FINITE-TIME BLOWUP FOR KELLER-SEGEL-NAVIER-STOKES SYSTEM IN THREE DIMENSIONS

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Abstract. While finite-time blowup solutions have been studied in depth for the Keller-Segel equation, a fundamental model describing chemotaxis, the existence of finite-time blowup solutions to chemotaxis-fluid models remains largely unexplored. To fill this gap in the literature, we use a quantitative method to directly construct a smooth finite-time blowup solution for the Keller-Segel-Navier-Stokes system with buoyancy in $3D$. The heart of the proof is to establish the non-radial finite-codimensional stability of an explicit self-similar blowup solution to $3D$ Keller-Segel equation with the abstract semigroup tool from [42], which partially generalizes the radial stability result [27] to the non-radial setting. Additionally, we introduce a robust localization argument to find blowup solutions with non-negative density and finite mass.

1. Introduction

Chemotaxis is a ubiquitous phenomenon observed in nature, where organisms like body cells and bacteria respond directionally to the chemical substance in the external environment. A fundamental mathematical model for chemotaxis is the Keller-Segel equation, formulated as follows:

$$\begin{cases} \partial_t \rho = \Delta\rho - \nabla\cdot(\rho\nabla c), \\ -\Delta c = \rho. \end{cases} \tag{KS}$$

Here, ρ represents the cell density, while c denotes the concentration of the self-emitted chemical substance. The system describes how cells exhibit random Brownian motion, resulting in dissipation. Simultaneously, the cells are attracted by chemical substances and move to the region with higher concentrations of chemical substances. For more background on chemotaxis and related models, interested readers can refer to [13, 33, 34].

On the other hand, many studies consider more practical chemotaxis models that are coupled with the ambient flow, and interested readers can refer to [32, 50, 53] for further motivations regarding the introduction of fluid-chemotaxis models. These models describe that cells are not solely attracted by self-emitted chemical substances, but are also transported by the ambient flow. In particular, this paper focuses on the $3D$ coupled Keller-Segel-Navier-Stokes system with buoyancy

$$\begin{cases} \partial_t \rho + u\cdot\nabla\rho = \Delta\rho - \nabla\cdot(\rho\nabla c), \\ -\Delta c = \rho, \\ \partial_t u + u\cdot\nabla u = \Delta u - \nabla\pi - \rho e_3, \\ \nabla\cdot u = 0, \end{cases} \tag{KS-NS}$$

where $e_3 = (0,0,1)$ and the vector field u, representing the ambient fluid flow, satisfies the Navier-Stokes equation with an additional force term $-\rho e_3$ on the right-hand side, which denotes the cell's reaction force to the buoyancy exerted by the fluid.

1.1. **Background.**

1.1.1. *Chemotaxis and singularity formation.* For Keller-Segel equation (KS), the total mass $M(t) = \int \rho(t,x)dx$ is conserved. In addition, the model (KS) satisfies the following scaling invariance: if (ρ, c) is the solution to (KS), so is

$$(\rho_\lambda, c_\lambda) = \left(\frac{1}{\lambda^2}\rho\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), c\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)\right), \quad \lambda > 0.$$

In two dimensions, the mass M plays a crucial role in determining whether a solution exhibits global existence or finite-time blowup, with $M = 8\pi$ serving as the threshold distinguishing between the two possibilities. Specifically, when $M > 8\pi$ and the initial data $\rho_0 \in L^1_+((1+|x|^2), dx)$, after tracking the evolution of the second moment $M_2[\rho(t)] := \int \rho(t,x)|x|^2 dx$, it can be confirmed that the solution would blow up in finite time (see [8, 21] for details). In addition, Blanchet, Dolbeault, and Perthame [8, 21] established a uniform L^∞ bound for the solution with mass $M < 8\pi$, proving the global existence.

In three dimensions, it is known that there is no critical mass: namely, there exist radial blowup solutions with arbitrarily small mass [46]. For general solutions that are not necessarily radial, Corrias, Perthame, and Zaag [19] proved blowup will happen as long as the second moment of the initial data is sufficiently small compared to its mass, whereas weak solutions exist globally if the initial data has small $L^{\frac{3}{2}}$ norm. Interested readers can refer to [6, 7, 49, 52] for more results.

When it comes to the singularity formation for the $2D$ Keller-Segel equation, with the model in the L^1 critical case, Naito and Suzuki [47] verified that any finite-time blowup solution to (KS) is of type II[1]. Later, in the radial setting, by using the tail analysis, Raphaël and Schweyer [51] precisely constructed a stable finite-blowup solution of the form

$$\rho(t,x) \simeq \frac{1}{\lambda^2(t)} U\left(\frac{x}{\lambda(t)}\right), \quad U(x) = \frac{8}{(1+|x|^2)^2},$$

with the blowup rate

$$\lambda(t) = \sqrt{T-t}\, e^{-\sqrt{\frac{|\ln(T-t)|}{2}} + O(1)}, \quad t \to T.$$

Subsequently, Collot, Ghoul, Masmoudi, and Nguyen [16] utilized spectral theory to extend stability to the nonradial setting and obtained a more precise blowup rate. Meanwhile, they also constructed countably many unstable blowup solutions. More recently, Buseghin, Davila, Del Pino, and Musso [12] employed the inner-outer gluing method to further extend these findings to multiple profiles.

Compared with $2D$ case, the singularity formation for the $3D$ Keller-Segel equation (KS) is much more diverse, with various blowup profiles and formations known to exist. Firstly, Brenner, Constantin, Kadanoff, Schenkel, and Venkataramani [9]

[1] The solution of (KS) exhibits type I blowup (or self-similar blowup) at $t = T$ if

$$\limsup_{t\to T} (T-t)\|\rho(t)\|_{L^\infty} < \infty,$$

otherwise, the blowup is of type II.

identified a countable family of self-similar blowup profiles. Moreover, they numerically verified the stability of an explicit self-similar solution given by

$$\rho(t,x) = \frac{1}{T-t} Q\left(\frac{x}{\sqrt{T-t}}\right), \quad \text{with} \quad Q(x) = \frac{4(6+|x|^2)}{(2+|x|^2)^2}. \tag{Profile}$$

Later, Glogić and Shörkhuber [27] rigorously established its stability in the radial case using semigroup theory. To the best of the authors' knowledge, it is still open for stability when extended to nonradial case. In addition, other non-self-similar blowup solutions to the $3D$ Keller-Segel equation have been identified. For example, Collot, Ghoul, Masmoudi, and Nguyen [17] discovered a type II blowup solution that concentrates in a thin layer outside the origin and collapses towards the center. More recently, Nguyen, Nouaili, and Zaag [48] proposed a type I-Log blowup solution for the $3D$ Keller-Segel equation.

1.1.2. *Suppression of blowup by fluids.*

When considering the chemotaxis-fluid models such as (KS-NS), the dynamics become more intricate. Upon cell aggregation, the buoyancy/friction-induced enhancement of the ambient flow might suppress the cell aggregation process. The literature on fluid-chemotaxis models is extensive, particularly regarding the suppression of system blowup.

Notably, significant progress has been made in the field of the suppression of chemotaxis blowup by the ambient fluid flow, building upon the pioneering work of Kiselev and Xu [38], where they identified the existence of passive flow[2], capable of suppressing finite-time blowup in both $2D$ and $3D$. Subsequently, Bedrossian and He [2, 3, 29] confirmed blowup suppression via strong shear flow in both $2D$ and $3D$.

Regarding active flow related to the Keller-Segel-Navier-Stokes system, Zeng, Zhang, and Zi [56] and He [30] demonstrated blowup suppression near large Couette flow in $2D$, while Li, Xiang, and Xu [39] revealed blowup suppression near strong Poiseuille flow in $2D$. Additionally, Cui, Wang, and Wang [20] identified blowup suppression near strong non-parallel shear flow in $3D$. These results all fall within the regime where the cell dynamics are perturbative with respect to the flow, so as to exploit the enhanced dissipation effect induced by fluid-mixing.

Beyond the perturbative regimes mentioned previously, it is natural to consider large data dynamics. Recently, Hu, Kiselev, and Yao [36] found a mechanism of suppression by buoyancy. This discovery led to the verification of global existence for arbitrary smooth data concerning the Keller-Segel equation coupled with a fluid flow adhering to Darcy's law for incompressible porous media via buoyancy force within a domain with a lower boundary within two dimensions. Furthermore, Hu and Kiselev [35] confirmed the suppression of blowup through sufficiently strong buoyancy within the context of Stokes-Boussinesq flow with a cold boundary both in $2D$ and $3D$. In addition, He and Gong [28] showed the global existence of solutions to the $2D$ coupled Keller-Segel-Navier-Stokes system with friction force when the total mass $M < 8\pi$.

Despite the abundant literature on the suppression of blowup, to the best of the authors' knowledge, there is limited literature to study whether the finite-time blowup would happen for fluid-chemotaxis models. Precisely speaking, in a bounded planar domain, when taking the advection of an arbitrary passive flow into account,

[2]Here, "passive" means that the flow is independent of the cells, otherwise it is called active flow.

Winkler [55] proved the existence of finite-time blowup as long as the cell is sufficiently concentrated initially. However, when it comes to the active flow into account, only numerical evidence [41] suggests that cell aggregation (i.e. blowup) is likely to happen.

1.2. **Main result.**

Our objective in this paper is to fill the gap in the literature regarding the existence of finite-time blowup solutions for anisotropic chemotaxis-fluid models. Specifically, our main result establishes the existence of a smooth finite-time blowup solution to (KS-NS) with negative density and finite mass, stated as follows.

Theorem 1.1 (Existence of smooth finite-time blowup solution with nonnegative density and finite mass)**.** *For any integer $s \geq 3$ and any divergence-free vector field $u_0 \in H^\infty_\sigma(\mathbb{R}^3)$ fixed, there exists non-negative $\rho_0 \in C_0^\infty(\mathbb{R}^3)$, such that the smooth solution to* (KS-NS) *with initial data (ρ_0, u_0) blows up at some time $t = T < \infty$. Moreover, we have*

$$\rho(t,x) = \frac{1}{T-t}\left[Q\left(\frac{x}{\sqrt{T-t}}\right) + \varepsilon\left(t, \frac{x}{\sqrt{T-t}}\right)\right], \qquad \text{for } x \in \mathbb{R}^3,\ t \in [0,T), \quad (1.1)$$

where Q is given by (Profile) *and $\lim_{t\to T^-} \|\varepsilon(t)\|_{H^s(\mathbb{R}^3)} = 0$.*

Comments to the main result.

1. Direct construction of blowup solution.

In the literature, most blowup results for chemotaxis equations are obtained either by tracking the evolution of some appropriate functional (such as the second moment) to obtain a contradiction [8, 19, 21], or by working in the radial setting where the mass accumulation function satisfies an ODE [26]. However, the coupling with fluid equations destroys the structures that these proofs rely on, making them inapplicable.

Therefore, we turn to the strategy of direct construction via stability analysis of an approximate blowup solution, often involving modulation or dynamical rescaling method. This strategy has found great success for parabolic, dispersive and fluid problems. In addition to works on Keller-Segel equation mentioned in Section 1.1.1, other instances include nonlinear heat equation [18, 45], nonlinear wave equation [22, 37], nonliear Schrödinger equation [42, 44], incompressible Euler equation [14, 15, 23, 24], compressible fluids [11, 43] and so forth.

2. Strategy for control of the self-similar flow in nonradial setting.

We choose the self-similar solution Q to (KS) given in (Profile) as an approximate solution to (KS-NS) and study the evolution of perturbation near Q in the parabolic zone (i.e. self-similar coordinate). The fluid part is treated perturbatively, based on the crucial observation that the Navier-Stokes equation is subcritical in scaling with respect to the Keller-Segel equation.[3] The main task is to obtain decay of linearized flow near Q for the Keller-Segel part.

In the radial setting, Glogić and Shörkhuber [27] have exploited the reduced-mass formulation of (KS) and the self-adjointness of the linearized operator in a certain weighted space to determine its spectrum and obtain semi-group decay. However, due to the anisotropic nature of buoyancy in (KS-NS), we have to extend to the

[3]This is different from the model in [36] where the fluid part is supercritical.

nonradial setting. Based on the abstract semigroup theory constructed by Merle-Raphaël-Rodnianski-Szeftel [42], the main idea is to show the linearized operator as the compact perturbation of a maximal accretive operator, which yields the semigroup decay modulo finite unstable directions (Proposition 2.7). This powerful method has found application in other various settings such as [11, 37, 43]. Another framework for semigroup decay (and spacetime estimates) without quantitative spectral analysis is presented in [40].

Finally, we integrate this semigroup estimate with a higher-order energy estimate to overcome the loss of derivative in the nonlinearity. Subsequently, we use a Brouwer argument to select initial data for decay in unstable directions.

3. On the stability of self-similar profile.

Following the standard approach in [18, Proposition 4.10], we can construct a finite-codimensional Lipschitz manifold of nonradial initial data such that the corresponding solution to system (KS-NS) blows up in finite time. After dropping the perturbative estimates for fluid part, the proof also works for (KS), implying the finite-codimensional asymptotic stability of Q in nonradial setting. To consider the nonradial stability problem, it requires to count the precise number of unstable directions of the linearized operator in the nonradial setting, which remains to be addressed. In particular, the radial case was done in [27].

4. Non-negativity of the density and finiteness of the mass.

In real-life scenarios, the density of the cell is always non-negative, and the total mass of the cell is finite. However, as indicated by (Profile), $Q \notin L^p(\mathbb{R}^3)$ for any $p \in \left[1, \frac{3}{2}\right]$, implying that the approximate solution has infinite mass.

To obtain finite-mass/energy blowup solution, one natural strategy is to truncate the profile far away and show it induces the analogous self-similar blowup behavior, which can be adapted for stable profile [24, 27]. A direct generalization for finite-codimensional stable blowup is when the instabilities are either generated by symmetry or of finite-mass [18]. However, it requires delicate spectral analysis to completely characterize the unstable spectrum. In addition, [42] (also see [37]) proposed another method to address the general finite-codimensional stability case by working in a weighted Sobolev space adapted to time-dependent dampened profile and modifying the choice of initial data suitably in the Brouwer argument.

In this paper, we propose a different approach to constructing compactly supported smooth initial data based on the finite-codimensional stability of the profile. We reparametrize the spectral decomposition to localize the unstable eigenmodes instead of the profile, which leads to both the compact support and non-negativity of the initial density. This modified spectral decomposition retains the diagonal part of the linear evolution, so it does not obstruct the bootstrap estimate (see Subsection 2.5 for details). Despite its apparent simplicity, this method has effectively addressed our problem and is robust enough to apply to other various models.

1.3. **Structure of the paper.**

In Section 2, we firstly introduce the parabolic variable and formulate the singularity formation as the perturbation of the self-similar profile Q, then consider the spectral properties of the linearized operator $\mathcal{L}$ and the semigroup decay modulo finite (modified) unstable directions. Section 3 is devoted to the nonlinear dynamics to complete the proof of Theorem 1.1. We leave the standard local well-posedness theory for (KS-NS) in the appendix.

1.4. **Notations.**

Firstly, for any $R > 0$, we define $B(0,R)$ as the ball centered at the origin with radial R in three dimensions. Then, we choose $0 \le \chi \le 1$ as a smooth cut-off function in $B(0,R)$ defined by

$$\chi(x) := \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \end{cases} \qquad \chi_R(x) := \chi\left(\frac{x}{R}\right). \tag{1.2}$$

Next, we call the vector of form $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ the multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For any given scalar function f, g, we define the partial derivative of f with respect to the multi-index α by

$$\partial^\alpha f(x) = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} f(x),$$

and for any $k \ge 0$, we denote D^k by

$$D^k f = (\partial^\alpha f)_{|\alpha|=k}.$$

In particular, for $k = 1$, we simplify the notation into $Df = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$, which is the gradient of f. Hence for any integer $k \ge 0$, we define the inner product on $\dot{H}^k$ and H^k by

$$(f,g)_{\dot{H}^k} = (D^k f, D^k g)_{L^2} := \sum_{|\alpha|=k} (\partial^\alpha f, \partial^\alpha g)_{L^2}, \quad (f,g)_{H^k} = (f,g)_{L^2} + (f,g)_{\dot{H}^k}.$$

We also write $H^\infty = \cap_{k \ge 0} H^k$. Next, we denote $H^k(\mathbb{R}^3;\mathbb{R})$, $H^k(\mathbb{R}^3;\mathbb{C})$ (or $\dot{H}^k(\mathbb{R}^3;\mathbb{R})$, $\dot{H}^k(\mathbb{R}^3;\mathbb{C})$ for the collection of all $\mathbb{R}$-valued or $\mathbb{C}$-valued functions respectively with finite H^k (or $\dot{H}^k$) norm. Out of the simplicity of notation, we will use H^k (or $\dot{H}^k$) to refer to $H^k(\mathbb{R}^3;\mathbb{C})$ (or $\dot{H}^k(\mathbb{R}^3;\mathbb{C})$) in Subsection 2.2-2.3, and to refer to $H^k(\mathbb{R}^3;\mathbb{R})$ (or $\dot{H}^k(\mathbb{R}^3;\mathbb{R})$) in Subsection 2.5 and Section 3.

Similarly, for any given vector-valued function $u = (u_1, u_2, u_3)$, we can also define $\partial^\alpha u$ by $\partial^\alpha u = (\partial^\alpha u_1, \partial^\alpha u_2, \partial^\alpha u_3)$, $D^k u$ by $D^k u = (\partial^\alpha u)_{|\alpha|=k}$, and thereafter the vector-valued $H^k(\mathbb{R}^3;\mathbb{R}^3)$ (or $\dot{H}^k(\mathbb{R}^3;\mathbb{R}^3)$) space similarly. In addition, we denote $H^k_\sigma(\mathbb{R}^3)$ the set of all divergence-free vector fields within the $H^k(\mathbb{R}^3;\mathbb{R}^3)$ space, namely

$$H^k_\sigma(\mathbb{R}^3) := \{u \in H^k(\mathbb{R}^3;\mathbb{R}^3) : \nabla \cdot u = 0\}.$$

Moreover, we denote C_0^∞ for the set of infinitely differentiable functions with compact support, and similarly $\mathscr{S}$ for Schwartz functions.

Finally, if A is a linear operator on a Hilbert space H, then we denote $\rho(A)$, $\sigma(A)$ to be the resolvent set and spectral set of A respectively.

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2. Linear theory in the parabolic zone

2.1. **Parabolic variables and renormalization.** Firstly, we renormalize the system (KS-NS) in the parabolic variables. Let

$$y = \frac{x}{\mu}, \quad \frac{d\tau}{dt} = \frac{1}{\mu^2}, \quad \tau\big|_{t=0} = 0, \quad \frac{\mu_\tau}{\mu} = -\frac{1}{2}, \tag{2.1}$$

and the corresponding renormalization

$$\rho(t,x)=\frac{1}{\mu^2}\Psi(\tau,y),\quad u(t,x)=\frac{1}{\mu}U(\tau,y),\quad \pi(t,x)=\frac{1}{\mu^2}\Pi(\tau,y). \tag{2.2}$$

If the initial data of μ is given by $\mu|_{\tau=0}=\mu_0>0$, then from the equation $\frac{\mu_\tau}{\mu}=-\frac{1}{2}$ determined in (2.1), we can explicitly solve it by

$$\mu(\tau)=\mu_0 e^{-\frac{1}{2}\tau},\quad \forall\tau\geq 0. \tag{2.3}$$

The original system (KS-NS) of (ρ,u,π) is mapped to the following renormalized system of (Ψ,U,Π),

$$\begin{cases}\partial_\tau\Psi+\frac{1}{2}\Lambda\Psi=\Delta\Psi+\nabla\cdot\left(\Psi\nabla\Delta^{-1}\Psi\right)-U\cdot\nabla\Psi,\\ \partial_\tau U+\frac{1}{2}\left(U+y\cdot\nabla U\right)+U\cdot\nabla U=\Delta U-\nabla\Pi-\mu\Psi e_3,\\ \nabla\cdot U=0,\end{cases} \tag{2.4}$$

where Λ is the scaling operator

$$\Lambda f:=2f+y\cdot\nabla f. \tag{2.5}$$

Plugging the ansatz $\Psi=Q+\varepsilon$, where Q is the self-similar profile to the Keller-Segel equation given in (Profile), the system is then transformed into:

$$\begin{cases}\partial_\tau\varepsilon=-\mathcal{L}\varepsilon+\nabla\cdot(\varepsilon\nabla\Delta^{-1}\varepsilon)-U\cdot\nabla\Psi,\\ \partial_\tau U+\frac{1}{2}\left(U+y\cdot\nabla U\right)+U\cdot\nabla U=\Delta U-\nabla\Pi-\mu\Psi e_3,\\ \nabla\cdot U=0,\end{cases} \tag{2.6}$$

where $-\mathcal{L}$ is the related linearized operator defined by

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}',\quad \text{with } \mathcal{L}_0 f=-\Delta f+\frac{1}{2}\Lambda f \ \text{ and } \ \mathcal{L}'f=-\nabla\cdot(f\nabla\Delta^{-1}Q)-\nabla\cdot(Q\nabla\Delta^{-1}f). \tag{2.7}$$

In the following content of Section 2, our focus will be on studying the spectral properties of the linearized operator $-\mathcal{L}$.

2.2. **Linear theory of $\mathcal{L}_0$.**

In this subsection, we study the spectral properties of the linear operator $\mathcal{L}_0$ in (2.7), which is the dominating term of the linearized operator $\mathcal{L}$.

Firstly, we study behavior of $\mathcal{L}_0$ on L^2_ω, where $L^2_\omega(\mathbb{R}^3)$ is an L^2 weighted space defined as follows:

$$(f,g)_{L^2_\omega(\mathbb{R}^3)}=\int_{\mathbb{R}^3}f\overline{g}\omega dy,\quad \omega(y)=e^{-\frac{|y|^2}{4}}.$$

Lemma 2.1 (Self-adjointness and semigroup of $\mathcal{L}_0$ in L^2_ω)**.** *The operator $\mathcal{L}_0: C_0^\infty(\mathbb{R}^3)\subset L^2_\omega(\mathbb{R}^3)\to L^2_\omega(\mathbb{R}^3)$ is essentially self-adjoint. Denote $\mathcal{L}_0\big|_{L^2_\omega}$ as its unique self-adjoint extension, then $\sigma\left(\mathcal{L}_0\big|_{L^2_\omega}\right)\subset\left[\frac{1}{4},\infty\right)$, and $-\mathcal{L}_0\big|_{L^2_\omega}$ generates a strongly continuous semigroup $(e^{-\tau\mathcal{L}_0})_{\tau\geq 0}$ of bounded operators on $L^2_\omega(\mathbb{R}^3)$. Explicitly, for $f\in C_0^\infty(\mathbb{R}^3)$,*

$$\left(e^{-\tau\mathcal{L}_0}f\right)(y)=e^{-\tau}(G_{1-e^{-\tau}}*f)\left(e^{-\frac{\tau}{2}}y\right), \tag{2.8}$$

where $G_\lambda=\lambda^{-\frac{3}{2}}G_1(\lambda^{-\frac{1}{2}}\cdot)$, with $G_1(y)=(4\pi)^{-\frac{3}{2}}e^{-\frac{|y|^2}{4}}$ being the heat kernel.

Proof. Conjugating $\mathcal{L}_0$ with the unitary map

$$U:L^2(\mathbb{R}^3)\to L^2_\omega(\mathbb{R}^3),\quad u\mapsto\omega^{-\frac{1}{2}}u, \tag{2.9}$$

we obtain

$$A_0 := U^{-1}\mathcal{L}_0 U = -\Delta + \frac{|y|^2}{16} + \frac{1}{4} : C_0^\infty(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3).$$

This is a dilated harmonic oscillator, which is essentially self-adjoint (see for example [31, Example 9.4]) and hence so is $\mathcal{L}_0$ on L^2_ω. Moreover,

$$\sigma\left(\mathcal{L}_0\big|_{L^2_\omega}\right) = \sigma\left(A_0\big|_{L^2}\right) \subset \left[\frac{1}{4}, \infty\right),$$

so from the Hille-Yosida theorem, $-\mathcal{L}_0$ generates a strongly continuous contraction semigroup on L^2_ω.

Now we compute (2.8). For $f_0 \in C_0^\infty(\mathbb{R}^3)$, the standard semigroup theory (see [25, Chap. 2 Proposition 6.4]) implies that $f(\tau) := e^{-\tau\mathcal{L}_0} f_0$ is the unique mild solution of the Cauchy problem

$$\begin{cases} \partial_\tau f + \mathcal{L}_0 f = 0, \\ f\big|_{t=0} = f_0, \end{cases} \tag{2.10}$$

namely $\int_0^\tau f(s)ds \in D\left(\mathcal{L}_0\big|_{L^2_\omega}\right)$ for all $\tau \geq 0$ and

$$f(\tau) = f_0 - \mathcal{L}_0 \int_0^\tau f(s)ds.$$

Now let $g(t) = e^{t\Delta} f_0 = G_t * f_0$. Then $f_0 \in C_0^\infty(\mathbb{R}^3)$ implies $g(t) \in \mathscr{S}(\mathbb{R}^3)$, and thus $g(t)$ is the classical solution of the linear heat equation

$$\begin{cases} \partial_t g - \Delta g = 0, \\ g\big|_{t=0} = f_0. \end{cases}$$

Taking the self-similar renormalization

$$\tau(t) = -\ln(1-t), \quad \lambda(t) = \sqrt{1-t}, \quad y = \frac{x}{\lambda(t)}, \quad v(\tau, y) = \lambda(t)^2 g(t, x),$$

we see $v(\tau) \in \mathscr{S}(\mathbb{R}^3)$ for $\tau \in [0, \infty)$ is a classical solution to (2.10), in particular a mild solution since $\mathscr{S}(\mathbb{R}^3) \subset D\left(\mathcal{L}_0\big|_{L^2_\omega}\right)$. By the uniqueness of mild solution, we have $f(\tau, y) = v(\tau, y)$, namely

$$\left(e^{-\tau\mathcal{L}_0} f_0\right)(y) = \lambda(t(\tau))^2 \left(G_{t(\tau)} * f_0\right)(\lambda(t(\tau))y) = e^{-\tau}\left(G_{1-e^{-\tau}} * f_0\right)(e^{-\frac{\tau}{2}} y).$$

□

Next, we study the spectral properties of $\mathcal{L}_0$ on Hilbert space H^k.

Lemma 2.2 (Closedness and smoothing resolvent estimate for $\mathcal{L}_0$ in H^k)**.** *Let $k \geq 0$. The operator $\mathcal{L}_0 : C_0^\infty(\mathbb{R}^3) \subset H^k(\mathbb{R}^3) \to H^k(\mathbb{R}^3)$ is closable. Define the closure as $\left(\mathcal{L}_0\big|_{H^k}, \mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right)\right)$. For any $\lambda \in \rho\left(\mathcal{L}_0\big|_{H^k}\right)$ with $\Re\lambda < \frac{1}{4}$, the resolvent satisfies a smoothing estimate for any $m \geq k$,*

$$\left\|(\mathcal{L}_0\big|_{H^k} - \lambda)^{-1} f\right\|_{H^{m+\frac{3}{2}}} \lesssim \left(\left|\frac{1}{4} - \Re\lambda\right|^{-\frac{1}{4}} + \left|\frac{1}{4} - \Re\lambda\right|^{-1}\right) \|f\|_{H^m}. \tag{2.11}$$

In particular, $\mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right) \subset H^{k+\frac{3}{2}}$.

Proof. Firstly, we consider the coercivity of $\mathcal{L}_0$ in H^k with domain $D(\mathcal{L}_0) = C_0^\infty(\mathbb{R}^3)$. For any $f \in C_0^\infty(\mathbb{R}^3)$, the $\dot{H}^k$ norm of $f_\lambda(y) = \lambda^2 f(\lambda y)$ satisifies

$$\|f_\lambda\|^2_{\dot{H}^k} = \int_{\mathbb{R}^3} |D^k f_\lambda|^2 dy = \lambda^{2k+1} \|D^k f\|^2_{L^2},$$

then taking derivative with respect to λ and choosing $\lambda = 1$, then

$$\Re(D^k(\Lambda f), D^k f)_{L^2} = \frac{2k+1}{2}\|f\|^2_{\dot{H}^k}. \tag{2.12}$$

Hence

$$\Re(\mathcal{L}_0 f, f)_{H^k} = \Re\left(\Delta f + \frac{1}{2}\Lambda f, f\right)_{L^2} + \Re\left(D^k\left(\Delta f + \frac{1}{2}\Lambda f\right), D^k f\right)_{L^2} \geq \frac{1}{4}\|f\|^2_{H^{k+1}}. \tag{2.13}$$

Then applying Cauchy-Schwarz inequality to (2.13), we easily obtain $\|\mathcal{L}_0 f\|_{H^k} \geq \frac{1}{4}\|f\|_{H^k}$ for $f \in C_0^\infty(\mathbb{R}^3)$. Also since $\mathcal{L}_0$ is densely defined on H^k, it is closable according to [25, Chap. II, Proposition 3.14 (iv)]. Besides, we have $\mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right) \subset \mathcal{D}\left(\mathcal{L}_0\big|_{L^2_\omega}\right)$, $\left(\mathcal{L}_0\big|_{H^k} - \lambda\right)^{-1} g = \left(\mathcal{L}_0\big|_{L^2_\omega} - \lambda\right)^{-1} g$ for $g \in H^k$, and $\mathcal{L}_0\big|_{H^k} f = \mathcal{L}_0\big|_{L^2_\omega} f$ for $f \in \mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right)$, which follows that H^k is embedded in L^2_ω and the uniqueness of closure $\left(\mathcal{L}_0, C_0^\infty(\mathbb{R}^3)\right)$ on L^2_ω and H^k.

From the closedness of $\mathcal{L}_0\big|_{H^k}$, it suffices to prove (2.11) for $f \in C_0^\infty(\mathbb{R}^3)$, for which we can replace $\mathcal{L}_0\big|_{H^k}$ by $\mathcal{L}_0\big|_{L^2_\omega}$ and apply (2.8). Indeed, noticing that for any $\zeta, \mu > 0$,

$$\|G_\zeta * f\|_{H^{k+\frac{3}{2}}} \sim \|\langle\cdot\rangle^{k+\frac{3}{2}}\hat{G}_\zeta \hat{f}\|_{L^2} \leq \|\langle\cdot\rangle^{\frac{3}{2}}\hat{G}_\zeta\|_{L^\infty}\|\langle\cdot\rangle^k f\|_{L^2} \lesssim \langle\zeta^{-1}\rangle^{\frac{3}{4}}\|f\|_{H^k},$$

thus for $\Re\lambda < \frac{1}{4}$ and $f \in C_0^\infty(\mathbb{R}^3)$, we obtain via Laplace transform and (2.8) that

$$(\mathcal{L}_0 - \lambda)^{-1} f = \int_0^\infty e^{-\tau(\mathcal{L}_0 - \lambda)} f d\tau = \int_0^\infty e^{-(1-\lambda)\tau}(G_{1-e^{-\tau}} * f)\left(e^{-\frac{\tau}{2}}\cdot\right) d\tau.$$

This implies

$$\begin{aligned}
\left\|(\mathcal{L}_0 - \lambda)^{-1} f\right\|_{H^{k+\frac{3}{2}}} &\leq \int_0^\infty e^{-(1-\lambda)\tau}\left\|(G_{1-e^{-\tau}} * f)\left(e^{-\frac{\tau}{2}}\cdot\right)\right\|_{H^{k+\frac{3}{2}}} d\tau \\
\lesssim \int_0^\infty e^{-(\frac{1}{4}-\lambda)\tau}\left\langle\frac{1}{1-e^{-\tau}}\right\rangle^{\frac{3}{4}}\|f\|_{H^k} d\tau &\lesssim \left(\left|\frac{1}{4} - \Re\lambda\right|^{-\frac{1}{4}} + \left|\frac{1}{4} - \Re\lambda\right|^{-1}\right)\|f\|_{H^k}.
\end{aligned}$$

□

2.3. **Perturbed maximal dissipativity of $-\mathcal{L}$.** This subsection focuses on the analysis of the linearized operator $-\mathcal{L}$. The main result is the following:

Proposition 2.3 (Perturbed maximal dissipativity of $-\mathcal{L}$). *For $k \geq 0$, there exist a maximally dissipative operator $A_0 : \mathcal{D}(A_0) \subset H^k(\mathbb{R}^3;\mathbb{C}) \to H^k(\mathbb{R}^3;\mathbb{C})$ with $\mathcal{D}(A_0) = \mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right)$, namely*

$$\forall f \in \mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right), \quad \Re(A_0 f, f)_{H^k} \leq 0, \tag{2.14}$$

$$\exists R > 0, \quad A_0 - R : \mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right) \to H^k(\mathbb{R}^3;\mathbb{C}) \text{ is surjective;} \tag{2.15}$$

and K compact on $H^k(\mathbb{R}^3;\mathbb{C})$ such that

$$-\mathcal{L} = A_0 - \frac{1}{16} + K. \tag{2.16}$$

Remark 2.4. We recall that maximal dissipative operators are closed. Hence $\mathcal{D}(\mathcal{L}\big|_{H^k}) = \mathcal{D}\left(\mathcal{L}_0\big|_{H^k}\right)$, and $\left(\mathcal{L}\big|_{H^k}, \mathcal{D}(\mathcal{L}\big|_{H^k})\right)$ is closed on H^k.

Recall $\mathcal{L}'$ from (2.7). We begin by showing its compactness. Decompose $\mathcal{L}'$ as

$$\begin{aligned}&\mathcal{L}' = \mathcal{L}'_1 + \mathcal{L}'_2 + \mathcal{L}'_3,\\ &\mathcal{L}'_1 f = -2Qu, \quad \mathcal{L}'_2 f = -\nabla Q \cdot \nabla \Delta^{-1} f, \quad \mathcal{L}'_3 f = -\nabla \Delta^{-1} Q \cdot \nabla f. \end{aligned} \tag{2.17}$$

We also define $\mathcal{C}'_{3,k}$ by

$$\mathcal{C}'_{3,k} = [D^k, \mathcal{L}'_3]. \tag{2.18}$$

Lemma 2.5 (Compactness of $\mathcal{L}'_1, \mathcal{L}'_2$, $\mathcal{L}'_3$ and $\mathcal{C}'_3$)**.** *For any $k \geq 0$, the linear operators $\mathcal{L}'_1 : H^{k+1}(\mathbb{R}^3) \to H^k(\mathbb{R}^3)$, $\mathcal{L}'_2 : H^k(\mathbb{R}^3) \to H^k(\mathbb{R}^3)$, $\mathcal{L}'_3 : H^{k+\frac{3}{2}}(\mathbb{R}^3) \to H^k(\mathbb{R}^3)$ and $\mathcal{C}'_{3,k} : H^{k+1}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ are all compact.*

Proof. Firstly of all, we consider the compactness of $\mathcal{L}'_1 : H^{k+1} \to H^k$. In fact, for any $R > 0$, since $2\chi_R Q \in C_0^\infty(\mathbb{R}^3) \subset \mathscr{S}(\mathbb{R}^3)$, by [1, Theorem 1.68], we see that the multiplier $f \mapsto \mathcal{L}'_{1,R} f := 2\chi_R Q f$ is compact from $H^{k+1} \to H^k$. In addition,

$$\begin{aligned}\|(1-\chi_R)Qf\|_{H^k} &= \|(1-\chi_R)Qf\|_{L^2} + \sum_{|\alpha|=k} \|\partial^\alpha ((1-\chi_R)Qf)\|_{L^2}\\ &\lesssim \|(1-\chi_R)Q\|_{W^{k,\infty}} \|f\|_{H^k} \lesssim R^{-p} \|f\|_{H^k}, \quad \forall f \in H^k,\end{aligned}$$

for some $p > 0$ uniformly in $R > 0$. Then by the standard diagonal method, $\mathcal{L}'_1$ is compact from H^{k+1} to H^k.

Secondly, for the commutator term $\mathcal{C}'_{3,k}$ defined in (2.16), it involves at most the k-th order derivative, then with the decay of the arbitrary derivative of ∇Q, we can repeat the argument above to conclude that $\mathcal{C}'_{3,k}$ is compact from $H^{k+1} \to H^k$.

Thirdly, we consider the compactness of $\mathcal{L}'_3 : H^{k+\frac{3}{2}} \to H^k$. Recall the definition of $\mathcal{L}'_3$ in (2.16), it suffices to prove that the multiplier $\mathcal{L}''_3 f := \nabla \Delta^{-1} Q f$ is compact from $H^{k+\frac{1}{2}} \to H^k$. In fact, similar to the argument above, by [1, Theorem 1.68] again, we can see that $\mathcal{L}''_{3,R} = \chi_R \nabla \Delta^{-1} Q$ is a compact multiplier from $H^{k+\frac{1}{2}}$ to H^k. In addition,

$$\left\| \left((1-\chi_R)\nabla\Delta^{-1}Q\right) f \right\|_{H^k} \lesssim \|(1-\chi_R)\nabla\Delta^{-1}Q\|_{W^{k,\infty}} \|f\|_{H^k} \lesssim R^{-p}\|f\|_{H^k}, \ \forall f \in H^k,$$

for some $p > 0$ uniformly in $R > 0$. Then by the standard diagonal argument again, $\mathcal{L}'_3$ is compact from $H^{k+\frac{3}{2}}$ to H^k.

Finally, let us consider the compactness of the nonlocal term $\mathcal{L}'_2 : H^k \to H^k$. For any $R > 0$, by Hölder's inequality and Sobolev inequality,

$$\begin{aligned}&\left\|(1-\chi_R)\nabla Q \cdot \nabla\Delta^{-1} f\right\|_{H^k}\\ &\lesssim \left\|(1-\chi_R)\nabla Q \cdot \nabla\Delta^{-1} f\right\|_{L^2} + \sum_{|\alpha|=k} \left\|\partial^\alpha((1-\chi_R)\nabla Q)\cdot\nabla\Delta^{-1} f\right\|_{L^2}\\ &\qquad + \sum_{|\alpha|<k, |\alpha|+|\beta|=k} \left\|\partial^\alpha((1-\chi_R)\nabla Q)\cdot\nabla\Delta^{-1}\partial^\beta f\right\|_{L^2}\\ &\lesssim \left\|(1-\chi_R)\nabla Q\right\|_{W^{k,3}} \|\nabla\Delta^{-1} f\|_{L^6} + \left\|(1-\chi_R)\nabla Q\right\|_{W^{k,\infty}} \|\nabla\Delta^{-1} f\|_{\dot{H}^1\cap\dot{H}^k}\\ &\lesssim \begin{cases} R^{-p}\|f\|_{H^{k-1}}, & k \geq 1,\\ R^{-p}\|f\|_{L^2}, & k = 0,\end{cases} \quad \forall f \in H^k,\end{aligned} \tag{2.19}$$

for some $p > 0$ independent of $R > 0$. By similar computation and $\nabla Q \in W^{k,3} \cap W^{k,\infty}$, we have the boundedness

$$\|\chi_R \mathcal{L}'_2 f\|_{H^{k+1}} \lesssim \|f\|_{H^k}, \quad \forall f \in H^k, \ \forall R > 0.$$

Then by Rellich-Kondrachov compactness theorem [10, Theorem 9.16], the operator $\chi_R \mathcal{L}_2'$ is compact from H^k to H^k. Consequently, using the diagonal argument again, we obtain that $\mathcal{L}_2' : H^k \to H^k$ is compact. □

Lemma 2.6 (Almost coercivity for $\mathcal{L}'$)**.** *For any $k \in \mathbb{N}$, and any $\delta > 0$, there exist finite $(q_j)_{1\le j\le N_0} \subset C_0^\infty(\mathbb{R}^3;\mathbb{C})$ such that for $f \in H^{k+1}(\mathbb{R}^3;\mathbb{C})$ with $f \perp_{H^k} q_j$ for $1 \le j \le N_0$, we have*

$$\Re(\mathcal{L}' f, f)_{H^k} \ge -\delta \|f\|_{H^{k+1}}^2. \tag{2.20}$$

Proof. By Lemma 2.5, $\mathcal{L}_1' + \mathcal{L}_2'$ is compact from H^{k+1} to H^k, then it yields a finite-rank operator $\tilde{\mathcal{T}}_{1,2,\delta} : H^{k+1} \to H^k$, such that such that $\tilde{\mathcal{R}}_{1,2,\delta} = \mathcal{L}_1' + \mathcal{L}_2' - \tilde{\mathcal{T}}_{1,2,\delta} : H^{k+1} \to H^k$ is bounded with $\|\tilde{\mathcal{R}}_{1,2,\delta}\|_{H^{k+1}\to H^k} \le \frac{\delta}{8}$. In particular, there exists $\{\tilde{q}_j\}_{1\le j\le N_{1,2}} \subset H^{k+1}$ and $\{p_j\}_{1\le j\le N_{1,2}} \subset H^k$ such that $\tilde{\mathcal{T}}_{1,2,\delta}$ can be represented by

$$\tilde{\mathcal{T}}_{1,2,\delta} = \sum_{j=1}^{N_{1,2}} (\cdot, \tilde{q}_j)_{H^{k+1}}\, p_j.$$

By the density of C_0^∞ in H^{k+1}, we introduce $\{\tilde{\tilde{q}}_j\}_{1\le j\le N_{1,2}} \subset C_0^\infty$ such that $\tilde{\tilde{\mathcal{T}}}_{1,2,\delta} = \sum_{j=1}^{N_{1,2}} (\cdot, \tilde{\tilde{q}}_j)_{H^{k+1}} p_j$ satisfies

$$\|\tilde{\mathcal{T}}_{1,2,\delta} - \tilde{\tilde{\mathcal{T}}}_{1,2,\delta}\|_{H^{k+1}\to H^k} \le \frac{\delta}{8}.$$

Then by integration by parts, Riesz's representative theorem and the density of C_0^∞ in H^k, we can find $\{q_j\}_{1\le j\le N_{1,2}} \subset C_0^\infty \subset H^k$ such that the finite-rank operator $\mathcal{T}_{1,2,\delta} = \sum_{j=1}^{N_{1,2}} (\cdot, q_j)_{H^k} p_j$ satisfies

$$\|\mathcal{T}_{1,2,\delta} - \tilde{\tilde{\mathcal{T}}}_{1,2,\delta}\|_{H^{k+1}\to H^k} \le \frac{\delta}{8}.$$

Hence, combining all of the estimates above and together with the triangle inequality, $\mathcal{R}_{1,2,\delta} = \mathcal{L}_1' + \mathcal{L}_2' - \mathcal{T}_{1,2,\delta}$ satisfies

$$\|\mathcal{R}_{1,2,\delta}\|_{H^{k+1}\to H^k} \le \frac{\delta}{2}.$$

Similarly, since $\mathcal{C}_{k,3}'$ is compact from H^{k+1} to L^2 derived from Lemma 2.5, repeating the argument above, we can find sequences $(q_j)_{N_{1,2}+1}^{N_{1,2}+N_3} \subset C_0^\infty$ and $(p_j)_{N_{1,2}+1}^{N_{1,2}+N_3} \subset L^2$, such that

$$\mathcal{T}_{3,\delta} = \sum_{N_{1,2}+1}^{N_{1,2}+N_3} (\cdot, q_j)_{H^k} p_j,$$

and the residual $\mathcal{R}_{3,\delta} = \mathcal{T}_{3,\delta} - \mathcal{C}_{3,k}'$ satisfies $\|\mathcal{R}_{3,\delta}\|_{H^{k+1}\to L^2} \le \frac{\delta}{2}$.

Additionally, we have the coercivity of $\mathcal{L}_3'$ under the L^2 inner product as follows:

$$\Re(\mathcal{L}_3' f, f)_{L^2} = -\Re \int \left(\nabla \Delta^{-1} Q \cdot \nabla f\right) \overline{f} dy = \frac{1}{2}\int Q|f|^2 dy \ge 0.$$

Consequently, combining all the argument above, for any $f \in H^{k+1}$ satisfying $f \perp_{H^k} q_j$ for any $1 \le j \le N_0 = N_{1,2} + N_3$, we have $\mathcal{T}_{1,2,\delta} f = \mathcal{T}_{3,\delta} f = 0$, hence

$$\begin{aligned}
&\Re(\mathcal{L}' f, f)_{H^k(\mathbb{R}^3)} = \Re((\mathcal{L}_1' + \mathcal{L}_2') f, f)_{H^k(\mathbb{R}^3)} + \Re(\mathcal{L}_3' f, f)_{L^2(\mathbb{R}^3)} + \Re(D^k \mathcal{L}_3' f, D^k f)_{L^2(\mathbb{R}^3)} \\
&= \Re((\mathcal{L}_1' + \mathcal{L}_2') f, f)_{H^k(\mathbb{R}^3)} + \Re(\mathcal{C}_{3,k}' f, D^k f)_{L^2(\mathbb{R}^3)} + \Re(\mathcal{L}_3' f, f)_{L^2(\mathbb{R}^3)} + \Re(\mathcal{L}_3' D^k f, D^k f)_{L^2(\mathbb{R}^3)} \\
&\ge \Re((\mathcal{L}_1' + \mathcal{L}_2') f, f)_{H^k(\mathbb{R}^3)} + \Re(\mathcal{C}_{3,k}' f, D^k f)_{L^2(\mathbb{R}^3)} \\
&= \Re(\mathcal{R}_{1,2,\delta} f, f)_{H^k(\mathbb{R}^3)} + \Re(\mathcal{R}_{3,\delta} f, D^k f)_{L^2(\mathbb{R}^3)} \ge -\delta \|f\|_{H^{k+1}}^2.
\end{aligned}$$

□

Now we are in place of proving Proposition 2.3.

Proof of Proposition 2.3. *Step 1. Dissipativity of $-\mathcal{L}$ modulo finite dimensions.* We claim that there exist $\{q_i\}_{i=1}^N \subset C_0^\infty(\mathbb{R}^3)$ with $(q_i, q_j)_{H^k} = \delta_{ij}$, and $C > 0$ such that for $f \in \mathcal{D}(\mathcal{L}_0\big|_{H^k})$,

$$\Re(\mathcal{L}f, f)_{H^k(\mathbb{R}^3)} \geq \frac{1}{8}\|f\|^2_{H^k(\mathbb{R}^3)} - C\sum_{j=1}^N |(f, q_j)_{H^k}|^2. \tag{2.21}$$

In fact, note that Lemma 2.5 and direct estimate of $\mathcal{L}_3'$ from (2.17) imply

$$\|\mathcal{L}'\|_{H^{k+1}\to H^k} \lesssim 1. \tag{2.22}$$

Thus the LHS of (2.21) is well-defined for $u \in \mathcal{D}(\mathcal{L}_0\big|_{H^k}) \subset H^{k+\frac{3}{2}}$ (from Lemma 2.2). Applying (2.13) and (2.20) with $\delta = \frac{1}{16}$, there exists $(q_j)_{j=1}^N \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\Re(\mathcal{L}f, f)_{H^k(\mathbb{R}^3)} \geq \frac{3}{16}\|u\|^2_{H^{k+1}(\mathbb{R}^3)}, \tag{2.23}$$

for any f satisfying $f \perp_{H^k} q_j$, $\forall\ 1 \leq j \leq N$. Without loss of generalization, we assume that the $\{q_j\}$ obtained previously satisfies the orthogonal condition $(q_i, q_j)_{H^k} = \delta_{ij}$, otherwise we can use the Gram-Schmidt orthonormalization process to realize it.

Thus for any $f \in H^k$, if we decompose f as

$$f = f_1 + f_2, \quad \text{where } f_2 = \sum_{j=1}^N (f, q_j)_{H^k} q_j,\ f_1 \in \operatorname{span}\{q_1, q_2, ..., q_N\}^\perp, \text{ and } f_1 \perp_{H^k} f_2,$$

then by Cauchy-Schwarz inequality, Young's inequality and the fact that $q_j \in C_0^\infty(\mathbb{R}^3)$, we conclude that

$$\begin{aligned}
\Re(\mathcal{L}f, f)_{H^k} &= \Re(\mathcal{L}f_1, f_1)_{H^k} + \Re(\mathcal{L}f_1, f_2)_{H^k} + \Re(\mathcal{L}f_2, f_1)_{H^k} + \Re(\mathcal{L}f_2, f_2)_{H^k} \\
&\geq \frac{3}{16}\|f_1\|^2_{H^k} - \frac{1}{16}\|f_1\|^2_{H^k} - C\left(\|\mathcal{L}f_2\|^2_{H^k} + \|\mathcal{L}^* f_2\|^2_{H^k} + \|f_2\|^2_{H^k}\right) \\
&= \frac{1}{8}\|f\|^2_{H^k} - \frac{1}{8}\|f_2\|^2_{H^k} - C\left(\|\mathcal{L}f_2\|^2_{H^k} + \|\mathcal{L}^* f_2\|^2_{H^k} + \|f_2\|^2_{H^k}\right) \\
&\geq \frac{1}{8}\|f\|^2_{H^k} - C\sum_{j=1}^N |(f, q_j)_{H^k}|^2,
\end{aligned}$$

for some universal $C > 0$, which is definitely (2.21).

Step 2. Construction of A_0 and end of proof. With $(q_j)_{1\leq j\leq N}$ and $C > 0$ from Step 1, we define A_0 and K as

$$K = C\sum_{j=1}^N (\cdot, q_j)_{H^k} q_j, \qquad A_0 = -\mathcal{L} + \frac{1}{16} - K : \mathcal{D}(\mathcal{L}_0\big|_{H^k}) \to H^k(\mathbb{R}^3; \mathbb{C}).$$

Immediately, K is compact on $H^k(\mathbb{R}^3; \mathbb{C})$ and A_0 is well-defined on $\mathcal{D}(\mathcal{L}_0\big|_{H^k})$ due to (2.22) and Lemma 2.2. The decomposition (2.16) comes from the definition of A_0, and the dissipativity follows (2.21) directly.

It remains to prove maximality (2.15). Noticing that

$$-A_0-\lambda = \mathcal{L}_0+\mathcal{L}'-\frac{1}{16}+K-\lambda = \left(I+(\mathcal{L}'+K)\left(\mathcal{L}_0-\frac{1}{16}-\lambda\right)^{-1}\right)\left(\mathcal{L}_0-\frac{1}{16}-\lambda\right),$$

then from resolvent estimate (2.11) and boundedness of $\mathcal{L}'$ (2.22), we can choose $\lambda \ll -1$ such that $\left\|(\mathcal{L}'+K)\left(\mathcal{L}_0-\frac{1}{16}-\lambda\right)^{-1}\right\|_{H^k\to H^k} \ll 1$. This implies $\left(I+(\mathcal{L}'+K)\left(\mathcal{L}_0-\frac{1}{16}-\lambda\right)^{-1}\right)$ is invertible on $H^k(\mathbb{R}^3;\mathbb{C})$, and thus $(-A_0-\lambda)^{-1}: H^k(\mathbb{R}^3;\mathbb{C}) \to D\left(\mathcal{L}\big|_{H^k}\right)$ is well-defined and bounded, which yields (2.15), then we have concluded the proof. □

2.4. **Spectral properties of $-\mathcal{L}$.** As a result of the decomposition (2.16), we conclude the following spectral properties of $-\mathcal{L}$ via the abstract semigroup theory from [42].

Proposition 2.7 ([42, Lemma 3.3, Lemma 3.4] Spectral properties of $-\mathcal{L}$ on $H^k(\mathbb{R}^3;\mathbb{C})$)**.** *For $k \geq 0$, consider $-\mathcal{L}: D(\mathcal{L}\big|_{H^k}) \subset H^k(\mathbb{R}^3;\mathbb{C}) \to H^k(\mathbb{R}^3;\mathbb{C})$ given in (2.7) and any $\delta \in (0, \frac{1}{16})$. There exists $0<\delta_g<\delta$ such that the following spectral properties of $-\mathcal{L}$ hold*[4]*:*

- *The set $\Lambda = \sigma(-\mathcal{L}) \cap \{\lambda \in \mathbb{C} : \Re\lambda > -\frac{\delta_g}{2}\}$ is finite and formed only by eigenvalues of $-\mathcal{L}$. Furthermore, the generalized eigenspace*

$$V = \bigoplus_{\lambda\in\Lambda} \ker(-\mathcal{L}-\lambda)^{\mu_\lambda},$$

is finite-dimensional, where μ_λ is the smallest integer such that $\ker(-\mathcal{L}-\lambda)^{\mu_\lambda} = \ker(-\mathcal{L}-\lambda)^{\mu_\lambda+1}$.

- *Consider $-\mathcal{L}^* = A_0^* - \frac{1}{16} + K^*$ and $\Lambda^* = \sigma(-\mathcal{L}^*) \cap \{\lambda \in \mathbb{C} : \Re\lambda > -\frac{\delta_g}{2}\}$. Similarly, we define the corresponding generalized eigenspace by*

$$V^* = \bigoplus_{\lambda\in\lambda^*} \ker(-\mathcal{L}^*-\lambda)^{\mu^*_\lambda},$$

with $\Lambda^* = \overline{\Lambda}$ and $\mu_\lambda = \mu^*_{\overline{\lambda}}$. Let $V^{*\perp}$ be the orgothogonal complement of V^* in $H^k(\mathbb{R}^3;\mathbb{C})$. Then V and $V^{*\perp}$ are invariant under $-\mathcal{L}$. And the whole space $H^k(\mathbb{R}^3;\mathbb{C})$ can be decomposed by $H^k(\mathbb{R}^3;\mathbb{C}) = V \oplus V^{*\perp}$.

- *Define the spectral projection of $-\mathcal{L}$ to the set Λ:*

$$P_u = \frac{1}{2\pi i}\int_\Gamma (z-(-\mathcal{L}))^{-1}dz, \tag{2.24}$$

where $\Gamma \subset \rho(-\mathcal{L})$ is an arbitrary anti-clockwise contour containing the set Λ. Then $V = \mathrm{Ran}P_u$ and $V^{*\perp} = \ker P_u = \mathrm{Ran}(1-P_u)$.[5]

- *(Stable part) $-\mathcal{L}$ generates a semigroup, which we denote by $e^{-\tau\mathcal{L}}$, and it satisfies*

$$\|e^{-\tau\mathcal{L}}v\|_{H^k} \lesssim e^{-\frac{1}{2}\delta_g\tau}\|v\|_{H^k}, \quad \forall v \in V^{*\perp}, \quad \tau \geq 0. \tag{2.25}$$

[4] The choice of δ_g is similar to the proof of [42, Lemma 3.4]. For instance, we can take

$$\epsilon_g = \inf\left(\left\{\Re\lambda+\frac{\delta}{2} : \lambda\in\sigma(-\mathcal{L}), 0>\Re\lambda>-\frac{\delta}{2}\right\}\cup\left\{\frac{\delta}{2}\right\}\right) \in \left(0,\frac{\delta}{2}\right],$$

to be the length of the spectral gap on the right of $-\frac{\delta}{2}$, and then let $-\frac{\delta_g}{2} = -\frac{\delta}{2}+\frac{\epsilon_g}{2}$.

[5] This statement is contained in the proof of [42, Lemma 3.3].

Notice that $\mathcal{L}$ is real-valued operator, we have

$$\overline{\mathcal{L}f} = \mathcal{L}\overline{f}, \quad f \in \mathcal{D}\left(\mathcal{L}\big|_{H^k(\mathbb{R}^3;\mathbb{C})}\right), \tag{2.26}$$

where $\overline{f}$ is the complex conjugation of f. This symmetry implies the following structure of unstable spectrum Λ.

Proposition 2.8 (Structure of unstable spectrum on $H^k(\mathbb{R}^3;\mathbb{R})$)**.** *For $k \geq 0$, and any $\delta_g > 0$ determined in Proposition 2.7, the following statements of $-\mathcal{L}$ are true.*

(1) *Symmetry of unstable spectrum: Λ is symmetric with respect to the real axis, namely we have*

$$\Lambda \cap \mathbb{R} = \{\lambda_j\}_{j=1}^N, \quad \Lambda \backslash \mathbb{R} = \{\xi_j, \overline{\xi}_j\}_{j=1}^J. \tag{2.27}$$

Moreover, there exists $\{v_j\}_{j=1}^{M_1} \subset H^k(\mathbb{R}^3;\mathbb{R})$ such that

$$\bigoplus_{\lambda \in \Lambda \cap \mathbb{R}} \ker(\mathcal{L} - \lambda)^{\mu_\lambda} = span_{\mathbb{C}}\{v_j\}_{j=1}^{M_1}.$$

(2) *Symmetry of Riesz projection on $H^k(\mathbb{R}^3;\mathbb{R})$: There exists $\varphi_j, \psi_j \in H^k(\mathbb{R}^3;\mathbb{R})$ for $j = 1, ..., N$ such that*

$$(\varphi_i, \psi_j)_{H^k} = \delta_{i,j}, \tag{2.28}$$

and for real-valued function $f \in H^k(\mathbb{R}^3;\mathbb{R})$,

$$P_u f = \sum_{i=1}^N (f, \psi_i)_{H^k} \varphi_j \in H^k(\mathbb{R}^3;\mathbb{R}). \tag{2.29}$$

(3) *Stable/unstable decomposition of $H^k(\mathbb{R}^3;\mathbb{R})$: Define*

$$H_u^k := \mathrm{Ran}\left(P_u\big|_{H^k(\mathbb{R}^3;\mathbb{R})}\right), \quad H_s^k := \mathrm{Ran}\left((1-P_u)\big|_{H^k(\mathbb{R}^3;\mathbb{R})}\right).$$

Then $H_u^k = V \cap H^k(\mathbb{R}^3;\mathbb{R})$, $H_s^k = V^{*\perp} \cap H^k(\mathbb{R}^3;\mathbb{R})$; they are invaraint under $-\mathcal{L}$; and thus $H^k(\mathbb{R}^3;\mathbb{R})$ can be decomposed by $H^k(\mathbb{R}^3;\mathbb{R}) = H_u^k \oplus H_s^k$.

(4) *Real Jordan normal form of $(-\mathcal{L})\big|_{H_u^k}$: The linear transformation $-\mathcal{L}|_{H_u^k}$ has all its eigenavlues lying in $\{z \in \mathbb{C} : \Re z > -\frac{\delta_g}{2}\}$. Moreover, there exists a basis of H_u^k such that it can be expressed as a real matrix $J = diag\{J_1, J_2, ..., J_l\}$, and here J_i must be one the following two forms:*

$$J_i^{Re} = \begin{bmatrix} \lambda_i & \frac{\delta_g}{10} & & \\ & \lambda_i & \ddots & \\ & & \ddots & \frac{\delta_g}{10} \\ & & & \lambda_i \end{bmatrix} \quad or \quad J_i^{Im} = \begin{bmatrix} C_i & \frac{\delta_g}{10} I_2 & & \\ & C_i & \ddots & \\ & & \ddots & \frac{\delta_g}{10} I_2 \\ & & & C_i, \end{bmatrix}, \tag{2.30}$$

where $C_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are the 2 by 2 real matrices, where $\lambda_i, a_i > -\frac{\delta_g}{2}$. In particular, we have

$$w^T J w \geq -\frac{6\delta_g}{10}|w|^2, \quad w \in \mathbb{R}^N. \tag{2.31}$$

Proof. (1) The symmetry of Λ and thereafter the decomposition (2.27) follow from that $(\mathcal{L} - \lambda)^n \varphi_\lambda = 0$ if and only if $(\mathcal{L} - \overline{\lambda})^n \overline{\varphi}_\lambda = 0$ due to the symmetry of $\mathcal{L}$ given in (2.26). Taking a (complex-valued) basis $\{\tilde{v}_j\}_{1 \leq j \leq M_1} \subset H^k(\mathbb{R}^3;\mathbb{C})$ of the real unstable generalized eigenspace $\bigoplus_{\lambda \in \Lambda \cap \mathbb{R}} \ker(-\mathcal{L} - \lambda)^{\mu_\lambda}$, the symmetry (2.26) implies that $\Re\tilde{v}_j, \Im\tilde{v}_j \in \bigoplus_{\lambda\in\Lambda\cap\mathbb{R}} \ker(-\mathcal{L}-\lambda)^{\mu_\lambda}$, from which we can choose a basis $\{v_j\}_{j=1}^{M_1} \subset H^k(\mathbb{R}^3;\mathbb{R})$.

(2) Denote the Riesz projection onto $\bigoplus_{\lambda\in\Lambda\cap\{\pm\Im z>0\}} \ker(\mathcal{L}-\lambda)^{\mu_\lambda}$ by $P_{Im\pm}$ and the one onto $\bigoplus_{\lambda\in\Lambda\cap\mathbb{R}} \ker(-\mathcal{L}-\lambda)^{\mu_\lambda}$ by P_{Re}. By the definition of P_{Im+} through the contour integration,

$$P_{Im+} = \frac{1}{2\pi i}\int_{\gamma_2} (\lambda - (-\mathcal{L}))^{-1} d\lambda,$$

where $\gamma_2 \subset \rho(-\mathcal{L})$ is an anti-clockwise closed curve enclosing $\Lambda \cap \{\Im z > 0\}$. So for real-valued f,

$$\begin{aligned}\overline{P_{Im+}f} &= \overline{\frac{1}{2\pi i}\int_{\gamma_2} (\lambda - (-\mathcal{L}))^{-1} f d\lambda} = -\frac{1}{2\pi i}\int_{\gamma_2} (\overline{\lambda} - (-\mathcal{L}))^{-1} f d\overline{\lambda} \\ &= -\frac{1}{2\pi i}\int_{\overline{\gamma}_2} (\lambda - (-\mathcal{L}))^{-1} f d\lambda = \frac{1}{2\pi i}\int_{-\overline{\gamma}_2} (\lambda - (-\mathcal{L}))^{-1} f d\lambda = P_{Im-}f \qquad (2.32)\end{aligned}$$

where $-\overline{\gamma}_2 \subset \rho(-\mathcal{L})$ is an anti-clockwise closed curve with its interior only containing $\Lambda \cap \{\Im z < 0\}$ thanks to the symmetry of Λ. Similarly, we have $\overline{P_{Re}f} = P_{Re}f$.

Now take $\{w_j\}_{j=1}^{M_2}$ to be the basis of $\bigoplus_{\lambda\in\Lambda\cap\{z\in\mathbb{C}:\Im z>0\}} \ker(-\mathcal{L}-\lambda)^{\mu_\lambda}$, then $\Re w_j, \Im w_j$ are non-trivial since $\lambda \notin \mathbb{R}$. By Riesz representative theorem, there exist $\{h_j\}_{j=1}^{M_1}, \{g_j\}_{j=1}^{M_2} \subset H^k(\mathbb{R}^3;\mathbb{C})$ such that

$$P_{Re} = \sum_{j=1}^{M_1} (\cdot, h_j)_{H^k} v_j \quad \text{ and } \quad P_{Im+} = \sum_{j=1}^{M_2} (\cdot, g_j)_{H^k} w_j,$$

with $\{v_j\}_{j=1}^{M_1} \subset H^k(\mathbb{R}^3;\mathbb{R})$ from (1). Note that the symmetries (2.32) and $\overline{P_{Re}f} = P_{Re}f$ also indicate that $P_{Im-} = \sum_{j=1}^{M_2} (\cdot, \overline{g}_j)_{H^k} \overline{w}_j$ on $H^k(\mathbb{R}^3;\mathbb{R})$ and $h_j \in H^k(\mathbb{R}^3;\mathbb{R})$. So the projection property of P_{Re} and P_{Im+} indicate that

$$(v_i, h_j)_{H^k} = (w_i, g_j)_{H^k} = \delta_{i,j},\ (v_i, g_j)_{H^k} = (w_i, h_j)_{H^k} = (\overline{w}_i, g_j) = 0, \quad \forall i, j. \qquad (2.33)$$

As a corollary, $\Re h_j, \Re g_j, \Im g_j$ are non-trivial from the non-triviality of $v_i, \Re w_i$ and $\Im w_i$. So with the symmetry of $P_{Im\pm}$ and P_{Re}, we compute for any real-valued function $f \in H^k(\mathbb{R}^3;\mathbb{R})$,

$$\begin{aligned}P_u f &= P_{Re}f + P_{Im+}f + P_{Im-}f = \frac{1}{2}\left(P_{Re} + \overline{P_{Re}}\right) f + \left(P_{Im+} + P_{Im-}\right) f \\ &= \sum_{j=1}^{M_1} (f, \Re h_j)_{H^k} v_j + \sum_{j=1}^{M_2} (f, g_j)_{H^k} w_j + \sum_{j=1}^{M_2} (f, \overline{g}_j)_{H^k} \overline{w}_j \\ &= \sum_{j=1}^{M_1} (f, \Re h_j)_{H^k} v_j + 2\sum_{j=1}^{M_2} \left[(f, \Re g_j)_{H^k} \Re w_j + (f, \Im g_j)_{H^k} \Im w_j\right].\end{aligned}$$

This implies the choice of $\{\varphi_j\}_{j=1}^N, \{\psi_j\}_{j=1}^N \subset H^k(\mathbb{R}^3;\mathbb{R})$ with $N = M_1 + 2M_2$ to realize (2.29), and the orthonormal relation (2.28) follows (2.33).

(3) For any $f \in H_u^k = \mathrm{Ran}(P_u\big|_{H^k(\mathbb{R}^3;\mathbb{R})}) \subset V$, there exists $g \in H^k(\mathbb{R}^3;\mathbb{R})$ such that $f = P_u g$, then by (2.29), $f \in H^k(\mathbb{R}^3;\mathbb{R})$, hence $f \in V \cap H^k(\mathbb{R}^3;\mathbb{R})$. Conversely, for any $f \in V \cap H^k(\mathbb{R}^3;\mathbb{R})$, then by (2.29) again, $f = P_u f \in V \cap H^k(\mathbb{R}^3;\mathbb{R})$. Hence $H_u^k = V \cap H^k(\mathbb{R}^3;\mathbb{R})$. Likewisely, we can derive $H_s^k = V^{*\perp} \cap H^k(\mathbb{R}^3;\mathbb{R})$.

When it comes to the invariance of $\mathcal{L}$, since P_u is commutable with $\mathcal{L}$ from the definition of P_u through the contour integral (2.24), for any $f \in H^k_u = V \cap H^k(\mathbb{R}^3;\mathbb{R})$,

$$\mathcal{L}f = \mathcal{L}P_u f = P_u \mathcal{L} f \in H^k_u,$$

hence H^k_u is invariant under $-\mathcal{L}$. Similarly, H^k_s is invariant under $-\mathcal{L}$.

(4) From (2) and (3), the operator $-\mathcal{L}\big|_{H^k_u}$ under the basis $\{\varphi_j\}_{1\le j\le N} \subset H^k(\mathbb{R}^3,\mathbb{R})$ is a real matrix denoted by A, and its eigenvalue set $\mathrm{eig}(A) = \Lambda \subset \{z \in \mathbb{C} : \Re z > -\frac{1}{2}\delta_g\}$. Hence its real Jordan normal form should be $\mathrm{diag}\{\tilde{J}_1, \tilde{J}_2, ..., \tilde{J}_l\}$, where $\tilde{J}_i$ is some blocks given by (2.30) but simply replacing $\frac{\delta_g}{10}$ by 1. If we introduce D_i to be the matrix $D^{Re}_i = \mathrm{diag}\{1, \delta_g/10, (\delta_g/10)^2 ...\}$ or $D^{Im}_i = \mathrm{diag}\{I_2, \frac{\delta_g}{10} I_2, \frac{\delta_g^2}{10^2} I_2, ...\}$, then

$$\left(D^{Re}_i\right)^{-1} \tilde{J}^{Re}_i D^{Re}_i = J^{Re}_i, \text{ and } \left(D^{Im}_i\right)^{-1} \tilde{J}^{Im}_i D^{Re}_i = J^{Im}_i,$$

which shows how we obtain (2.30). To show (2.31), it suffices to prove each block $\tilde{J}_i$ satisfies (2.30). In fact, for the block corresponding to real-valued eigenvalue, given any $w \in \mathbb{R}^{k_i}$ for $k_j = \mathrm{Rank} J_i$, by Cauchy-Schwarz inequality,

$$w^T \left(J^{Re}_i + \frac{6\delta_g}{10}\right) w \ge \sum_{j=1}^{k_i} \frac{\delta_g}{10} w_i^2 - \frac{\delta_g}{10} \sum_{j=1}^{k_i - 1} |w_j||w_{j+1}| \ge 0.$$

For the block corresponding to complex eigenvalue, notice that for any $v \in \mathbb{R}^2$,

$$v^T C_i v = [v_1, v_2] \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix} [v_1, v_2]^T = a_i |v|^2,$$

For any $w = [w_1^T, w_2^T, ..., w_{k_i}^T]^T \in \mathbb{R}^{2k_i}$, with $w_i \in \mathbb{R}^2$, we compute

$$w^T \left(J^{Im}_i + \frac{6\delta_g}{10} I\right) w = \sum_{j=1}^{k_i} w_j^T C_i w_j + \sum_{j=1}^{k_i-1} \frac{\delta_g}{10} w_j^T w_{j+1} + \frac{6\delta_g}{10} w^T w$$

$$= \sum_{j=1}^{k_i} \left(a_i + \frac{6\delta_g}{10}\right) |w_j|^2 + \sum_{j=1}^{k_i-1} \frac{\delta_g}{10} w_j^T w_{j+1} \ge \frac{\delta_g}{10} \sum_{j=1}^{k_i} w_j^T w_j - \sum_{j=1}^{k_i-1} \frac{\delta_g}{10} |w_j||w_{j+1}| \ge 0,$$

then (2.31) has been verified. □

Lemma 2.9 (Smoothing effect of P_u)**.** *For $k \ge 0$, and any $\delta_g > 0$ determined in Proposition 2.7. For every $m \ge k$, the Riesz projection $P_u : H^k(\mathbb{R}^3;\mathbb{C}) \to V$ defined by* (2.24) *satisfies the bound*

$$\|P_u f\|_{H^{m+1}} \lesssim \|f\|_{H^m}. \tag{2.34}$$

In particular, $V \subset H^\infty(\mathbb{R}^3;\mathbb{C})$ and $H^k_u \subset H^\infty(\mathbb{R}^3;\mathbb{R})$.

Proof. Smoothing of P_u (2.34). Recall the definition of the Riesz projection (2.24) with contour $\Gamma \subset \{z \in \mathbb{C} : \Re z > -\frac{\delta_g}{2}\} \cap \rho(-\mathcal{L})$. Note that

$$\mathcal{L} + \lambda = \mathcal{L}_0 + \mathcal{L}' + \lambda = (I + \mathcal{L}'(\mathcal{L}_0 + \lambda)^{-1})(\mathcal{L}_0 + \lambda).$$

From Lemma 2.2, it suffices to show that $I + \mathcal{L}'(\mathcal{L}_0 + \lambda)^{-1} : H^m \to H^m$ has an inversion uniformly bounded for $\lambda \in \Gamma$.

First of all, we claim that $\mathcal{L}'(\mathcal{L}_0 + \lambda)^{-1} : H^m \to H^m$ is compact. In fact, as shown in Lemma 2.5, we get that $\mathcal{L}'_1, \mathcal{L}'_2 : H^{m+1} \to H^m$ is compact, which indicates that $(\mathcal{L}'_1 + \mathcal{L}'_2)(\mathcal{L}_0 + \lambda)^{-1} : H^m \to H^m$ is compact. Then it suffices to prove that $\mathcal{L}'_3(\mathcal{L}_0 + \lambda)^{-1} : H^m \to H^m$ is compact, and it can be verified with the two facts that $(\mathcal{L}_0+\lambda)^{-1}: H^m \to H^{m+\frac{3}{2}}$ is bounded and that $\mathcal{L}_3': H^{m+\frac{3}{2}} \to H^m$ is compact, which has been proved in (2.11) and Lemma 2.5 respectively.

Next we show that $I+\mathcal{L}'(\mathcal{L}_0+\lambda)^{-1}$ has a bounded inverse on H^m for all $\lambda \in \Gamma \subset \{z \in \mathbb{C} : \Re z > -\frac{\delta_g}{2}\} \cap \rho(-\mathcal{L})$. By contradiction, if we assume that the result is not true, then by Fredholm's alternative, there exists a non-trivial $f \in H^m$ such that

$$\left(I+\mathcal{L}'(\mathcal{L}_0+\lambda)^{-1}\right) f = 0, \quad \Rightarrow \quad -\mathcal{L}f = \lambda f,$$

a contradiction to $\lambda \in \rho(-\mathcal{L})$.

Finally, since the resolvent identity $(\mathcal{L}_0+\lambda)^{-1} - (\mathcal{L}_0+\lambda')^{-1} = (\lambda'-\lambda)(\mathcal{L}_0+\lambda)^{-1}(\mathcal{L}_0+\lambda')^{-1}$ implies that $I+\mathcal{L}'(\mathcal{L}_0+\lambda)^{-1}$ is continuous with respect to λ in the operator norm, the uniformly bounded inversion of $I+\mathcal{L}'(\mathcal{L}_0+\lambda)^{-1}$ follows the invertibility of each $\lambda \in \Gamma$ and compactness of Γ. Hence we have verified (2.34).

Smoothness of elements in V. Since $f \in V \subset H^m$, and by the fact that $P_u : H^m \to V$ is a projection, it means that $f = P_u f$. Then by (2.34), $f \in H^{m+1}$. Iterating this argument again and again, the regularity of f can then be improved to $f \in H^\infty$. $\square$

2.5. **Modified unstable space.** In this section, we construct modified unstable space $\tilde{H}_u^k \subset L^1(\mathbb{R}^3;\mathbb{R}) \subset C_0^\infty(\mathbb{R}^3;\mathbb{R})$ and design the related modified unstable projection $\tilde{P}_u$, with the linear diagonal dynamics preserved (see Proposition 2.12).

Besides, from now on, we confine our attention to the real-valued functions and shall represent H^k and $\dot{H}^k$ as $H^k(\mathbb{R}^3;\mathbb{R})$ and $\dot{H}^k(\mathbb{R}^3;\mathbb{R})$ respectively.

Fix $k \geq 0$, and $\delta_g > 0$ determined in Proposition 2.7. For $\{\varphi_j, \psi_j\}_{1\leq j\leq N}$ given in Proposition 2.8 (2), we define the matrix by

$$M_R = \{(\chi_R\varphi_i, \psi_j)_{H^k}\}_{1\leq i,j\leq N}, \tag{2.35}$$

where χ_R the cut-off function defined in (1.2). Noticing that (2.28) implies

$$M_R = \{(\varphi_i,\psi_j)_{H^k} + ((1-\chi_R)\varphi_i,\psi_j)_{H^k}\}_{1\leq i,j\leq N} = Id + o_R(1),$$

we make the following definition.

Definition 2.10 (Modified unstable space)**.** *For any $k \geq 0$, and any $\delta_g > 0$ determined in Proposition 2.7, there exists $R = R_{k,\delta_g} \gg 1$ such that the matrix M_R from* (2.35) *is invertible, and we define the modified unstable (generalized) eigenmodes $\{\tilde{\varphi}_i\}_{i=1}^N$ by*

$$(\tilde{\varphi}_1, ..., \tilde{\varphi}_N)^T = M_R^{-1}(\chi_R\varphi_1, \chi_R\varphi_2, ..., \chi_R\varphi_N)^T, \tag{2.36}$$

In addition, on the real-valued Sobolev space H^k, we define the modified spectral projections as

$$\tilde{P}_u = \sum_{j=1}^N (\cdot,\psi_j)_{H^k}\tilde{\varphi}_j, \ \text{and } \tilde{P}_s = I - \tilde{P}_u, \tag{2.37}$$

and accordingly, we define the subspaces of H^k by

$$\tilde{H}_u^k = \mathrm{Ran}(\tilde{P}_u) = \mathrm{span}(\tilde{\varphi}_i)_{i=1}^N, \quad \text{and } \tilde{H}_s^k = \mathrm{Ran}(\tilde{P}_s). \tag{2.38}$$

Immediately, the following basic properties hold.

Lemma 2.11. *For any $k \geq 0$, and any $\delta_g > 0$ determined in Proposition 2.7, the projection $\tilde{P}_u$, $\tilde{P}_s$ and subspaces $\tilde{H}_u^k$, $\tilde{H}_s^k$ from Definition 2.10 satisfy the following statements.*

- *$\tilde{P}_u$ is a projection from H^k to $\tilde{H}^k_u$, namely $\tilde{P}_u^2 = \tilde{P}_u$. Moreover, $H^k = \tilde{H}^k_u \oplus \tilde{H}^k_s$.*
- *$H^k_u \subset C_0^\infty(\mathbb{R}^3)$. For any $m \geq k$, $\tilde{P}_u : H^k \to H^m$ and $\tilde{P}_s : H^m \to H^m$ are bounded.*
- *$\tilde{P}_u$ and P_u satisfy the following identity:*

$$\tilde{P}_u P_u = \tilde{P}_u, \quad P_u \tilde{P}_u = P_u. \tag{2.39}$$

- *$\tilde{P}_s$ and P_s satisfy the following identity:*

$$P_s \tilde{P}_s = \tilde{P}_s, \quad \tilde{P}_s P_s = P_s. \tag{2.40}$$

Hence $\tilde{H}^k_s = H^k_s$.

Proof. From (2.35) and (2.36), we have the orthogonality

$$(\tilde{\varphi}_i, \psi_j)_{H^k} = \delta_{i,j} \text{ for any } 1 \leq i, j \leq N. \tag{2.41}$$

which implies the first statement and the decomposition of H^k into direct sum. The choice of $\{\tilde{\varphi}_i\}$ (2.36) and smoothness of φ_i from Lemma 2.9 also yield $\tilde{H}^k_u = \mathrm{span}(\tilde{\varphi}_i)_{i=1}^N = \mathrm{span}(\chi_R \varphi_i)_{i=1}^N \subset C_0^\infty$. The boundedness of $\tilde{P}_u$, $\tilde{P}_s$ follows immediately.

From Proposition 2.8, for any $f \in H^k(\mathbb{R}^3; \mathbb{R})$,

$$\tilde{P}_u P_u f = \sum_{j=1}^N \sum_{i=1}^N (f, \psi_i)_{H^k} (\varphi_i, \psi_j)_{H^k} \tilde{\varphi}_j = \sum_{j=1}^N (f, \psi_j)_{H^k} \tilde{\varphi}_j = \tilde{P}_u f,$$

and $$P_u \tilde{P}_u f = \sum_{j=1}^N \sum_{i=1}^N (f, \psi_i)_{H^k} (\tilde{\varphi}_i, \psi_j)_{H^k} \varphi_j = \sum_{j=1}^N (f, \psi_j)_{H^k} \varphi_j = P_u f,$$

hence (2.39) has been verified. And (2.40) is a simple result of (2.39).

Furthermore, for any $f \in \tilde{H}^k_s$, (2.40) implies

$$f = \tilde{P}_s f = P_s \tilde{P}_s f \subset \mathrm{Ran} P_s = H^k_s,$$

so $\tilde{H}^k_s \subset H^k_s$. The other side of inclusion follows similarly. □

Next, we discuss the spectral properties of $-\mathcal{L}$ under this modified spectral decomposition.

Proposition 2.12 (Modified spectral properties of $-\mathcal{L}$)**.** *For any $k \geq 0$, and $\delta_g > 0$ determined in Proposition 2.7, for $\tilde{P}_u$, $\tilde{P}_s$ defined in (2.37) and $\tilde{H}^k_u$, $\tilde{H}^k_s$ defined in (2.38), we have the following properties:*
- *(Modified unstable part) $-\tilde{P}_u \mathcal{L} P_u$ maps $\tilde{H}^k_u$ to $\tilde{H}^k_u$. Moreover, there is some real basis $\{\tilde{\phi}_i\}_{1 \leq i \leq N}$ such that it can be also expressed as $J = diag\{J_1, J_2, ..., J_l\}$ with J_i given in (2.30). In addition, J satisfies (2.31).*
- *(Modified stable part)*

$$\|e^{-\tau \mathcal{L}} f\|_{H^k} \lesssim e^{-\frac{1}{2}\delta_g \tau} \|f\|_{H^k}, \quad \forall f \in \tilde{H}^k_s. \tag{2.42}$$

Proof. The second conclusion (2.42) directly follows from (2.25) and $\tilde{H}^k_s = H^k_s$ from Lemma 2.11. Now we focus on the modified unstable part. In fact, for any $1 \leq i \leq N$, we see that

$$P_u \tilde{\varphi}_i = \sum_{j=1}^N (\tilde{\varphi}_i, \psi_j)_{H^k} \varphi_j = \sum_{j=1}^N \delta_{ij} \varphi_j = \varphi_i,$$

and

$$\tilde{P}_u \varphi_i = \sum_{j=1}^N (\varphi_i, \psi_j)_{H^k} \tilde{\varphi}_j = \sum_{j=1}^N \delta_{ij} \tilde{\varphi}_j = \tilde{\varphi}_i.$$

If we denote A as the representation matrix of $-\mathcal{L}$ under the basis $\{\varphi_1, \varphi_2, ..., \varphi_N\}$ of H_u^k, then according to Proposition 2.8 and Lemma 2.11, for any $f \in \tilde{H}_u^k$, expressed as $f = \sum_{i=1}^N f^i \tilde{\varphi}_i$ for some $\mathbf{f} = (f^1, f^2, ..., f^N)^T \in \mathbb{R}^N$,

$$-\tilde{P}_u \mathcal{L} P_u f = -\tilde{P}_u \mathcal{L} \sum_{j=1}^N f^j \varphi_j = -\tilde{P}_u \sum_{j=1}^N (A\mathbf{f})^j \varphi_j = \sum_{j=1}^N (A\mathbf{f})^j \tilde{\varphi}_j,$$

which implies that the representation matrix of $-\tilde{P}_u \mathcal{L} P_u$ under the basis $\{\tilde{\varphi}_i\}_{1\le i\le N} \subset \tilde{H}_u^k$ should also be A. According to Proposition 2.8, there exists a real-valued transition matrix Q such that $Q^{-1}AQ = J$. Consequently, there exists a real basis $\{\tilde{\phi}_i\}_{1\le i\le N} \subset \tilde{H}_u^k$ such that the related representative matrix of $-\tilde{P}_u \mathcal{L} P_u$ is J. □

Corollary 2.13. *For any $k \ge 0$, and $\delta_g > 0$ determined in Proposition 2.7, for $\tilde{P}_u$ defined in (2.37) and $\tilde{H}_u^k$ defined in (2.38), there exists an inner product on $\tilde{H}_u^k$, which we denote by $(\cdot,\cdot)_{\tilde{B}}$, such that*

$$(-\tilde{P}_u \mathcal{L} P_u f, f)_{\tilde{B}} \ge -\frac{6\delta_g}{10} \|f\|_{\tilde{B}}^2, \quad \forall\, f \in \tilde{H}_u^k. \tag{2.43}$$

Moreover, $(\cdot,\cdot)_{\tilde{B}}$ induces a norm $\|\cdot\|_{\tilde{B}}$ on $\tilde{H}_u^k$ and $\|\cdot\|_{\tilde{B}} \simeq \|\cdot\|_{H^k}$.

Proof. For any $f, g \in \tilde{H}_u^k$, from Proposition 2.12, they can be uniquely decomposed into $f = \sum_{i=1}^N \tilde{f}^i \tilde{\phi}_i$ and $g = \sum_{i=1}^N \tilde{g}^i \tilde{\phi}_i$ for some $\tilde{\mathbf{f}} = (\tilde{f}^1, ...\tilde{f}^N)^T \in \mathbb{R}^N$ and $\tilde{\mathbf{g}} = (\tilde{g}^1, ...\tilde{g}^N)^T \in \mathbb{R}^N$, then we define

$$(f, g)_{\tilde{B}} = \tilde{\mathbf{f}}^{\perp} \tilde{\mathbf{g}}. \tag{2.44}$$

It can be easily verified that (2.44) defines an inner product $(\cdot,\cdot)_{\tilde{B}}$ on $\tilde{H}_u^k$, inducing a norm $\|\cdot\|_{\tilde{B}}$ on $\tilde{H}_u^k$, then (2.43) follows from Proposition 2.12 and Proposition 2.8 (4). Finally, the equivalence $\|\cdot|_{\tilde{B}} \simeq \|\cdot\|_{H^k}$ is established based on the well-known fact that any two norms on a finite-dimensional space are equivalent. □

3. Existence of Stable Trajectory

In this section, we come back to the nonlinear system (2.6) and discuss its dynamics to prove Theorem 1.1. We fix $k \ge 2$ and $\delta_g < \frac{1}{16}$ chosen from Proposition 2.7. Applying the modified spectral decomposition $I = \tilde{P}_s + \tilde{P}_u$ (see Definition 2.10) to (2.6), the real-valued variables $(\tilde{\varepsilon}_s, \tilde{\varepsilon}_u, U, \Pi) := (\tilde{P}_s\varepsilon, \tilde{P}_u\varepsilon, U, \Pi)$ solve the system

$$\begin{cases} \partial_\tau \tilde{\varepsilon}_u = -\tilde{P}_u \mathcal{L}\varepsilon + \tilde{P}_u\left(-U\cdot\nabla\Psi + N(\varepsilon)\right), \\ \partial_\tau \tilde{\varepsilon}_s = -\tilde{P}_s \mathcal{L}\varepsilon + \tilde{P}_s\left(-U\cdot\nabla\Psi + N(\varepsilon)\right), \\ \partial_\tau U + \frac{1}{2}\left(U + y\cdot\nabla U\right) + U\cdot\nabla U = \Delta U - \nabla\Pi - \mu\Psi e_3, \\ \nabla\cdot U = 0, \end{cases}$$

where $\mu(\tau) = \mu_0 e^{-\frac{1}{2}\tau}$ for some $\mu_0 > 0$ and $N(\varepsilon)$ is the nonlinear term

$$N(\varepsilon) = \nabla\cdot(\varepsilon\nabla\Delta^{-1}\varepsilon).$$

In addition, by (2.39) and the commutability between $\mathcal{L}$ and P_u, we obtain that

$$-\tilde{P}_u\mathcal{L}\varepsilon = -\tilde{P}_u P_u \mathcal{L}\varepsilon = -\tilde{P}_u \mathcal{L} P_u \varepsilon = -\tilde{P}_u \mathcal{L} P_u \tilde{P}_u \varepsilon = -\tilde{P}_u \mathcal{L} P_u \tilde{\varepsilon}_u,$$

and similarly, from (2.40) and the fact that $\tilde{H}_s^k = H_s^k$, one gets that

$$-\tilde{P}_s\mathcal{L}\varepsilon = -\tilde{P}_s\mathcal{L}\tilde{P}_s\varepsilon - \tilde{P}_s\mathcal{L}\tilde{P}_u\varepsilon = -\mathcal{L}\tilde{\varepsilon}_s - \tilde{P}_s\mathcal{L}\tilde{\varepsilon}_u,$$

hence the system can be rewritten as

$$\begin{cases} \partial_\tau \tilde{\varepsilon}_u = -\tilde{P}_u \mathcal{L} P_u \tilde{\varepsilon}_u + \tilde{P}_u \left(-U \cdot \nabla \Psi + N(\varepsilon)\right), \\ \partial_\tau \tilde{\varepsilon}_s = -\mathcal{L}\tilde{\varepsilon}_s - \tilde{P}_s \mathcal{L} \tilde{\varepsilon}_u + \tilde{P}_s \left(-U \cdot \nabla \Psi + N(\varepsilon)\right), \\ \partial_\tau U + \frac{1}{2}\left(U + y \cdot \nabla U\right) + U \cdot \nabla U = \Delta U - \nabla \Pi - \mu \Psi e_3, \\ \nabla \cdot U = 0. \end{cases} \tag{3.1}$$

Remark 3.1. The local wellposedness of (KS-NS) given in Theorem A.1 implies the local wellposedness of renormalized system (2.6) with $(\varepsilon, U) \in H^m \times H^m_\sigma$ for any $m \geq 2$, and thereafter of this spectrally decoupled system (3.1) with $(\tilde{\varepsilon}_s, \tilde{\varepsilon}_u, U) \in \left(\tilde{H}^k_s \cap H^m\right) \times \tilde{H}^k_u \times H^m_\sigma$ for $m \geq k \geq 2$.

Proposition 3.2 (Bootstrap)**.** *For any fixed $k \geq 2$, we take $\delta_g < \frac{1}{16}$ from Proposition 2.7 and let $\tilde{P}_s, \tilde{P}_u$, $\tilde{H}^k_s, \tilde{H}^k_u$ and $R = R_{k,\delta_g}$ be given by Definition 2.10. Then there exist $0 < \delta_i \ll 1 (0 \leq i \leq 4)$ with*

$$\delta_0 \ll \delta_4 \ll \delta_3 \ll \delta_1 \ll \delta_2 \ll \delta_3^{\frac{1}{2}} \ll \min\{\delta_g, Q(2R)\}, \tag{3.2}$$

such that for any initial datum $\mu(0) = \mu_0$, $\tilde{\varepsilon}_s(0) = \tilde{\varepsilon}_{s0} \in \tilde{H}^k_s \cap H^{k+2}$ and $U(0) = U_0 \in H^{k+2}_\sigma$ satisfying

$$|\mu_0| + \|\tilde{\varepsilon}_{s0}\|_{H^{k+1}} + \|U_0\|_{\dot{H}^1 \cap \dot{H}^{k+1}} \leq \delta_0, \tag{3.3}$$

there exists $\tilde{\varepsilon}_{u0} \in \tilde{H}^k_u$ such that the solution to the (2.4) starting from $(\Psi_0, U_0) = (Q + \tilde{\varepsilon}_{s0} + \tilde{\varepsilon}_{u0}, U_0)$ globally exists and satisfies the followings for all $\tau \geq 0$:

- *(Control of the modified stable part $\tilde{\varepsilon}_s$)*

$$\|\tilde{\varepsilon}_s(\tau)\|_{H^k} < \delta_1 e^{-\frac{1}{2}\delta_g \tau}, \tag{3.4}$$

- *(Control of the higher regularity of the $\tilde{\varepsilon}_s$)*

$$\|\tilde{\varepsilon}_s(\tau)\|_{\dot{H}^{k+1}} < \delta_2 e^{-\frac{1}{2}\delta_g \tau}, \tag{3.5}$$

- *(Control of the modified unstable part $\tilde{\varepsilon}_u$)*

$$\|\tilde{\varepsilon}_u(\tau)\|_{\tilde{B}} \leq \delta_3 e^{-\frac{7}{10}\delta_g \tau}, \tag{3.6}$$

- *(Control of the flow)*

$$\|U(\tau)\|_{\dot{H}^1 \cap \dot{H}^{k+1}} < \delta_4 e^{-\frac{1}{8}\tau}. \tag{3.7}$$

Proposition 3.2 is the center of the paper, and it implies Theorem 1.1. As in [18, 42], the Proposition 3.2 will be proven via contradiction using a topological argument as follows: given $(\mu_0, \tilde{\varepsilon}_{s0}, U_0)$ satisfying (3.3) with $\nabla \cdot U_0 = 0$, we assume that for any $\tilde{\varepsilon}_u \in \tilde{H}^k_u$ satisfies (3.6) when $\tau = 0$, the exit time

$$\tau^* = \sup\{\tau \geq 0 : (\mu, \tilde{\varepsilon}_s, \tilde{\varepsilon}_u, U) \text{ satisfy (2.3)-(3.7) simultaneously on } [0, \tau]\}, \tag{3.8}$$

is finite and then we look for a contradiction under the assumptions of the coefficients given in (3.2). Subsequently, we therefore study the flow on $[0, \tau^*]$ where (3.4)-(3.7) holds. Precisely, we will verify that the bounds in (3.4), (3.5), and (3.7) can be further improved within the bootstrap regime, ensuring that $(\tilde{\varepsilon}_s, U)$ will not exit the bootstrap regime at time $\tau = \tau^*$ and the only potential scenario for the solution to exit the bootstrap regime is if (3.6) fails as $\tau > \tau^*$, thus together with the outgoing flux property of $\tilde{\varepsilon}_u$ at the exit time, we can conclude a contradiction via Brouwer's fixed point theorem.

3.1. **A priori estimates.** In this subsection, we consider the a priori estimates. It is worth noting that throughout our discussion, the constant $C>0$ of the estimates may vary from line to line as required.

3.1.1. *Energy estimate of the flow U.*

Lemma 3.3 (L^∞ smallness of U)**.** *Under the bootstrap assumptions in Proposition 3.2, there exists a universal constant $C>0$ such that*

$$\|U(\tau)\|_\infty \le C\|U\|_{\dot H^1\cap \dot H^{k+1}} \le C\delta_4 e^{-\frac{1}{8}\tau}, \quad \forall \tau\in[0,\tau^*]. \tag{3.9}$$

Proof. With the initial data $\rho_0\in H^{k+2}$ and $U_0\in H^{k+2}_\sigma$, the local well-posedness of system (KS-NS) implies that $U(\tau)\in H^{k+2}_\sigma$ for any $\tau\in[0,\tau^*]$. Consequently, by Sobolev embedding and Gagliardo-Nirenberg interpolation inequality,

$$\|U\|_{L^\infty} \lesssim \|U\|_{L^6}^{\frac12}\|U\|_{\dot H^2}^{\frac12} \lesssim \|U\|_{\dot H^1}^{\frac12}\|U\|_{\dot H^2}^{\frac12} \lesssim \|U\|_{\dot H^1\cap\dot H^{k+1}},$$

then (3.9) follows (3.7). □

Lemma 3.4 (Energy estimates of U)**.** *Under the assumptions of Proposition 3.2, there exists a universal constant $C>0$ such that*

$$\frac{d}{d\tau}\|U\|_{\dot H^1}^2 \le -\frac14\|U\|_{\dot H^1}^2 + C\delta_4^3 e^{-\frac38\tau} + C\mu_0^2 e^{-\tau}, \quad \forall\tau\in[0,\tau^*], \tag{3.10}$$

and

$$\frac{d}{d\tau}\|U\|_{\dot H^{k+1}}^2 \le -\frac14\|U\|_{\dot H^{k+1}}^2 + C\delta_4^3 e^{-\frac38\tau} + C\mu_0^2 e^{-\tau}, \quad \forall\tau\in[0,\tau^*]. \tag{3.11}$$

Furthermore, the bootstrap assumption (3.7) can be improved to

$$\|U(\tau)\|_{\dot H^1\cap\dot H^{k+1}} \le \frac12\delta_4 e^{-\frac18\tau}, \quad \forall\tau\in[0,\tau^*]. \tag{3.12}$$

Proof. Estimate of (3.10). By (3.1), we have

$$\begin{aligned}
\frac{d}{d\tau}\frac12\|U\|_{\dot H^1}^2 &= \sum_{|\alpha|=1}\int \partial^\alpha U\cdot\partial^\alpha(\partial_\tau U)dy\\
&= \sum_{|\alpha|=1}\int \partial^\alpha U\cdot\left(\partial^\alpha\left(-\frac12(U+y\cdot\nabla U) - U\cdot\nabla U + \Delta U - \nabla\Pi - \mu\Psi e_3\right)\right)dy\\
&= -\frac12\sum_{|\alpha|=1}\int\partial^\alpha U\cdot\partial^\alpha(U+y\cdot\nabla U)dy - \sum_{|\alpha|=1}\int\partial^\alpha U\cdot(\partial^\alpha(U\cdot\nabla U))\,dy\\
&\quad - \|U\|_{\dot H^2}^2 - \sum_{|\alpha|=1}\mu\int\partial^\alpha U\cdot\partial^\alpha\Psi e_3 dy,
\end{aligned}$$

where the last equality uses the incompressible condition $\nabla\cdot U=0$. Similar to the argument for proving (2.12), we can get that

$$\frac12\sum_{|\alpha|=1}\int\partial^\alpha U\cdot\partial^\alpha(U+y\cdot\nabla U)dy = \frac12\|U\|_{\dot H^1}^2. \tag{3.13}$$

When it comes to the nonlinear term, by interpolation inequality and Young's inequality, one obtains that

$$\|f\|_{L^3}^3 \le C\|f\|_{L^2}^{\frac32}\|\nabla f\|_{L^2}^{\frac32} \le \frac14\|\nabla f\|_{L^2}^2 + C\|f\|_{L^2}^6, \quad \text{for any } f\in H^1,$$

thus it implies that the nonlinear term can be estimated by

$$\begin{aligned}
&\Big|\sum_{|\alpha|=1}\int \partial^\alpha U\cdot(\partial^\alpha(U\cdot\nabla U))\,dy\Big| \\
=&\Big|\sum_{|\alpha|=1}\int \partial^\alpha U\cdot((U\cdot\nabla\partial^\alpha U))\,dy+\sum_{|\alpha|=1}\int \partial^\alpha U\cdot((\partial^\alpha U\cdot\nabla U))\,dy\Big| \\
=&\Big|\sum_{|\alpha|=1}\int \partial^\alpha U\cdot(\partial^\alpha U\cdot\nabla U)dy\Big|\le C\|U\|^3_{\dot W^{1,3}}\le\frac14\|U\|^2_{\dot H^2}+C\|U\|^6_{\dot H^1}.
\end{aligned}$$

Additionally, for the buoyancy term, from bootstrap assumption (3.4) and (3.6),

$$\|\mu D\Psi\|_{L^2}=\mu\|DQ+D\tilde\varepsilon_s+D\tilde\varepsilon_u\|_{L^2}\le C\mu.$$

In summary, combing all the estimates above, for any $\tau\in[0,\tau^*]$, by the bootstrap assumptions given in Proposition 3.2, one obtains that

$$\begin{aligned}
\frac{d}{d\tau}\frac12\|U\|^2_{\dot H^1}&\le-\frac14\|U\|^2_{\dot H^1}+C\|U\|^6_{\dot H^1}+C\mu\|U\|_{\dot H^1}\\
&\le-\frac18\|U\|^2_{\dot H^1}+C\|U\|^6_{\dot H^1}+C\mu^2\le-\frac18\|U\|^2_{\dot H^1}+C\delta_4^6e^{-\frac34\tau}+C\mu_0^2e^{-\tau},
\end{aligned}$$

which yields (3.10).
Estimate of (3.11). Similarly, from (3.1), one obtains that

$$\begin{aligned}
&\frac{d}{d\tau}\frac12\|U\|^2_{\dot H^{k+1}}=\sum_{|\alpha|=k+1}\int\partial^\alpha U\cdot\partial^\alpha(\partial_\tau U)dy\\
=&\sum_{|\alpha|=k+1}\int(\partial^\alpha U)\cdot\left(\partial^\alpha\left(-\frac12(U+y\cdot\nabla U)-U\cdot\nabla U+\Delta U-\nabla\Pi-\mu\Psi e_3\right)\right)dy\\
=&-\frac{2k+1}{4}\|U\|^2_{\dot H^{k+1}}-\|U\|^2_{\dot H^{k+2}}-\sum_{|\alpha|=k+1}\int\partial^\alpha U\cdot(\partial^\alpha(U\cdot\nabla U))\,dy-\sum_{|\alpha|=k+1}\mu\int\partial^\alpha U\cdot\partial^\alpha\Psi e_3dy,
\end{aligned}$$

here we used $U(\tau)\in H^{k+2}_\sigma$ under the bootstrap assumption by the local wellposedness Theorem A.1.

For the nonlinear term, from interpolation inequality,

$$\|D^jf\|_{L^4}\lesssim\|Df\|_{L^2}^{\frac{k+\frac54-j}{k+1}}\|D^{k+2}f\|_{L^2}^{1-\frac{k+\frac54-j}{k+1}},\quad\forall 1\le j\le k+1,$$

we have

$$\begin{aligned}
&\sum_{0\le i\le k}\Big\||D^{k+1-i}U|\cdot|D^{i+1}U|\Big\|_{L^2}\le\sum_{0\le i\le k}\|D^{k+1-i}U\|_{L^4}\|D^{i+1}U\|_{L^4}\\
\lesssim&\sum_{0\le i\le k}\left(\|DU\|_{L^2}^{\frac{k+\frac54-(k+1-i)}{k+1}}\|D^{k+2}U\|_{L^2}^{1-\frac{k+\frac54-(k+1-i)}{k+1}}\right)\left(\|DU\|_{L^2}^{\frac{k+\frac54-(i+1)}{k+1}}\|D^{k+2}U\|_{L^2}^{1-\frac{k+\frac54-(i+1)}{k+1}}\right)\\
\lesssim&\|U\|_{\dot H^1}^{\frac{k+\frac12}{k+1}}\|U\|_{\dot H^{k+2}}^{2-\frac{k+\frac12}{k+1}},
\end{aligned}$$

then by integration by parts and $\nabla\cdot U=0$, it implies that the nonlinear term satisfies

$$\sum_{|\alpha|=k+1}\int \partial^\alpha U\cdot(\partial^\alpha(U\cdot\nabla U))\,dy \le C\|U\|_{\dot H^{k+1}}\sum_{0\le i\le k}\Big\||D^{k+1-i}U|\cdot|D^{i+1}U|\Big\|_{L^2}$$

$$\le C\|U\|_{\dot H^{k+1}}\|U\|_{\dot H^1}^{\frac{k+\frac12}{k+1}}\|U\|_{\dot H^{k+2}}^{2-\frac{k+\frac12}{k+1}} \le C\|U\|_{\dot H^1\cap\dot H^{k+1}}^{\frac{2k+\frac32}{k+1}}\|U\|_{\dot H^{k+2}}^{\frac{k+\frac32}{k+1}} \le \frac14\|U\|_{\dot H^{k+2}}^2 + C\|U\|_{\dot H^1\cap\dot H^{k+1}}^{\frac{4k+3}{k+\frac12}}.$$

In addition, for the buoyancy term, by bootstrap assumption (3.5) and (3.6),

$$\|D^{k+1}\Psi\|_{L^2} = \|D^{k+1}(Q+\tilde\varepsilon_s+\tilde\varepsilon_u)\|_{L^2}\lesssim 1.$$

In summary, together with (3.9),

$$\begin{aligned}\frac{d}{d\tau}\frac12\|U\|_{\dot H^{k+1}}^2 &\le -\frac{2k+1}{4}\|U\|_{\dot H^{k+1}}^2 + C\|U\|_{\dot H^1\cap\dot H^{k+1}}^{\frac{4k+3}{k+\frac12}} + C\mu\|U\|_{\dot H^{k+1}}\\ &\le -\frac18\|U\|_{\dot H^{k+1}}^2 + C\|U\|_{\dot H^1\cap\dot H^{k+1}}^3 + C\mu^2\\ &\le -\frac18\|U\|_{\dot H^{k+1}}^2 + C\delta_4^3 e^{-\frac38\tau} + C\mu_0^2e^{-\tau}.\end{aligned}$$

which is (3.11).

Estimate of (3.12). Applying Gronwall's inequality onto (3.10) and (3.11), then together with the conditions given in Proposition 3.2,

$$\begin{aligned}\|U\|_{\dot H^1\cap\dot H^{k+1}}^2 &= \|U(\tau)\|_{\dot H^1}^2 + \|U(\tau)\|_{\dot H^{k+1}}^2\\ &\le e^{-\frac14\tau}\|U(0)\|_{\dot H^1\cap\dot H^{k+1}}^2 + C\int_0^\tau e^{-\frac14(\tau-s)}\left(\delta_4^3e^{-\frac38 s}+\mu_0^2e^{-s}\right)ds\\ &\le C(\delta_0^2+\delta_4^3+\mu_0^2)e^{-\frac14\tau} \le \frac14\delta_4^2e^{-\frac14\tau}.\end{aligned}$$

□

3.1.2. *Semigroup estimate for* $\|\tilde\varepsilon_s\|_{H^k}$.

Lemma 3.5 (Semigroup estimate of $\|\tilde\varepsilon_s\|_{H^k}$)**.** *Under the assumptions of Proposition 3.2, the bootstrap assumption (3.4) can be improved to*

$$\|\tilde\varepsilon_s(\tau)\|_{H^k} \le \frac12\delta_1 e^{-\frac{\delta_g}{2}\tau},\quad \forall\,\tau\in[0,\tau^*]. \tag{3.14}$$

Proof. By Duhamel's principle, $\tilde\varepsilon_s$ can be solved as

$$\tilde\varepsilon_s(\tau) = e^{-\tau\mathcal L}\tilde\varepsilon_s(0) + \int_0^\tau e^{-(\tau-s)\mathcal L}\left(-\tilde P_s\mathcal L\tilde\varepsilon_u + \tilde P_s\left(N(\varepsilon)-U\cdot\nabla\Psi\right)\right)(s)ds,$$

then by Proposition 2.12, this yields that

$$\|\tilde\varepsilon_s(\tau)\|_{H^k} \le Ce^{-\frac{\delta_g\tau}{2}}\|\tilde\varepsilon_s(0)\|_{H^k} + \int_0^\tau e^{-\frac{\delta_g}{2}(\tau-s)}\|-\tilde P_s\mathcal L\tilde\varepsilon_u + \tilde P_s\left(N(\varepsilon)-U\cdot\nabla\Psi\right)\|_{H^k}ds.$$

For $-\tilde P_s\mathcal L\tilde\varepsilon_u$, using the fact that $\mathcal L: H^{k+2}\to H^k$ is bounded, the boundedness of $\tilde P_s$ in H^k (Lemma 2.11) and the bootstrap assumption (3.6),

$$\|-\tilde P_s\mathcal L\tilde\varepsilon_u\|_{H^k}\lesssim\|\mathcal L\tilde\varepsilon_u\|_{H^k}\lesssim\|\tilde\varepsilon_u\|_{H^{k+2}}\lesssim\|\tilde\varepsilon_u\|_{\tilde B}\lesssim\delta_3e^{-\frac{7\delta_g}{10}\tau}.$$

For the nonlinear term $\tilde P_sN(\varepsilon)$, we apply the bilinear estimate (A.3) and bootstrap assumptions (3.4), (3.5), (3.6) to find

$$\begin{aligned}\|\tilde P_sN(\varepsilon)\|_{H^k} &\lesssim \|N(\varepsilon)\|_{H^k}\lesssim\|\tilde\varepsilon_s\|_{H^{k+1}}^2+\|\tilde\varepsilon_u\|_{H^{k+1}}^2\lesssim\|\tilde\varepsilon_s\|_{H^{k+1}}^2+\|\tilde\varepsilon_u\|_{\tilde B}^2\\ &\lesssim(\delta_1+\delta_2)^2e^{-\delta_g\tau}+\delta_3^2e^{-\frac75\delta_g\tau}.\end{aligned}\tag{3.15}$$

When it comes to the advection term, by interpolation inequality, bootstrap assumptions (3.4), (3.5) and (3.6) together with (3.9), the $\dot{H}^k$ norm can be controlled as follows:

$$\begin{aligned}
\|D^k\left(U\cdot\nabla\Psi\right)\|_{L^2} &\lesssim \sum_{i=0}^{k}\left\||D^iU|\cdot|D^{k+1-i}\Psi|\right\|_{L^2}\\
&\lesssim \|U\|_{L^\infty}\|D^{k+1}\Psi\|_{L^2}+\sum_{i=1}^{k}\|D^iU\|_{L^4}\|D^{k+1-i}\Psi\|_{L^4}\\
&= \|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\|D^{k+1}\Psi\|_{L^2}+\sum_{i=1}^{k}\|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\|\Psi\|_{L^2\cap\dot{W}^{k,4}}\\
&\lesssim \|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\|\Psi\|_{H^{k+1}}\lesssim \|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}.
\end{aligned}$$

A similar argument also implies that

$$\|U\cdot\nabla\Psi\|_{L^2}\le\|U\|_{L^\infty}\|\nabla\Psi\|_{L^2}\lesssim\|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\|\Psi\|_{H^{k+1}}\lesssim\|U\|_{\dot{H}^1\cap\dot{H}^{k+1}},$$

so together with Lemma 2.11, we have obtained the H^k estimate of the advection term by

$$\|\tilde{P}_s\left(-U\cdot\nabla\Psi\right)\|_{H^k}\le\|U\cdot\nabla\Psi\|_{H^k}\lesssim\|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\lesssim\delta_4e^{-\frac{1}{8}\tau},\quad\forall\tau\in[0,\tau^*]. \tag{3.16}$$

In summary, with $\delta_g<\frac{1}{16}$, we conclude that

$$\begin{aligned}
\|\tilde{\varepsilon}_s(\tau)\|_{H^k} &\le Ce^{-\frac{\delta_g\tau}{2}}\|\tilde{\varepsilon}_s(0)\|_{H^k}+C\int_0^\tau e^{-\frac{\delta_g}{2}(\tau-s)}\left(\delta_3e^{-\frac{7\delta_g}{10}s}+(\delta_1+\delta_2)^2e^{-\delta_gs}+\delta_3^2e^{-\frac{7}{5}\delta_gs}+\delta_4e^{-\frac{1}{8}s}\right)ds\\
&\le Ce^{-\frac{\delta_g\tau}{2}}\left(\delta_0+(\delta_1+\delta_2+\delta_3)^2+\delta_3+\delta_4\right)\le\frac{1}{2}\delta_1e^{-\frac{\delta_g\tau}{2}},\quad\forall\tau\in[0,\tau^*].
\end{aligned}$$

□

3.1.3. *Energy estimate* $\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}}$.

Lemma 3.6 (Energy estimate $\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}}$)**.** *Under the assumptions of Proposition 3.2, there exists a constant $C>0$ such that*

$$\frac{d}{d\tau}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}}\le-\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}}+C\left(\delta_1+\delta_3+\delta_4\right)^2e^{-\delta_g\tau},\quad\forall\tau\in[0,\tau^*]. \tag{3.17}$$

Furthermore, the constant in (3.5) can be improved to

$$\|\tilde{\varepsilon}_s(\tau)\|_{\dot{H}^{k+1}}\le\frac{1}{2}\delta_2e^{-\frac{1}{2}\delta_g\tau},\quad\forall\tau\in[0,\tau^*]. \tag{3.18}$$

Proof. Step 1. Linear estimates. By integration by parts, interpolation inequality together with (2.12),

$$\begin{aligned}
(-\mathcal{L}_0\tilde{\varepsilon}_s,\tilde{\varepsilon}_s)_{\dot{H}^{k+1}} &= (\Delta\tilde{\varepsilon}_s,\tilde{\varepsilon}_s)_{\dot{H}^{k+1}}-\left(\frac{1}{2}\Lambda\tilde{\varepsilon}_s,\tilde{\varepsilon}_s\right)_{\dot{H}^{k+1}}=-\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}}-\frac{2k+3}{4}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}},\\
(\nabla\cdot(\tilde{\varepsilon}_s\nabla\Delta^{-1}Q),\tilde{\varepsilon}_s)_{\dot{H}^{k+1}} &= (D^{k+1}\nabla\cdot(\tilde{\varepsilon}_s\nabla\Delta^{-1}Q),D^{k+1}\tilde{\varepsilon}_s)_{L^2}=-(D^{k+1}(\tilde{\varepsilon}_s\nabla\Delta^{-1}Q),\nabla D^{k+1}\tilde{\varepsilon}_s)_{L^2}\\
&= -(D^{k+1}\tilde{\varepsilon}_s\nabla\Delta^{-1}Q,\nabla D^{k+1}\tilde{\varepsilon}_s)_{L^2}+O(\|\tilde{\varepsilon}_s\|_{H^k}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}})\\
&= \frac{1}{2}\int Q|D^{k+1}\tilde{\varepsilon}_s|^2dx+O(\|\tilde{\varepsilon}_s\|_{H^k}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}})\le C\|\tilde{\varepsilon}_s\|_{H^k}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}},\\
(\nabla\cdot(Q\nabla\Delta^{-1}\tilde{\varepsilon}_s),\tilde{\varepsilon}_s)_{\dot{H}^{k+1}} &= -(D^{k+1}(Q\nabla\Delta^{-1}\tilde{\varepsilon}_s),\nabla D^{k+1}\tilde{\varepsilon}_s)_{L^2}\le C\|\tilde{\varepsilon}_s\|_{H^k}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}},
\end{aligned}$$

which implies that

$$
\begin{aligned}
-(\mathcal{L}\tilde{\varepsilon}_s,\tilde{\varepsilon}_s)_{\dot{H}^{k+1}} &\leq -\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}} - \frac{2k+3}{4}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + C\|\tilde{\varepsilon}_s\|_{H^k}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}} \\
&\leq -\frac{7}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}} - \frac{2k+3}{4}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + C\|\tilde{\varepsilon}_s\|^2_{H^k}.
\end{aligned}
$$

Here we used $\tilde{\varepsilon}_s(\tau) \in H^{k+2}$ under the bootstrap assumption again by the local wellposedness Theorem A.1.

Step 2. Nonlinear estimates. For $-\tilde{P}_s\mathcal{L}\tilde{\varepsilon}_u$, by Cauchy-Schwarz inequality, the definition of $\tilde{B}$ norm,

$$
\left(-\tilde{P}_s\mathcal{L}\tilde{\varepsilon}_u,\tilde{\varepsilon}_s\right)_{\dot{H}^{k+1}} \leq C\|\mathcal{L}\tilde{\varepsilon}_u\|_{H^{k+1}}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}} \leq C\|\tilde{\varepsilon}_u\|_{\tilde{B}}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}} \leq \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + C\|\tilde{\varepsilon}_u\|^2_{\tilde{B}}.
$$

When it comes to the nonlinear term $\tilde{P}_sN(\varepsilon)$, by Sobolev embedding inequality and bilinear estimate (A.3), one gets

$$
\|\tilde{P}_sN(\varepsilon)\|_{\dot{H}^{k+1}} \lesssim \|\varepsilon\|^2_{H^{k+1}} + \|\varepsilon\|_{H^k}\|\varepsilon\|_{\dot{H}^{k+2}} \lesssim \|\tilde{\varepsilon}_u\|^2_{\tilde{B}} + \|\tilde{\varepsilon}_s\|^2_{H^k} + \|\varepsilon\|_{H^k}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}},
$$

which, by bootstrap assumptions (3.4), (3.5), (3.6), Cauchy-Schwarz inequality and Young's inequality, indicates that

$$
\begin{aligned}
\left(\tilde{P}_sN(\varepsilon),\tilde{\varepsilon}_s\right)_{\dot{H}^{k+1}} &\leq C\left(\|\tilde{\varepsilon}_u\|^2_{\tilde{B}} + \|\tilde{\varepsilon}_s\|^2_{H^k} + \|\varepsilon\|_{H^k}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}}\right)\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}} \\
&\leq C\left(\|\tilde{\varepsilon}_u\|^2_{\tilde{B}} + \|\tilde{\varepsilon}_s\|^2_{H^k}\right)\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}} + C\left(\|\tilde{\varepsilon}_s\|_{H^k} + \|\tilde{\varepsilon}_u\|_{H^k}\right)^{\frac{3}{2}}\|\tilde{\varepsilon}_s\|^{\frac{3}{2}}_{\dot{H}^{k+2}} \\
&\leq \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}} + C\left(\|\tilde{\varepsilon}_u\|^4_{\tilde{B}} + \|\tilde{\varepsilon}_s\|^4_{H^k}\right).
\end{aligned}
$$

Moreover, as for the advection term, similar to the argument of (3.16), under the bootstrap assumptions in Proposition 3.2, we conclude that

$$
\|\tilde{P}_s(-U\cdot\nabla\Psi)\|_{\dot{H}^{k+1}} \lesssim \|U\cdot\nabla\Psi\|_{H^{k+1}} \lesssim \|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\left(1+\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}}\right),
$$

which yields that

$$
\begin{aligned}
&\left|(\tilde{P}_s(U\cdot\nabla\Psi),\tilde{\varepsilon}_s)_{\dot{H}^{k+1}}\right| \leq C\|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}}\left(1+\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+2}}\right) \\
&\leq C\|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\|\tilde{\varepsilon}_s\|_{\dot{H}^{k+1}} + C\|U\|_{\dot{H}^1\cap\dot{H}^{k+1}}\|\tilde{\varepsilon}_s\|^{\frac{1}{2}}_{H^k}\|\tilde{\varepsilon}_s\|^{\frac{3}{2}}_{\dot{H}^{k+2}} \\
&\leq \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}} + C\|U\|^2_{\dot{H}^1\cap\dot{H}^{k+1}} + C\|\tilde{\varepsilon}_s\|^2_{H^k}.
\end{aligned}
$$

Step 3. Conclusion. In summary, from (3.1), combining all the estimates given above,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} &= -(\mathcal{L}\tilde{\varepsilon}_s,\tilde{\varepsilon}_s)_{\dot{H}^{k+1}} - \left(\tilde{P}_s\mathcal{L}\tilde{\varepsilon}_u,\tilde{\varepsilon}_s\right)_{\dot{H}^{k+1}} + \left(\tilde{P}_s\left(N(\varepsilon)-U\cdot\nabla\Psi\right),\tilde{\varepsilon}_s\right)_{\dot{H}^{k+1}} \\
&\leq -\frac{7}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}} - \frac{2k+3}{4}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + C\|\tilde{\varepsilon}_s\|^2_{H^k} + \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + C\|\tilde{\varepsilon}_u\|^2_{\tilde{B}} \\
&\quad + \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}} + C\left(\|\tilde{\varepsilon}_u\|^4_{\tilde{B}} + \|\tilde{\varepsilon}_s\|^4_{H^k}\right) \\
&\quad + \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + \frac{1}{8}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+2}} + C\|U\|^2_{\dot{H}^1\cap\dot{H}^{k+1}} + C\|\tilde{\varepsilon}_s\|^2_{H^k} \\
&\leq -\frac{k}{2}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + C\left(\|U\|^2_{\dot{H}^1\cap\dot{H}^{k+1}} + \|\tilde{\varepsilon}_s\|^2_{H^k} + \|\tilde{\varepsilon}_u\|^2_{\tilde{B}}\right) \\
&\leq -\frac{k}{2}\|\tilde{\varepsilon}_s\|^2_{\dot{H}^{k+1}} + C\left(\delta_1+\delta_3+\delta_4\right)^2 e^{-\delta_g\tau},
\end{aligned}
$$

which is (3.17). Hence by Gronwall's inequality,

$$\begin{aligned}\|\tilde{\varepsilon}_s(\tau)\|^2_{\dot{H}^{k+1}} &\leq e^{-\tau}\|\tilde{\varepsilon}_s(0)\|^2_{\dot{H}^{k+1}} + C\int_0^\tau e^{-(\tau-s)}\left(\delta_1+\delta_3+\delta_4\right)^2 e^{-\delta_g s}ds \\ &\leq C(\delta_0+\delta_1+\delta_3+\delta_4)^2 e^{-\delta_g\tau} \leq \frac{1}{4}\delta_2^2 e^{-\delta_g\tau},\end{aligned}$$

where the last inequality holds from the conditions of $\{\delta_i\}$ given in Proposition 3.2. □

3.2. **Control of unstable mode.**

Proof of Proposition 3.2.
Improvement of the bootstrap assumptions. By contradiction, we assume that there exists $(\mu_0, \tilde{\varepsilon}_{s0}, U_0)$ satisfying (3.3), such that for any $\tilde{\varepsilon}_{u0}$ with $\|\tilde{\varepsilon}_{u0}\|_{\tilde{B}} \leq \delta_3$, the exit time τ^* defined in (3.8) is finite. Then according to Lemma 3.4, Lemma 3.5, and Lemma 3.6, combined with the continuity of the solution to system (3.1) with respect to τ, there exists $0 < \tau_\epsilon \ll 1$ such that (3.4), (3.5), and (3.7) hold for all $\tau \in [0, \tau^* + \tau_\epsilon]$.

Outgoing flux property. With the argument above, we see that the only scenario for the solution to exit the bootstrap regime is when the bootstrap assumption for the unstable part $\tilde{\varepsilon}_u$ given in (3.6) does not hold for $\tau > \tau^*$. Besides, from the local wellposedness of (KS-NS) given in Theorem A.1, we can see that $\tau \mapsto \tilde{\varepsilon}_u(\tau)$ is continuous. Hence, if we define

$$\tilde{\mathcal{B}}(\tau) = \{v \in \tilde{H}^k_u : \|v\|_{\tilde{B}} \leq \delta_3 e^{-\frac{7}{10}\delta_g\tau}\},$$

then from the previous contradiction assumption,

$$\begin{cases}\tilde{\varepsilon}_u(\tau) \in \tilde{\mathcal{B}}(\tau), & \forall\, \tau \in [0,\tau^*], \\ \tilde{\varepsilon}_u(\tau^*) \in \partial\mathcal{B}(\tau^*), & \tau = \tau^*.\end{cases}$$

Moreover, from (2.42) and (3.1),

$$\begin{aligned}\frac{d}{d\tau}\frac{1}{2}\|\tilde{\varepsilon}_u\|^2_{\tilde{B}} &= \left(-\tilde{P}_u\mathcal{L}\tilde{P}_u\tilde{\varepsilon}_u, P_u\right)_{\tilde{B}} + \left(\tilde{P}_u\left(-U\cdot\nabla\Psi + N(\varepsilon)\right), \tilde{\varepsilon}_u\right)_{\tilde{B}} \\ &\geq -\frac{6\delta_g}{10}\|\tilde{\varepsilon}_u\|^2_{\tilde{B}} + \left(\tilde{P}_u\left(-U\cdot\nabla\Psi + N(\varepsilon)\right), \tilde{\varepsilon}_u\right)_{\tilde{B}}.\end{aligned}$$

Similar to the estimate of (3.15), by bootstrap assumptions (3.4), (3.5) and (3.6), the nonlinear part can be controlled by

$$\begin{aligned}|(\tilde{P}_u N(\varepsilon), \tilde{\varepsilon}_u)_{\tilde{B}}| &\lesssim \|\tilde{P}_u N(\varepsilon)\|_{\tilde{B}}\|\tilde{\varepsilon}_u\|_{\tilde{B}} \lesssim \|N(\varepsilon)\|_{H^k}\|\tilde{\varepsilon}_u\|_{\tilde{B}} \\ &\lesssim \left(\|\tilde{\varepsilon}_s\|^2_{H^{k+1}} + \|\tilde{\varepsilon}_u\|^2_{\tilde{B}}\right)\|\tilde{\varepsilon}_u\|_{\tilde{B}} \\ &\lesssim \left((\delta_1^2+\delta_2^2)e^{-\delta_g\tau} + \delta_3^2 e^{-\frac{7}{5}\delta_g\tau}\right)\delta_3 e^{-\frac{7}{10}\delta_g\tau} \lesssim \delta_2^2\delta_3 e^{-\frac{17}{10}\delta_g\tau}, \quad \forall\tau\in[0,\tau^*].\end{aligned}$$

For the advection term, from the estimate obtained in (3.16), by Cauchy-Schwarz inequality and bootstrap assumption (3.6), one concludes that

$$|(\tilde{P}_u\left(U\cdot\nabla\Psi\right), \tilde{\varepsilon}_u)_{\tilde{B}}| \lesssim \|U\cdot\nabla\Psi\|_{H^k}\|\tilde{\varepsilon}_u\|_{\tilde{B}} \lesssim \delta_3\delta_4 e^{-\left(\frac{1}{8}+\frac{7\delta_g}{10}\right)\tau}, \quad \forall\tau\in[0,\tau^*].$$

In summary, we conclude the outer going flux property of $\tilde{\varepsilon}_u$ as follows:

$$\begin{aligned}
&\frac{d}{d\tau}\Big|_{\tau=\tau^*}\left(e^{\frac{7}{5}\delta_g\tau}\|\tilde{\varepsilon}_u\|_{\tilde{B}}^2\right)\\
&\geq \frac{\delta_g}{5}e^{\frac{7}{5}\delta_g\tau}\|\tilde{\varepsilon}_u\|_{\tilde{B}}^2 - C\delta_3\left(\delta_2^2 e^{-\frac{3\delta_g}{10}\tau}+\delta_4 e^{-\frac{1}{16}\tau}\right)\Big|_{\tau=\tau^*}\\
&\geq \frac{\delta_g}{5}\delta_3^2 - C\delta_3(\delta_2^2+\delta_4) \geq \frac{\delta_g}{10}\delta_3^2 > 0. \qquad (3.19)
\end{aligned}$$

Brouwer's topological argument. By the local well-posedness theory and (3.19), we have the continuity of the mapping $\tilde{\varepsilon}_{u0} \mapsto \tilde{\varepsilon}_u(\tau^*(\tilde{\varepsilon}_{u0}))$, where $\tau^*(\tilde{\varepsilon}_{u0})$ denotes the exit time with initial data $\tilde{\varepsilon}_{u0}$. Subsequently, we define a continuous map $\Phi : \overline{B_{\tilde{B}}(0,1)} \to \partial B_{\tilde{B}}(0,1)$ as follows:

$$f \in \overline{B_{\tilde{B}}(0,1)} \mapsto \tilde{\varepsilon}_{u0} := \delta_3 f \in \overline{B_{\tilde{B}}(0,\delta_3)} \mapsto \tilde{\varepsilon}_u(\tau^*) \in \partial\tilde{\mathcal{B}}(\tau^*) \mapsto \frac{\tilde{\varepsilon}_u(\tau^*)}{\|\tilde{\varepsilon}_u(\tau^*)\|_{\tilde{B}}} \in \partial B_{\tilde{B}}(0,1),$$

where $B_{\tilde{B}}(0,1)$ represents the unit ball centered at the origin in the finite-dimensional space $(\tilde{H}_u^k, |\cdot|_{\tilde{B}})$. Particularly, when $f \in \partial B_{\tilde{B}}(0,1)$, i.e., $\delta_3 f \in \partial B_{\tilde{B}}(0,\delta_3)$, according to the outer going flux property of the flow as indicated in (3.19), $\tau = 0$ is the exit time, thus leading to $\Phi(f) = f$. In essence, $\Phi = Id$ on the boundary $\partial B_{\tilde{B}}(0,1)$. However, upon applying Brouwer's fixed-point theorem to $-\Phi$ on $B_{\tilde{B}}(0,1)$, we conclude the existence of a fixed point of $-\Phi$ on the boundary $\partial B_{\tilde{B}}(0,1)$, contradicting the assertion that $\Phi = Id$ on the boundary. Consequently, we have established Proposition 3.2. □

3.3. **Existence of a finite blowup solution to (KS-NS) with finite mass.**

Proof of Theorem 1.1.
Construction of initial datum. Firstly, we choose an arbitrary integer $s \geq 3$, an arbitrary divergence-free vector field $u_0 \in H_\sigma^\infty$. If we set $k = s-1$ and $\delta_g \ll 1$ in Proposition 3.2, then there exist constants $R \gg 1$ from Definition 2.10 and $\{\delta_i\}_{0\leq i\leq 4}$ satisfying (3.2), such that Proposition 3.2 holds true.

For the initial velocity field $u_0 \in H_\sigma^\infty$ and the constant $0 < \delta_0 \ll 1$ determined above, we choose the initial data of scaling parameter $\mu_0 \in (0, \frac{1}{8}\delta_0)$ sufficiently small, such that the initial renormalized velocity field $U_0 := \mu_0 u_0(\mu_0\cdot)$ satisfies

$$\|U_0\|_{\dot{H}^1\cap\dot{H}^s}^2 = \|\mu_0 u_0(\mu_0\cdot)\|_{\dot{H}^1\cap\dot{H}^s}^2 = \mu_0\|u_0\|_{\dot{H}^1}^2 + \mu_0^{2s-1}\|u_0\|_{\dot{H}^s}^2 \leq \left(\frac{1}{8}\delta_0\right)^2.$$

In addition, for Q given by (Profile), we choose $R_0 \gg R \gg 1$ sufficiently large such that

$$\begin{aligned}
&\left\|\tilde{P}_s\left((1-\chi_{R_0})Q\right)\right\|_{H^s} + \left\|\tilde{P}_u\left((1-\chi_{R_0})Q\right)\right\|_{H^s}\\
&\leq C\|(1-\chi_{R_0})Q\|_{H^s} \leq \frac{1}{8}\delta_0 \ll Q(2R), \qquad (3.20)
\end{aligned}$$

hence Proposition 3.2 holds with the initial datum

$$(\mu_0, \tilde{\varepsilon}_{s0}, U_0) = \left(\mu_0, -\tilde{P}_s\left((1-\chi_{R_0})Q\right), \mu_0 u_0(\mu_0\cdot)\right), \qquad (3.21)$$

which yields that there exists $\tilde{\varepsilon}_{u0} \in \tilde{H}_u^k$ such that the solution $(\mu, \tilde{\varepsilon}_s, \tilde{\varepsilon}_u, U)$ to (3.1) globally exists and satisfies (3.4), (3.5), (3.6) and (3.7) on $[0,+\infty)$. In other words, in the parabolic zone, the solution (μ, Ψ, U) to (2.4) with initial datum (μ_0, Ψ_0, U_0), where Ψ_0 is defined by

$$\Psi_0 = Q + \tilde{\varepsilon}_{s0} + \tilde{\varepsilon}_{u0} = Q - \tilde{P}_s\left((1-\chi_{R_0})Q\right) + \tilde{\varepsilon}_{u0} = \chi_{R_0}Q + \tilde{P}_u\left((1-\chi_{R_0})Q\right) + \tilde{\varepsilon}_{u0}, \tag{3.22}$$

globally exists and the corresponding solution $\Psi(\tau)$ can be decomposed into $\Psi(\tau) = Q + \varepsilon(\tau)$ with

$$\begin{aligned}\|\varepsilon(\tau)\|_{H^s} &\le \|\tilde{\varepsilon}_s(\tau)\|_{H^s} + \|\tilde{\varepsilon}_u(\tau)\|_{H^s}\\ &\le C\left(\|\tilde{\varepsilon}_s(\tau)\|_{H^s} + \|\tilde{\varepsilon}_u(\tau)\|_{\tilde{B}}\right) \le C(\delta_1+\delta_2+\delta_3)e^{-\frac{1}{2}\delta_g\tau}, \quad \forall\, \tau \ge 0,\end{aligned}$$

for some universal constant $C > 0$.

<u>*Non-negativity and finite-mass of the solution.*</u> Observing from (3.22) and recalling Lemma 2.11 that $\tilde{\varepsilon}_{u0} \in \tilde{H}^{s-1}_u \subset C_0^\infty$, the initial data Ψ_0 constructed in (3.22) satisfies $\Psi_0 \in C_0^\infty \subset L^1$. Furthermore, according (3.2), (3.6) and (3.20), by Sobolev embedding inequality, there exists a universal constant $C > 0$ such that

$$\|\tilde{P}_u\left((1-\chi_{R_0})\,Q\right) + \tilde{\varepsilon}_{u0}\|_{L^\infty} \le \|\tilde{P}_u\left((1-\chi_{R_0})\,Q\right)\|_{H^s} + \|\tilde{\varepsilon}_{u0}\|_{H^{s-1}} \le C(\delta_0+\delta_3) \ll Q(2R).$$

Hence with the previous argument, the radial decreasing of profile Q and the fact that $R_0 \gg R$, within the $2R$-ball $B(0, 2R)$, which contains the support of $\tilde{P}_u\left((1-\chi_{R_0})\,Q\right) + \tilde{\varepsilon}_{u0}$, we obtain that

$$\begin{aligned}\Psi_0(y) &= \chi_{R_0}Q + \tilde{P}_u\left((1-\chi_{R_0})\,Q\right) + \tilde{\varepsilon}_{u0}\\ &\ge Q(2R) - \|\tilde{P}_u\left((1-\chi_{R_0})\,Q\right) + \tilde{\varepsilon}_{u0}\|_{L^\infty} \ge \frac{1}{2}Q(2R) > 0, \quad \forall\, |y| \le 2R.\end{aligned}$$

In addition, outside of $2R$-ball, we have $\Psi_0(y) = \chi_{R_0}Q \ge 0$ when $|y| \ge 2R$. Consequently, we conclude the non-negativity of the initial density Ψ_0.

If we go back to the original system (KS-NS), by (2.1) and (2.2), then the related initial datum become as follows:

$$(\rho_0, u_0) = \left(\frac{1}{\mu_0^2}\Psi_0\left(\frac{x}{\mu_0}\right), u_0\right),$$

where $\rho_0 \in C_0^\infty \in L^1$. And by the local-wellposedness theory (see Theorem A.1), this ensures that the corresponding solution $(\rho(t), u(t)) \in H^\infty \times H^\infty_\sigma$ throughout its lifespan.

<u>*Finite blowup rate.*</u> From (2.1), by chain rule,

$$\frac{d\mu}{dt} = \frac{d\mu}{d\tau}\frac{d\tau}{dt} = -\frac{\mu}{2}\frac{1}{\mu^2} = -\frac{1}{2\mu}, \quad \Rightarrow \quad \frac{d}{dt}\mu^2 = -1.$$

This yields that the solution would blow up at finite time $T = \mu_0^2$ and μ can be explicitly solved by

$$\mu(t) = \sqrt{\mu_0^2 - t} = \sqrt{T-t}, \quad \forall\, t \in [0, T).$$

Hence we have finished the proof of Theorem 1.1.

$\square$

Appendix A. Local-wellposedness to Keller-Segel-Navier-Stokes system

Theorem A.1 (Local wellposedness)**.** *Fix $(\rho_0, u_0) \in H^m(\mathbb{R}^3) \times H^m_\sigma(\mathbb{R}^3)$ for $m \geq 2$. Then there exist a small constant $T_1 = T_1(\|\rho_0\|_{H^{\max\{2,m-1\}}}, \|u_0\|_{H^{\max\{2,m-1\}}}) > 0$ such that there exists a unique solution $(\rho, u) \in C([0,T_1], H^m) \times C([0,T_1], H^m_\sigma)$ to* (KS-NS)*. Moreover, if ρ_0 is nonnegative, then $\rho(t,x)$ stays nonnegative on $[0,T_1]$. If the maximal existence time $T^* = T^*(\rho_0, u_0) < \infty$, then*

$$\lim_{t\to T^*} \left(\|\rho(t)\|_{H^2} + \|u(t)\|_{\dot H^1\cap \dot H^2}\right) = \infty. \tag{A.1}$$

Proof. This is a standard application of semigroup estimate and contraction mapping principle, see for [5, Chapter 5] for the Navier-Stokes case.

We write (KS-NS) in the Duhamel form

$$\begin{cases} \rho = e^{t\Delta}\rho_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot\left(-u\rho + \rho\nabla\Delta^{-1}\rho\right) ds, \\ u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\left(-\nabla\cdot(u\otimes u) + \rho e_3\right) ds, \end{cases} \tag{A.2}$$

where we use the incompressiblity to rewrite $u\cdot\nabla\rho = \nabla\cdot(u\rho)$ and to apply the Leray projection $\mathbb{P}$ (see [5, Section 1.3.5]) to remove the pressure term. Noticing that for $m \geq 2$, with $m_* := \max\{2, m-1\}$, the aggregation term of Keller-Segel equation can be controlled by

$$\begin{aligned} \|\rho\nabla\Delta^{-1}\rho\|_{H^m} &\lesssim \|\rho\|_{H^m}\|\nabla\Delta^{-1}\rho\|_{L^\infty} + \|\rho\|_{L^\infty}\|\nabla\Delta^{-1}\rho\|_{\dot H^1\cap\dot H^m} \\ &\quad + \sum_{\substack{|\alpha|+|\beta|=m,\\ 1\leq|\alpha|,|\beta|\leq m-1}} \|\partial^\alpha\rho\|_{L^4}\|\partial^\beta\nabla\Delta^{-1}\rho\|_{L^4} \\ &\lesssim \|\rho\|_{H^m}\|\rho\|_{H^{m_*}}, \end{aligned} \tag{A.3}$$

for any $\rho \in H^m$ with $m \geq 2$. And similarly

$$\|fg\|_{H^m} \lesssim_m \|f\|_{H^m}\|g\|_{\dot H^1\cap\dot H^{m_*}} + \|g\|_{H^m}\|f\|_{\dot H^1\cap\dot H^{m_*}},$$

for any $f, g \in H^m$ with $m \geq 2$.

We apply the boundedness of $\mathbb{P}$ in H^m, smoothing estimate of the heat kernel $\|e^{t\Delta}\|_{H^m\to H^{m+1}} \lesssim t^{-\frac12}$, together with the boundedness of Leray projector $\mathbb{P}$ in H^m (see [4, Lemma 1.15]) to obtain

$$\begin{aligned} &\left\|\int_0^t e^{(t-s)\Delta}\nabla\cdot\left(-u\rho + \rho\nabla\Delta^{-1}\rho\right) ds\right\|_{H^m} \\ \lesssim\ &\int_0^t (t-s)^{-\frac12}\left(\|u\rho\|_{H^m} + \|\rho\nabla\Delta^{-1}\rho\|_{H^m}\right)(s)ds \\ \lesssim\ &t^{\frac12}\left[\left(\|\rho\|_{L^\infty_s H^m_x} + \|u\|_{L^\infty_s H^m_x}\right)\|\rho\|_{L^\infty_s H^{m_*}_x} + \|\rho\|_{L^\infty_s H^m_x}\|u\|_{L^\infty_s(\dot H^1_x\cap\dot H^{m_*}_x)}\right] \end{aligned}$$

and

$$\begin{aligned} &\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\left(-\nabla\cdot(u\otimes u) + \rho e_3\right) ds\right\|_{H^m} \\ \lesssim\ &\int_0^t (t-s)^{-\frac12}\left(\|u\otimes u\|_{H^m} + \|\rho\|_{H^m}\right)(s)ds \\ \lesssim\ &t^{\frac12}\left(\|u\|_{L^\infty_s H^m_x}\|u\|_{L^\infty_s(\dot H^1_x\cap\dot H^{m_*}_x)} + \|\rho\|_{L^\infty_s H^m_x}\right). \end{aligned}$$

It is then easy to verify that (A.2) has a unique solution for short time by contraction mapping principle, with the time interval only depend on $\|u\|_{L^\infty_t(\dot H^1_x\cap\dot H^{m_*}_x)} + \|\rho\|_{L^\infty_t H^{m_*}_x} + 1$. With $m \geq 2$, this becomes a classical solution to (KS-NS), and the persistence of non-negativity of the density ρ is ensured by the strong maximum principle (see [54, Lemma 2.1]). Noticing that the lifespan depending on one less order regularity of the solution, the blowup criterion (A.1) now follows this local wellposedness and induction on $m \geq 2$. $\square$

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