BOUNDEDNESS OF LOG FANO CONE SINGULARITIES AND DISCRETENESS OF LOCAL VOLUMES

CHENYANG XU AND ZIQUAN ZHUANG

Abstract. We prove that in any fixed dimension, K-semistable log Fano cone singularities whose volumes are bounded from below by a fixed positive number form a bounded set. As a consequence, we show that the set of local volumes of klt singularities of a fixed dimension has zero as the only accumulation point.

1. Introduction

In recent years, there has been remarkable progress in the algebro-geometric study of K-stability. Besides the global theory for Fano varieties, a local stability theory has also been introduced for Kawamata log terminal (klt) singularities, see [LLX20, Zhu23b] for an overview. Notably, the stable degeneration conjecture is settled in [XZ22] (see also [Blu18, LX18, Xu20, XZ21]). It provides for any klt singularity $x \in (X = \mathrm{Spec}(R), \Delta)$, a canonical degeneration to a K-semistable log Fano cone singularity $x_0 \in (X_0, \Delta_0; \xi_v)$. More precisely, let v be a valuation minimizing the normalized volume function

$$\widehat{\mathrm{vol}}_{X,\Delta}\colon \mathrm{Val}_{X,x} \to \mathbb{R}_{>0} \cup \{+\infty\}$$

defined as in [Li18], then the corresponding degeneration is obtained as $X_0 = \mathrm{Spec}(\mathrm{Gr}_v R)$ with a cone vertex $x_0 \in X_0$, Δ_0 is the corresponding degeneration of Δ and ξ_v the Reeb vector induced by the valuation v. This degeneration is volume-preserving, i.e. it satisfies

$$\widehat{\mathrm{vol}}(x, X, \Delta) = \widehat{\mathrm{vol}}_{X,\Delta}(v) = \widehat{\mathrm{vol}}_{X_0,\Delta_0}(\mathrm{wt}_{\xi_v}) = \widehat{\mathrm{vol}}(x_0, X_0, \Delta_0)\,. \tag{1.1}$$

To complete the picture of the local stability theory, one central open problem (see e.g. [XZ22, Conjecture 1.7]) is the boundedness of K-semistable n-dimensional log Fano cone singularities $(X, \Delta; \xi)$, assuming we fix a lower bound of the local volume

$$\widehat{\mathrm{vol}}(x, X, \Delta) = \widehat{\mathrm{vol}}_{X,\Delta}(\mathrm{wt}_\xi)\,.$$

The aim of this paper is to settle this local boundedness question. Our main theorem is the following. The relevant definitions are recalled in Section 2.

Theorem 1.1. *Let $n \in \mathbb{N}$ and let $I \subseteq [0, 1]$ be a finite set. Let $\varepsilon > 0$ be a positive real number. Let $\mathcal{K}$ be the set of K-semistable n-dimensional polarized log Fano cone singularities $x \in (X, \Delta; \xi)$ with coefficients in I such that*

$$\widehat{\mathrm{vol}}(x, X, \Delta) \geq \varepsilon.$$

Then $\mathcal{K}$ is bounded.

By the volume-preserving property of the degeneration (1.1), Theorem 1.1 implies that all n-dimensional klt singularities with a positive lower bound on their volumes, are bounded up to degeneration. In particular, we have the following consequence which gives a positive answer to [LLX20, Question 6.12].

Theorem 1.2. *Fix $n \in \mathbb{N}$ and a finite set $I \subseteq [0,1]$. Then the set*

$$\widehat{\mathrm{Vol}}_{n,I} = \left\{ \widehat{\mathrm{vol}}(x, X, \Delta) \mid x \in (X, \Delta) \text{ is klt, } \dim(X) = n, \operatorname{Coeff}(\Delta) \subseteq I \right\}$$

has 0 as the only accumulation point.

Philosophically, one can compare Theorem 1.2 with [HMX14, Theorem 1.3] which deals with the global case of log general type pairs, where volumes might accumulate even when the coefficients belong to a finite set, though they still satisfy the descend chain condition (DCC) property. Special cases of Theorem 1.2 have been previously established, including when X is bounded [HLQ23], (X, Δ) is of complexity at most one [MS21, LMS23], X is of dimension at most three [LMS23, Zhu23a], or assuming (X, Δ) admits a Kollár component with bounded log discrepancy [Zhu23a].

To prove Theorem 1.1, we need to sort out a birational geometric condition that is more flexible than K-semistability so that it is preserved under small perturbations of the polarization. In particular, we want to include the case when ξ is a rational perturbation of the Reeb vector coming from the K-semistable log Fano cone structure. This is why instead of proving Theorem 1.1 directly, we aim to prove the following more general statement.

Theorem 1.3. *Let $n \in \mathbb{N}$ and let $I \subseteq [0,1]$ be a finite set. Let $\varepsilon, \theta > 0$. Let $\mathcal{S}$ be the set of n-dimensional polarized log Fano cone singularities $x \in (X, \Delta; \xi)$ with coefficients in I such that*

$$\widehat{\mathrm{vol}}(x, X, \Delta) \geq \varepsilon, \quad \text{and} \quad \Theta(X, \Delta; \xi) \geq \theta.$$

Then $\mathcal{S}$ is bounded.

Here $\Theta(X, \Delta; \xi)$ is the volume ratio of the log Fano cone singularity, see Definition 2.7. It serves as a local analog of the alpha invariants of Fano varieties. A polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ is K-semistable if and only if $\Theta(X, \Delta; \xi) = 1$, thus Theorem 1.3 implies Theorem 1.1. An observation from [Zhu23a] is that the condition of volume ratio having a lower bound should be the right generalization of the K-semistable condition to guarantee boundedness.

Under the assumption that $\Theta(X, \Delta; \xi) \geq \theta$, the condition that $\widehat{\mathrm{vol}}(x, X, \Delta)$ has a uniform positive lower bound is equivalent to $\widehat{\mathrm{vol}}_{X,\Delta}(\mathrm{wt}_\xi)$ being uniformly bounded from below by a positive number. When ξ is rational, the latter condition is equivalent to a global condition on the orbifold base. This leads to a special case for this version of Theorem 1.3.

Corollary 1.4. *Fix a positive integer n, and two positive numbers α_0, ε. Consider all $(n-1)$-dimensional log Fano pairs (V, Δ_V) such that there exist some $r > 0$ and some Weil divisor L on V which satisfy*

$$\alpha(V, \Delta_V) \geq \alpha_0, \ -(K_V + \Delta_V) \sim_{\mathbb{Q}} rL \quad \text{and} \quad r(-K_V - \Delta_V)^{n-1} \geq \varepsilon\,.$$

Let $\mathcal{S}$ be the set of log Fano cone singularities given by

$$X = \operatorname{Spec} \bigoplus_m H^0(V, mL)$$

for all possible (V, Δ, L) as above and Δ the closure of the pull back of Δ_V on X. Then $\mathcal{S}$ is bounded.

We note that, somewhat surprisingly, in Corollary 1.4 all such (V, Δ, L) themselves are not bounded.

1.5 (Strategy of the proof)**.** The corresponding global result of Theorem 1.3 has been proved in [Jia20], where it is shown that Fano varieties whose volume and alpha invariant are bounded away from zero form a bounded set. To prove the boundedness of the log Fano cone singularities $x \in (X, \Delta; \xi)$, one wants to reduce it to some boundedness question for projective Fano type varieties. A natural candidate would be the quotient

$$(V, \Delta_V) := ((X, \Delta) \setminus \{x\})/\langle \xi \rangle$$

(we may assume ξ is rational after a perturbation and hence it generates a $\mathbb{G}_m$-action). While one can show that the alpha invariant of the log Fano pair (V, Δ_V) is bounded from below (see [Zhu24, Zhu23a]), however, the volume $(-K_V - \Delta_V)^{n-1}$ could be arbitrarily small (see Example 2.8). In particular, (V, Δ_V) is not necessarily bounded, which posts a major challenge in the proof.

A better candidate, first proposed in [Zhu23a], is the projective orbifold cone compactification $(\overline{X}, \overline{\Delta})$ of (X, Δ), which adds a divisor isomorphic to V at infinity. One piece of evidence from [Zhu23a] is that if the local volume of the singularity $x \in (X, \Delta)$ is bounded from below, then so is the volume of log Fano pair $(\overline{X}, \overline{\Delta})$. On the other hand, the alpha invariant of $(\overline{X}, \overline{\Delta})$ can be arbitrarily small and the log Fano pairs $(\overline{X}, \overline{\Delta})$ still do not form a bounded family as we vary the Reeb vector ξ. To get around this problem, the arguments in [Zhu23a] rely on an additional subtle assumption on the Kollár components of the singularities, and it is not clear if this extra assumption is always satisfied. In this paper, we take a different path and follow a strategy which in spirit is closer to [Bir19, Jia20].

The first step is to prove a birational version of [Jia20]. More specifically, we first show (see Section 2.4) that if the volume ratio $\Theta(X, \Delta; \xi)$ is bounded from below, then away from the divisor at infinity, the alpha invariant of $(\overline{X}, \overline{\Delta})$ is also bounded from below, i.e. there is some uniform $\alpha_0 > 0$ such that $\alpha_{x_1}(\overline{X}, \overline{\Delta}) \geq \alpha_0$ for all $x_1 \in X$. By adapting the arguments of [Jia20], we then show (see Section 3.1) that this together with boundedness of the volume imply that the set of projective orbifold cones $(\overline{X}, \overline{\Delta})$ is log birationally bounded (Definition 2.19). In particular, they are birational to a bounded set of pairs.

The next step is to improve the log birational boundedness to boundedness in codimension one, see Section 3.2. Inspired by [Bir19], and using the fact that $\alpha_{x_1}(\overline{X}, \overline{\Delta}) \geq \alpha_0$ away from the divisor at infinity, we show that there is a uniform way to modify the bounded birational model Y obtained from the previous step, so that the only exceptional divisor of the induced birational map $\overline{X} \dashrightarrow Y$ is the divisor at infinity. In other words, $Y \dashrightarrow \overline{X}$ is close to a birational contraction except possibly over one divisor. This is the best we can hope for, as the divisor at infinity depends on the Reeb vector, and in general cannot be extracted on a bounded model. The main ingredient for this step is the construction of sub-klt bounded complements of $(\overline{X}, \overline{\Delta})$ with certain control on the negative part. This in turn relies on the boundedness of complements proved in [Bir19, Theorem 1.7], as well as the birationally bounded model constructed in the previous step.

Finally, to finish the argument, we recover (X, Δ) by running a carefully chosen minimal model program on Y and using the affineness of X to show that X is embedded as an open set of the ample model obtained from the minimal model program sequence. See Section 3.3.

Acknowledgement. CX is partially supported by NSF DMS-2139613, DMS-2201349 and a Simons Investigator grant. ZZ is partially supported by the NSF Grants DMS-2240926, DMS-2234736, a Clay research fellowship, as well as a Sloan fellowship.

2. Preliminaries

2.1. **Notation and conventions.** We work over an algebraically closed field $\mathbb{k}$ of characteristic 0. We follow the standard terminology from [KM98, Kol13].

A sub-pair (X, Δ) consists of a normal variety X together with an $\mathbb{R}$-divisor Δ on X (a priori, we do not require that $K_X + \Delta$ is $\mathbb{R}$-Cartier). It is called a pair if Δ is effective. A log smooth pair (Y, Σ) consists of a smooth variety Y and a simple normal crossing divisor Σ on Y. We say that a pair (X, Δ) is log Fano if (X, Δ) is klt ([KM98, Definition 2.34]) and $-(K_X + \Delta)$ is $\mathbb{R}$-Cartier and ample.

A singularity $x \in (X, \Delta)$ consists of a pair (X, Δ) and a closed point $x \in X$. We will always assume that X is affine and $x \in \mathrm{Supp}(\Delta)$ (whenever $\Delta \neq 0$). We say that the singularity is klt if (X, Δ) is klt in a neighbourhood of x.

Given an $\mathbb{R}$-divisor Δ on X and a birational map $\varphi\colon Y \dashrightarrow X$, we denote the strict transform of Δ on the birational model Y by Δ_Y, i.e. $\Delta_Y = \varphi_*^{-1}\Delta$. If Δ is $\mathbb{R}$-Cartier, the birational pullback $\varphi^*\Delta$ is defined as the $\mathbb{R}$-divisor $f_*g^*\Delta$ where $f\colon W \to Y$, $g\colon W \to X$ is a common resolution.

When we refer to a constant C as $C = C(n, \varepsilon, \cdots)$ it means C only depends on $n, \varepsilon, \cdots$, etc.

2.2. **Local volumes.** We first briefly recall the definition of the local volumes of klt singularities [Li18]. For this we need the notion of valuations. A *valuation* over a singularity $x \in X$ is an $\mathbb{R}$-valued valuation $v : K(X)^* \to \mathbb{R}$ (where $K(X)$ denotes the function field of X) such that v is centered at x (i.e. if $f \in \mathcal{O}_{X,x}$, then $v(f) > 0$ if and only if $f \in \mathfrak{m}_x$) and $v|_{\mathbb{k}^*} = 0$. The set of such valuations is denoted as $\mathrm{Val}_{X,x}$. Let $x \in (X, \Delta)$ be a singularity and assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The *log discrepancy* function

$$A_{X,\Delta}\colon \mathrm{Val}_{X,x} \to \mathbb{R} \cup \{+\infty\},$$

is defined as in [JM12] and [BdFFU15, Theorem 3.1]. It generalizes the usual log discrepancies of divisors; in particular, for divisorial valuations, i.e. valuations of the form $\lambda \cdot \mathrm{ord}_F$ where $\lambda > 0$ and F is a prime divisor on some proper birational model $\pi\colon Y \to X$, we have

$$A_{X,\Delta}(\lambda \cdot \mathrm{ord}_F) = \lambda \cdot A_{X,\Delta}(F) = \lambda \cdot (1 + \mathrm{ord}_F(K_Y - \pi^*(K_X + \Delta))).$$

For klt singularities, one has $A_{X,\Delta}(v) > 0$ for all $v \in \mathrm{Val}_{X,x}$. We denote by $\mathrm{Val}^*_{X,x}$ the set of valuations $v \in \mathrm{Val}_X$ with center x and $A_{X,\Delta}(v) < +\infty$. The *volume* of a valuation $v \in \mathrm{Val}_{X,x}$ is defined as

$$\mathrm{vol}(v) = \mathrm{vol}_{X,x}(v) = \limsup_{m\to\infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!},$$

where $n = \dim X$ and $\mathfrak{a}_m(v)$ denotes the valuation ideal, i.e.

$$\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\}.$$

Definition 2.1. Let $x \in (X, \Delta)$ be an n-dimensional klt singularity. For any $v \in \mathrm{Val}_{X,x}$, we define the *normalized volume* of v as

$$\widehat{\mathrm{vol}}_{X,\Delta}(v) := \begin{cases} A_{X,\Delta}(v)^n \cdot \mathrm{vol}_{X,x}(v) & \text{if } A_{X,\Delta}(v) < +\infty \\ +\infty & \text{if } A_{X,\Delta}(v) = +\infty \end{cases}.$$

The *local volume* of $x \in (X, \Delta)$ is defined as

$$\widehat{\mathrm{vol}}(x, X, \Delta) := \inf_{v \in \mathrm{Val}^*_{X,x}} \widehat{\mathrm{vol}}_{X,\Delta}(v).$$

By [Li18, Theorem 1.2], the local volume of a klt singularity is always positive.

2.3. **Log Fano cone singularities.** Next we recall the definition of log Fano cone singularities and their K-semistability.

Definition 2.2. Let $X = \mathrm{Spec}(R)$ be a normal affine variety and $\mathbb{T}$ an algebraic torus (i.e. $\mathbb{T} \cong \mathbb{G}_m^r$ for some $r > 0$). We say that a $\mathbb{T}$-action on X is *good* if it is effective and there is a unique closed point $x \in X$ that is in the orbit closure of any $\mathbb{T}$-orbit. We call x the vertex of the $\mathbb{T}$-variety X.

Let $N := N(\mathbb{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ be the co-weight lattice and $M = N^*$ the weight lattice. We have a weight decomposition

$$R = \oplus_{\alpha \in M} R_\alpha,$$

and the action being good implies that $R_0 = \mathbb{k}$ and every R_α is finite dimensional. For $f \in R$, we denote by f_α the corresponding component in the above weight decomposition.

Definition 2.3. A *Reeb vector* on X is a vector $\xi \in N_\mathbb{R}$ such that $\langle \xi, \alpha \rangle > 0$ for all $0 \neq \alpha \in M$ with $R_\alpha \neq 0$. The set $\mathfrak{t}^+_\mathbb{R}$ of Reeb vectors is called the Reeb cone.

For any $\xi \in \mathfrak{t}^+_\mathbb{R}$, we can define a valuation wt_ξ by setting

$$\mathrm{wt}_\xi(f) := \min\{\langle \xi, \alpha \rangle \mid \alpha \in M, f_\alpha \neq 0\}$$

where $f \in R$. It is not hard to verify that $\mathrm{wt}_\xi \in \mathrm{Val}_{X,x}$.

Definition 2.4. A log Fano cone singularity is a klt singularity that admits a nontrivial good torus action. A polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ consists of a log Fano cone singularity $x \in (X, \Delta)$ together with a Reeb vector ξ (called a polarization).

By abuse of convention, a good $\mathbb{T}$-action on a klt singularity $x \in (X, \Delta)$ means a good $\mathbb{T}$-action on X such that x is the vertex and Δ is $\mathbb{T}$-invariant. Using terminology from Sasakian geometry, we say a polarized log Fano cone $x \in (X, \Delta; \xi)$ is *quasi-regular* if ξ generates a $\mathbb{G}_m$-action (i.e. ξ is a real multiple of some element of N); otherwise, we say that $x \in (X, \Delta; \xi)$ is *irregular*. The torus generated by ξ will be denoted by $\langle \xi \rangle$.

Definition 2.5 ([CS18, CS19], [LX18, Theorem 2.34])**.** We say that a polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ is *K-semistable* if

$$\widehat{\mathrm{vol}}(x, X, \Delta) = \widehat{\mathrm{vol}}_{X,\Delta}(\mathrm{wt}_\xi).$$

Log Fano cone singularities play a special role in the local stability theory of klt singularities, due to the following statement. It was originally known as the Stable Degeneration Conjecture.

Theorem 2.6. *Every klt singularity $x \in (X = \mathrm{Spec}(R), \Delta)$ has a special degeneration to a K-semistable log Fano cone singularity $x_0 \in (X_0, \Delta_0; \xi_v)$ with*

$$\widehat{\mathrm{vol}}(x, X, \Delta) = \widehat{\mathrm{vol}}(x_0, X_0, \Delta_0).$$

More precisely, up to rescaling, there is a unique valuation v minimizing $\widehat{\mathrm{vol}}_{X,\Delta}$, and $X_0 = \mathrm{Spec}(\mathrm{Gr}_v(R))$. In addition, denote by Δ_0 the degeneration of Δ, and ξ_v the Reeb vector induced by v, then $(X_0, \Delta_0; \xi_v)$ is a K-semistable log Fano cone.

Proof. See [Blu18, LX18, Xu20, XZ21, XZ22] for the case of rational coefficients and the extension to real coefficients in [Zhu23a]. □

Definition 2.7. The *volume ratio* of a polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ is defined to be

$$\Theta(X, \Delta; \xi) := \frac{\widehat{\mathrm{vol}}(x, X, \Delta)}{\widehat{\mathrm{vol}}_{X,\Delta}(\mathrm{wt}_\xi)}.$$

By definition, $0 < \Theta(X, \Delta; \xi) \le 1$ and $\Theta(X, \Delta; \xi) = 1$ if and only if $x \in (X, \Delta; \xi)$ is K-semistable.

2.4. **Orbifold cones.** Every quasi-regular polarized log Fano cone singularity has a natural projective orbifold cone compactification. This provides a convenient way to think about these singularities. In this subsection, we recall this construction and relate some invariants of the singularities with those of the projective orbifold cones. For more background, see [Kol04] or [Zhu23a, Section 3.1].

Let $x \in (X, \Delta; \xi)$ be a quasi-regular log Fano cone singularity. By definition, the $\langle \xi \rangle \cong \mathbb{G}_m$-action on $X \setminus \{x\}$ has finite stabilizers, hence the quotient map

$$X \setminus \{x\} \to V := (X \setminus \{x\})/\langle \xi \rangle$$

is a Seifert $\mathbb{G}_m$-bundle (in the sense of [Kol04]), and there is an ample $\mathbb{Q}$-divisor L on V such that (see [Kol04, Theorem 7])

$$X = \mathrm{Spec} \bigoplus_{m \in \mathbb{N}} H^0(V, \lfloor mL \rfloor)\,.$$

The zero section V_0 of this Seifert $\mathbb{G}_m$-bundle gets contracted to the closed point $x \in X$ (as a valuation, ord_{V_0} is proportional to wt_ξ), thus we can compactify X to $\overline{X}$ by adding the infinity section V_∞. Let $\overline{\Delta}$ be the closure of Δ on $\overline{X}$. We call $(\overline{X}, \overline{\Delta} + V_\infty)$ the (*projective*) *orbifold cone compactification* of $x \in (X, \Delta; \xi)$ (*cf.* [Zhu23a, Section 3.1]).

Note that $(\overline{X}, \overline{\Delta} + V_\infty)$ is plt and $-(K_{\overline{X}} + \overline{\Delta} + V_\infty)$ is ample, see [Zhu23a, Lemma 3.3] or [Kol04]. By adjunction along $V_\infty \cong V$, we may write

$$(K_{\overline{X}} + \overline{\Delta} + V_\infty)|_{V_\infty} = K_V + \Delta_V$$

for some effective divisor Δ_V. Then (V, Δ_V) is a klt log Fano pair. We call (V, Δ_V) the orbifold base of the singularity $x \in (X, \Delta; \xi)$. There exists some $r > 0$ such that $-(K_V + \Delta_V) \sim_{\mathbb{R}} rL$, and we have

$$-(K_{\overline{X}} + \overline{\Delta} + V_\infty) \sim_{\mathbb{R}} rV_\infty\,. \tag{2.1}$$

A subtle feature of the local boundedness problem is that the orbifold bases do not belong to a bounded set; already their volumes can be arbitrarily small when we fix the singularity $x \in (X, \Delta)$ and vary the Reeb vector ξ.

Example 2.8. Let $(x \in X) = (0 \in \mathbb{A}^n)$ and $\xi = (\xi_1, \dots, \xi_n)$ for some pairwise coprime positive integers ξ_i $(1 \le i \le n)$. Assume that $n \ge 3$ and $\xi_1 \le \cdots \le \xi_n$. Then

$$\overline{X} = \mathbb{P}(1, \xi_1, \dots, \xi_n), \quad V \cong \mathbb{P}(\xi_1, \dots, \xi_n), \quad \Delta_V = 0, \quad \text{and} \quad L = \mathcal{O}(1).$$

We can easily compute

$$\Theta(\mathbb{A}^n; \xi) = \frac{n^n}{\widehat{\mathrm{vol}}_{\mathbb{A}^n}(\mathrm{wt}(\xi))} = \frac{\xi_1 \cdots \xi_n \cdot n^n}{(\xi_1 + \cdots + \xi_n)^n}.$$

So $\Theta(\mathbb{A}^n, \xi)$ has a positive lower bound if and only if $\frac{\xi_n}{\xi_1}$ has an upper bound. Using [BJ20, Corollary 7.16], one can show that this is also equivalent to the condition that the α-invariant $\alpha(V)$ defined below in Definition 2.10 has a positive lower bound.

On the other hand,

$$\mathrm{vol}(-K_V) = \frac{(\xi_1 + \cdots + \xi_n)^{n-1}}{\xi_1 \cdots \xi_n}.$$

So if $\frac{\xi_n}{\xi_1}$ is bounded from above, then $\mathrm{vol}(-K_V)$ is bounded away from zero if and only if all the weights ξ_i are bounded from above.

A key observation from [Zhu23a], following a direct calculation using (2.1), is that the volume of a log Fano cone singularity is more closely related to the global volume of its projective orbifold cone compactification (rather than the orbifold base).

Lemma 2.9. *Notation as above. Then we have*

$$\widehat{\mathrm{vol}}_{X,\Delta}(\mathrm{wt}_\xi) = \mathrm{vol}(-(K_{\overline{X}} + \overline{\Delta} + V_\infty)).$$

In particular, if $\Theta(X, \Delta; \xi) \ge \theta > 0$, *then* $\mathrm{vol}(-(K_{\overline{X}} + \overline{\Delta} + V_\infty)) \le n^n \theta^{-1}$.

Proof. The equality is [Zhu23a, Lemma 3.4]. The other implication then follows from [LX19, Theorem 1.6]. □

We next relate the volume ratio with the α-invariants of the orbifold base or the projective orbifold cone. First we recall some definitions.

Definition 2.10. Let (X, Δ) be a projective klt pair and let L be a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We define the α-invariant $\alpha(X, \Delta; L)$ as

$$\alpha(X, \Delta; L) := \inf \{\mathrm{lct}(X, \Delta; D) \mid 0 \le D \sim_{\mathbb{R}} L\},$$

where $\mathrm{lct}(X, \Delta; D)$ denotes the log canonical threshold, i.e. the largest number λ such that $(X, \Delta + \lambda D)$ is log canonical. For any projective pair (X, Δ) that is klt at a closed point $x \in X$, we can similarly define the log canonical threshold $\mathrm{lct}_x(X, \Delta; D)$ at x and the local α-invariant

$$\alpha_x(X, \Delta; L) := \inf \{\mathrm{lct}_x(X, \Delta; D) \mid 0 \le D \sim_{\mathbb{R}} L\}.$$

When the pair (X, Δ) is clear from the context, we will just write $\alpha(L)$ and $\alpha_x(L)$. For a log Fano pair (X, Δ), we define $\alpha(X, \Delta) := \alpha(X, \Delta; -K_X - \Delta)$ and similarly $\alpha_x(X, \Delta)$.

While at a point on the infinity divisor V_∞, the α-invariant of the projective orbifold cone $(\overline{X}, \overline{\Delta})$ could be very small when r is large in (2.1), the following result roughly says that for any point outside the infinity divisor, the local α-invariant of $(\overline{X}, \overline{\Delta})$ is bounded by the (global) α-invariant of the orbifold base.

Lemma 2.11. *Let $x \in (X, \Delta; \xi)$ be a quasi-regular polarized log Fano cone singularity. Let $(\overline{X}, \overline{\Delta} + V_\infty)$ be its projective orbifold cone compactification, and let (V, Δ_V) be the orbifold base. Then we have*

$$\alpha_{x_1}(\overline{X}, \overline{\Delta} + V_\infty) \geq \min\{1, \alpha(V, \Delta_V)\}$$

for all closed point $x_1 \in X$. In particular, $\alpha_x(\overline{X}, \overline{\Delta} + V_\infty) = \min\{1, \alpha(V, \Delta_V)\}$.

Proof. Let $D_V \sim_{\mathbb{R}} -(K_V + \Delta_V)$ be an effective $\mathbb{R}$-divisor and let D be the closure in $\overline{X}$ of its pullback to $X \setminus \{x\}$. Then we have $D \sim_{\mathbb{R}} -(K_{\overline{X}} + \overline{\Delta} + V_\infty)$ and $\mathrm{ord}_{V_0}(D) = A_{X,\Delta}(V_0)$ (for usual cones see [Kol13, Proposition 3.14], the general case follows from the computations in [Kol04, Section 4]). In particular, $\alpha_x(\overline{X}, \overline{\Delta} + V_\infty) \leq 1$. Since $X \setminus \{x\} \to V$ is a Seifert $\mathbb{G}_m$-bundle, the pair $(V, \Delta_V + tD_V)$ is log canonical if and only if $(\overline{X}, \overline{\Delta} + tD)$ is log canonical on $X \setminus \{x\}$. Thus we also get $\alpha_x(\overline{X}, \overline{\Delta} + V_\infty) \leq \alpha(V, \Delta_V)$.

Suppose that there exists some effective $\mathbb{R}$-divisor $D \sim_{\mathbb{R}} -(K_{\overline{X}} + \overline{\Delta} + V_\infty)$ such that $t := \mathrm{lct}_{x_1}(\overline{X}, \overline{\Delta} + V_\infty; D) < \min\{1, \alpha(V, \Delta_V)\}$ for some $x_1 \in X$. Since $x_1 \notin V_\infty$ and V_∞ is ample, we may assume that $V_\infty \notin \mathrm{Supp}(D)$. The non-klt locus of the pair $(\overline{X}, \overline{\Delta} + V_\infty + tD)$ thus contains at least x_1 and V_∞. Since $-(K_{\overline{X}} + \overline{\Delta} + V_\infty + tD)$ is ample, Kollár-Shokurov's connectedness lemma implies that $(\overline{X}, \overline{\Delta} + V_\infty + tD)$ is not plt along V_∞. It then follows from adjunction that $(V, \Delta_V + tD|_{V_\infty})$ (we identify V with V_∞) is not klt, and hence $\alpha(V, \Delta_V) \leq t$, a contradiction. In other words, we have

$$\alpha_{x_1}(\overline{X}, \overline{\Delta} + V_\infty) \geq \min\{1, \alpha(V, \Delta_V)\}.$$

Combined with the upper bounds of $\alpha_x(\overline{X}, \overline{\Delta} + V_\infty)$ we obtain above, this also gives $\alpha_x(\overline{X}, \overline{\Delta} + V_\infty) = \min\{1, \alpha(V, \Delta_V)\}$. □

We now relate the volume ratio with the α-invariant of the orbifold base.

Lemma 2.12. *There exists some constant $c > 0$ depending only on the dimension such that for any quasi-regular polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ of dimension n with orbifold base (V, Δ_V), we have*

$$c \cdot \alpha(V, \Delta_V) \geq \Theta(X, \Delta; \xi) \geq \min\{1, \alpha(V, \Delta_V)\}^n.$$

Proof. This essentially follows from [Zhu23a, Lemma 3.12 and Remark 3.13]. We provide a (slightly different) proof for the reader's convenience. Let $(\overline{X}, \overline{\Delta} + V_\infty)$ be the associated projective orbifold cone as before. Let $D_V \sim_{\mathbb{R}} -(K_V + \Delta_V)$ be an effective $\mathbb{R}$-divisor and let D be the closure in X of its pullback to $X \setminus \{x\}$. Then $\mathrm{wt}_\xi(D) = A_{X,\Delta}(\mathrm{wt}_\xi)$. The uniform Izumi inequality in [Zhu24, Lemma 3.4] thus implies that

$$\mathrm{lct}_x(X, \Delta; D) \geq c_0 \cdot \Theta(X, \Delta; \xi)$$

for some constant $c_0 = c_0(n) > 0$. But we also have $\mathrm{lct}(V, \Delta_V; D_V) \geq \mathrm{lct}_x(X, \Delta; D)$ as in the proof of Lemma 2.11. As D_V is arbitrary, this gives the first inequality with $c = c_0^{-1}$.

By Lemma 2.11 and the following Lemma 2.13, we have

$$\begin{aligned}\widehat{\mathrm{vol}}(x, X, \Delta) &\geq \alpha_x(\overline{X}, \overline{\Delta} + V_\infty)^n \cdot \mathrm{vol}(-(K_{\overline{X}} + \overline{\Delta} + V_\infty)) \\ &= \min\{1, \alpha(V, \Delta_V)\}^n \cdot \mathrm{vol}(-(K_{\overline{X}} + \overline{\Delta} + V_\infty))\,.\end{aligned}$$

On the other hand, we have $\widehat{\mathrm{vol}}_{X,\Delta}(\mathrm{wt}_\xi) = \mathrm{vol}(-(K_{\overline{X}} + \overline{\Delta} + V_\infty))$ by Lemma 2.9. This gives the second inequality. □

We have used the following statement, which is well-known to experts.

Lemma 2.13. *Let (X, Δ) be a pair of dimension n that is klt at a closed point x, and let L be a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then we have*

$$\widehat{\mathrm{vol}}(x, X, \Delta) \geq \alpha_x(X, \Delta; L)^n \cdot \mathrm{vol}(L).$$

Proof. Let $t = \alpha(X, \Delta; L)$. Suppose that $x \in X$ is a smooth point and $\mathrm{vol}(L) > \frac{n^n}{t^n}$. Then it is well-known, by a simple dimension count, that there exists some effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ such that $\mathrm{mult}_x D > \frac{n}{t}$; in particular,

$$\alpha_x(X, \Delta; L) \leq \mathrm{lct}_x(\overline{X}, \overline{\Delta}; D) < t\,,$$

a contradiction. We can apply the same dimension counting argument at a singular point $x \in X$, as long as we replace mult_x by the minimizing valuation of the normalized volume function, and n^n by the local volume $\widehat{\mathrm{vol}}(x, X, \Delta)$. □

2.5. **Bounded family.** In this subsection we define various notions of boundedness.

Definition 2.14. We call $(\mathcal{X}, \mathcal{D}) \to B$ a family of pairs if $\mathcal{X}$ is flat over B, the fibers $\mathcal{X}_b$ are connected, normal and not contained in $\mathrm{Supp}(\mathcal{D})$.

We call $B \subseteq (\mathcal{X}, \mathcal{D}) \to B$ an $\mathbb{R}$-Gorenstein family of klt singularities (over a normal but possibly disconnected base B) if

(1) $(\mathcal{X}, \mathcal{D}) \to B$ is a family of pairs, and $B \subseteq \mathcal{X}$ is a section,
(2) $K_{\mathcal{X}/B} + \mathcal{D}$ is $\mathbb{R}$-Cartier and $b \in (\mathcal{X}_b, \mathcal{D}_b)$ is a klt singularity for all $b \in B$.

Definition 2.15. We say that a set $\mathcal{C}$ of sub-pairs is bounded if there exists a family $(\mathcal{X}, \mathcal{D}) \to B$ of pairs over a finite type base B such that for any $(X, D) \in \mathcal{C}$, there exists a closed point $b \in B$ and an isomorphism $(X, \mathrm{Supp}(D)) \cong (\mathcal{X}_b, \mathrm{Supp}(\mathcal{D}_b))$.

Definition 2.16 ([Zhu23a, Definition 2.16])**.** We say that a set $\mathcal{S}$ of polarized log Fano cone singularities is bounded if there exists finitely many $\mathbb{R}$-Gorenstein families $B_i \subseteq (\mathcal{X}_i, \mathcal{D}_i) \to B_i$ of klt singularities over finite type bases, each with a fiberwise good $\mathbb{T}_i$-action for some nontrivial algebraic torus $\mathbb{T}_i$, such that every $x \in (X, \Delta; \xi)$ in $\mathcal{S}$ is isomorphic to $b \in (\mathcal{X}_{i,b}, \mathcal{D}_{i,b}; \xi_b)$ for some i, some $b \in B_i$ and some $\xi_b \in N(\mathbb{T}_i)_{\mathbb{R}}$.

A priori, it may happen that a set of log Fano cone singularities is bounded as a set of sub-pairs, but becomes unbounded when we take into account the log Fano cone structure. Nonetheless, the two boundedness notions coincide if the volume ratios are bounded away from zero and the coefficients belong to a fixed finite set.

Lemma 2.17. *Let $\theta > 0$ and let $I \subseteq [0, 1]$ be a finite set. Let $\mathcal{S}$ be a set of polarized log Fano cone singularities $x \in (X, \Delta; \xi)$ with coefficients in I and $\Theta(X, \Delta; \xi) \geq \theta$. Assume that the underlying set of pairs is bounded. Then $\mathcal{S}$ is bounded.*

Proof. Let $f\colon (\mathcal{X},\mathcal{D})\to B$ be a family of pairs over a finite type base such that for any $x\in (X,\Delta;\xi)$ in $\mathcal{S}$, we have an isomorphism $(X,\mathrm{Supp}(\Delta))\cong(\mathcal{X}_b,\mathrm{Supp}(\mathcal{D}_b))$ for some $b\in B$. After base change along $\mathcal{X}\to B$ and possibly stratifying B, we may assume that f admits a section $\sigma\colon B\to\mathcal{X}$ so that the above isomorphism induces an isomorphism

$$(x\in (X,\mathrm{Supp}(\Delta)))\cong(\sigma(b)\in(\mathcal{X}_b,\mathrm{Supp}(\mathcal{D}_b)))\,.$$

Since the coefficients belong to the finite set I, we may also assign coefficients to $\mathcal{D}$ and assume that $(X,\Delta)\cong(\mathcal{X}_b,\mathcal{D}_b)$. After these reductions, by [Kol23, Lemma 4.44] (or rather its proof) and inversion of adjunction, we know that there exists a finite collection of locally closed subset B_i of B such that the family $(\mathcal{X},\mathcal{D})$ becomes $\mathbb{R}$-Gorenstein after base change to $\sqcup_i B_i$ and enumerates exactly all the klt fibers of $B\subseteq(\mathcal{X},\mathcal{D})\to B$ (*cf.* the last part of the proof of [Zhu23a, Theorem 3.1]). Thus by replacing B with $\sqcup_i B_i$, we may assume that $B\subseteq(\mathcal{X},\mathcal{D})\to B$ is an $\mathbb{R}$-Gorenstein family of klt singularities to begin with. Since $\Theta(X,\Delta;\xi)\ge\theta$ by assumption, together with [Zhu23a, Lemma 2.15 and Theorem 3.1], we then see that the set $\mathcal{S}$ is bounded as a set of log Fano cone singularities. $\square$

We also recall the definition of log birational boundedness. For an $\mathbb{R}$-divisor G, we denote its positive part by G^+ and negative part by G^-, i.e. $G=G^+-G^-$ where G^+, G^- are effective without common components.

Definition 2.18. Let (X,G) and (Y,Σ) be projective sub-pairs. We say that (Y,Σ) log birationally dominates (X,G) if there exist a birational map $\varphi\colon Y\dashrightarrow X$ such that $\mathrm{Supp}(\Sigma)$ contains the birational transform of $\mathrm{Supp}(G)$ and the exceptional divisors of φ, i.e. $\mathrm{Supp}(\Sigma)\supseteq\mathrm{Supp}(\varphi^{-1}_*G)+\mathrm{Ex}(\varphi)$. We say that (Y,Σ) log birationally dominates (X,G) effectively if in addition the φ^{-1}-exceptional divisors are contained in $\mathrm{Supp}(G^-)$.

We will say (Y,Σ) log birationally dominates (X,G) (effectively) through φ if we want to specify the birational map $\varphi\colon X\dashrightarrow Y$.

Note that if (Y,Σ) log birationally dominates (X,G) with G being $\mathbb{R}$-Cartier, and G' is the birational pullback of G, then $\mathrm{Supp}(G')\subseteq\mathrm{Supp}(\Sigma)$.

Definition 2.19. Let $\mathcal{C}$ be a set of projective sub-pairs and let $\mathcal{P}$ be a set of projective pairs. We say that $\mathcal{P}$ log birationally dominates $\mathcal{C}$ (resp. log birationally dominates $\mathcal{C}$ effectively) if any $(X,G)\in\mathcal{C}$ is log birationally (resp. log birationally and effectively) dominated by some $(Y,\Sigma)\in\mathcal{P}$.

We say that $\mathcal{C}$ is log birationally bounded if there exists a bounded set $\mathcal{P}$ of pairs that log birationally dominates $\mathcal{C}$ (*cf.* [HMX13, Definition 2.4.1]).

The following criterion for log birational boundedness is a special case of [HMX13, Lemma 3.2] or [Bir19, Proposition 4.4].

Proposition 2.20. *Let n be a positive integer and let $c_0,c_1>0$. Let $\mathcal{C}$ be the set of pairs $(X,\Delta+\Gamma)$ of dimension n such that*

- $-(K_X+\Delta)$ *is ample,*
- *the non-zero coefficients of Δ are at least c_0,*
- Γ *is a $\mathbb{Q}$-Cartier, effective, nef Weil divisor,*
- $|\Gamma|$ *defines a birational map and* $\mathrm{vol}(\Gamma)\le c_1$.

Then $\mathcal{C}$ is log birationally bounded. More precisely, there exists a bounded set $\mathcal{P}$ of projective log smooth pairs (Y,Σ) depending only on n,c_0,c_1 such that the following are satisfied:

For any $(X, \Delta+\Gamma) \in \mathcal{C}$, there exist some log smooth pair $(Y, \Sigma) \in \mathcal{P}$ and a birational map $\varphi\colon Y \dashrightarrow X$ such that:

(1) *(Y, Σ) log birationally dominates $(X, \Delta + \Gamma)$ through φ.*
(2) *There exists some effective and big Cartier divisor $A \leq \Sigma$ on Y such that $|A|$ is base point free and $|\Gamma - (\varphi^{-1})^* A| \neq \emptyset$.*

Proof. Log birational boundedness follows from [Bir19, Proposition 4.4(1)], which also gives the property (1). Property (2) follows from [Bir19, Proposition 4.4(3)] (or from the construction of the bounded set $\mathcal{P}$ in *loc. cit.*, as A is simply the birational transform of the movable part of $|\Gamma|$). $\square$

3. Boundedness

In this section, we give the proof of our main theorems. The main statement is Theorem 1.3, and we divide its proof into three parts, as outlined in 1.5.

3.1. **Log birational boundedness.** To prove Theorem 1.3, we first aim to show that the log Fano cone singularities have log birationally bounded projective orbifold cone compactifications. From Section 2.4, we have seen that the local alpha invariants of the projective orbifold cones are bounded from below away from the divisor at infinity, and their volumes are also bounded. The situation is thus somewhat similar to those of [Jia20]. Our first step is to refine some of the arguments in [Jia20] to prove an effective birationality result. Log birationally boundedness is then an immediate consequence.

In the global (Fano) setting, [Jia20] proceeds as follow. In order to show that $|-mK_X|$ defines a birational map for some fixed integer m, one aims to create isolated non-klt centers on the Fano variety X. The main observation from [Jia20] is that if both the alpha invariant and the volume are bounded from below, then the volumes of any covering family of subvarieties are also bounded from below, and this allows one to cut down the dimension of the non-klt centers. The next two lemmas show that this strategy still work if we replace the global alpha invariant by the local one.

Lemma 3.1 (*cf.* [Jia20, Lemma 3.1])**.** *Let X be a normal projective variety of dimension n and L a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let $f\colon Y \to T$ be a projective morphism and $\mu\colon Y \to X$ a surjective morphism. Assume that a general fiber F of f is of dimension k and is mapped birationally onto its image G in X. Then for any general smooth point $x \in X$, we have*

$$\mathrm{vol}(L|_G) \geq \frac{\alpha_x(L)^{n-k}}{\binom{n}{k}(n-k)^{n-k}} \mathrm{vol}(L).$$

Proof. This follows from [Jia20, Lemma 3.1] with some small modifications. We sketch the argument for the reader's convenience. By perturbing the coefficients and rescaling, we may assume that L is Cartier. Replacing f by its Stein factorization, we may assume that F is connected. We also assume that Y and T are smooth by taking log resolution. Moreover, by the Bertini Theorem we may replace T by a general complete intersection subvariety and assume that μ is generically finite. In particular, it is étale at the generic point of F (since F is a general fiber). We may also choose F so that $x \in G$. Clearly it suffices to consider the case when $k < n$.

Let $t = f(F) \in T$ and $l \in \mathbb{Q}_+$. By a direct calculation (using that F has trivial normal bundle in Y), we have

$$h^0(Y, \mu^* L^{\otimes m} \otimes \mathcal{O}_Y / \mathcal{I}_F^{lm}) \leq h^0(F, \mu^* L^{\otimes m}) \cdot h^0(\mathcal{O}_T / \mathfrak{m}_t^{lm}) + O(m^{n-1})$$

for sufficiently large and divisible integers m. Hence if

$$\frac{\mathrm{vol}(L)}{n!} > \frac{\mathrm{vol}(L|_G) \cdot l^{n-k}}{k! \cdot (n-k)!}, \tag{3.1}$$

then $h^0(X, mL) > h^0(Y, \mu^* L^{\otimes m} \otimes \mathcal{O}_Y / \mathcal{I}_F^{lm})$ for $m \gg 0$. It follows that there exists some effective divisor $D \sim_{\mathbb{Q}} L$ such that $\mathrm{mult}_F(\mu^* D) \geq l$; as μ is étale at the generic point of F, this also implies that $\mathrm{mult}_G D \geq l$ and therefore

$$\alpha_x(L) \leq \mathrm{lct}_x(D) \leq \frac{n-k}{l}$$

as G has codimension $n - k$ in X. This holds for every l that satisfies (3.1); the lemma then follows. □

Lemma 3.2 (*cf.* [Jia20, Theorem 1.5])**.** *Let $\varepsilon, \alpha > 0$. Let X be a normal projective variety of dimension n, and let L be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $(L^n) \geq \varepsilon$ and $\alpha_x(L) \geq \alpha$ for a general point $x \in X$. Then there exists some positive integer $m_0 = m_0(n, \varepsilon, \alpha)$ such that $|K_X + \lceil mL \rceil|$ defines a birational map for all $m \geq m_0$.*

Proof. The assumptions and Lemma 3.1 imply that there exists some $m_0 = m_0(n, \varepsilon, \alpha) > 0$ such that $\mathrm{vol}(mL|_G) > (2k)^k$ (where $k = \dim G$) for any general member G of a covering family of positive dimensional subvarieties of X and all $m \geq m_0$. The argument in [Bir19, 2.31(2)] implies that mL is potentially birational ([HMX14, Definition 3.5.3]), and then the lemma follows from [HMX13, Lemma 2.3.4]. □

We can now prove the effective birationality of the orbifold cone compactifications.

Proposition 3.3. *Fix some positive integer n, a finite coefficient set $I \subseteq [0,1] \cap \mathbb{Q}$, and some positive real numbers $\varepsilon, \theta > 0$. Then there exist some positive integer $m = m(n, \varepsilon, \theta, I)$ such that $m \cdot I \subseteq \mathbb{N}$ and for any quasi-regular polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ with*

$$\dim X = n, \quad \mathrm{Coef}(\Delta) \subseteq I, \quad \widehat{\mathrm{vol}}(X, \Delta; \xi) \geq \varepsilon \quad \text{and} \quad \Theta(X, \Delta; \xi) \geq \theta, \tag{3.2}$$

the following statements hold for its orbifold cone compactification $(\overline{X}, \overline{\Delta} + V_\infty)$:

1. *The pair $(\overline{X}, \overline{\Delta} + V_\infty)$ has an m-complement.*
2. *The linear system $|-m(K_{\overline{X}} + \overline{\Delta} + V_\infty)|$ defines a birational map.*

Here we define an m-complement of a pair (X, D) as an effective $\mathbb{Q}$-divisor Γ such that $(X, D + \Gamma)$ is log canonical and $m(K_X + D + \Gamma) \sim 0$.

Proof. Item (1) is the boundedness of complements proved in [Bir19, Theorem 1.7]. Let us prove (2). Let $L = -(K_{\overline{X}} + \overline{\Delta} + V_\infty)$. By Lemma 2.9 and our assumption on the local volume, we have $\mathrm{vol}(L) \geq \varepsilon$. By Lemmas 2.11 and 2.12, there exists some positive number $\alpha = \alpha(n, \theta) > 0$ such that $\alpha_{x_1}(L) = \alpha_{x_1}(\overline{X}, \overline{\Delta} + V_\infty) \geq \alpha$ for all $x_1 \in X$. Thus

Lemma 3.2 guarantees the existence of some positive integer $m = m(n, \varepsilon, \theta, I)$ such that $m\Delta$ has integer coefficients and

$$|K_{\overline{X}} + \lceil (m+1)L \rceil| = |mL - V_\infty|$$

defines a birational map. It follows that $|mL|$ also defines a birational map. By taking common multiples, we get a positive integer $m = m(n, \varepsilon, \theta, I) > 0$ such that (1) and (2) simultaneously hold. □

From Lemmas 2.9, we know that

$$\mathrm{vol}(M) \leq m^n \mathrm{vol}(-(K_{\overline{X}} + \overline{\Delta} + V_\infty)) \leq (mn)^n \theta^{-1}.$$

Thus by Proposition 2.20, this immediately implies that the set of orbifold cone compactifications of quasi-regular log Fano cone singularities satisfying (3.2) is log birationally bounded. Choose some (log bounded) birational model $\varphi\colon (Y, \Sigma) \dashrightarrow \overline{X}$ of the projective orbifold cone $\overline{X}$. Our next task is to reconstruct X from Y.

3.2. **Boundedness in codimension one.** To reconstruct X, we need to first understand the exceptional divisors of the birational map $\varphi^{-1}\colon \overline{X} \dashrightarrow Y$. Note that the infinity divisor V_∞ is typically φ^{-1}-exceptional, and since it depends on the choice of the Reeb vector ξ, we will not have much control over it. The next result shows that other than V_∞, the remaining φ^{-1}-exceptional divisors are essentially "bounded". To state it precisely let us make one more definition.

Definition 3.4. Let (X, Δ) be a pair and let N be a positive integer. A sub-klt N-complement of (X, Δ) is a (not necessarily effective) $\mathbb{Q}$-divisor G on X such that $N(K_X + \Delta + G) \sim 0$ and $(X, \Delta + G)$ is sub-klt.

Proposition 3.5. *Fix some positive integer n, a finite coefficient set $I \subseteq [0,1] \cap \mathbb{Q}$, and some positive real numbers $\varepsilon, \theta > 0$. There exist a bounded set $\mathcal{P}$ of projective log smooth pairs (Y, Σ) and a positive integer $N = N(n, \varepsilon, \theta, I)$, such that the following holds for any quasi-regular polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ satisfying* (3.2)*:*

(1) *$(\overline{X}, \overline{\Delta})$ has a sub-klt N-complement G such that* $\mathrm{Supp}(G^-) \subseteq V_\infty \subseteq \mathrm{Supp}(G)$.
(2) *There exists some $(Y, \Sigma) \in \mathcal{P}$ that log birationally dominates $(\overline{X}, \overline{\Delta} + G)$ effectively* (*Definition 2.18*).

Informally, the implication (2) means that the log Fano cone singularities are bounded in codimension one. The existence of a sub-klt bounded complement will later be used to ensure that certain MMPs exist and terminate.

The proof of the proposition is inspired by the proof of [HMX14, Theorem 1.6] and [Bir19, Proposition 7.13]. The main technical part is to construct a bounded sub-klt complement of $(\overline{X}, \overline{\Delta})$ satisfying certain conditions. We first discuss how the existence of such a complement affects boundedness in codimension one.

Lemma 3.6. *Let N be a positive integer and let $\mathcal{C}$ be a set of projective sub-klt sub-pairs (X, G) satisfying $N(K_X + G) \sim 0$. Assume that $\mathcal{C}$ is log birationally bounded. Then there exists a bounded set $\mathcal{P}$ of projective log smooth pairs that log birationally dominates $\mathcal{C}$ effectively* (*Definition 2.19*).

Proof. By assumption, we may choose a bounded set $\mathcal{P}$ of projective pairs that log birationally dominates $\mathcal{C}$. After passing to a log resolution, we may assume that $\mathcal{P}$ is bounded set of log smooth pairs. For any $(X, G) \in \mathcal{C}$, let $(Y, \Sigma) \in \mathcal{P}$ be a log smooth pair that log birationally dominates (X, G) through a birational map $\varphi \colon Y \dashrightarrow X$. Write $\varphi^*(K_X + G) = K_Y + G_Y$. Then G_Y is a sub-klt N-complement of Y supported on Σ (see the remark after Definition 2.18). In particular, $G_Y \le (1 - \frac{1}{N})\Sigma$. The discrepancy of any φ^{-1}-exceptional divisor F must satisfy

$$a(F; Y, (1 - \frac{1}{N})\Sigma) \le a(F; Y, G_Y) = a(F; X, G) \le 0,$$

unless $F \subseteq \mathrm{Supp}(G^-)$. Since (Y, Σ) is log smooth, the pair $(Y, (1-\frac{1}{N})\Sigma)$ is klt, hence there are only finitely many exceptional divisors with discrepancy at most 0, and these can be extracted via successive blowups along the strata of Σ. In other words, up to replacing the bounded set $\mathcal{P}$ of log smooth pairs, we may assume that the only φ^{-1}-exceptional divisors are among the components of G^-, thus $\mathcal{P}$ also dominates $\mathcal{C}$ effectively. □

We next construct the sub-klt bounded complements on the projective orbifold cones.

Lemma 3.7. *There exists a positive integer $N = N(n, \varepsilon, \theta, I)$ such that for any quasi-regular polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ satisfying* (3.2)*, there exists a sub-klt N-complement G of $(\overline{X}, \overline{\Delta})$ such that*

(1) $\mathrm{Supp}(G^-) \subseteq V_\infty \subseteq \mathrm{Supp}(G)$*, and*
(2) $-(K_{\overline{X}} + \overline{\Delta} + V_\infty) - \frac{1}{2}G^+$ *is ample.*

Proof. We follow the argument of [Bir19, Proposition 7.13]. Let $m = m(n, \varepsilon, \theta, I) > 0$ be the integer given by Proposition 3.3. In particular, there exists an m-complement $\Gamma \in \frac{1}{m}|M|$ where $M = -m(K_{\overline{X}} + \overline{\Delta} + V_\infty)$. By Proposition 2.20 as in the remark right after Proposition 3.3, we find a bounded set $\mathcal{P}$ of projective log smooth pairs (Y, Σ) depending only on $n, \varepsilon, \theta, I$, such that for any $x \in (X, \Delta; \xi)$ satisfying (3.2) and any m-complement Γ as above, there exists some log smooth pair $(Y, \Sigma) \in \mathcal{P}$ and some birational map $\varphi \colon \overline{X} \dashrightarrow Y$ such that:

(1) (Y, Σ) log birationally dominates $(\overline{X}, \overline{\Delta} + V_\infty + \Gamma)$ through φ.
(2) There exists some effective and big Cartier divisor $A \le \Sigma$ on Y such that $|A|$ is base point free and $|M - (\varphi^{-1})^*A| \neq \emptyset$.

Define Γ_Y by the crepant pullback formula

$$K_Y + \Gamma_Y = \varphi^*(K_{\overline{X}} + \overline{\Delta} + V_\infty + \Gamma) \sim_{\mathbb{Q}} 0.$$

Since (Y, Σ) belongs to a bounded family, We can also choose some positive integer m_0 depending only on $\mathcal{P}$, and some $\mathbb{Q}$-divisor $B = B^+ - B^-$ in a bounded family where

$$B^+ \in |A| \quad \text{and} \quad m_0 B^- \in |m_0 A|,$$

such that B^+ is in a general position (by Bertini theorem) and $\Sigma \subseteq \mathrm{Supp}(B^-)$ (this is possible since A is big). By construction, $\mathrm{Supp}(\Gamma_Y) \subseteq \Sigma$ and (Y, Γ_Y) is sub-lc. Hence the pair $(Y, \Gamma_Y + B)$ is sub-klt and $K_Y + \Gamma_Y + B \sim_{\mathbb{Q}} 0$. Its crepant pullback to $\overline{X}$ is $(\overline{X}, \overline{\Delta} + V_\infty + \Gamma + B_X)$ where

$$B_X^+ = (\varphi^{-1})^* B^+ \quad \text{and} \quad B_X^- = (\varphi^{-1})^* B^- .$$

In particular, as B^+ and $m_0 B^-$ are both Cartier, the coefficients of B_X belongs to $\frac{1}{m_0}\mathbb{Z}$.

Choose some $R \in |M - (\varphi^{-1})^*A|$. We may write

$$B_X^- + R = \lambda V_\infty + mC$$

where $V_\infty \not\subseteq \operatorname{Supp}(C)$. Note that $B_X^- \sim_{\mathbb{Q}} (\varphi^{-1})^*A$ and thus

$$B_X^- + R \sim_{\mathbb{Q}} M = -m(K_{\overline{X}} + \overline{\Delta} + V_\infty)\,,$$

hence $C \sim_{\mathbb{Q}} -\mu(K_{\overline{X}} + \overline{\Delta} + V_\infty)$ for some $\mu \leq 1$. We also note that the coefficients of C are contained in $\frac{1}{mm_0}\mathbb{Z}$.

By Lemma 2.12, the orbifold base satisfies $\alpha(V, \Delta_V) \geq \alpha_0$ for some positive constant $\alpha_0 = \alpha_0(n, \varepsilon, \theta, I) > 0$. We may assume that $\alpha_0 < 1$. By adjunction, this implies that $(\overline{X}, \overline{\Delta} + V_\infty + \alpha_0 C)$ is log canonical in a neighbourhood of V_∞. By Lemma 2.11, we also know that $(\overline{X}, \overline{\Delta} + V_\infty + \alpha_0 C)$ is log canonical away from V_∞. Hence the pair $(\overline{X}, \overline{\Delta} + V_\infty + \alpha_0 C)$ is log canonical everywhere. By [Bir19, Theorem 1.7], it has an N-complement $C' \geq 0$ for some positive integer N that only depends on the dimension and the coefficients; tracing through the construction above, this in turn means that N only depends on n, ε, θ and the finite set I. Now consider the linear combination

$$G := V_\infty + t(\Gamma + B_X) + (1-t)(\alpha_0 C + C').$$

for some fixed rational number $t \in (0,1)$ such that $mt \leq (1-t)\alpha_0 < 1$ and $\frac{m_0}{t} \notin \mathbb{Z}$. As $\operatorname{mult}_{V_\infty} B_X \in \frac{1}{m_0}\mathbb{Z}$ and $V_\infty \not\subseteq \operatorname{Supp}(\Gamma + C + C')$, the second condition on t simply guarantees that $\operatorname{mult}_{V_\infty} G \neq 0$ and hence $V_\infty \subseteq \operatorname{Supp}(G)$. Since G is a convex combination of bounded complements of $(\overline{X}, \overline{\Delta})$ and $(X, V_\infty + \Gamma + B_X)$ is sub-klt, we see that G is a sub-klt N-complement of $(\overline{X}, \overline{\Delta})$ after possibly enlarging N. Moreover, as $mt \leq (1-t)\alpha_0$ by our choice of t and $B_X^- \leq mC$ away from V_∞, we have $tB_X^- \leq (1-t)\alpha_0 C$ away from V_∞ and therefore $\operatorname{Supp}(G^-) \subseteq V_\infty$. In particular, the resulting sub-klt complement G satisfies (1).

By construction, $G^- \leq tB_X^-$ and $M - B_X^-$ is pseudo-effective. Thus

$$-(K_{\overline{X}} + \overline{\Delta} + V_\infty) - G^- \sim_{\mathbb{Q}} \left(\frac{1}{m} - t\right) M + t(M - B_X^-) + (tB_X^- - G^-)$$

is big. But since both $K_{\overline{X}} + \overline{\Delta}$ and G^- are proportional to the ample divisor V_∞, this implies the left hand side above is in fact ample. As

$$G = G^+ - G^- \sim_{\mathbb{Q}} -(K_{\overline{X}} + \overline{\Delta} + V_\infty)\,,$$

it follows that $-(K_{\overline{X}} + \overline{\Delta} + V_\infty) - \frac{1}{2}G^+$ is also ample, proving (2). $\square$

We may now return to the proof of Proposition 3.5.

Proof of Proposition 3.5. Let $N = N(n, \varepsilon, \theta, I)$ be the positive integer from Lemma 3.7. Then for any $x \in (X, \Delta; \xi)$ satisfying (3.2), there exists a sub-klt N-complement G of $(\overline{X}, \overline{\Delta})$ such that

(a) $\operatorname{Supp}(G^-) \subseteq V_\infty \subseteq \operatorname{Supp}(G)$, and
(b) $-(K_{\overline{X}} + \overline{\Delta} + V_\infty) - \frac{1}{2}G^+$ is ample.

In particular, part (1) of the proposition is satisfied. Possibly replacing N by a larger multiple, we may assume, by Proposition 3.3, that $|-N(K_{\overline{X}}+\overline{\Delta}+V_\infty)|$ defines a birational map. It follows that

$$|NG^+| = |-N(K_{\overline{X}}+\overline{\Delta}+V_\infty)+NG^-|$$

also defines a birational map.

Condition (b) above together with Lemma 2.9 implies that

$$\mathrm{vol}(G^+) \le 2^n \mathrm{vol}(-(K_{\overline{X}}+\overline{\Delta}+V_\infty)) \le (2n)^n\theta^{-1}.$$

By Proposition 2.20, we deduce that there exists a bounded set $\mathcal{P}$ of projective log smooth pairs, such that for any quasi-regular log Fano cone singularity $x\in(X,\Delta;\xi)$ satisfying (3.2), there exists some $(Y,\Sigma)\in\mathcal{P}$ that log birationally dominates $(\overline{X},\overline{\Delta}+V_\infty+G^+)$. Using condition (a), we see that (Y,Σ) also log birationally dominates $(\overline{X},\overline{\Delta}+G)$. In particular, the sub-klt pair $(\overline{X},\overline{\Delta}+G)$ belongs to a log birationally bounded set. But then by Lemma 3.6, after possibly replacing the bounded set $\mathcal{P}$, we may further assume that (Y,Σ) log birationally dominates $(\overline{X},\overline{\Delta}+G)$ effectively. This implies part (2) of the proposition. □

3.3. **From boundedness in codimension one to boundedness.** Finally, we shall recover the log Fano cone singularity X from its (modified) birational model Y given by the previous subsection (it is important to note that we will not attempt to recover the projective orbifold cone $\overline{X}$, which does not belong to a bounded family). The basic strategy is as follows. Since X is affine, it suffices to recover its section ring from Y. Using the birational model Y, we will identify a big open subset of X with an open subset U of Y, and the question is to find the section ring $\Gamma(\mathcal{O}_U)$. If D is an effective divisor with support $Y\setminus U$, we may try to run a D-MMP on Y and construct its ample model $(\overline{Y},\overline{D})$. Then $U'=\overline{Y}\setminus\overline{D}$ is affine since $\overline{D}$ is ample, and $\Gamma(\mathcal{O}_U)$ is simply the section ring of U'.

Turning to more details, we begin with some general setup. Let X be an affine normal variety, let $\overline{X}$ be a normal projective compactification, and let V_∞ be the divisorial part of $\overline{X}\setminus X$ (a typical example is the orbifold cone compactifications we consider in previous sections). Let $\varphi\colon Y \dashrightarrow \overline{X}$ be a birational map with Y proper. Let Σ_0 be the sum of $\varphi_*^{-1}V_\infty$ and the exceptional divisors of φ, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that $\mathrm{Supp}(D)=\Sigma_0$. A birational contraction $g\colon Y\dashrightarrow \overline{Y}$ is called an ample model of D if $\overline{Y}$ is proper, g_*D is $\mathbb{Q}$-Cartier ample, and $D\ge g^*g_*D$.

Lemma 3.8. *Assume that all the φ^{-1}-exceptional divisors are contained in V_∞, and the ample model $g\colon Y\dashrightarrow\overline{Y}$ of D exists. Then the composition $\psi=g\circ\varphi^{-1}\colon\overline{X}\dashrightarrow\overline{Y}$ induces an isomorphism $X\cong\overline{Y}\setminus g_*\Sigma_0$.*

Proof. Let $\psi\colon\overline{X}\dashrightarrow\overline{Y}$, $U'=\overline{Y}\setminus g_*\Sigma_0$ and let $U=Y\setminus\Sigma_0$. Since g_*D is ample, its complement U' is affine. In order to prove the lemma, it suffices to show that ψ induces an isomorphism $\Gamma(\mathcal{O}_X)\cong\Gamma(\mathcal{O}_{U'})$. For this we first show that the induced birational map $(\varphi^{-1})|_X\colon X\dashrightarrow U$ is an isomorphism over some big open sets of both X and U.

To see this, note that the exceptional divisors of φ are contained in Σ_0, while the exceptional divisors of φ^{-1} are contained in V_∞. We also have $\mathrm{Supp}(\varphi_*\Sigma_0)\subseteq V_\infty$ and $\mathrm{Supp}(\varphi_*^{-1}V_\infty)\subseteq\Sigma_0$ by construction. Thus the complement of all the exceptional locus contains a big open subset of both X and U, and φ is an isomorphism over this open set. In particular, we have $\Gamma(\mathcal{O}_X) \cong \Gamma(\mathcal{O}_U)$.

Let $f\colon W \to Y$, $h\colon W \to \overline{Y}$ be a common resolution. Since $g_*D = h_*f^*D$ is ample and $D \geq g^*g_*D$ by assumption, we have $f^*D \geq h^*g_*D$ by the negativity lemma. This implies that $\mathrm{Supp}(f^*D) = h^{-1}(\mathrm{Supp}(g_*D))$ and therefore the induced morphism

$$W \setminus \mathrm{Supp}(f^*D) \to \overline{Y} \setminus \mathrm{Supp}(g_*D) = U'$$

is proper, hence they have the same global sections. Similarly the morphism

$$W \setminus \mathrm{Supp}(f^*D) \to Y \setminus \mathrm{Supp}(D) = U$$

is proper as well. Thus they induce isomorphisms

$$\Gamma(\mathcal{O}_U) \cong \Gamma(\mathcal{O}_{W\setminus\mathrm{Supp}(f^*D)}) \cong \Gamma(\mathcal{O}_{U'}).$$

Combined with the previous established isomorphism $\Gamma(\mathcal{O}_X) \cong \Gamma(\mathcal{O}_U)$, this proves that ψ induces an isomorphism $\Gamma(\mathcal{O}_X) \cong \Gamma(\mathcal{O}_{U'})$ and hence $X \cong \overline{Y} \setminus g_*\Sigma_0$. $\square$

Now we can put things together to prove the main theorems.

Proof of Theorem 1.3. By [Zhu23a, Lemma 2.18], after possibly replacing the positive constants ε, θ and the finite set I, we may assume that $I \subseteq [0,1]\cap\mathbb{Q}$. By [Zhu23a, Lemma 2.11], after perturbing the Reeb vector ξ and decreasing ε, θ, we may further assume that the all the polarized log Fano cone singularities in $\mathcal{S}$ are quasi-regular.

By Proposition 3.5, there exist a bounded set $\mathcal{P}$ of projective log smooth pairs (Y, Σ) and a positive integer N, depending only on n, ε, θ and I, such that the following holds for the projective orbifold cone compactification $(\overline{X}, \overline{\Delta}+V_\infty)$ of any log Fano cone singularity $x \in (X, \Delta; \xi)$ in $\mathcal{S}$:

(1) $(\overline{X}, \overline{\Delta})$ has a sub-klt N-complement G such that $\mathrm{Supp}(G^-) \subseteq V_\infty \subseteq \mathrm{Supp}(G)$.
(2) There exists some $(Y, \Sigma) \in \mathcal{P}$ that log birationally dominates $(\overline{X}, \overline{\Delta}+G)$ effectively.

Let $\Sigma_0 \subseteq Y$ be the sum of the birational transform of V_∞ and the exceptional divisors of the birational map $\varphi\colon Y \dashrightarrow \overline{X}$. Note that $\Sigma_0 \subseteq \Sigma$. In order to apply Lemma 3.8, let us show that there exists some effective divisor D with $\mathrm{Supp}(D) = \Sigma_0$ such that the ample model of D exists. Once this is achieved, the remaining step is to run a D-MMP in the bounded family $\mathcal{P}$ for some uniform choice of D.

Let (Y, G_Y) be the crepant pull back of $(X, \overline{\Delta}+G)$. Then G_Y is a sub-klt N-complement of Y which satisfies $\mathrm{Supp}(G_Y) \subseteq \Sigma$ and $\mathrm{Supp}(G_Y^-) \subseteq \Sigma_0$. In particular, the coefficients of G_Y are at most $1 - \frac{1}{N}$.

Consider a new boundary divisor Γ on Y as follows: if F is a prime divisor on Y but is not a component of Σ_0, then we set

$$\mathrm{mult}_F(\Gamma) := \mathrm{mult}_F(G_Y) \geq 0;$$

if F is an irreducible component of Σ_0, then set

$$\mathrm{mult}_F(\Gamma) := \max\left\{0, \mathrm{mult}_F(G_Y) + \frac{1}{2N}\right\}.$$

Let $D := \Gamma - G_Y$. By construction, both $\mathbb{Q}$-divisors D and Γ are effective, $\mathrm{Supp}(\Gamma) \subseteq \Sigma$, $\mathrm{Supp}(D) = \Sigma_0$, the coefficients of Γ are contained in

$$\Lambda := \left\{0, \frac{1}{2N}, \frac{2}{2N}, \dots, 1 - \frac{1}{2N}\right\}$$

(in particular, as (Y, Σ) is log smooth, the pair (Y, Γ) is klt), and we have

$$K_Y + \Gamma \sim_{\mathbb{Q}} (K_Y + \Gamma) - (K_Y + G_Y) \sim_{\mathbb{Q}} D.$$

As $\mathrm{Supp}(\varphi^* V_\infty) \subseteq \Sigma_0$ and V_∞ is ample on $\overline{X}$, we know that Σ_0 is big and the same holds for D as $\mathrm{Supp}(D) = \Sigma_0$. In particular, $K_Y + \Gamma$ is big. Thus by [BCHM10, Theorem 1.2], the ample model $g \colon Y \dashrightarrow \overline{Y}$ of $K_Y + \Gamma \sim_{\mathbb{Q}} D$ exists. By Lemma 3.8, the composition $\psi \colon \overline{X} \dashrightarrow \overline{Y}$ induces an isomorphism $X \cong \overline{Y} \setminus g_* \Sigma_0$. Since Σ contains the birational transform of $\overline{\Delta}$ in its support, we also see that $\mathrm{Supp}(\Delta) = (g_* \Sigma_1)|_X$ for some reduced divisor $\Sigma_1 \leq \Sigma$.

To summarize, we have proved the following. For any log Fano cone singularity $x \in (X, \Delta; \xi)$ in $\mathcal{S}$, there exist a log smooth pair (Y, Σ) from the bounded set $\mathcal{P}$, two reduced divisors $\Sigma_0, \Sigma_1 \leq \Sigma$, and an effective divisor Γ supported on Σ with coefficients in Λ, such that $K_Y + \Gamma$ is big and

$$(X, \mathrm{Supp}(\Delta)) \cong (\overline{Y} \setminus g_* \Sigma_0, (g_* \Sigma_1)|_{\overline{Y} \setminus g_* \Sigma_0}),$$

where $g \colon Y \dashrightarrow \overline{Y}$ is the ample model of $K_Y + \Gamma$.

Since $\mathcal{P}$ is bounded, all the pairs (Y, Σ) in $\mathcal{P}$ arise as the fibers of some bounded family $(\mathcal{Y}, \Sigma_{\mathcal{Y}}) \to B$ of pairs over a finite type base B, i.e. there exists $b \in B$ such that

$$(Y, \Sigma) \cong (\mathcal{Y}_b, \Sigma_{\mathcal{Y}_b}) := (\mathcal{Y}, \Sigma_{\mathcal{Y}}) \times_B b.$$

After stratifying B and performing a base change, we may assume that B is smooth, $(\mathcal{Y}, \Sigma_{\mathcal{Y}}) \to B$ is log smooth, and every irreducible component of Σ is the restriction of some component of $\Sigma_{\mathcal{Y}}$. In particular, there are $\mathbb{Q}$-divisors $\Sigma_{0,\mathcal{Y}}, \Sigma_{1,\mathcal{Y}}$ and $\Gamma_{\mathcal{Y}}$ supported on $\Sigma_{\mathcal{Y}}$ that restricts to Σ_0, Σ_1 and Γ on the fiber (Y, Σ).

To conclude the proof of the boundedness, we note that by [HMX13, Theorem 1.8], the volume of $K_{\mathcal{Y}_b} + \Gamma_{\mathcal{Y}_b}$ is locally constant in $b \in B$; moreover, over the components of B where $K_{\mathcal{Y}_b} + \Gamma_{\mathcal{Y}_b}$ is big, the relative ample model $h \colon \mathcal{Y} \dashrightarrow \overline{\mathcal{Y}}$ of $K_{\mathcal{Y}} + \Gamma_{\mathcal{Y}}$ over B exists, whose restriction over b yields the ample model $g_b \colon \mathcal{Y}_b \dashrightarrow \overline{\mathcal{Y}}_b$ of $K_{\mathcal{Y}_b} + \Gamma_{\mathcal{Y}_b}$. By Noetherian induction, after possibly stratifying B again, we may assume that the restriction of $h_* \Sigma_{0,\mathcal{Y}}$ (resp. $h_* \Sigma_{1,\mathcal{Y}}$) to the fiber $\overline{\mathcal{Y}}_b$ is exactly $(g_b)_* \Sigma_{0,\mathcal{Y}_b}$ (resp. $(g_b)_* \Sigma_{1,\mathcal{Y}_b}$). Set

$$\mathcal{X} := \overline{\mathcal{Y}} \setminus h_* \Sigma_{0,\mathcal{Y}} \quad \text{and} \quad \Delta_{\mathcal{X}} := h_* \Sigma_{1,\mathcal{Y}}|_{\mathcal{X}}.$$

There are only finitely many choices of $\Sigma_{0,\mathcal{Y}}, \Sigma_{1,\mathcal{Y}}$ and $\Gamma_{\mathcal{Y}}$, as their coefficients belong to the finite set $\Lambda \cup \{1\}$. This leads to finitely many families $(\mathcal{X}, \Delta_{\mathcal{X}}) \to B$ as above. By the previous discussion, for any log Fano cone singularity $x \in (X, \Delta; \xi)$ in $\mathcal{S}$, the pair $(X, \mathrm{Supp}(\Delta))$ appears as a fiber of one of the families $(\mathcal{X}, \Delta_{\mathcal{X}}) \to B$; therefore, the set of pairs underlying $\mathcal{S}$ is bounded. By Lemma 2.17, this implies that $\mathcal{S}$ is also a bounded set of log Fano cone singularities. $\square$

Proof of Corollary 1.4. It suffices to prove $x \in (X, \Delta; \xi)$ satisfies the condition of Theorem 1.3, where x is the vertex of the cone, and ξ corresponds to the $\mathbb{G}_m$-action given by the cone structure.

By Lemma 2.12, $\Theta(X, \Delta; \xi) \geq (\min\{\alpha_0, 1\})^n$. One can directly calculate

$$\widehat{\mathrm{vol}}_{X,\Delta}(\mathrm{wt}_\xi) = r(-K_V - \Delta_V)^{n-1} \geq \varepsilon$$

(see e.g. [Zhu23a, Lemma 3.4]), which implies that $\widehat{\mathrm{vol}}(x, X, \Delta) \geq \varepsilon \cdot (\min\{\alpha_0, 1\})^n$. Thus the set $\mathcal{S}$ is bounded by Theorem 1.3. □

Proof of Theorem 1.1. This directly follows from Theorem 1.3 when $\Theta(X, \Delta; \xi) = 1$. □

Proof of Theorem 1.2. By Theorem 2.6, every klt singularity $x \in (X, \Delta)$ has a special degeneration to a K-semistable log Fano cone singularity $x_0 \in (X_0, \Delta_0; \xi_v)$ with $\widehat{\mathrm{vol}}(x, X, \Delta) = \widehat{\mathrm{vol}}(x_0, X_0, \Delta_0)$. Moreover, the coefficients of Δ_0 belong to the finite set

$$I^+ := \{\sum_i m_i a_i \mid m_i \in \mathbb{N}, a_i \in I\} \cap [0, 1].$$

Therefore, Theorem 1.2 follows from Theorem 1.1 and the constructibility of the local volume function in bounded families [Xu20, Theorem 1.3] (see also [HLQ23, Theorem 3.5] for the real coefficient case). □

Remark 3.9. It is important to understand more about the set $\widehat{\mathrm{Vol}}_{n,I}$. When $I = \{0\}$, it is proved in [LX19] that the maximal number in $\widehat{\mathrm{Vol}}_n := \widehat{\mathrm{Vol}}_{n,\{0\}}$ is n^n, the local volume of a smooth point. It is conjectured that the second largest number in $\widehat{\mathrm{Vol}}_n$ is $2(n-1)^n$ (the local volume of an ordinary double point), but for now this is known only when $n \leq 3$.

References

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[BdFFU15] S. Boucksom, T. de Fernex, C. Favre, and S. Urbinati, *Valuation spaces and multiplier ideals on singular varieties*, Recent advances in algebraic geometry, 2015, pp. 29–51.

[Bir19] Caucher Birkar, *Anti-pluricanonical systems on Fano varieties*, Ann. of Math. (2) **190** (2019), no. 2, 345–463.

[BJ20] Harold Blum and Mattias Jonsson, *Thresholds, valuations, and K-stability*, Adv. Math. **365** (2020), 107062.

[Blu18] Harold Blum, *Existence of valuations with smallest normalized volume*, Compos. Math. **154** (2018), no. 4, 820–849.

[CS18] Tristan C. Collins and Gábor Székelyhidi, *K-semistability for irregular Sasakian manifolds*, J. Differential Geom. **109** (2018), no. 1, 81–109.

[CS19] ———, *Sasaki-Einstein metrics and K-stability*, Geom. Topol. **23** (2019), no. 3, 1339–1413.

[HLQ23] Jingjun Han, Yuchen Liu, and Lu Qi, *ACC for local volumes and boundedness of singularities*, J. Algebraic Geom. **32** (2023), no. 3, 519–583.

[HMX13] Christopher D. Hacon, James McKernan, and Chenyang Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. (2) **177** (2013), no. 3, 1077–1111.

[HMX14] ———, *ACC for log canonical thresholds*, Ann. of Math. (2) **180** (2014), no. 2, 523–571.

[Jia20] Chen Jiang, *Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 5, 1235–1250.

[JM12] Mattias Jonsson and Mircea Mustaţă, *Valuations and asymptotic invariants for sequences of ideals*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 6, 2145–2209 (2013).

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol04] János Kollár, *Seifert G_m-bundles* (2004). arXiv:0404386.

[Kol13] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[Kol23] ______, *Families of varieties of general type*, Cambridge Tracts in Mathematics, vol. 231, Cambridge University Press, Cambridge, 2023. With the collaboration of Klaus Altmann and Sándor J. Kovács.

[Li18] Chi Li, *Minimizing normalized volumes of valuations*, Math. Z. **289** (2018), no. 1-2, 491–513.

[LLX20] Chi Li, Yuchen Liu, and Chenyang Xu, *A guided tour to normalized volumes*, Geometry Analysis, In Honor of Gang Tian's 60th Birthday, 2020, pp. 167–219.

[LMS23] Yuchen Liu, Joaquín Moraga, and Hendrik Süß, *On the boundedness of singularities via normalized volume* (2023). arXiv:2205.12326v3.

[LX18] Chi Li and Chenyang Xu, *Stability of Valuations: Higher Rational Rank*, Peking Math. J. **1** (2018), no. 1, 1–79.

[LX19] Yuchen Liu and Chenyang Xu, *K-stability of cubic threefolds*, Duke Math. J. **168** (2019), no. 11, 2029–2073.

[MS21] Joaquín Moraga and Hendrik Süß, *Bounding toric singularities with normalized volume* (2021). To appear in Bull. Lond. Math. Soc., arXiv:2111.01738.

[Xu20] Chenyang Xu, *A minimizing valuation is quasi-monomial*, Ann. of Math. (2) **191** (2020), no. 3, 1003–1030.

[XZ21] Chenyang Xu and Ziquan Zhuang, *Uniqueness of the minimizer of the normalized volume function*, Camb. J. Math. **9** (2021), no. 1, 149–176.

[XZ22] ______, *Stable degenerations of singularities* (2022). arXiv:2205.10915.

[Zhu23a] Ziquan Zhuang, *On boundedness of singularities and minimal log discrepancies of Kollár components, II* (2023). To appear in Geom. Topol. arXiv:2302.03841.

[Zhu23b] ______, *Stability of klt singularities* (2023). To appear in Handbook of Geometry and Topology of Singularities, Volume VII. arXiv:2307.10525.

[Zhu24] ______, *On boundedness of singularities and minimal log discrepancies of Kollár components*, J. Algebraic Geom. **33** (2024), no. 3, 521–565.

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
Email address: chenyang@princeton.edu

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
Email address: zzhuang@jhu.edu