

Notes on a conjecture by Paszkiewicz on an ordered product of positive contractions

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ABSTRACT. Paszkiewicz's conjecture asserts that given a decreasing sequence $T_1 \geq T_2 \geq \dots$ of positive contractions on a separable infinite-dimensional Hilbert space H , the product $S_n = T_n T_{n-1} \cdots T_1$ converges in the strong operator topology. In these notes, we give an equivalent, more precise formulation of his conjecture. Moreover, we show that the conjecture is true for the following two cases: (1) 1 is not in the essential spectrum of T_n for some $n \in \mathbb{N}$. (2) The von Neumann algebra generated by $\{T_n \mid n \in \mathbb{N}\}$ admits a faithful normal tracial state. We also remark that the analogous conjecture for the weak convergence is true.

1. INTRODUCTION AND STATEMENT OF THE RESULT

In the problem session of 2018 workshop “Noncommutative Harmonic Analysis”, at Będlewo, Paszkiewicz announced the following conjecture about a product of positive contractions.

Conjecture 1.1 (Adam Paszkiewicz, 2018). *Let H be a separable infinite-dimensional Hilbert space, $T_1 \geq T_2 \geq \dots$ be a sequence of positive linear contractions on H . Then the sequence $S_n := T_n T_{n-1} \cdots T_1$ converges strongly.*

In fact, it is possible to make a guess about what the limit of S_n should be, if it exists. Note that since $T_1 \geq T_2 \geq \dots$ is a non-increasing sequence of positive operators, the limit $T := \lim_{n \rightarrow \infty} T_n$ (SOT) exists (SOT stands for the strong operator topology). We will use the notation that for a Borel subset A of $\mathbb{R}$, $1_A(T)$ denotes the spectral projection of T corresponding to A . Let $P := 1_{\{1\}}(T)$. Now consider the following

Conjecture 1.2. *Let $T_1 \geq T_2 \geq \dots$ be as in Conjecture 1.1. Then $\lim_{n \rightarrow \infty} S_n = P$ ($*$ -strongly).*

The purpose of these notes is to show that the above two conjectures are actually equivalent. Moreover, we verify the conjecture for some class of operators:

Theorem 1.3. *The following statements hold.*

- (1) *Conjecture 1.1 and Conjecture 1.2 are equivalent.*
- (2) *Conjecture 1.1 is true if the operators $T_1, T_2, \dots$ satisfy one of the following two conditions.*
 - (2-i) *1 is not in the essential spectrum of T_n for some $n \in \mathbb{N}$.*
 - (2-ii) *$\{T_1, T_2, \dots\}$ generates a finite von Neumann algebra.*

In Proposition 2.3 (1), which will be used to prove the equivalence of the above conjectures, we show that $\lim_{n \rightarrow \infty} S_n^* = P$ (SOT) holds. Thus, (2-ii) is an immediate consequence of this convergence. Note also that because the $*$ -operation is continuous on norm-bounded sets in the WOT (weak operator topology), the WOT analogue of Paszkiewicz's conjecture is true:

Proposition 1.4. *Let $T_1 \geq T_2 \geq \dots$ be as in Conjecture 1.1. Then $\lim_{n \rightarrow \infty} S_n = P$ weakly.*

However, on an infinite-dimensional Hilbert space, the $*$ -operation is highly discontinuous on norm-bounded sets in the SOT. In this context, we remark that by the classical Amemiya–Ando's theorem, random products of projections always converge in the WOT (see [1] for details

and more general results), while the SOT-convergence of random products of projections was shown not to hold in general by Paszkiewicz [4] for random products of 5 projections, and later by Kopecká and Müller [2] for random products of 3 projections (see also [3]). Thus, the difference between the SOT and the WOT is significant.

2. PROOF OF THEOREM 1.3

For general facts about self-adjoint operators and spectral theory, we refer the reader to [5]. In the sequel, we fix a separable infinite-dimensional Hilbert space H . The set of all bounded linear operators on the Hilbert space H is denoted by $\mathbb{B}(H)$.

2.1. Proof of Theorem 1.3 (1) and (2-ii). First, we show Theorem 1.3 (1) and (2-ii). The following elementary lemma will be useful.

Lemma 2.1. *Let $T \in \mathbb{B}(H)$ be a positive contraction. Then for $\xi \in H$, the following three conditions are equivalent.*

- (1) $T\xi = \xi$.
- (2) $\|T\xi\| = \|\xi\|$.
- (3) $\langle T\xi, \xi \rangle = \|\xi\|^2$.

In particular, if $T, T' \in \mathbb{B}(H)_+$ satisfies $0 \leq T' \leq T \leq 1$, then $1_{\{1\}}(T') \leq 1_{\{1\}}(T)$ holds.

Proof. For the first part, (1) $\implies$ (2) and (1) $\implies$ (3) are clear. We show (2) $\implies$ (1). Assume (2). For $0 < \varepsilon < 1$, let $p_\varepsilon = 1_{[1-\varepsilon, 1]}(T)$. Then

$$\begin{aligned} \|\xi\|^2 &= \|T\xi\|^2 = \|Tp_\varepsilon\xi\|^2 + \|Tp_\varepsilon^\perp\xi\|^2 \\ &\leq \|p_\varepsilon\xi\|^2 + (1-\varepsilon)^2\|p_\varepsilon^\perp\xi\|^2 \end{aligned}$$

This implies, because $\|\xi\|^2 = \|p_\varepsilon\xi\|^2 + \|p_\varepsilon^\perp\xi\|^2$, that $p_\varepsilon^\perp\xi = 0$. Thus $\xi = p_\varepsilon\xi$. Since $\lim_{\varepsilon \rightarrow +0} p_\varepsilon = 1_{\{1\}}(T)$ strongly, we obtain $\xi = 1_{\{1\}}(T)\xi$, whence $T\xi = \xi$ holds.

Finally, we show (3) $\implies$ (1). By (3), we have $\|T^{\frac{1}{2}}\xi\| = \|\xi\|$, which by (2) $\implies$ (1) applied to $T^{\frac{1}{2}}$ implies that $T^{\frac{1}{2}}\xi = \xi$. Thus $T\xi = \xi$ holds.

For the last part, if $\xi \in H$ satisfies $T'\xi = \xi$, then by $T' \leq T$, $\|\xi\|^2 = \langle T'\xi, \xi \rangle \leq \langle T\xi, \xi \rangle \leq \|\xi\|^2$. Therefore, $\langle T\xi, \xi \rangle = \|\xi\|^2$ holds. By (3) $\implies$ (1), we obtain $T\xi = \xi$. This shows that $1_{\{1\}}(T') \leq 1_{\{1\}}(T)$. $\square$

Corollary 2.2. *Let $T_1 \geq T_2 \geq \dots$ be a decreasing sequence of positive contractions on a Hilbert space H . Then $P_n = 1_{\{1\}}(T_n)$ converges to $P = 1_{\{1\}}(T)$ in SOT, where $T = \lim_{n \rightarrow \infty} T_n$ (SOT).*

Proof. By Lemma 2.1, we know that $P_1 \geq P_2 \geq \dots$ is a non-increasing sequence of projections. Therefore the SOT-limit $P' = \lim_{n \rightarrow \infty} P_n$ exists and P' is also a projection. Since $T_n \geq T$ for every $n \in \mathbb{N}$, we have $P_n \geq P$ by Lemma 2.1, whence $P' \geq P$ holds. If $\xi \in P'(H)$, then for every $n \in \mathbb{N}$, $P_n \geq P'$, whence $P_n\xi = \xi$. Therefore $T_n\xi = \xi$. Letting $n \rightarrow \infty$, we obtain $T\xi = \xi$, which shows that $\xi \in P(H)$. Therefore $P' \leq P$, and $P' = P$ holds. $\square$

Proof of Theorem 1.3 (1) follows from parts (1) and (2) of the next proposition (part (3) is not needed, but we include the observation which might be useful for further study).

Proposition 2.3. *Let $T_1 \geq T_2 \geq \dots$ be a sequence of positive contractions on H and let $S_n := T_n \cdots T_1$. The following statements hold (WOT stands for the weak operator topology):*

- (1) $\lim_{n \rightarrow \infty} S_n^* = P$ (SOT). In particular, $\lim_{n \rightarrow \infty} S_n = P$ (WOT) holds.
- (2) Let $\xi \in H$. If the set $\{S_n\xi \mid n \in \mathbb{N}\}$ is totally bounded, then $\lim_{n \rightarrow \infty} \|S_n\xi - P\xi\| = 0$ holds.
- (3) For every $\xi \in H$ and every $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \|S_{n+k}\xi - S_n\xi\| = 0$ holds.

Proof. (1) Let $\xi \in H$ and $k \in \mathbb{N}$. Then for each $n \in \mathbb{N}$,

$$\|S_{n+k}^* P^\perp \xi\| = \|T_1 \cdots T_n (T_{n+1} \cdots T_{n+k} P^\perp \xi)\| \leq \|T_{n+1} \cdots T_{n+k} P^\perp \xi\|.$$

Since $T_{n+j} \xrightarrow{n \rightarrow \infty} T$ ($1 \leq j \leq k$) (SOT) and since the operator multiplication is jointly SOT-continuous on the unit ball of $\mathbb{B}(H)$, $T_{n+1} \cdots T_{n+k} \xrightarrow{n \rightarrow \infty} T^k$ (SOT) holds. Thus

$$\limsup_{n \rightarrow \infty} \|S_n^* P^\perp \xi\| = \limsup_{n \rightarrow \infty} \|S_{n+k}^* P^\perp \xi\| \leq \|T^k P^\perp \xi\|.$$

By letting $k \rightarrow \infty$, we obtain

$$\|T^k P^\perp \xi\|^2 = \int_{[0,1)} t^{2k} d\|e_T(t)\xi\|^2 \xrightarrow{k \rightarrow \infty} 0$$

by the Lebesgue dominated convergence theorem. Here, $e_T(\cdot)$ is the spectral resolution of T . This shows that $\lim_{n \rightarrow \infty} \|S_n^* P^\perp \xi\| = 0$. On the other hand, $T_n P\xi = P\xi$, for every $n \in \mathbb{N}$, whence $\lim_{n \rightarrow \infty} S_n^* P\xi = P\xi$. Hence $\lim_{n \rightarrow \infty} \|S_n^* \xi - P\xi\| = 0$.

(2) Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Since $\{S_n \xi \mid n \in \mathbb{N}\}$ is totally bounded and since $S_n P\xi = P\xi$ ($n \in \mathbb{N}$), the set $\{S_n P^\perp \xi \mid n \in \mathbb{N}\}$ is totally bounded. Thus, there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, there exists $k_n \in \{1, \dots, n_0\}$ such that $\|S_n P^\perp \xi - S_{k_n} P^\perp \xi\| < \varepsilon$ holds. Then

$$\begin{aligned} \|S_{n+k} P^\perp \xi\| &= \|T_{n+k} \cdots T_{n+1} S_n P^\perp \xi\| \\ &\leq \|T_{n+k} \cdots T_{n+1} (S_n P^\perp \xi - S_{k_n} P^\perp \xi)\| + \|T_{n+k} \cdots T_{n+1} S_{k_n} P^\perp \xi\| \\ &\leq \|S_n P^\perp \xi - S_{k_n} P^\perp \xi\| + \|T_{n+k} \cdots T_{n+1} S_{k_n} P^\perp \xi\|. \end{aligned}$$

By a similar reasoning as in (i), we get

$$\limsup_{n \rightarrow \infty} \|S_n P^\perp \xi\| = \limsup_{n \rightarrow \infty} \|S_{n+k} P^\perp \xi\| \leq \varepsilon + \max_{1 \leq j \leq n_0} \|T^k S_j P^\perp \xi\|.$$

Since $S_j P^\perp \xi \in P^\perp(H)$ ($1 \leq j \leq n_0$), it follows that $\lim_{k \rightarrow \infty} \max_{1 \leq j \leq n_0} \|T^k S_j P^\perp \xi\| = 0$. Thus $\limsup_{n \rightarrow \infty} \|S_n P^\perp \xi\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\lim_{n \rightarrow \infty} \|S_n P^\perp \xi\| = 0$. Therefore $\lim_{n \rightarrow \infty} \|S_n \xi - P\xi\| = 0$ holds.

(3) It suffices to show that $\lim_{n \rightarrow \infty} \|S_{n+1} \xi - S_n \xi\| = 0$. Define $a_n = \langle S_{n+1} \xi, S_n \xi \rangle$ and $b_n = \langle S_n \xi, S_n \xi \rangle$ ($n \in \mathbb{N}$). Since T_{n+1} is a positive contraction, we have $0 \leq a_n \leq b_n$. Since T_n is a contraction, $(b_n)_{n=1}^\infty$ is positive and non-increasing. Therefore, the limit $\beta = \lim_{n \rightarrow \infty} b_n$ exists. On the other hand, for every $n \in \mathbb{N}$,

$$\begin{aligned} b_{n+1} &= \langle S_{n+1} \xi, S_{n+1} \xi \rangle = \langle T_{n+1}^2 S_n \xi, S_n \xi \rangle \\ &\leq \langle T_{n+1} S_n \xi, S_n \xi \rangle = a_n \leq \langle S_n \xi, S_n \xi \rangle = b_n. \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} a_n = \beta$ holds. Then

$$\begin{aligned} \|S_{n+1} \xi - S_n \xi\|^2 &= \|S_{n+1} \xi\|^2 + \|S_n \xi\|^2 - 2\operatorname{Re}\langle S_{n+1} \xi, S_n \xi \rangle \\ &\xrightarrow{n \rightarrow \infty} \beta + \beta - 2\beta = 0. \end{aligned}$$

This shows the claim. $\square$

Remark 2.4. Let $(\xi_n)_{n=1}^\infty$ be a sequence in H with the following properties.

- (i) $\lim_{n \rightarrow \infty} \xi_n = 0$ weakly in H .
- (ii) $(\|\xi_n\|)_{n=1}^\infty$ is non-increasing (hence it is convergent).
- (iii) For every $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \|\xi_{n+k} - \xi_n\| = 0$ holds.

If it follows that $\{\xi_n \mid n \in \mathbb{N}\}$ is totally bounded, then $\lim_{n \rightarrow \infty} S_n = P$ (SOT) by Proposition 2.3

(ii) (put $\xi_n = S_n P^\perp \xi$). We remark, however, that the set $\{\xi_n \mid n \in \mathbb{N}\}$ satisfying (i), (ii) and (iii) need not be totally bounded in general.

Non-example 2.5. Let $\theta_n = \frac{\pi}{2n}$ ($n \in \mathbb{N}$). Fix a CONS $(e_n)_{n=1}^\infty$ for H . We will construct a sequence $\mathcal{S} = \{\eta_{n,j} \mid n \in \mathbb{N}, 1 \leq j \leq n\}$ of unit vectors in H with the following properties:

- (a) $\|\eta_{n,j+1} - \eta_{n,j}\| = \|\eta_{n+1,1} - \eta_{n,n}\| = 2 \sin \frac{\theta_n}{2}$ ($n \geq 2, 1 \leq j \leq n-1$).
- (b) $\eta_{n,1} = e_n$ ($n \in \mathbb{N}$). In particular, $\{e_n \mid n \in \mathbb{N}\} \subset \mathcal{S}$ holds.

(c) $\eta_{n,j} \in \text{span}\{e_n, e_{n+1}\}$ ($n \in \mathbb{N}$, $1 \leq j \leq s_n$).

Define a linear ordering $<$ on $I = \{(n, j) \mid n \in \mathbb{N}, 1 \leq j \leq n\}$ by $(n, j) < (n', j')$ if $n < n'$ or $n = n'$ and $j < j'$. Fix the order-preserving bijection $\langle \cdot, \cdot \rangle: I \rightarrow \mathbb{N}$ given by

$$\langle n, j \rangle = j + \frac{(n-1)n}{2}, \quad 1 \leq j \leq n.$$

We set

$$\eta_{n,j} := \cos((j-1)\theta_n)e_n + \sin((j-1)\theta_n)e_{n+1}, \quad n \in \mathbb{N}, 1 \leq j \leq n$$

By construction, (b) and (c) hold. Both the angle between $\eta_{n,j+1}$ and $\eta_{n,j}$ and the angle between $\eta_{n+1,1}$ and $\eta_{n,n}$ are θ_n if $n \geq 2$ and $1 \leq j \leq n$. Therefore (a) holds.

Then the sequence $(\xi_n)_{n=1}^\infty$ given by $\xi_{\langle n,j \rangle} = \eta_{n,j}$ ($(n, j) \in I$) does the job. Indeed, by (a), (iii) holds. It is obvious that (ii) holds. (i) holds because of (c). However, by (b), $\{\xi_n\}_{n=1}^\infty$ is not relatively compact, hence it is not totally bounded. Note that the sequence $(\xi_n)_{n=1}^\infty$ we constructed is of the form

$$\xi_n = U_n U_{n-1} \cdots U_1 e_1,$$

where each U_n is a unitary such that $\text{rank}(U_n - 1) = 2$ for every $n \in \mathbb{N}$.

Proof of Theorem 1.3 (1) and (2-ii). It is clear that Conjecture 1.2 $\implies$ Conjecture 1.1 holds. Conversely, assume that Conjecture 1.1 holds and $S = \lim_{n \rightarrow \infty} S_n$ (SOT). Then for each $\xi \in H$, $(S_n \xi)_{n=1}^\infty$ converges, whence it is totally bounded. Therefore by Proposition 2.3 (i) and (ii), $\lim_{n \rightarrow \infty} S_n = P$ (S*OT). Therefore Conjecture 1.2 holds. This proves (1). (2-ii) follows from Proposition 2.3 (i) and the fact that the map $V \mapsto V^*$ is strongly continuous on the unit ball of a finite von Neumann algebra. $\square$

2.2. Proof of Theorem 1.3 (2). Here we prove Theorem 1.3 (2).

Definition 2.6. Let $T_1 \geq T_2 \geq \dots$ be a decreasing sequence of positive contractions on a Hilbert space H . We say that it has uniform spectral gap at 1, if there exist $\delta \in (0, 1)$ and $N \in \mathbb{N}$ such that $\sigma(T_n) \cap (1 - \delta, 1) = \emptyset$ for all $n \geq N$.

Theorem 1.3 (2) follows from the next Proposition.

Proposition 2.7. Let $T_1 \geq T_2 \geq \dots$ be a decreasing sequence of positive contractions on a Hilbert space H .

- (1) If $T_1 \geq T_2 \geq \dots$ has uniform spectral gap at 1. Then the Conjecture 1.1 holds for $T_1 \geq T_2 \geq \dots$.
- (2) If 1 is not in the essential spectrum of T_n for some $n \in \mathbb{N}$, then $T_1 \geq T_2 \geq \dots$ has uniform spectral gap at 1.

Proof of Proposition 2.7 (1). Fix δ and N witnessing the uniform spectral gap at 1 of $T_1 \geq T_2 \geq \dots$. Let $\xi \in H$. Assume first that $\xi \in P(H)$. Then for all $n \in \mathbb{N}$, $T_n \xi = \xi$, so that $S_n \xi = \xi$. Thus $\lim_{n \rightarrow \infty} S_n \xi = \xi = P\xi$. Next, assume that $\xi \in P^\perp(H)$ and $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $n_0 \geq N$ and $\|\xi - P_{n_0}^\perp \xi\| < \varepsilon$. Let $\eta := T_{n_0} \cdots T_1 \xi$ and $\eta' := T_{n_0} \cdots T_1 P_{n_0}^\perp \xi$. Then because all T_i 's are contractions, we have $\|\eta - \eta'\| \leq \|\xi - P_{n_0}^\perp \xi\| < \varepsilon$. Note also that $\eta' \in P_{n_0}^\perp(H)$, because all $T_1, \dots, T_{n_0}$ leave the range of $P_{n_0}^\perp$ invariant, hence the range of $P_{n_0}^\perp$ invariant. Since $P_{n_0}^\perp \leq P_{n_0+1}^\perp$, we have $\eta' \in P_{n_0+1}^\perp(H)$. Thus $T_{n_0+1} \eta' \in P_{n_0+1}^\perp(H) \subset P_{n_0+2}^\perp(H)$, so that $T_{n_0+2} T_{n_0+1} \eta' \in P_{n_0+3}^\perp(H)$. By induction, we obtain

$$\eta'_j := T_{n_0+j} T_{n_0+j-1} \cdots T_{n_0+1} \eta' \in P_{n_0+j+1}^\perp(H) = 1_{[0, 1-\delta]}(T_{n_0+j+1})(H), \quad j \in \mathbb{N}.$$

Therefore

$$\begin{aligned}
\|T_{n_0+j}(\underbrace{T_{n_0+j-1} \cdots T_{n_0+1}}_{=\eta'_{j-1} \in P_{n_0+j}^\perp(H)} \eta')\| &\leq \|T_{n_0+j}|_{P_{n_0+j}^\perp(H)}\| \|T_{n_0+j-1} \eta'_{j-2}\| \\
&\leq \cdots \leq \prod_{k=1}^j \|T_{n_0+k}|_{P_{n_0+k}^\perp(H)}\| \|\eta'\| \\
&\leq (1-\delta)^j \|\eta'\|.
\end{aligned}$$

This shows that

$$\begin{aligned}
\|S_{n_0+j} \xi\| &= \|T_{n_0+j} \cdots T_{n_0+1} \eta\| \\
&\leq \|T_{n_0+j} \cdots T_{n_0+1}(\eta - \eta')\| + (1-\delta)^j \|\eta'\| \\
&< \varepsilon + (1-\delta)^j \|\eta'\|.
\end{aligned}$$

Thus $\limsup_{j \rightarrow \infty} \|S_{n_0+j} \xi\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\lim_{n \rightarrow \infty} \|S_n \xi\| = 0$. Therefore for general $\xi \in H$, we have

$$\|S_n \xi - P \xi\| \leq \|S_n(P \xi) - P \xi\| + \|S_n P^\perp \xi\| \xrightarrow{n \rightarrow \infty} 0.$$

This shows that $\lim_{n \rightarrow \infty} S_n = P$ (SOT). $\square$

For the proof of Proposition 2.7 (2), we need the following lemma. We denote by $\sigma_e(T)$ the essential spectrum of an operator T .

Lemma 2.8. *Let T, T' be positive contractions on a Hilbert space H such that $T \geq T'$.*

- (1) *If $\delta \in (0, 1)$ satisfies $\sigma(T) \cap (1 - \delta, 1) = \emptyset$ and $\sigma(T') \cap (1 - \delta, 1) \neq \emptyset$, then $P' \not\leq P$ holds, where $P := 1_{\{1\}}(T)$ and $P' := 1_{\{1\}}(T')$.*
- (2) *If $1 \in \sigma_e(T')$, then $1 \in \sigma_e(T)$ holds.*

Proof. (1) By $0 \leq T' \leq T \leq 1$ and Lemma 2.1, we know that $P' \leq P$ holds. Assume by contradiction that $P' = P$ holds. Let $t \in \sigma(T') \cap (1 - \delta, 1)$ and $\varepsilon > 0$ be such that $1 - \delta < t - \varepsilon$ and $t + \varepsilon < 1$. Then there exists a nonzero vector $\xi \in 1_{(t-\varepsilon, t+\varepsilon)}(T')(H)$. Since $P' = 1_{\{1\}}(T')$ and $1_{(t-\varepsilon, t+\varepsilon)}(T')$ are orthogonal, we have $P' \xi = P \xi = 0$. This implies that $\xi = P^\perp \xi = 1_{[0, 1-\delta]}(T) \xi$. Then by $T' \leq T$, we obtain

$$(1) \quad (t - \varepsilon) \|\xi\|^2 \leq \langle T' \xi, \xi \rangle \leq \langle T \xi, \xi \rangle = \langle T 1_{[0, 1-\delta]}(T) \xi, \xi \rangle \leq (1 - \delta) \|\xi\|^2,$$

which contradicts the condition $t - \varepsilon > 1 - \delta$. Therefore $P' \not\leq P$ holds.

(2) By $1 \in \sigma_e(T')$, there exists an orthonormal sequence $(\xi_n)_{n=1}^\infty$ in H such that $\lim_{n \rightarrow \infty} \|T' \xi_n - \xi_n\| = 0$. Thus $\lim_{n \rightarrow \infty} \|T' \xi_n\| = 1 = \lim_{n \rightarrow \infty} \langle T' \xi_n, \xi_n \rangle$ holds. Then by $0 \leq T' \leq T \leq 1$, we have

$$\begin{aligned}
\|T \xi_n - \xi_n\|^2 &= \|T \xi_n\|^2 - 2 \langle T \xi_n, \xi_n \rangle + \|\xi_n\|^2 \\
&\leq 2 - 2 \langle T \xi_n, \xi_n \rangle \\
&\leq 2 - 2 \langle T' \xi_n, \xi_n \rangle \xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

Thus, by Weyl's criterion for the essential spectrum (see e.g., [5, Proposition 8.11]), $1 \in \sigma_e(T)$ holds. $\square$

Proof of Proposition 2.7 (2). Assume that $1 \notin \sigma_e(T_{n_0})$. Let $P_{n_0} = 1_{\{1\}}(T_{n_0})$. By $1 \notin \sigma_e(T_{n_0})$, 1 is not an accumulation point of the spectrum $\sigma(T_{n_0})$ of T_{n_0} , and it is not an eigenvalue of T_{n_0} of infinite multiplicity either. Thus there exists $\delta_0 \in (0, 1)$ such that $\sigma(T_{n_0}) \cap (1 - \delta_0, 1) = \emptyset$, and $d = \text{rank}(P_{n_0})$ is finite (possibly $d = 0$). If there is no $n > n_0$ such that $\sigma(T_n) \cap (1 - \delta_0, 1) \neq \emptyset$, then $\delta = \delta_0$ and $N = n_0$ work. If there is such an $n > n_0$, let n_1 be the smallest such number. Then by $\sigma(T_{n_1}) \cap (1 - \delta_0, 1) \neq \emptyset$, $T_{n_1} \leq T_1$ and Lemma 2.8 (1), we have $\text{rank}(P_{n_1}) < \text{rank}(P_{n_0}) = d < \infty$ (thus $d \neq 0$ if such n_1 exists). By Lemma 2.8 (2), $1 \notin \sigma_e(T_n)$ for every $n \geq n_0$. Thus, by the above argument, we may find $0 < \delta_1 < \delta_0$ such that $\sigma(T_{n_1}) \cap (1 - \delta_1, 1) = \emptyset$. If there is no $n > n_1$ such that $\sigma(T_n) \cap (1 - \delta_1, 1) \neq \emptyset$, we set $N = n_1$ and $\delta = \delta_1$. If there is such an $n > n_1$,

let n_2 be the smallest such number, and find $0 < \delta_2 < \delta_1$ such that $\sigma(T_{n_2}) \cap (1 - \delta_2, 1) = \emptyset$. Then $\text{rank}(P_{n_2}) < \text{rank}(P_{n_1})$. Inductively, we find a sequence $n_1 < n_2 < \cdots$ and $\delta_1 > \delta_2 > \cdots$. These sequences must have the same length at most d . Let k be the length of these sequences. Then $N = n_k$ and $\delta = \delta_k$ work. $\square$

As a corollary, we obtain the following result, which was in fact the earliest and motivational result in this project, shown to us by Yasumichi Matsuzawa. The author would like to thank him for sharing his proof.

Corollary 2.9 (Matsuzawa). *Conjecture 1.1 holds if T_n is compact for some $n \in N$.*

ACKNOWLEDGMENTS

The current work started as a joint work with Yasumichi Matsuzawa (Shinshu University) in 2018. However, he declined to be a coauthor and suggested that the current author write the paper alone.

Nevertheless, we would like to emphasize that it was exactly his proof of Corollary 2.9 which motivated us to generalize his proof, and the outcome is the main result of this paper. Therefore, his contribution to this work is essential. The author is supported by Japan Society for the Promotion of Sciences KAKENHI 20K03647.

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