomials $p_1, \dots, p_\ell$ of countably many variables, there exists some tuple $\mathbf{z}$ from the unit ball in $\mathcal{S}$ such that

$$|\operatorname{tr}^{\mathcal{S}}(p_j(\mathbf{x},\mathbf{z})) - \operatorname{tr}^{\mathcal{S}^{\mathcal{W}}}(p_j(\mathbf{x},\mathbf{y}))| < \epsilon.$$

In particular, there exists $\mathbf{w}$ in $\mathcal{Q}$ (namely $\mathbf{w} = \pi(\mathbf{z})$) such that

$$|\operatorname{tr}^{\mathcal{Q}}(p_j(\pi(\mathbf{x}),\mathbf{w})) - \operatorname{tr}^{\mathcal{S}^{\mathcal{W}}}(p_j(\mathbf{x},\mathbf{y}))| < \epsilon.$$

Hence, by countable saturation of $\mathcal{Q}$ (Lemma 3.5), there exists $\mathbf{w}$ in $\mathcal{Q}$ such that

$$\operatorname{tr}^{\mathcal{Q}}(p(\pi(\mathbf{x}),\mathbf{w})) = \operatorname{tr}^{\mathcal{M}^{\mathcal{V}}}(p(\mathbf{x},\mathbf{y}))$$

for all non-commutative polynomials p. Hence, there is a trace-preserving embedding $\widehat{\pi} : \widehat{\mathcal{S}} \to \mathcal{Q}$ such that $\widehat{\pi}(\mathbf{x}) = \pi(\mathbf{x})$ and $\widehat{\pi}(\mathbf{y}) = \mathbf{w}$.

Now

$$h(\widehat{\pi}(\mathcal{B}_i) : \mathcal{Q}) \leq h(\widehat{\pi}(\mathcal{B}_i) : \widehat{\pi}(\widehat{\mathcal{S}})) = h(\mathcal{B}_i : \widehat{\mathcal{S}}) = h(\mathcal{B}_i : \mathcal{S}^{\mathcal{W}}) \leq h(\mathcal{P}_i : \mathcal{S}^{\mathcal{W}}) = 0;$$

here the equality $h(\mathcal{B}_i : \widehat{\mathcal{S}}) = h(\mathcal{B}_i : \mathcal{S}^{\mathcal{W}})$ follows because $h(\mathcal{A}_i : \widehat{\mathcal{M}})$ only depends on the existential type of generators of $\mathcal{A}_i$ in $\widehat{\mathcal{M}}$, and this is the same as its existential type in $\mathcal{M}^{\mathcal{V}}$ since $\widehat{\mathcal{M}}$ is an elementary substructure. Since $\widehat{\pi}(\mathcal{B}_i)$ by construction contains $\operatorname{ad}_{v_i} \circ \pi_i(\mathcal{R}_i)$ and hence contains $v_i a v_i^* = u_i$, we can apply Theorem 4.4 to obtain that the algebras $\pi_i(\mathcal{B}_i)$ are freely independent in $\mathcal{Q}$. This means also that the $\mathcal{B}_i$'s are freely independent in $\mathcal{S}^{\mathcal{W}}$.

Therefore, we have shown that the Pinsker algebras $\mathcal{P}_i$ are freely independent. The claims of the theorem now follow quickly from this, together with the properties of h:

(1) If $\mathcal{A}_i \subseteq \mathcal{M}^{\mathcal{V}} \subseteq \mathcal{S}^{\mathcal{W}}$ and $\mathcal{A}_i \cap \mathcal{M}_i$ is diffuse and $h(\mathcal{A}_i : \mathcal{M}^{\mathcal{V}}) = 0$, then we obtain $h(\mathcal{A}_i : \mathcal{S}^{\mathcal{W}}) \leq h(\mathcal{A}_i : \mathcal{M}^{\mathcal{V}}) = 0$, and therefore $\mathcal{A}_i$ is contained inside the Pinsker algebra $\mathcal{P}_i$. Thus, the $\mathcal{A}_i$'s are freely independent.
(2) Similar to point (1), it suffices to note that $\mathcal{C}_i$ is contained in the Pinsker algebra $\mathcal{P}_i$, which follows from [33, Fact 2.9] as noted in the introduction at Corollary E.
(3) Note that the wq-normalizer $\mathcal{N}_i$ of $\mathcal{M}_i$ in $\mathcal{M}$ is contained in the wq-normalizer $\tilde{\mathcal{N}_i}$ of $\mathcal{M}_i$ in $\mathcal{S}^{\mathcal{W}}$. By Properties 3 and 9 in Section 2.3 of [25], we have $h(\tilde{\mathcal{N}_i} : \mathcal{S}^{\mathcal{W}}) = h(\mathcal{A}_i : \mathcal{S}^{\mathcal{W}}) \leq h(\mathcal{R}_i^{\mathcal{V}'} : \mathcal{S}^{\mathcal{W}}) = 0$. Since $\tilde{\mathcal{N}_i} \cap \mathcal{M}_i$ is diffuse, we have that $\tilde{\mathcal{N}_i}$ is contained in the Pinsker algebra $\mathcal{P}_i$. Thus, the $\mathcal{N}_i$'s are freely independent as desired. □

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Department of Mathematics and Statistics, York University
N520 Ross, 4700 Keele Street, Toronto, ON M3J 1P3

Email address: davidjekel@gmail.com

URL: http://davidjekel.com

Department of Mathematics, University of California, San Diego,
9500 Gilman Drive # 0112, La Jolla, CA 92093

Email address: srivatsav.kunnawalkam.elayavalli@vanderbilt.edu

URL: https://sites.google.com/view/srivatsavke