A concentration phenomenon for h-extra edge-connectivity reliability analysis of enhanced hypercubes $Q_{n,2}$ with exponentially many faulty links[1]

Yali Sun[a], Mingzu Zhang[a,*], Xing Feng[b], Xing Yang[c]

[a]*College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China*
[b]*School of Science, Jimei University, Xiamen, Fujian 361021, PR China*
[c]*Department of Electrical and Computer Engineering,The University of Mississippi, Oxford MS,38677,USA*

Abstract

Reliability assessment of interconnection networks is critical to the design and maintenance of multiprocessor systems. The (n,k)-enhanced hypercube $Q_{n,k}$ as a variation of the hypercube Q_n, was proposed by Tzeng and Wei in 1991. As an extension of traditional edge-connectivity, h-extra edge-connectivity of a connected graph G, $\lambda_h(G)$, is an essential parameter for evaluating the reliability of interconnection networks. This article intends to study the h-extra edge-connectivity of the $(n,2)$-enhanced hypercube $Q_{n,2}$. Suppose that the link malfunction of an interconnection network $Q_{n,2}$ does not isolate any subnetwork with no more than $h-1$ processors, the minimum number of these possible faulty links concentrate on a constant 2^{n-1} for each integer $\lceil \frac{11\times 2^{n-1}}{48} \rceil \leq h \leq 2^{n-1}$ and $n \geq 9$. That is, for about 77.083 percent values of $h \leq 2^{n-1}$, the corresponding h-extra edge-connectivity of $Q_{n,2}$, $\lambda_h(Q_{n,2})$, presents a concentration phenomenon. Moreover, the above lower and upper bounds of h are both tight.

Keywords: Interconnection networks, Reliability and links fault tolerance, Concentration phenomenon, Enhanced hypercubes, h-Extra edge-connectivity.

1. Introduction

The growing need to process and store massive amounts of data has led to increase more interest in multiprocessor systems. The advent of multiprocessor systems with a large number of processors and links meets this requirement [11, 21, 27]. As the scale of such these systems continues to increase, so does the probability of links malfunctioning or failing. In addition, finding an appropriate parameter to measure the reliability of the system is crucial to the design and maintenance of the multiprocessor system. It is well known that the underlying topology of an interconnection network can be modelled by a connected graph $G = (V, E)$, with vertex set V representing processors and edge set E representing the communication links between processors.

The performance of the interconnection network can usually be reflected by the topological parameters of its underlying connected graph G. The connectivity and edge-connectivity are two essential parameters for the reliability and fault tolerance of interconnection networks. The connectivity $\kappa(G)$ or the edge-connectivity $\lambda(G)$ of the connected graph G is defined as the minimum number of vertices or edges whose removal from G makes the remaining disconnected. To overcome this deficiency, Harary [6] proposed conditional connectivity and conditional edge-connectivity in 1983. Due to the closed interconnection between various local parts of G, when some malfunction of links and processors occurs, part of the local structure cannot be destroyed completely. The edges in a forbidden faulty edge set cannot fail simultaneously. By restricting the forbidden faulty edge set to the sets of neighboring edges of any

[1] This work was supported by the National Natural Science Foundation of China (Grant Nos. 12101528 and 12001250), Basic scientific research in universities of Xinjiang Uygur Autonomous Region (Grant No. 202401120001) and Science and Technology Project of Xinjiang Uygur Autonomous Region (Grant No. 2020D01C069).

* Corresponding author
Email addresses: y_lisun@163.com (Yali Sun), mzuzhang@163.com (Mingzu Zhang), fengxing_fm@163.com (Xing Feng), xyang4@go.olemiss.edu (Xing Yang)

Table 1: Previous known and current results on the h-extra edge-connectivity for some classes of interconnection networks.

Graph	h	λ_h	Author
Q_n	$1 \le h \le 2^{\lfloor \frac{n}{2} \rfloor}$	$nh - ex_h(Q_n), n \ge 4$	Li and Yang [10] in 2013
Q_n^3	$3^{\lceil \frac{n}{2} \rceil + r} - \lfloor \frac{3^{2r+e+1}}{2} \rfloor \le h \le 3^{\lceil \frac{n}{2} \rceil + r}$	$2(\lfloor \frac{n}{2} - r)3^{\lceil \frac{n}{2} \rceil + r}, n \ge 3$	Ma et al. [12] in 2021
FQ_n	1	$n + 1, n \ge 2$	El-Amawy and Latifi [4] in 1991
	2	$2n, n \ge 2$	Zhu and Xu [30] in 2006
	3	$3n - 1, n \ge 5$	Zhu et al. [31] in 2007
	4	$4n - 4, n \ge 5$	Chang et al. [2] in 2014
	$\le n$	$\xi_h(FQ_n), n \ge 6$	Yang and Li [18] in 2014
	$\le 2^{\lceil \frac{n}{2} \rceil + 1} - 4$, for odd n	$\xi_h(FQ_n), n \ge 4$	Zhang et al. [25] in 2016
	$\le 2^{\lceil \frac{n}{2} \rceil + 1} - 2$, for even n	$\xi_h(FQ_n), n \ge 4$	Zhang et al. [25] in 2016
	$2^{\lceil \frac{n}{2} \rceil + 1} - d_r \le h \le 2^{\lceil \frac{n}{2} \rceil + 1}$ £	$\lfloor \frac{n}{2} \rfloor 2^{\lceil \frac{n}{2} \rceil + 1}, n \ge 4$	Zhang et al. [25] in 2016
	$2^{\lfloor \frac{n}{2} \rfloor + r} - l_r \le h \le 2^{\lfloor \frac{n}{2} \rfloor + r}$ ‡	$(\lceil \frac{n}{2} \rceil - r + 1)2^{\lfloor \frac{n}{2} \rfloor + r}$	Zhang et al. [25] in 2016
	$1 \le h \le 2^{n-1}$	Algorithm	Zhang et al. [26] in 2018
B_n	1	n	Chen et al. [3] in 2003
	2	$2n - 2$	Chen et al. [3] in 2003
	3	$3n - 5$	Zhu et al. [32] in 2006
	4	$4n - 8$	Hong and Hsieh [8] in 2013
	$\frac{2^{n-1}+2^f}{3} \le h \le 2^{n-1}$ §	2^{n-1}	Zhang et al. [24] in 2014
$Q_{n,k}$	1	$2n, 5 \le k \le n - 1$	Sabir et al. [13] in 2019
	2	$3n - 1, 5 \le k \le n - 1$	Sabir et al. [13] in 2019
	$1 \le h \le 2^{\lceil \frac{n}{2} \rceil} - d_r, n \le 2k + 3, k \ge 3$ £	$(n + 1)h - \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i}$	Xu et al. [17] in 2021
	$2^{\lceil \frac{n}{2} \rceil} - d_r \le h \le 2^{\lceil \frac{n}{2} \rceil}, 2k \le n \le 2k + 3$	$(n + 1)h - \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i}$	Xu et al. [17] in 2021
	$2^{\lceil \frac{n}{2} \rceil} - d_r \le h \le 2^{\lceil \frac{n}{2} \rceil}, k + 2 \le n \le 2k - 1$	$\lfloor \frac{n}{2} \rfloor 2^{\lceil \frac{n}{2} \rceil}$	Xu et al. [17] in 2021
	$1 \le h \le 2^{\lceil \frac{n}{2} \rceil + 1} - d_r, n \ge 2k + 4$	$(n + 1)h - \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i}$	Xu et al. [17] in 2021
	$2^{\lceil \frac{n}{2} \rceil + 1} - d_r \le h \le 2^{\lceil \frac{n}{2} \rceil + 1}$	$\lfloor \frac{n}{2} \rfloor 2^{\lceil \frac{n}{2} \rceil + 1}$	Xu et al. [17] in 2021
$\boldsymbol{Q_{n,2}}$	$\boldsymbol{\lceil \frac{11 \times 2^{n-1}}{48} \rceil \le h \le 2^{n-1}}$	$\boldsymbol{2^{n-1}, (k = 2)}$	**Current**

‡ where $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$ and $l_r = \frac{2^{2r-1}}{3}$ if n is odd and $l_r = \frac{2^{2r-2}}{3}$ if n is even.

§ where $f = 0$ if n is even, and $f = 1$ if n is odd.

£ where $d_r = 2$ if n is even, and $d_r = 4$ if n is odd.

spanning subgraph with no more than $h - 1$-vertices in the faulty networks, Fàbrega and Fiol [5] proposed the h-extra edge-connectivity in 1996. Given a positive integer h, an h-extra edge-cut of a connected graph G is defined as a set of edges whose deletion yields a disconnected graph with all its components having at least h vertices. The h-extra edge-connectivity, denoted as $\lambda_h(G)$, is defined as the minimum cardinality of all h-extra edge-cuts of G. Given a vertex set $X \subset V(G)$, the complement of a vertex set X is $\overline{X} = V(G) \setminus X$. $G[X]$ and $[X, \overline{X}]$ can be defined as the subgraph induced by the vertex set X and the set of edges of G in which each edge contains one end vertex in X and the other end in $\overline{X}$, respectively. Let $\xi_m(G) = \min\{|[X, \overline{X}]| : |X| = m \le \lfloor |V(G)|/2 \rfloor, G[X] \text{ is connected}\}$. If $\lambda_h(G) = \xi_h(G)$, it is called λ_h-optimal; otherwise, it is not λ_h-optimal. Many authors studied exact values of the h-extra edge-connectivity of some important classes of the interconnection network, such as hypercubes [10], folded hypercubes [2, 4, 18, 25, 26, 30, 31], BC network [3, 8, 19, 24, 32], and 3-ary n-cube [12]. The specific conclusions are shown in Table 1.

The enhanced hypercube is a variant of the hypercube. Based on n-dimensional hypercube Q_n, Tzeng and Wei [15] proposed the concept of (n, k)-enhanced hypercube $Q_{n,k}$ for $1 \le k \le n - 1$, by adding different types of complement edges. Compared to Q_n, by adding various kinds of k-complement edge on Q_n, the (n, k)-enhanced hypercube $Q_{n,k}$ performs very well in many measurements such as mean internode distance, diameter, traffic density, connectivity, fault tolerance, cost-effective[15], communication ability and diagnosability [16].

Undoubtedly, the enhanced hypercubes $Q_{n,k}$ require more hardware to build than hypercubes Q_n. However, when n is huge, the expense is minimal, and the benefits of the structural advantages are substantial. With such attractive properties, the (n, k)-enhanced hypercube has been widely studied.

Recently, the h-extra edge-connectivity and h-extra connectivity of $Q_{n,k}$ are widely investigated. For the edge versions,

Sabir et al. [13] investigated $\lambda_h(Q_{n,k})$ for $h = 1, 2$ in 2019 (see Table 1); Xu et al. [17] studied $\lambda_h(Q_{n,k})$ for $1 \le h \le 2^{\lceil \frac{n}{2} \rceil + 1}, n \ge 2k + 4$ and $1 \le h \le 2^{\lceil \frac{n}{2} \rceil}, n \le 2k + 3, k \ge 3$ in 2021 (see Table 1). For the vertex versions, Li et al. [9] determined that $\kappa_1(Q_{n,k})$ for $n = k + 1$ and $k \ge 1$, $\kappa_2(Q_{n,k})$ for $n = k + 1$ and $k \ge 3$ and $\kappa_3(Q_{n,k})$ for $n = k + 1$ and $k \ge 3$ in 2020. Sabir et al. [13] also determined $\kappa_1(Q_{n,k})$ for $n \ge 7, 2 \le k \le n - 5$ and $\kappa_2(Q_{n,k})$ for $n \ge 9$ and $2 \le k \le n - 7$ in 2019. Yin and Xu [22] proved $\kappa_g(Q_{n,k})$ for $0 \le g \le n - k - 1, 4 \le k \le n - 5$ and $n \ge 9$ in 2022. In particular, for $k = 1$, the (n, k)-enhanced hypercubes $Q_{n,k}$ is n-dimensional folded hypercubes FQ_n.

In 2013, Li and Yang investigated $\lambda_h(Q_n)$ for $1 \le h \le 2^{\lfloor \frac{n}{2} \rfloor}$ and $n \ge 4$. In 2014, Yang and Li [18] determined $\lambda_h(FQ_n)$ for $h \le n$ and $n \ge 6$. In 2014, Zhang et al. [24] studied $\lambda_h(B_n)$ for $1 \le h \le 2^{\lfloor \frac{n}{2} \rfloor + 1}$ and $n \ge 4$. In 2014, Yang and Meng [20] investigated $\kappa_g(Q_n)$ for $0 \le g \le n - 4$. In 2017, Zhou [29] determined $\kappa_g(HL_n)$ for $0 \le g \le n - 3$ and $n \ge 5$. Compared to classical Menger theory, both h-extra edge-connectivity and g-extra connectivity significantly improved the fault tolerance and reliability of interconnection networks. Since h or g is very small, they usually satisfy the λ_h-optimality $\lambda_h(G) = \xi_h(G)$ or κ_g-optimality $\kappa_g(G) = \xi_g^v(G)$.

They also allow a linear number of malfunctions. It is not enough. We want to go further. For every integer $h_1 \le h \le h_2$, the value of the function $\lambda_h(G)$ is a constant, one then says that the h-extra edge-connectivity of a graph G is concentrated for the interval $h_1 \le h \le h_2$, and represents a concentration phenomenon. If the bounds h_1 and h_2 are sharp, $\lambda_{h_1-1}(G) < \lambda_{h_1}(G) = \lambda_h(G) = \lambda_{h_2}(G) < \lambda_{h_2+1}(G)$, it means that this interval $h_1 \le h \le h_2$ is maximal. In particular, for $h_1 = h_2$, $\lambda_h(G) = \xi_h(G)$ is λ_h-optimal. Recently, Zhang et al. (2016) [25] studied the values of $\lambda_h(FQ_n)$ concentrate on $\lfloor \frac{n}{2} \rfloor 2^{\lceil \frac{n}{2} \rceil + 1}$ for $2^{\lceil \frac{n}{2} \rceil + 1} - d_r \le h \le 2^{\lceil \frac{n}{2} \rceil + 1}$, where $d_r = 4$ if n is odd and $d_r = 2$ if n is even. Xu et al. (2021) [17] investigated the values of $\lambda_h(Q_{n,k})$ concentrate on $\lfloor \frac{n}{2} \rfloor 2^{\lceil \frac{n}{2} \rceil}$ for $2^{\lceil \frac{n}{2} \rceil} - d_r \le h \le 2^{\lceil \frac{n}{2} \rceil}$, where $d_r = 4$ if n is odd and $d_r = 2$ if n is even. With the increase of n, the concentration phenomenon also becomes obvious. Zhang et al. (2016) [25] also determined the values of $\lambda_h(FQ_n)$ concentrate on $(\lceil \frac{n}{2} \rceil - r + 1)2^{\lfloor \frac{n}{2} \rfloor + r}$ for $2^{\lfloor \frac{n}{2} \rfloor + r} - l_r \le h \le 2^{\lfloor \frac{n}{2} \rfloor + r}$, where $r = 1, 2, \dots, \left\lceil \frac{n}{2} \right\rceil - 1$ and $l_r = \frac{2^{2r}-1}{3}$ if n is odd and $l_r = \frac{2^{2r+1}-2}{3}$ if n is even. As far as we know, the study of the concentration phenomenon of $\lambda_h(Q_{n,k})$ has just started. Inspired by the above results, this paper mainly focuses on the most obvious concentration phenomenon of $\lambda_h(Q_{n,2})$ in the subinterval $\lceil \frac{11 \times 2^{n-1}}{48} \rceil \le h \le 2^{n-1}$. For example, the values of $\xi_h(Q_{n,2})$ are marked in blue, the values of $\lambda_h(Q_{n,2})$ are marked in red, and the subinterval we examines is marked in green (see Fig. 1).

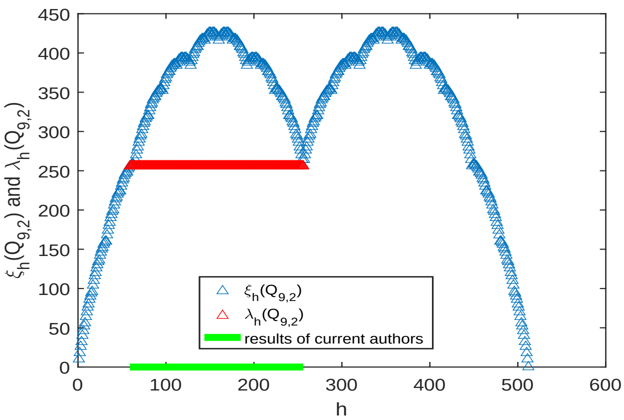

Fig. 1: The values of $\xi_h(Q_{9,2})$ and $\lambda_h(Q_{9,2})$.

Theorem 1. *For three integers $n \ge 9$, $\lceil \frac{11 \times 2^{n-1}}{48} \rceil \le h \le 2^{n-1}$ and $1 \le r \le \lceil \frac{n}{2} \rceil - 1$, the results are as follows:*
(a) $\lambda_h(Q_{n,2}) = \xi_{2^{n-1}}(Q_{n,2}) = 2^{n-1}$;
(b) It is λ_h-optimal ($\lambda_h(Q_{n,2}) = \xi_h(Q_{n,2}) = 2^{n-1}$) if and only if $h = m_{n,r}$ or $h = m_{n,r+1}$.

The rest of this paper is organized as follows. Section 2 introduces some related definitions and lemmas. Section 3 gives several lemmas about the properties of the function $\xi_m(Q_{n,2})$. Section 4 determines that the value of the h-extra edge-connectivity of $Q_{n,2}$ concentrates on a constant 2^{n-1}. The last section concludes our results.

2. Preliminaries

Recall that the h-extra edge-connectivity of a connected graph G, $\lambda_h(G)$, is the minimum number of an edge-cut of the graph G whose removal separates the graph G with all resulting components having at least h vertices. Given a vertex set $X \subset V(G)$, we denote the set of edges of G in which each edge contains exactly one end in X and the other in $\overline{X} = V(G) \setminus X$ by $[X, \overline{X}]$. If F be a minimum h-extra edge-cut of connected graph G, then there is a fact that $G - F$ has exactly two components. In fact, if F is the minimum h-extra edge-cut of the connected graph G, $G - F$ has p components $C_1, C_2, \cdots, C_p$ with at least h vertices, $p \geq 3$. Since the graph G is connected, there must exist integer i, j with $[V_{C_i}, V_{C_j}] \neq \emptyset, |F_1| = |F \setminus [V_{C_i}, V_{C_j}]| < |F|$. Thus, F_1 is also a minimum h-extra edge-cut of G, which contradicts the minimality of F. Hence, $G - F$ has exactly two components. Although the original definition of $\xi_m(G)$ only requires that $G[X]$ is connected, we do need that both $G[X]$ and $G[\overline{X}]$ are connected in this paper. The function $\xi_m(Q_{n,2})$ of $(n, 2)$-enhanced hypercubes $Q_{n,2}$ does have the same result after modifying this condition. Let

$$\xi_m(G) = \min\{|[X, \overline{X}]| : |X| = m \leq \lfloor |V(G)|/2 \rfloor, \text{and both } G[X] \text{ and } G[\overline{X}] \text{ are connected}\}. \tag{1}$$

For a d-regular graph,

$$\xi_m(G) = dm - ex_m(G), \tag{2}$$

where $ex_m(G)$ is twice the maximum number of edges among all m vertices induced subgraphs for each $m \leq \lfloor |V(G)|/2 \rfloor$. Actually, if we can find $X_m^* \subseteq V(G), |X_m^*| = m$, with $ex_m(G) = 2|E(G[X_m^*)]|$, and so that, both $G[X_m^*]$ and $G[\overline{X_m^*}]$ are connected. Then

$$\xi_m(G) = |[X_m^*, \overline{X_m^*}]| = dm - ex_m(G) = dm - 2|E(G[X_m^*]|.$$

By the definition of the h-extra edge-connectivity of G, $\xi_m(G)$ offers the upper bound for the $\lambda_h(G)$ for all $1 \leq h \leq \lfloor |V(G)|/2 \rfloor$. So, the function $\lambda_h(G)$ (by Zhang et al. [26] page 299),

$$\lambda_h(G) = \min\{\xi_m(G) : h \leq m \leq \lfloor |V(G)|/2 \rfloor\}. \tag{3}$$

Let n, k be positive integers. The definitions of the n-dimensional hypercube Q_n, folded hypercube FQ_n and (n, k)-enhanced hypercube $Q_{n,k}$ are stated as follows.

Definition 1. *[14] For an integer $n \geq 1$, the n-dimensional hypercube, denoted by Q_n, is a graph with 2^n vertices. The vertex set $V(Q_n) = \{x_n x_{n-1} \ldots x_1 : x_i \in \{0, 1\}, 1 \leq i \leq n\}$ is the set of all n-bit binary strings. Two vertices $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = y_n y_{n-1} \cdots y_2 y_1$ of Q_n are adjacent if and only if they differ in exactly one position.*

For any vertices $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = y_n y_{n-1} \cdots y_2 y_1$, the edge $e = xy$ is called k-complementary edges $(1 \leq k \leq n - 1)$ if and only if $y_i = x_i$ for $n - k + 1 < i \leq n$, and $y_j = \overline{x_j}$ for $1 \leq j \leq n - k + 1$.

As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, first proposed by EL-Amawy and Latifi [4], is a graph obtained from the hypercube Q_n by adding an edge between every pair of vertices $x_n x_{n-1} \cdots x_1$ and $\overline{x_n}\ \overline{x_{n-1}} \cdots \overline{x_1}$, where $\overline{x_i} = 1 - x_i$ for all $1 \leq i \leq n$. The FQ_n is to add complement edges in two $(n-1)$-dimensional sub-cubes. Motivated by this, by adding k-complementary edges between two paired lower-dimensional sub-cubes, in 1991, Tzeng and Wei [15] introduced the (n, k)-enhanced hypercube $Q_{n,k}$.

Definition 2. *For two integers n and k with $1 \leq k \leq n - 1$, the (n, k)-enhanced hypercube, denoted by $Q_{n,k}$, is defined to be a graph with the vertex set $V(Q_{n,k}) = \{x_n x_{n-1} \ldots x_2 x_1 : x_i \in \{0, 1\}, 1 \leq i \leq n\}$. Two vertices $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = y_n y_{n-1} \cdots y_2 y_1$ are adjacent if y satisfies one of the following two conditions:*
(1) $y = x_n x_{n-1} \cdots x_{i+1} \bar{x}_i x_{i-1} \cdots x_2 x_1$ for $1 \leq i \leq n$, where $\bar{x}_i = 1 - x_i$ or
(2) $y = x_n x_{n-1} \cdots \bar{x}_{n-k+1} \bar{x}_{n-k} \cdots \bar{x}_2 \bar{x}_1$.

Note that $Q_{n,1}$ is the n-dimensional folded hypercube FQ_n. The $(n, 2)$-enhanced hypercube $Q_{n,2}$ is obtained from the hypercube Q_n by adding 2-complementary edges between two pairs of vertices $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = x_n \bar{x}_{n-1} \cdots \bar{x}_2 \bar{x}_1$ in two $(n-1)$-dimensional sub-cubes.

The $(n, 2)$-enhanced hypercube $Q_{n,2}$ is $(n + 1)$-regular $(n + 1)$-connected with 2^n vertices and $(n + 1)2^{n-1}$ edges [15, 16].The enhanced hypercubes $Q_{3,1}$, $Q_{3,2}$ and $Q_{4,2}$ are illustrated in Fig. 2, where the complementary edges are

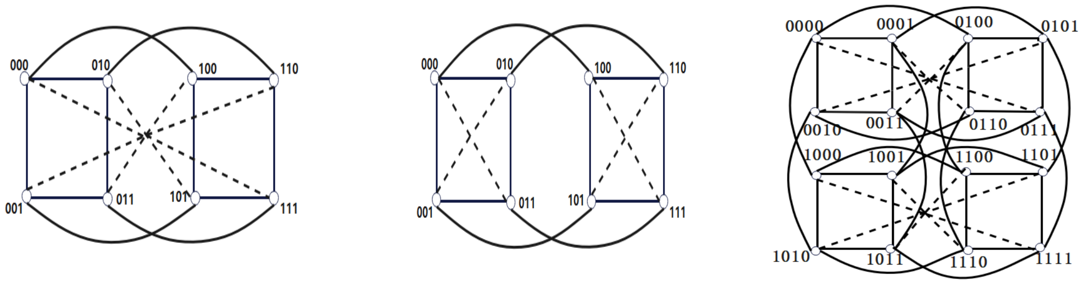

Fig. 2: $Q_{3,1}$ (i.e. FQ_3), $Q_{3,2}$ and $Q_{4,2}$.

represented by a short dotted line. As the integer n grows, the scale of $Q_{n,2}$ expands exponentially, and the topological structure of $Q_{n,2}$ becomes more and more complicated. Thus, the bitmaps of the adjacency matrix of $Q_{n,2}$ represent the adjacent relationship between vertices of $Q_{n,2}$. These figures of the adjacency matrix of $Q_{4,2}$, $Q_{5,2}$, $Q_{6,2}$ and $Q_{7,2}$ are shown in Fig. 3 (in four figures, the dark pixel at location (x, y) corresponds to the edges between vertices x and y). The bitmaps of the adjacency matrix of $Q_{n,2}$ have high symmetry, iterative fractal, and recursive structure.

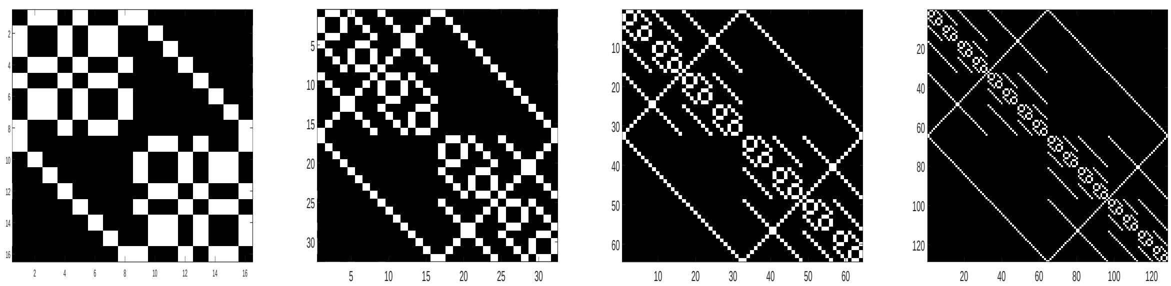

Fig. 3: The bitmap of adjacency matrix of $Q_{n,2}$ for $4 \le n \le 7$.

For a positive integer $1 \le m \le 2^{n-1}$, there exists a unique binary representation $m = \sum_{i=0}^{s} 2^{t_i}$, where $t_0 = \lfloor \log_2 m \rfloor$, $t_i = \left\lfloor \log_2 \left(m - \sum_{r=0}^{i-1} 2^{t_r} \right) \right\rfloor$ for $i \ge 1$, and $t_0 > t_1 > \cdots > t_s$. These conditions are used throughout the article when not causing ambiguity. If $x = x_n x_{n-1} \ldots x_1$ is a vertex of the $(n, 2)$-enhanced hypercubes $Q_{n,2}$, every vertex can be denoted by decimal number $\sum_{i=1}^{n} x_i 2^{i-1}$, $x_i \in \{0, 1\}$. Let S_m be the set $\{0, 1, 2, \ldots, m-1\}$ (under decimal representation). And L_m^n denotes the corresponding set represented by n-binary strings. By the construction of $Q_{n,2}$, L_m^n is the subset of $V(Q_{n,2})$ and $Q_{n,2}[L_m^n]$ is the subgraph induced by L_m^n in $Q_{n,2}$. Both $Q_{n,2}[L_m^n]$ and $Q_{n,2}\left[\overline{L_m^n}\right]$ are connected. The subgraphs induced by L_m^4 in $Q_{n,2}$ for $m = 4, 6$ and 8 are shown in Fig. 4.

Happer [7], Li and Yang [10] independently obtained the exact expression of the function $ex_m(Q_n)$.

Lemma 1. *For a positive integer* $m = \sum_{i=0}^{s} 2^{t_i} \le 2^n$, $\xi_m(Q_n) = nm - ex_m(Q_n)$, *where* $ex_m(Q_n) = 2|E(Q_n[L_m^n])| = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i}$.

Arockiaraj et al. [1] obtained the exact expression of the function $ex_m(Q_{n,k})$ in 2019, which was rewritten by Xu et al. in 2021 [17].

In the following, we let $[x]^+ = \begin{cases} x, & \text{if } x \ge 0; \\ 0, & \text{if } x < 0. \end{cases}$

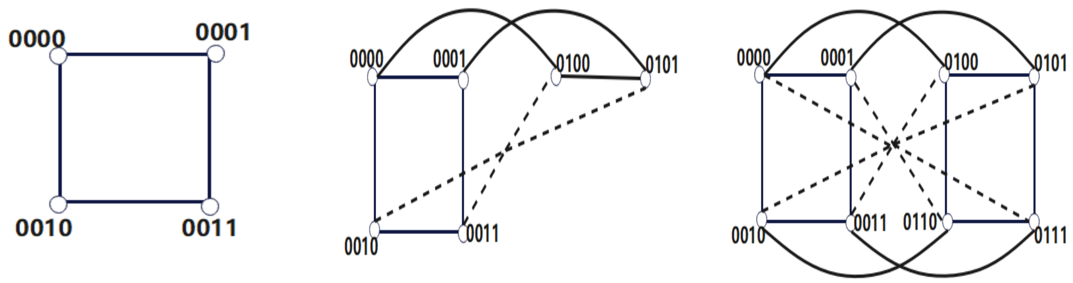

Fig. 4: Induced subgraph $Q_{4,2}[L_4^4]$, $Q_{4,2}[L_6^4]$ and $Q_{4,2}[L_8^4]$.

Lemma 2. *For each integer $1 \le m \le 2^n$ and $m = \sum_{i=0}^{s} 2^{t_i}$, $\xi_m(Q_{n,2}) = (n+1)m - ex_m(Q_{n,2})$, where*

$$\begin{aligned} ex_m(Q_{n,2}) &= 2\left|E(Q_{n,2}[L_m^n])\right| \\ &= 2\left|E(Q_n[L_m^n])\right| + \lfloor \tfrac{m}{2^{n-1}} \rfloor 2^{n-1} + 2\left[m - \lfloor \tfrac{m}{2^{n-1}} \rfloor 2^{n-1} - 2^{n-2}\right]^+ \\ &= \begin{cases} ex_m(Q_n) & \text{if } 1 \le m \le 2^{n-2}; \\ ex_m(Q_n) + 2m - 2^{n-1} & \text{if } 2^{n-2} < m \le 2^{n-1}; \\ ex_m(Q_n) + 2^{n-1} & \text{if } m > 2^{n-1} \text{ and } m = 2^{n-1} + x, \\ & 0 \le x < 2^{n-2}; \\ ex_m(Q_n) + 2x & \text{if } m > 2^{n-1} \text{ and } m = 2^{n-1} + x, \\ & 2^{n-2} \le x < 2^{n-1}. \end{cases} \end{aligned} \tag{4}$$

Then several specific examples are used to illustrate the calculation of $ex_m(Q_{n,2})$. For example, for $n = 4$ and $m = 4$. $ex_m(Q_{n,2}) = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i}$. Note that $S_4 = \{0, 1, 2, 3\}$ and $L_4^4 = \{0000, 0001, 0010, 0011\}$. Since $4 = 2^2$, it can be seen that $t_0 = 2$ and $ex_4(Q_{4,2}) = 2|E(Q_{n,2}[L_4^4])| = 2\times2^2+2\times0\times2^2 = 8$; for $n = 4$ and $m = 8$. $ex_m(Q_{n,2}) = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i} + 2m - 2^{n-1}$. There are $S_8 = \{0, 1, \ldots, 7\}$ and $L_8^4 = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\}$. Since $8 = 2^3$, it can be obtained that $t_0 = 3$ and $ex_8(Q_{4,2}) = 2|E(Q_{n,2}[L_8^4])| = 3\times2^3 + 2\times0\times2^3 + 2\times8 - 2^3 = 32$. The induced graph $Q_{4,2}[L_4^4]$ and $Q_{4,2}[L_8^4]$ are shown in Fig. 4.

Lemma 3. *([17]) For positive integers $1 \le m \le 2^t$ and $0 \le t \le n$, $ex_m(Q_n) \le tm$ and $ex_m(Q_{n,k}) \le (t+1)m$.*

Lemma 4. *([17]) For positive integers $h \le m = \sum_{i=0}^{s} 2^{t_i} \le 2^{n-1}$,*

$$\lambda_h(Q_{n,2}) = \min\left\{\xi_m(Q_{n,2}) : h \le m \le 2^{n-1}\right\},$$

satisfying that

$$\xi_m(Q_{n,2}) = (n+1)m - ex_m(Q_{n,2}). \tag{5}$$

For $m \le 2^{n-1}$, the following two iterative properties of the expression of $ex_m(Q_{n,2})$ depend on whether $Q_{n,2}$ matches complementary edges in the sub-network and how many complementary edges there are.

Lemma 5. *Let m, n be two integers, $n \ge 4, 1 \le m = \sum_{i=0}^{s} 2^{t_i} \le 2^{n-1}$. For $m_1 = \sum_{i=0}^{a} 2^{t_i}$, $m = m_1 + m_2$, and $t_0 > t_1 \cdots > t_a > t_{a+1} > t_{a+2} > \cdots > t_s, a < s$,*
(a) $ex_m(Q_{n,2}) = ex_{m_1}(Q_{n,2}) + ex_{m_2}(Q_{n,2}) + 2(a+1)m_2$ for $1 \le m \le 2^{n-2}$;
(b) $ex_m(Q_{n,2}) = ex_{m_1}(Q_{n,2}) + ex_{m_2}(Q_{n,2}) + 2m_1 + 2(a+1)m_2$ for $2^{n-2} < m \le 2^{n-1}$.

Proof. Note that $m_2 = m - m_1 = 2^{t_{a+1}} + 2^{t_{a+2}} + \cdots + 2^{t_s} = \sum_{i=a+1}^{s} 2^{t_i} = \sum_{i=0}^{s-a-1} t_{i+a+1}$ $2^{t_{i+a+1}}$. Since the expression of $ex_m(Q_{n,2})$ strongly depends on the binary decomposition of m and the domain of m, it can be divided into the following two cases according to its two different forms.

(a). For $1 \le m \le 2^{n-2}$, by Lemma 2, it can be obtained $ex_{m_1}(Q_{n,2}) = \sum_{i=0}^{a} t_i 2^{t_i} + \sum_{i=0}^{a} 2i2^{t_i}$ and $ex_{m_2}(Q_{n,2}) = \sum_{i=0}^{s-a-1} t_{i+a+1} 2^{t_{i+a+1}} + \sum_{i=0}^{s-a-1} 2i2^{t_{i+a+1}}$. Note that

$$
\begin{aligned}
ex_m(Q_{n,2}) &= \textstyle\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i} \\
&= (\textstyle\sum_{i=0}^{a} t_i 2^{t_i} + \sum_{i=0}^{s-a-1} t_{i+a+1} 2^{t_{i+a+1}}) + (\sum_{i=0}^{a} 2i2^{t_i} + \sum_{i=0}^{s-a-1} 2(a+1+i)2^{t_{i+a+1}}) \\
&= ex_{m_1}(Q_{n,2}) + ex_{m_2}(Q_{n,2}) + 2\textstyle\sum_{i=0}^{s-a-1}(a+1)2^{t_{i+a+1}} \\
&= ex_{m_1}(Q_{n,2}) + ex_{m_2}(Q_{n,2}) + 2(a+1)m_2.
\end{aligned}
$$

(b). For $2^{n-2} < m \leq 2^{n-1}$, by Lemma 2, it is sufficient to show that $ex_{m_1}(Q_{n,2}) = \sum_{i=0}^{a} t_i 2^{t_i} + \sum_{i=0}^{a} 2i2^{t_i} + 2m_1 - 2^{n-1}$ and $ex_{m_2}(Q_{n,2}) = \sum_{i=0}^{s-a-1} t_{i+a+1} 2^{t_{i+a+1}} + \sum_{i=0}^{s-a-1} 2i2^{t_{i+a+1}}$. Note that

$$
\begin{aligned}
ex_m(Q_{n,2}) &= \textstyle\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i} + 2m - 2^{n-1} \\
&= (\textstyle\sum_{i=0}^{a} t_i 2^{t_i} + \sum_{i=0}^{s-a-1} t_{i+a+1} 2^{t_{i+a+1}}) + (\sum_{i=0}^{a} 2i2^{t_i} + \sum_{i=0}^{s-a-1} 2(a+1+i)2^{t_{i+a+1}}) + 2m - 2^{n-1} \\
&= ex_{m_1}(Q_{n,2}) + ex_{m_2}(Q_{n,2}) + 2m_2 + 2\textstyle\sum_{i=0}^{s-a-1}(a+1)2^{t_{i+a+1}} \\
&= ex_{m_1}(Q_{n,2}) + ex_{m_2}(Q_{n,2}) + 2(a+2)m_2.
\end{aligned}
$$

To sum up, the proof is completed. □

3. Some properties of the function $\xi_m(Q_{n,2})$

The exact value of the function $\lambda_h(Q_{n,2})$ highly depends on the monotonic interval and fractal structure of the function $\xi_m(Q_{n,2})$. Then we introduce several lemmas of the properties of function $\xi_m(Q_{n,2})$.

Let $f = 0$ if n is even, and $f = 1$ if n is odd. To deal with the interval $\lceil \frac{11\times 2^{n-1}}{48} \rceil \leq m \leq 2^{n-1}$, by inserting $\lceil \frac{n}{2} \rceil$ numbers of $m_{n,r}$ satisfying

$$
\lceil \frac{11 \times 2^{n-1}}{48} \rceil = m_{n,1} < m_{n,2} < \cdots < m_{n,r} < m_{n,r+1} < \cdots < m_{n,\lceil \frac{n}{2} \rceil - 1} = 2^{n-1}.
$$

This interval will be divided into $\lceil \frac{n}{2} \rceil - 1$ numbers of integer subintervals. The expression of $m_{n,r}$ is defined as follows:

$$
m_{n,r} = \begin{cases}
\sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-1-f} & \text{if } 1 \leq r \leq \lceil \frac{n}{2} \rceil - 4; (e) \\
\sum_{i=0}^{3} 2^{n-4-i} & \text{if } r = \lceil \frac{n}{2} \rceil - 3; (f) \\
2^{n-3} & \text{if } r = \lceil \frac{n}{2} \rceil - 2; (g) \\
2^{n-1} & \text{if } r = \lceil \frac{n}{2} \rceil - 1, (h)
\end{cases}
$$

for $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$. By calculation, it can be obtained that $\lceil \frac{11\times 2^{n-1}}{48} \rceil = m_{n,1} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{1-f}$.

Actually, if $1 \leq r \leq \lceil \frac{n}{2} \rceil - 4$ and n is even, $m_{n,r} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-1}$. $m_{n,1} = 2^{n-4} + 2^{n-5} + 2^{n-6} + 2^{n-8} + 2^{n-10} + \cdots + 2^2 + 2^1$ and $3m_{n,1} = 2m_{n,1} + m_{n,1} = 2^{n-3} + 2^{n-4} + 2^{n-5} + 2^{n-6} + 2^{n-7} + \cdots + 2^3 + 2^2 + (2^{n-4} + 2^{n-5} + 2^2)$, so $m_{n,1} = \frac{11\times 2^{n-5} + 2^{1-f}}{3} = \lceil \frac{11\times 2^{n-1}}{48} \rceil$. If n is odd, $m_{n,r} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-2}$. $m_{n,1} = 2^{n-4} + 2^{n-5} + 2^{n-6} + 2^{n-8} + 2^{n-10} + \cdots + 2^1 + 2^0$ and $3m_{n,1} = m_{n,1} + 2m_{n,1} = 2^{n-3} + 2^{n-4} + 2^{n-5} + 2^{n-6} + 2^{n-7} + \cdots + 2^2 + 2^1 + +2^0 + (2^{n-4} + 2^{n-5} + 2^1)$, thus $m_{n,1} = \frac{11\times 2^{n-5} + 2^{1-f}}{3} = \lceil \frac{11\times 2^{n-1}}{48} \rceil$.

For some small cases $4 \leq n \leq 8$, not all of these four situations occur, see Table 2. Throughout this paper only the situation of $n \geq 9$.

Table 2: The variability of r, and $m_{n,r}$ for $4 \leq n \leq 8$.

n	$m_{n,1}$	$m_{n,2}$	$m_{n,3}$	$\cdots$	$m_{n,\lceil \frac{n}{2} \rceil - 1}$
4	$1, (h)$				
5	$4, (g)$	$16, (h)$			
6	$8, (g)$	$32, (h)$			
7	$15, (g)$	$16, (h)$	$64, (f)$		
8	$30, (g)$	$32, (h)$	$128, (f)$		

The variety of n, r, and $m_{n,r}$ for $n = 9$ or 10 are illustrated in Table 3.

Table 3: The variability of r, and $m_{n,r}$ for $n = 9$ or 10.

$n=9$			$n=10$		
r	$m_{n,r}$	$m_{n,r}$	r	$m_{n,r}$	$m_{n,r}$
1	59	$2^5+2^4+2^3+2^1+2^0$	1	118	$2^6+2^5+2^4+2^2+2^1$
2	60	$2^5+2^4+2^3+2^2$	2	120	$2^6+2^5+2^4+2^3$
3	64	2^6	3	128	2^7
4	256	2^8	4	512	2^9

Lemma 6. *[17] Let c, n and m be three integers, $n \geq 4, 0 \leq c \leq n-2$ and $2^c \leq m \leq 2^{n-1}$. Then $\xi_m(Q_{n,2}) \geq \xi_{2^c}(Q_{n,2})$.*

Lemma 7. *Let n, r be two integers, $n \geq 9$, $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$. Then $\xi_{m_{n,r}}(Q_{n,2}) = 2^{n-1}$.*

Proof. According to different expressions of $m_{n,r}$, the proof will be divided into four cases.

Case 1. For $1 \leq r \leq \lceil \frac{n}{2} \rceil - 4$, $m_{n,r} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-1-f}$, by Lemma 2 and formula (5), it can be obtained that

$$\begin{aligned}
\xi_{m_{n,r}}(Q_{n,2}) &= (n+1)m_{n,r} - ex_{m_{n,r}}(Q_{n,2}) \\
&= (n+1)[\textstyle\sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-1-f}] - \{\sum_{i=0}^{2}[(n-4-i)2^{n-4-i} + 2i2^{n-4-i})] \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r}[(n \\
&\quad -8-2i)2^{n-8-2i} + 2(3+i)2^{n-8-2i}] + [(2r-1-f)2^{2r-1-f} + 2(\lceil \tfrac{n}{2} \rceil - r)2^{2r-1-f}]\} \\
&= (n+1-n+4-i) \textstyle\sum_{i=0}^{2} 2^{n-4-i} + (n+1-n+8) \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + (n+2+f-2\lceil \frac{n}{2} \rceil)2^{2r-1-f} \\
&= (5-i) \textstyle\sum_{i=0}^{2} 2^{n-4-i} + 3 \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-f} \\
&= 5 \cdot 2^{n-4} + 4 \cdot 2^{n-5} + 4 \cdot 2^{n-6} - 2^{2r-f} + 2^{2r-f} \\
&= 3 \cdot 2^{n-3} + 2^{n-3} \\
&= 2^{n-1}.
\end{aligned}$$

Case 2. For $r = \lceil \frac{n}{2} \rceil - 3$, $m_{n,r} = \sum_{i=0}^{3} 2^{n-4-i}$, by Lemma 2 and the formula (5), $\xi_{m_{n,r}}(Q_{n,2}) = (n+1)m_{n,r} - ex_{m_{n,r}}(Q_{n,2}) = \sum_{i=0}^{3}(n-4-i)2^{n-4-i} + \sum_{i=0}^{3} 2i2^{n-4-i} = (5-i)\sum_{i=0}^{3} 2^{n-4-i} = 5 \times 2^{n-4} + 4 \times 2^{n-5} + 3 \times 2^{n-6} + 2 \times 2^{n-7} = 2^{n-1}$.

Case 3. For $r = \lceil \frac{n}{2} \rceil - 2$, $m_{n,r} = 2^{n-3}$, by Lemma 2 and the formula (5), it is not difficult to see that $\xi_{2^{n-3}}(Q_{n,2}) = (n+1) \times 2^{n-3} - (n-3) \times 2^{n-3} = 2^{n-1}$.

Case 4. For $r = \lceil \frac{n}{2} \rceil - 1$, $m_{n,r} = 2^{n-1}$, by the formula (5) and Lemma 2, then $\xi_{2^{n-1}}(Q_{n,2}) = (n+1) \times 2^{n-1} - [(n-1) \times 2^{n-1} + 2 \times 2^{n-1} - 2^{n-1}] = 2^{n-1}$.

From the above four cases, it can conclude that $\xi_{m_{n,r}}(Q_{n,2}) = 2^{n-1}$ for $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$. The proof is completed. □

Lemma 8. *Given two integers $n \geq 9, \lceil \frac{11 \times 2^{n-1}}{48} \rceil \leq m \leq 2^{n-1}$. There exists a positive integer r, satisfying $m_{n,r} < m < m_{n,r+1}$. $\xi_m(Q_{n,2}) > \xi_{m_{n,r}}(Q_{n,2}) = \xi_{m_{n,r+1}}(Q_{n,2}) = \cdots = \xi_{m_{n,\lceil \frac{n}{2} \rceil - 1}}(Q_{n,2}) = \xi_{2^{n-1}}(Q_{n,2}) = 2^{n-1}$.*

Proof. According to different expressions of $ex_m(Q_{n,2})$, the proof will be divided into two cases.

Case 1. $\lceil \frac{11 \times 2^{n-1}}{48} \rceil \leq m \leq 2^{n-2}$.

One can check that $m_{n,r+1} - m_{n,r} = 2^{2r-1-f}$ for $1 \leq r \leq \lceil \frac{n}{2} \rceil - 3$. By Lemma 7, $\xi_{m_{n,r}}(Q_{n,2}) = \xi_{2^{n-1}}(Q_{n,2}) = 2^{n-1}$ for $1 \leq r \leq \lceil \frac{n}{2} \rceil - 1$. Let $m = m_{n,r} + 2^{2r-1-f} + p$, where $m_{n,r} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-1-f}, 0 \leq p < 2^{2r-1-f}, p = \sum_{i=0}^{s} 2^{t'_i} < m_{n,r+1} - m_{n,r}, 2r-1-f > t'_0 > t'_1 > \cdots > t'_s$. If $1 \leq r < \lceil \frac{n}{2} \rceil - 4$, then $m_{n,r} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 4 - r} 2^{n-8-2i} + 2^{2r-1-f}$; if $r = \lceil \frac{n}{2} \rceil - 3$, then $m_{n,r} = \sum_{i=0}^{3} 2^{n-4-i}$. By the equation (5) and Lemma 5, one can deduce that

$$\begin{aligned}
&\xi_m(Q_{n,2}) - \xi_{m_{n,r}+2^{2r-1-f}}(Q_{n,2}) \\
&= \xi_{m_{n,r}+2^{2r-1-f}+p}(Q_{n,2}) - \xi_{m_{n,r}+2^{2r-1-f}}(Q_{n,2}) \\
&= (n+1)m - ex_m(Q_{n,2}) - (n+1)(m_{n,r} + 2^{2r-1-f}) + ex_{m_{n,r}+2^{2r-1-f}}(Q_{n,2}) \\
&= (n+1)(m_{n,r} + 2^{2r-1-f} + p) - ex_{m_{n,r}+2^{2r-1-f}+p}(Q_{n,2}) - (n+1)(m_{n,r} + 2^{2r-1-f}) + ex_{m_{n,r}+2^{2r-1-f}}(Q_{n,2}) \\
&= (n+1)p - ex_{m_{n,r}+2^{2r-1-f}+p}(Q_n) + ex_{m_{n,r}+2^{2r-1-f}}(Q_n) \quad \textbf{(Lemma 2)} \\
&= (n+1)p - [ex_{m_{n,r}+2^{2r-1-f}}(Q_n) + ex_p(Q_n) + 2(\lceil \tfrac{n}{2} \rceil - r + 1)p] + ex_{m_{n,r}+2^{2r-1-f}}(Q_n) \\
&= (2r-f-1)p - ex_p(Q_{2r-f-1}) \\
&= \xi_p(Q_{2r-f-1}).
\end{aligned}$$

For $p < 2^{2r-1-f}$, the value of $ex_p(Q_n)$ is uniquely determined by the binary representation of p. Therefore, $ex_p(Q_n) = ex_p(Q_{2r-f-1})$. By Lemma 1, $ex_p(Q_{2r-f-1}) = 2|E(Q_{2r-1-f}[L_p^{2r-f-1}])|$. $[L_p^{2r-f-1}, \overline{L_p^{2r-f-1}}]$ be an edge cut of Q_{2r-f-1}. Since Q_{2r-f-1} is connected graph, and if one deletes the edge cut $[L_p^{2r-f-1}, \overline{L_p^{2r-f-1}}]$, two induced subgraphs by L_p^{2r-f-1} and $\overline{L_p^{2r-f-1}}$ are connected, the edge cut $[L_p^{2r-f-1}, \overline{L_p^{2r-f-1}}]$ of Q_{2r-f-1} does exist. By Lemma 3, it is sufficient to show that $ex_p(Q_{2r-f-1}) \leq (2r-1-f)p$, and $\xi_m(Q_{n,2}) - \xi_{m_{n,r}}(Q_{n,2}) = (2r-f-1)p - ex_p\left(Q_{2r-f-1}\right) > 0$.

If $r = \lceil \frac{n}{2} \rceil - 2$, then $m_{n,r} = 2^{n-3}$. There exists a positive integer $p' = \sum_{i=0}^{s} 2^{t_i'}$, satisfying $0 \leq p' < 2^{n-3}$, $m_1 = 2^{n-3} + p'$ and $n-3 > t_0' > t_1' > \cdots > t_s'$. The proof of $\xi_{m_1}(Q_{n,2}) > \xi_{m_{n,r}}(Q_{n,2})$ is the same as the above proof of $\xi_m(Q_{n,2}) > \xi_{m_{n,r}}(Q_{n,2})$.

Case 2. $2^{n-2} \leq m \leq 2^{n-1}$.

If $r = \lceil \frac{n}{2} \rceil - 2$, then $m_{n,r} = 2^{n-3}$. There exists a positive integer $m'' = \sum_{i=0}^{s} 2^{t_i'}$, satisfying $0 \leq m'' < 2^{n-2}$, $m = m_{n,r} + 2^{n-3} + m'' = 2^{n-2} + m''$ and $n-2 > t_0' > t_1' > \cdots > t_s'$. By the equation (5) and Lemma 5,

$$\begin{aligned}
&\xi_m(Q_{n,2}) - \xi_{2^{n-3}}(Q_{n,2}) \\
&= \xi_m(Q_{n,2}) - \xi_{2^{n-2}}(Q_{n,2}) + \xi_{2^{n-2}}(Q_{n,2}) - \xi_{2^{n-3}}(Q_{n,2}) \\
&= (n+1)(2^{n-2} + m'') - (n+1)2^{n-2} - (ex_m(Q_{n,2}) - ex_{2^{n-2}}(Q_{n,2})) + 2^{n-2} \\
&= (n+1)m'' - (ex_{2^{n-2}+m''}(Q_n) + 2m'') + ex_{2^{n-2}}(Q_n) + 2^{n-2} \quad \textbf{(Lemma 2)} \\
&= (n+1)m'' - ex_{2^{n-2}}(Q_n) - ex_{m''}(Q_n) - 4m'' + ex_{2^{n-2}}(Q_n) + 2^{n-2} \\
&= (n+1)m'' - ex_{m''}(Q_n) - 4m'' + 2^{n-2} \\
&= (n-3)m'' - ex_{m''}(Q_n) + 2^{n-2} \\
&= (n-3)m'' - ex_{m''}(Q_{n-3}) + 2^{n-2} \\
&= \xi_{m''}(Q_{n-3}) + 2^{n-2} \\
&> 0.
\end{aligned}$$

For $0 \leq m'' \leq 2^{n-2}$, the value of $ex_{m''}(Q_n)$ is uniquely determined by the binary representation of m''. Thus, $ex_{m''}(Q_n) = ex_{m''}(Q_{n-3})$. By Lemma 3, $(n-3)m'' - ex_m''(Q_{n-3}) > 0$ for $0 \leq m'' \leq 2^{n-2}$. Thus, $\xi_m(Q_{n,2}) > \xi_{m_{n,r}}(Q_{n,2})$.

Combining the above two cases, $\xi_m(Q_{n,2}) > \xi_{m_{n,r}}(Q_{n,2}) = \xi_{m_{n,r+1}}(Q_{n,2}) = 2^{n-1}$ for $1 \leq r \leq \lceil \frac{n}{2} \rceil - 2$. So the proof is completed. □

4. The h-extra edge-connectivity of $Q_{n,2}$ concentrates on 2^{n-1} for $\lceil \frac{11\times 2^{n-1}}{48} \rceil \leq h \leq 2^{n-1}$

Proof. **The proof of Theorem** 1 (a). Given each integer h, for $\lceil \frac{11\times 2^{n-1}}{48} \rceil \leq h \leq m_{n,\lceil \frac{n}{2} \rceil - 1}$, there exists an integer r, $1 \leq r \leq \lceil \frac{n}{2} \rceil - 1$, satisfying $m_{n,r} \leq h \leq m_{n,r+1}$. By Lemma 4 and Lemma 8, $\lambda_h(Q_{n,2}) = \min\{\xi_m(Q_{n,2}) : m_{n,r} \leq h \leq m < m_{n,r+1}\} = \xi_{m_{n,r}}(Q_{n,2})$ for $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$. So for any $\lceil \frac{11\times 2^{n-1}}{48} \rceil \leq h \leq 2^{n-1}$, $\lambda_h(Q_{n,2}) = \min\{\xi_m(Q_{n,2}) : \lceil \frac{11\times 2^{n-1}}{48} \rceil \leq h \leq m \leq 2^{n-1}\} = \xi_{2^{n-1}}(Q_{n,2})$.

$$\begin{aligned}
\lambda_h(Q_{n,2}) &= \min\{\xi_m(Q_{n,2}) : h \leq m \leq 2^{n-1}\} \{\textbf{Lemma 4}\} \\
&= \min\{\xi_m(Q_{n,2}) : h \leq m \leq m_{n,\lceil \frac{n}{2} \rceil - 1}\} \{\textbf{Lemmas 8 and 6}\} \\
&= \xi_{2^{n-1}}(Q_{n,2}) \{\textbf{Lemmas 7 and 8}\} \\
&= 2^{n-1}.
\end{aligned}$$

The proof of Theorem 1 (b). If $h = m_{n,r}$ or $h = m_{n,r+1}$, by Lemma 4 and Lemma 7, $\lambda_h(Q_{n,2}) = \xi_h(Q_{n,2}) = 2^{n-1}$. If $m_{n,r} < h < m_{n,r+1}$, by Lemma 8, $\xi_h(Q_{n,2}) > \xi_{m_{n,r+1}}(Q_{n,2})$, by Lemma 4 and Lemma 7, $\lambda_h(Q_{n,2}) = \min\{\xi_m(Q_{n,2}) : h \leq m \leq m_{n,r+1}\} = \xi_{m_{n,r+1}}(Q_{n,2}) = \xi_{m_{n,r+2}}(Q_{n,2}) = \cdots = \xi_{m_{n,\lceil \frac{n}{2} \rceil - 1}}(Q_{n,2}) = 2^{n-1}$. So, one can get $\lambda_h(Q_{n,2}) = \xi_h(Q_{n,2}) = 2^{n-1}$ for $h = m_{n,r}$ or $h = m_{n,r+1}$, $1 \leq r < \lceil \frac{n}{2} \rceil - 1$.

The proof is completed. □

Remark 1. For $\lceil \frac{11\times 2^{n-1}}{48} \rceil \leq h \leq 2^{n-1}$ and $n \geq 9$, the lower and upper bounds of h in $\lambda_h(Q_{n,2})$ are both tight.

(1) In fact, if n is even, then $m_{n,1} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 5} 2^{n-8-2i} + 2$, $m_{n,1} - 1 = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 5} 2^{n-8-2i} + 1$. By Lemma 4, $ex_{m_{n,1}}(Q_{n,2}) = ex_{m_{n,1}-1}(Q_{n,2}) + n - 2$. So, $\xi_{m_{n,1}}(Q_{n,2}) - \xi_{m_{n,1}-1}(Q_{n,2}) = (n+1)m_{n,1} - ex_{m_{n,1}}(Q_{n,2}) - (n+1)(m_{n,1} - 1) + ex_{m_{n,1}-1}(Q_{n,2}) = 3$. Note that $\lambda_{m_{n,1}-1}(Q_{n,2}) = \min\{\xi_h(Q_{n,2}) : m_{n,1} - 1 \leq h \leq m_{n,1}\} = \xi_{m_{n,1}-1}(Q_{n,2}) = 2^{n-1} - 1 < 2^{n-1} = \lambda_{m_{n,1}}(Q_{n,2}) = \xi_{m_{n,1}}(Q_{n,2})$. If n is odd, then $m_{n,1} = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 5} 2^{n-8-2i} + 1$, $m_{n,1} - 1 = \sum_{i=0}^{2} 2^{n-4-i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 5} 2^{n-8-2i}$. By Lemma 4, $ex_{m_{n,1}}(Q_{n,2}) = ex_{m_{n,1}-1}(Q_{n,2}) + n - 3$. So, $\xi_{m_{n,1}}(Q_{n,2}) - \xi_{m_{n,1}-1}(Q_{n,2}) = (n+1)m_{n,1} - ex_{m_{n,1}}(Q_{n,2}) - (n+

Table 4: Examples of $\xi_h(Q_{n,2})$ and $\lambda_h(Q_{n,2})$ for $4 \le n \le 9$.

h	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\xi_h(Q_{4,2})$	5	8	11	12	13	12	11	8																								
$\lambda_h(Q_{4,2})$	5	8	8	8	8	8	8	8																								
$\xi_h(Q_{5,2})$	6	10	14	16	20	22	24	24	26	26	26	24	24	22	20	16																
$\lambda_h(Q_{5,2})$	6	10	14	16	16	16	16	16	16	16	16	16	16	16	16	16																
$\xi_h(Q_{6,2})$	7	12	17	20	25	28	31	32	37	40	43	44	47	48	49	48	51	52	53	52	53	52	51	48	49	48	47	44	43	40	37	32
$\lambda_h(Q_{6,2})$	7	12	17	20	25	28	31	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32
$\xi_h(Q_{7,2})$	8	14	20	24	30	34	38	40	46	50	54	56	60	62	64	64	70	74	78	80	84	86	88	88	92	94	96	96	98	98	98	96
$\lambda_h(Q_{7,2})$	8	14	20	24	30	34	38	40	46	50	54	56	60	62	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64
$\xi_h(Q_{8,2})$	9	16	23	28	35	40	45	48	55	60	65	68	73	76	79	80	87	92	97	100	105	108	111	112	117	120	123	124	127	128	129	128
$\lambda_h(Q_{8,2})$	9	16	23	28	35	40	45	48	55	60	65	68	73	76	79	80	87	92	97	100	105	108	111	112	117	120	123	124	127	128	128	128
$\xi_h(Q_{9,2})$	10	18	26	32	40	46	52	56	64	70	76	80	86	90	94	96	104	110	116	120	126	130	134	136	142	146	150	152	156	158	160	160
$\lambda_h(Q_{9,2})$	10	18	26	32	40	46	52	56	64	70	76	80	86	90	94	96	104	110	116	120	126	130	134	136	142	146	150	152	156	158	160	160
h	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
$\xi_h(Q_{7,2})$	100	102	104	104	106	106	106	104	106	106	106	104	104	102	100	96	98	98	98	96	96	94	92	88	88	86	84	80	78	74	70	64
$\lambda_h(Q_{7,2})$	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64	64
$\xi_h(Q_{8,2})$	135	140	145	148	153	156	159	160	165	168	171	172	175	176	177	176	181	184	187	188	191	192	193	192	195	196	197	196	197	196	195	192
$\lambda_h(Q_{8,2})$	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128
$\xi_h(Q_{9,2})$	168	174	180	184	190	194	198	200	206	210	214	216	220	222	224	224	230	234	238	240	244	246	248	248	252	254	256	256	258	258	258	256
$\lambda_h(Q_{9,2})$	168	174	180	184	190	194	198	200	206	210	214	216	220	222	224	224	230	234	238	240	244	246	248	248	252	254	256	256	256	256	256	256
h	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96
$\xi_h(Q_{8,2})$	197	200	203	204	207	208	209	208	211	212	213	212	213	212	211	208	211	212	213	212	213	212	211	208	209	208	207	204	203	200	197	192
$\lambda_h(Q_{8,2})$	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128
$\xi_h(Q_{9,2})$	264	270	276	280	286	290	294	296	302	306	310	312	316	318	320	320	326	330	334	336	340	342	344	344	348	350	352	352	354	354	354	352
$\lambda_h(Q_{9,2})$	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256
h	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128
$\xi_h(Q_{8,2})$	195	196	197	196	197	196	195	192	193	192	191	188	187	184	181	176	177	176	175	172	171	168	165	160	159	156	153	148	145	140	135	128
$\lambda_h(Q_{8,2})$	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128
$\xi_h(Q_{9,2})$	358	362	366	368	372	374	376	376	380	382	384	384	386	386	386	384	388	390	392	392	394	394	394	392	394	394	394	392	392	390	388	384
$\lambda_h(Q_{9,2})$	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256
h	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160
$\xi_h(Q_{9,2})$	390	394	398	400	404	406	408	408	412	414	416	416	418	418	418	416	420	422	424	424	426	426	426	424	426	426	426	424	424	422	420	416
$\lambda_h(Q_{9,2})$	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256
h	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192
$\xi_h(Q_{9,2})$	420	422	424	424	426	426	426	424	426	426	426	424	424	422	420	416	418	418	418	416	416	414	412	408	408	406	404	400	398	394	390	384
$\lambda_h(Q_{9,2})$	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256
h	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220	221	222	223	224
$\xi_h(Q_{9,2})$	388	390	392	392	394	394	394	392	394	394	394	392	392	390	388	384	386	386	386	384	384	382	380	376	376	374	372	368	366	362	358	352
$\lambda_h(Q_{9,2})$	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256
h	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256
$\xi_h(Q_{9,2})$	354	354	354	352	352	350	348	344	344	342	340	336	334	330	326	320	320	318	316	312	310	306	302	296	294	290	286	280	276	270	264	256
$\lambda_h(Q_{9,2})$	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256

$1)(m_{n,1}-1) + ex_{m_{n,1}-1}(Q_{n,2}) = 4$. Similarly it can see that $\lambda_{m_{n,1}-1}(Q_{n,2}) = \min\{\xi_h(Q_{n,2}) : m_{n,1}-1 \le h \le m_{n,1}\} = \xi_{m_{n,1}-1}(Q_{n,2}) = 2^{n-1}-2 < 2^{n-1} = \lambda_{m_{n,1}}(Q_{n,2}) = \xi_{m_{n,1}}(Q_{n,2})$. Therefore, the lower bound is sharp.

(2) As $|V(Q_{n,2})| = 2^n$, by the definition of h-extra edge-connectivity, there are at least two components with at least h vertices. So, the upper bound of the above interval is 2^{n-1}. Therefore, the upper bound is sharp.

There are some cases when $4 \le n \le 9$ and $h \le 2^{n-1}$, the data of $\lambda_h(Q_{n,2})$ and $\xi_h(Q_{n,2})$ have been presented in Table 4, where the values of $\lambda_h(Q_{n,2})$ do not satisfy the equality $\lambda_h(Q_{n,2}) = \xi_h(Q_{n,2})$ are marked in red, otherwise are marked in black. Based on these data, the scatter plots of $\xi_h(Q_{n,2})$ and $\lambda_h(Q_{n,2})$ are plotted. We plot the $\xi_h(Q_{n,2})$ marked in " Δ " scatters and the $\lambda_h(Q_{n,2})$ marked in " $*$ " scatters for $4 \le n \le 12$ in Fig. 5. On the X-axis in Fig. 5, the result of this article is represented by the green lines.

We make a simulation of computing the possible sizes of the edge-cuts of $Q_{n,2}$ for $n = 5$. In the first figure of Fig. 6, the simulink results for the edge-cuts $[X, \overline{X}]_{Q_{5,2}}$ of $Q_{5,2}$ with one component having h vertices and the function $\xi_h(Q_{5,2})$ for $1 \le h \le 2^5$ are displayed. The possible sizes of the edge-cuts $[X, \overline{X}]_{Q_{5,2}}$ of $Q_{5,2}$ for $h = 6$ are $22, 24, 26, 28, 30, 32$, and 34 according to the distribution of the first figure of Fig. 6. The lower bound for these values is $\xi_6(Q_{5,2}) = 22$. The scatter plot of the function $\xi_h(Q_{5,2})$ (depicted in blue " $\triangle$ " scatters) is symmetric with regard to $h = 2^4$ because $|[X, \overline{X}]| = |[\overline{X}, X]|$. In general, the theoretical function $\xi_h(Q_{5,2})$ lower bounds our simulation on the sizes of all the edge-cuts $[X, \overline{X}]_{Q_{5,2}}$ with one component containing h vertices for each $0 \le h \le 2^4$.

The sizes of the h-extra edge-cuts of $Q_{5,2}, \xi_h(Q_{5,2})$ and $\lambda_h(Q_{5,2})$ for $h \le 2^4$ are are shown in the second figure of Fig. 6. According to Lemma 4, $\lambda_h(Q_{5,2}) = \min\{\xi_m(Q_{5,2}) : 1 \le h \le m \le 2^4\}$. We also find that the h-extra edge-connectivity of the $(5,2)$-enhanced hypercube $Q_{5,2}$ presents a concentration phenomenon on the value 16 for $4 \le h \le 16$. The results of the simulation are in consistent with those of theoretical analysis.

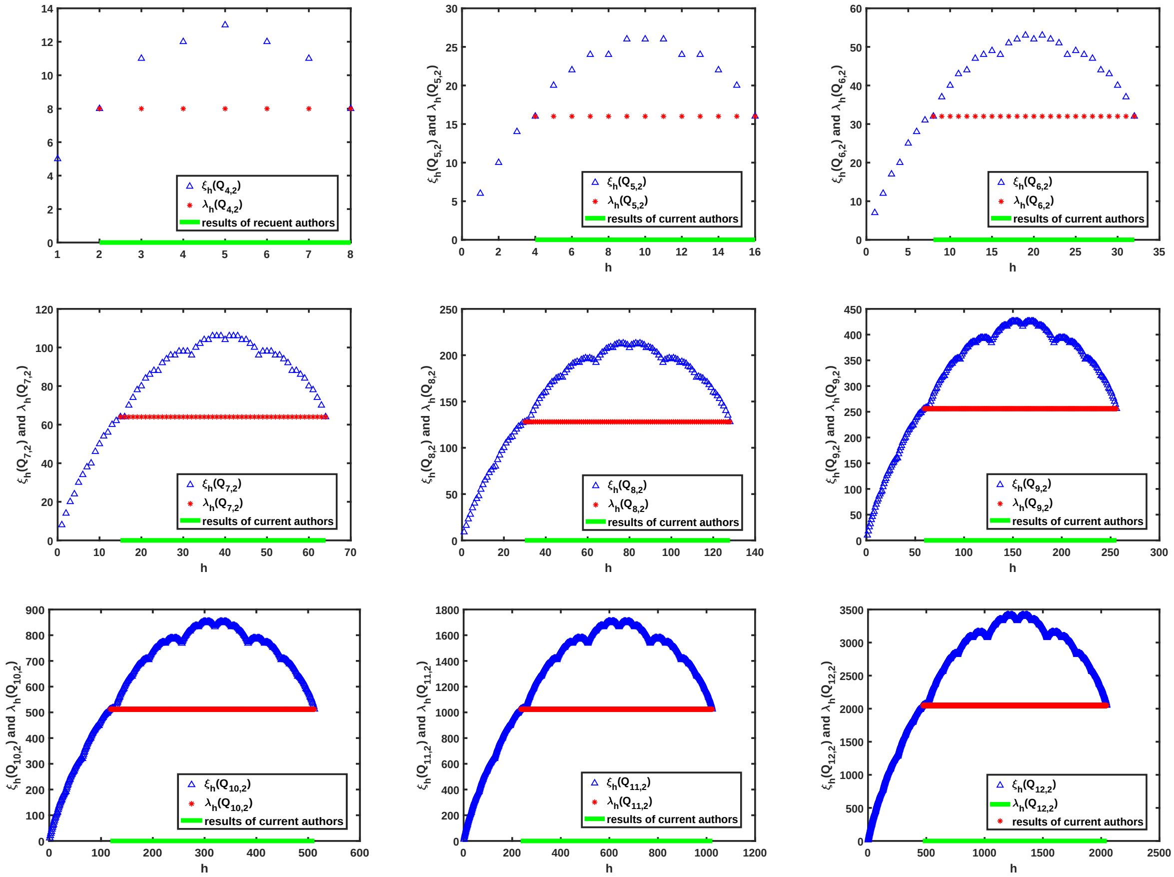

Fig. 5: The scatter plot of $\lambda_h(Q_{n,2})$ and $\xi_h(Q_{n,2})$ for case $4 \le n \le 12$.

Table 5: The values $g(n)$ and $R(n)$ for $4 \le n \le 31$.

n	$g(n)$	R	n	$g(n)$	R
4	7	87.5%	18	101035	77.083587%
5	13	81.25%	19	202070	77.083587%
6	25	78.125%	20	404139	77.083396%
7	50	78.125%	21	808278	77.083396%
8	99	77.34375%	22	1616555	77.083349%
9	198	77.34375%	23	3233110	77.083349%
10	395	77.148437%	24	6466219	77.083337%
11	790	77.148437%	25	12932438	77.083337%
12	1579	77.099609%	26	25864875	77.083334%
13	3158	77.099609%	27	51729750	77.083334%
14	6315	77.087402%	28	103459499	77.083333%
15	12630	77.087402%	29	206918998	77.083333%
16	25259	77.084350%	30	413837995	77.083333%
17	50518	77.084350%	31	827675990	77.083333%

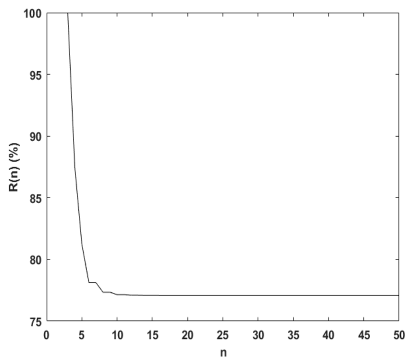

Fig. 7: The plot of function $R(n)$.

Unexpectedly, we find that the h-extra edge-connectivity of $Q_{n,2}$ exists a concentration phenomenon for some exponentially large h on the interval of $\lceil \frac{11\times 2^{n-1}}{48} \rceil \le h \le 2^{n-1}$. Let $g(n) = |\{h \ : \ \lambda_h(Q_{n,2}) = 2^{n-1}, h \le 2^{n-1}\}|$. So $g(n) = 2^{n-1} - \lceil \frac{11\times 2^{n-1}}{48} \rceil + 1$. Due to $|V(Q_{n,2})| = 2^n$, $\lambda_h(Q_{n,2})$ is well-defined for any integer $1 \le h \le 2^{n-1}$. Let $R(n) = \frac{g(n)}{2^{n-1}}$ be the percentage of the number of integer h with the corresponding $\lambda_h(Q_{n,2}) = \xi_h(Q_{n,2}) = 2^{n-1}$ for $1 \le h \le 2^{n-1}$. For

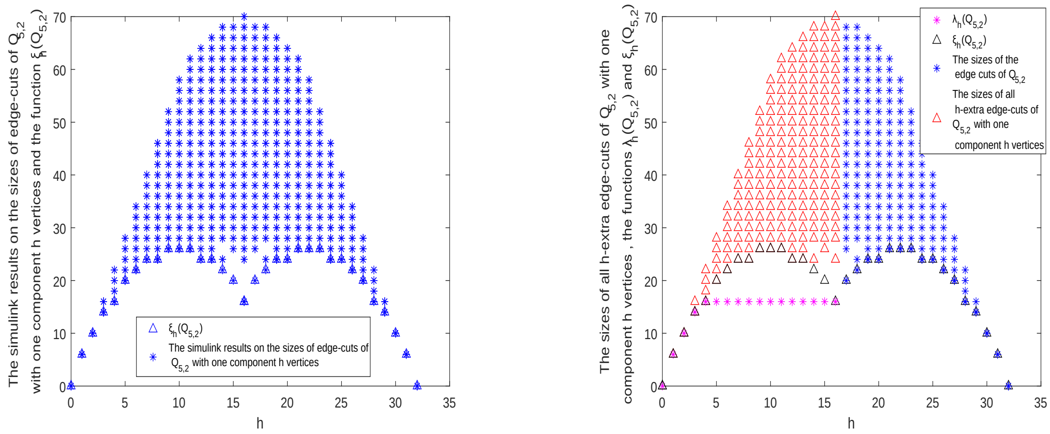

Fig. 6: The comparison of the sizes of h-extra edge-cuts in $Q_{5,2}$ between the simulation and our results.

the sake of simplicity, Table 5 lists some exact values of the function $R(n)$ for $4 \leq n \leq 31$. Then $R(n) = \frac{2^{n-1} - \lceil \frac{11\times 2^{n-1}}{48} \rceil + 1}{2^{n-1}}$, $\lim_{n\to\infty} R(n) = \frac{37}{48}$. The function $R(n)$ is shown in Fig. 7. The ratio of the length of the $\lambda_h(Q_{n,2}) = 2^{n-1}$ subinterval to the $0 \leq h \leq 2^{n-1}$ interval gets infinitely closer to $\frac{37}{48}$ as n grows. For $n \to \infty$, 77.083% of $\lambda_h(Q_{n,2})$ is 2^{n-1}, which shows the concentration phenomenon of $\lambda_h(Q_{n,2})$. Furthermore, similar results can be obtained, even if the lower bound of h is not $\lceil \frac{11\times 2^{n-1}}{48} \rceil$ for $4 \leq n \leq 8$.

5. Conclusions

It is well known that the h-extra edge-connectivity is an important indicator for measuring the fault tolerance and reliability of interconnection networks. This paper shows that the h-extra edge-connectivity of $(n, 2)$-enhanced hypercubes $Q_{n,2}$ presents a concentration phenomenon in the subinterval $\lceil \frac{11\times 2^{n-1}}{48} \rceil \leq h \leq 2^{n-1}$ for $n \geq 9$. For about 77.083% values of $h \leq 2^{n-1}$, the minimum number of link malfunctions is 2^{n-1}, and these link malfunctions disconnect $(n, 2)$-enhanced hypercube $Q_{n,2}$ and keep each resulting connected subnetworks with at least h processors. Our results provide a more accurate measure for evaluating a large-scale $Q_{n,2}$ network reliability and availability. In order to completely solve the h-extra edge-connectivity of the remaining intervals, we will give an algorithm to determine the exact value and the optimality of the h-extra edge-connectivity of $Q_{n,2}$ for each integer $h \leq 2^{n-1}$. Moreover, for the general network $Q_{n,k}$ an attempt to design an algorithm to solve the exact value and the optimality of $\lambda_h(Q_{n,k})$ also can be made.

Acknowledgements

The authors would like to thank anonymous referees and editors for their help. Their valuable comments and suggestions help to improve the quality of this paper.

CRediT authorship contribution statement

Yali Sun: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Software, Supervision, Validation, Visualization, Writing - original draft, Writing - review & editing. **Mingzu Zhang:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing - original draft, Writing - review & editing. **Xing Feng and Xing Yang:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Resources, Supervision, Validation, Visualization, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data available.

References

References

[1] M. Arockiaraj, J.-B. Liu, A.J. Shalini, Vertex decomposition method for wirelength problem and its applications to enhanced hypercube networks, IET Comput. Digit. Tech. 13 (2019) 87-92.

[2] N.-W. Chang, C.-Y. Tsai, S.-Y. Hsieh, On 3-extra connectivity and 3-extra edge connectivity of folded hypercubes, IEEE Trans. Comput. 63 (6) (2014) 1594-1600.

[3] Y.C. Chen, J.M. Tan, L.-H. Hsu, S.-S. Kao, Super-connectivity and super edge-connectivity for some interconnection networks, Appl. Math. Comput. 140 (2003) 245-254.

[4] A. El-Amawy, S. Latifi, Properties and performance of folded hypercubes, IEEE Trans. Parallel Distrib. Syst. 2 (1) (1991) 31-42.

[5] J. Fàbrega, M.A. Fiol, On the extraconnectivity of graphs, Discrete Math. 155 (1996) 49-57.

[6] F. Harary, Conditional connectivity, Networks 13 (3) (1983) 347-357.

[7] L.H. Harper, Optimal assignments of numbers to vertices, J. Soc. Ind. Appl. Math. 12 (1) (1964) 131135.

[8] W.-S. Hong, S.-Y. Hsieh, Extra edge connectivity of hypercube-like networks, Int. J. Parallel Emergent Distrib. Syst. 28 (2) (2013) 123-133.

[9] J. Li, Y.Z. Huang, L.M. Lin, H. Yu, R.Q. Chen, The extra connectivity of enhanced hypercubes, Int. J. Parallel Emergent Distrib. Syst. 35 (1) (2020) 91-102.

[10] H. Li, W.H. Yang, Bounding the size of the subgraph induced by m vertices and extra edge-connectivity of hypercubes, Discrete Appl. Math. 161 (2013) 2753-2757.

[11] M.J. Lv, J.X. Fan, B.L. Cheng, J. Yu, X.J. Jia, Construction algorithms of fault-tolerant paths and disjoint paths in k-ary n-cube networks, J. Parallel Distrib. Comput. 183 (2024) 104761.

[12] W.H. Ma, M.Z. Zhang, J.X. Meng, T.L. Ma, Exponential type of many-to-many edge disjoint paths on ternary n-cubes, J. Parallel Distrib. Comput. 158 (2021) 67-79.

[13] E. Sabir, A. Mamut, E. Vumar, The extra connectivity of the enhanced hypercubes, Theoret. Comput. Sci. 799 (2019) 22-31.

[14] H. Sullivan, T.R. Bashkow, A large scale, homogeneous, fully distributed parallel machine, I, Proceeding 4th Annual Sympo. Comput. archit. (1977) 105-117.

[15] N.-F. Tzeng, S.Z. Wei, Enhanced hypercubes, IEEE Trans. Comput. 40 (1991) 284-294.

[16] D.J. Wang, Diagnosability of enhanced hypercubes, IEEE Trans. Comput. 43 (9) (1994) 1054-1061.

[17] L.Q. Xu, S.M. Zhou, J.F. Liu, S.S. Yin, Reliability measure of multiprocessor system based on enhanced hypercubes, Discrete Appl. Math. 289 (2021) 125-138.

[18] W.H. Yang, H. Li, On reliability of the folded hypercubes in terms of the extra edge-connectivity, Inform. Sci. 272 (2014) 238-243.

[19] W.H. Yang, H.Q. Lin, Reliability evaluation of BC networks in terms of the extra vertex- and edge-connectivity, IEEE Trans. Comput. 63 (10) (2014) 2540-2548.

[20] W.H. Yang, J.X. Meng, Extraconnectivity of Folded Hypercubes, Ars Comb. 116 (2014) 121-127.

[21] Y.Y. Yang, M.Z. Zhang, J.X. Meng, Fault tolerance analysis for hamming graphs with large-scale faulty links based on k-component edge-connectivity, J. Parallel Distrib. Comput. 173 (2023) 107-114.

[22] S.S. Yin, L.Q. Xu, On the g-extra connectivity of the enhanced hypercubes, Comput. J. 65 (9) (2022) 2339-2346.

[23] M.Z. Zhang, Edge isoperimetric problem on graphs and the related applications[D]Xia, men: Univ of Xiamen (2018) 68-77.

[24] M.Z. Zhang, J.X. Meng, W.H. Yang, Y.Z. Tian, Reliability analysis of bijective connection networks in terms of the extra edge-connectivity, Inform. Sci. 279 (2014) 374-382.

[25] M.Z. Zhang, L.Z. Zhang, X. Feng, Reliability measures in relation to the h-extra edge-connectivity of folded hypercubes, Theoret. Comput. Sci. 615 (2016) 71-77.

[26] M.Z. Zhang, L.Z. Zhang, X. Feng, H.-J. Lai, An $O(log_2(N))$ algorithm for reliability evaluation of h-extra edge-connectivity of folded hypercubes, IEEE Trans. Reliab. 67 (1) (2018) 297-307.

[27] H. Zhang, S.M. Zhou, Characterization of matroidal connectivity of regular networks, J. Parallel Distrib. Comput. 186 (2024) 104818.

[28] S.L. Zhao, W.H. Yang, Conditional connectivity of folded hypercubes, Discrete Appl. Math. 257 (2019) 388-392.

[29] J.-X. Zhou, On g-extra connectivity of hypercube-like networks, J. Comput. Syst. Sci. 88 (2017) 208-219.

[30] Q. Zhu, J.-M. Xu, On restricted edge connectivity and extra edge connectivity of hypercubes and folded hypercubes, J. Univ. Sci. Technol. China 36 (2006) 246-253.

[31] Q. Zhu, J.-M. Xu, X.M. Hou, M. Xu, On reliability of the folded hypercubes, Inform. Sci. 177 (2007) 1782-1788.
[32] Q. Zhu, J.-M. Xu, M. Lv, Edge fault tolerance analysis of a class of interconnection networks, Appl. Math. Comput. 172 (2006) 111-121.