

Network shell structure based on hub and non-hub nodes

Gaogao Dong,^{1,2,3} Nannan Sun,¹ Fan Wang,⁴ and Renaud Lambiotte^{1,5}

¹*School of Mathematical Sciences, Jiangsu University,
212013 Zhenjiang, Jiangsu, People's Republic of China*

²*Emergency Management Institute, Jiangsu University,
212013 Zhenjiang, Jiangsu, People's Republic of China*

³*Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems,
Nanjing Normal University, 210023 Zhenjiang, Jiangsu, People's Republic of China*

⁴*Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel*

⁵*Mathematical Institute, University of Oxford, Woodstock Rd, Oxford OX2 6GG, UK*

Abstract

The shell structure holds significant importance in various domains such as information dissemination, supply chain management, and transportation. This study focuses on investigating the shell structure of hub and non-hub nodes, which play important roles in these domains. Our framework explores the topology of Erdős-Rényi (ER) and Scale-Free (SF) networks, considering source node selection strategies dependent on the nodes' degrees. We define the shell l in a network as the set of nodes at a distance l from a given node and represent r_l as the fraction of nodes outside shell l . Statistical properties of the shells are examined for a selected node, taking into account the node's degree. For a network with a given degree distribution, we analytically derive the degree distribution and average degree of nodes outside shell l as functions of r_l . Moreover, we discover that r_l follows an iterative functional form $r_l = \phi(r_{l-1})$, where ϕ is expressed in terms of the generating function of the original degree distribution of the network.

1. INTRODUCTION

Complex systems, spanning diverse domains such as social networks, biological networks, and information propagation networks, are effectively abstracted and represented using network models. In these networks, nodes serve as representations of individuals or entities in the system, while links capture their interconnections and relationships. It is notable that network structures frequently demonstrate a hierarchical organization, wherein nodes are systematically grouped into shells or layers based on their connectivity patterns [1–6]. This hierarchical arrangement offers a profound

insight into the inherent organizational principles governing the network's behavior [7–10]. The investigation of shell structures becomes paramount in this context, enabling researchers to uncover critical details regarding how information, influence, or disruptions traverse through distinct layers of the network [11–14]. The network's shell structure, defined as the set of nodes at varying distances from a randomly selected root node, holds paramount importance in diverse domains like information dissemination, supply chain management, and transportation [15–19]. Particularly in processes involving the propagation of information or viruses from a root node to subsequent nodes, the structure of the shell is intricately linked to the network's degree distribution and diameter [20–23].

Root nodes, as foundational elements in the shell structure of complex networks, hold paramount significance in delineating the structural hierarchy [16]. They typically initiate network growth or information dissemination, influencing subsequent development and connectivity patterns within the network. The hierarchical structure in the network under random root nodes was studied by Shao et al [12]. However, as the starting point for the formation of subsequent shells or layers, the targeted selection of root nodes holds undeniable significance. In contexts such as information dissemination or epidemiological networks, targeted selection of hub nodes can accelerate information flow and curb disease transmission [24, 25]. In biological networks, key proteins or genes may influence tumor cell proliferation, invasion, and metastasis, offering insights into disease pathogenesis and treatment [26]. In economic networks, entities with significant economic scale and influence serve as bridges connecting different markets, industries, or regions, facilitating the rapid flow of information, capital, goods, and services, while also necessitating careful supervision to prevent market dominance or systemic risks [27]. However, smaller-degree non-hub nodes are often overlooked, yet their presence is crucial for a comprehensive understanding of the diversity and complexity of network structures. Proteins or metabolites with specific biological functions play a critical role in understanding biological processes and disease mechanisms based on their positions and connectivity patterns in biological network [28]. Similarly, stations located in remote or low-population-density areas ensure the universality and accessibility of transportation services [29]. In these real networks, hub nodes facilitate global connectivity and fast propagation, while non-hub nodes focus on local services and resilience. This underscores the importance of in-depth understanding of characteristics of different node types to design more robust and efficient networks [30, 31]. Therefore, choosing the root node as the starting point for the hierarchical structure of the network should not be overlooked [32].

Building upon the influence of the root node, our investigation will explore the implications of selecting different root nodes on the network's various hierarchical structures and overall topology.

In this paper, we develop a mathematical framework to understand the shell structure of hub and non-hub root nodes, which depends on the degree of nodes. We denote the number of nodes l away from the root node as B_l , then the fraction of nodes greater than l is $r_l \equiv 1 - \frac{1}{N} \sum_{m=0}^l B_m$, and we define r external as the r_N nodes with the largest distance from a root node. Here, we put forth a theory to elucidate the behavior of the degree distribution $P_r(k)$ in E_r and the behaviour of the average degree $\langle k(r^\alpha) \rangle$ as a function of r in connected networks. Furthermore, we analytically derive r_l as a function of r_{l+1} with different point selection strategies. We also observe that in SF networks, as both r and α decrease, the degree distribution of the remaining nodes gradually transitions from a power-law distribution to a Poisson distribution.

2. DEGREE DISTRIBUTION OF NODES IN THE r EXTERIOR E_r

A. Branching process

According to the probability formula $W_\alpha(k_i) = \frac{k_i^\alpha}{\sum_{i=1}^N k_i^\alpha}$, the root node is selected to generate the network. Fig. 1 illustrates the neighbour nodes and the generation process of network structure in each layer. By examining this process, we can gain insights into how the network evolves based on the selection of the root node. It can be observed that when the root node is a hub node (red dot in Fig. 1 A), or a non-hub root node (red dot in Fig. 1 B), there are significant differences in the topology of the two. These differences not only impact the distribution of nodes within each layer but also influence the overall connectivity and structure of the network. Understanding these variations is crucial for comprehending how different types of nodes contribute to the network's functionality and resilience.

B. Degree distribution and average degree of nodes in the r exterior E_r

After the root node is selected, as the growth process unfolds, the degree distribution among the remaining nodes undergoes alterations through the incessant incorporation of nodes into the aggregation. In this section, we present and address differential equations to elucidate these dynamic transformations. Let $A_r(k)$ be the number of nodes with degree k , and $P_r(k)$ signify the new degree distribution of nodes within fraction r of the network,

$$P_r(k) = \frac{A_r(k)}{rN}. \quad (1)$$

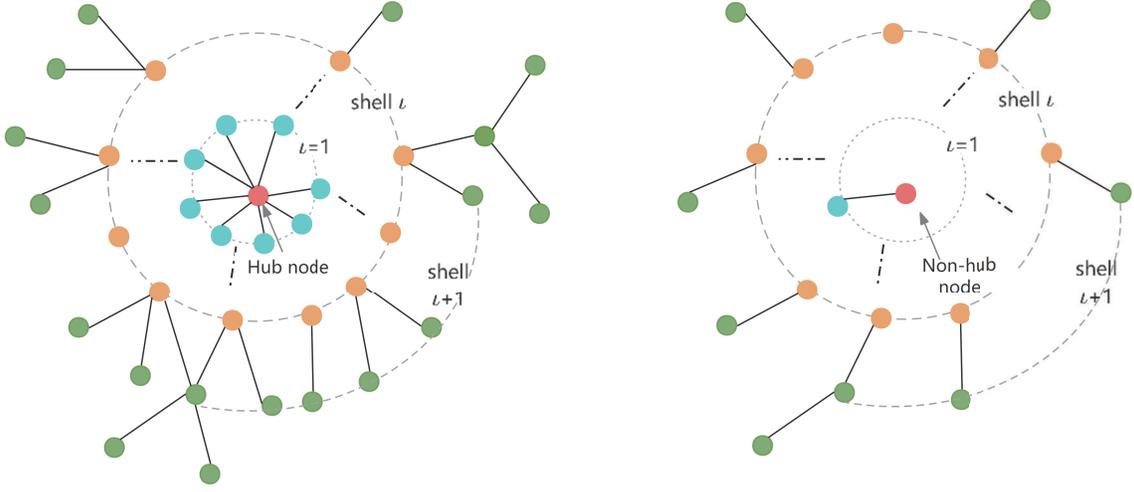


FIG. 1: Schematic diagrams of each layer structure with different types of root nodes. Red dots are root nodes, blue dots are first level neighbour nodes, orange dots are nodes l away from root node. (A) Root node is a hub node, (B) root node is a non-hub node.

When an link is established from the aggregate to a free node, the alteration in $A_r(k)$ is encapsulated by the expression $A_{r-1/N}(k) = A_r(k) - P_r(k)k^\alpha / \langle k(r)^\alpha \rangle$. In the limit as $N \rightarrow \infty$, the above equation can be reformulated in terms of the derivative of $A_r(k)$ with respect to r ,

$$\frac{dA_r(k)}{dr} \approx N[A_r(k) - A_{r-1/N}(k)] = N \frac{P_r(k)k^\alpha}{\langle k(r)^\alpha \rangle}, \quad (2)$$

where $\langle k(r)^\alpha \rangle \equiv \sum_{k=0}^{\infty} P_r(k)k^\alpha$ Upon differentiating Eq. 1 with respect to r and incorporating Eq. 2, we derive the following expression

$$-r \frac{dP_r(k)}{dr} = P_r(k) - \frac{P_r(k)k^\alpha}{\langle k(r)^\alpha \rangle}, \quad (3)$$

which strictly requires $N \rightarrow \infty$. To solve Eq. 3, one defines the function $G_\alpha(x) = \sum_{k=0}^{\infty} P(k)x^{k^\alpha}$, and substitutes $f = G_\alpha^{-1}(r)$. Hence, we can get the degree distribution of nodes within fraction of r

$$P_r(k) = P(k) \frac{f^{k^\alpha}}{G_\alpha(f)} = \frac{1}{r} P(k) f^{k^\alpha}, \quad (4)$$

and we can get the average degree of the r fraction of the node

$$\langle k(r^\alpha) \rangle = \sum_{k=0}^{\infty} P_r(k)k = \frac{1}{r} \sum_{k=0}^{\infty} P(k)k f^{k^\alpha}. \quad (5)$$

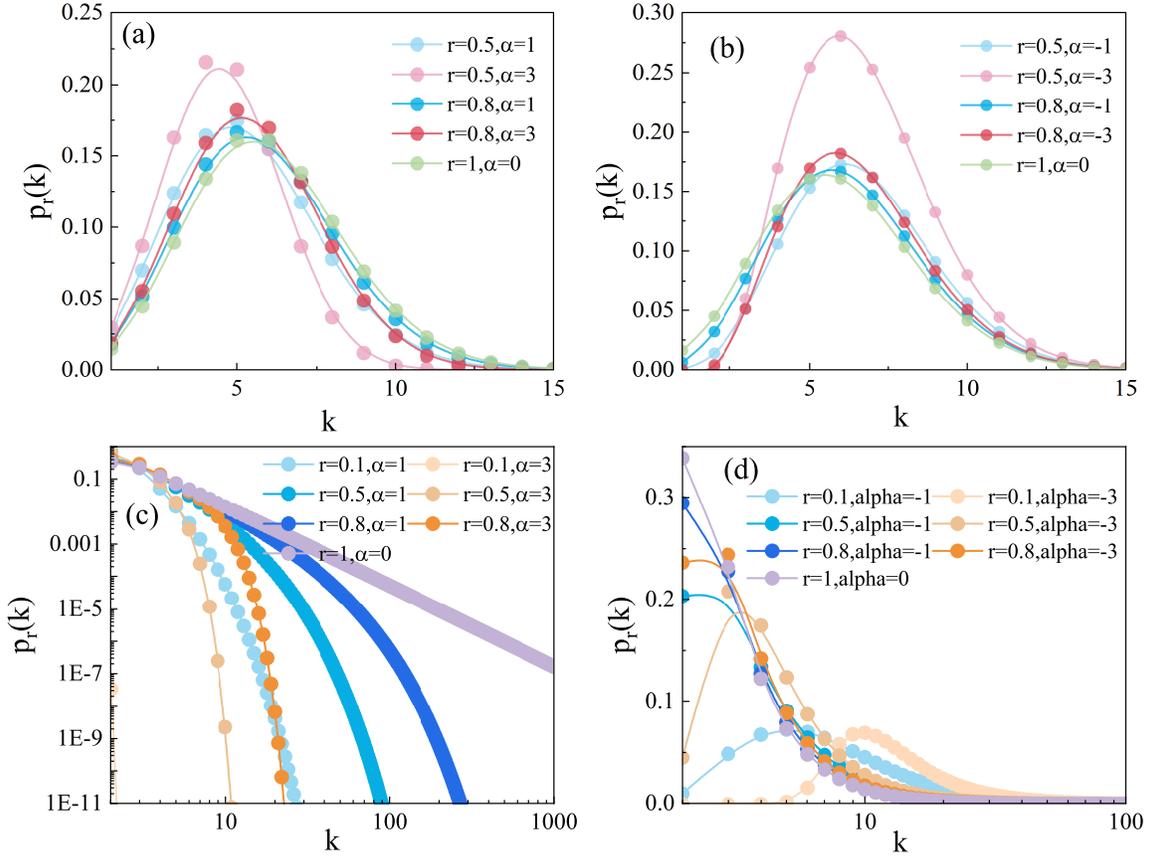


FIG. 2: The degree distribution $P_r(k)$ of r fraction nodes for different r , α and networks. ER network with $N = 10^6$, $\langle k \rangle = 6$ and $r = 0.5, 0.8$ and 1 . SF network with $N = 10^6$, $\lambda = 2.5$, $\langle k \rangle = 6$, $k_{min} = 2$ and $r = 0.1, 0.5, 0.8$ and 1 . (a) $\alpha = 0, 1, 3$, (b) $\alpha = 0, -1, -3$. (c) $\alpha = 0, 1, 3$, (d) $\alpha = 0, -1, -3$. The simulation results in symbols agree very well with the theoretical prediction lines of Eq. 4.

We conducted numerical tests on the theoretical Erdős-Rényi (ER) network with $N = 10^6$ nodes, varying the values of $\langle k \rangle$ which is the average degree. In order to receive $P_r(k)$, we initiated the process by selecting a root node which depends on the degree of the node and identifying the nodes within fraction r along with their degree distribution $P_r(k)$. This procedure was iterated across different roots and network realizations. The results of the calculations are shown in Fig. 2. The analytical results (full lines) are computed using Eq. 4.

We can find that the numerical values and simulations match well, and from Fig. 2(a) we can find that when $\alpha > 0$ that is according to the W_α probability to select the node with larger degree value as the source node, we can find that different r corresponding to the $P_r(k)$ all obey Poisson

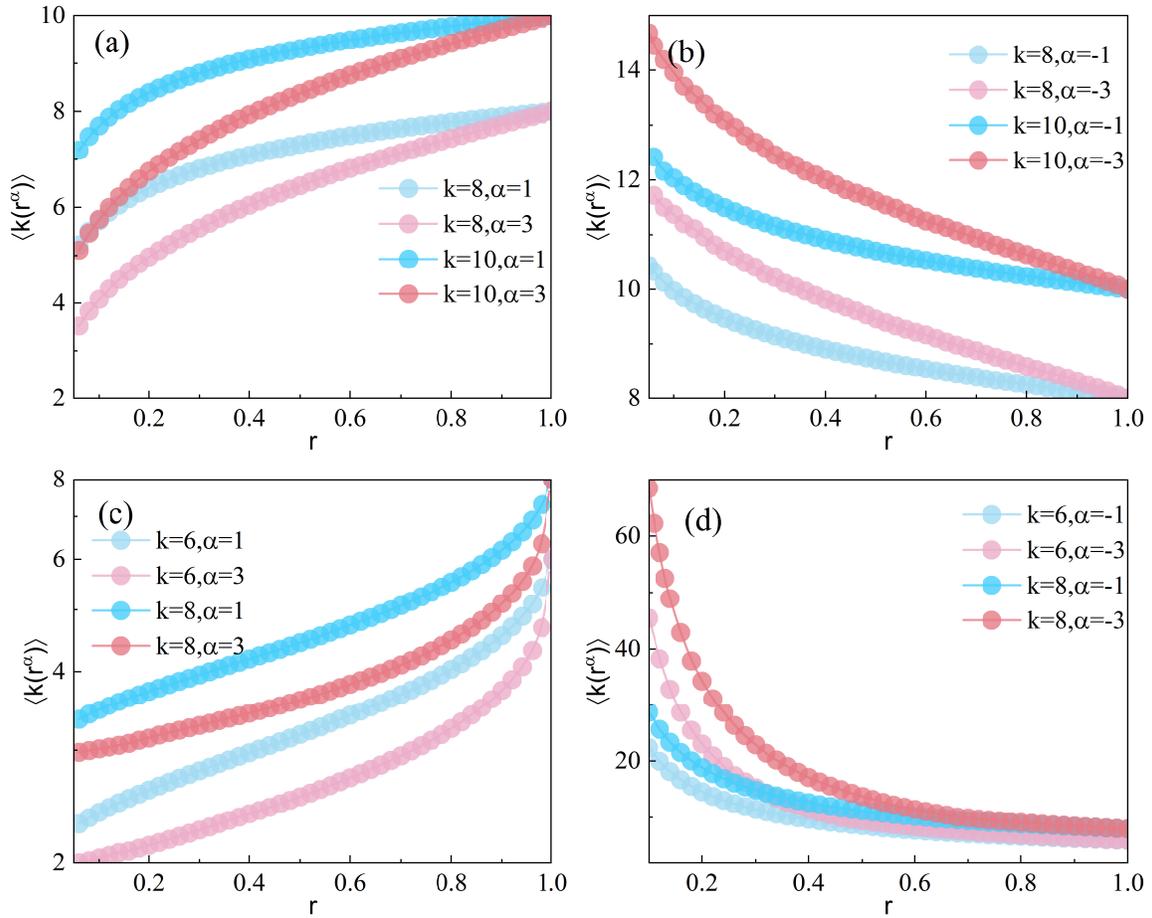


FIG. 3: Comparison between the simulation result and the theoretical prediction for the degree distribution $\langle k(r^\alpha) \rangle$ in E_r fraction of nodes. ER network with $\langle k \rangle = 6$. (a) $\alpha > 0$, (b) $\alpha < 0$. SF network with $\lambda = 2.5$. (c) $\alpha > 0$, (d) $\alpha < 0$. The simulation results symbols agree very well with the theoretical predictions lines of Eq. 5.

distribution. With the same r , it can be clearly found that the larger α corresponds to the smaller average degree of the network, and at the same time the larger average degree corresponds to the larger $P_r(k)$. Comparing Fig. 2(a) to 2(b), we can find that the change of α has little effect on the average degree of the network for different r fractions. However, as r and α decrease, the degree distribution $P_r(k)$ corresponding to the network average degree $\langle k \rangle$ becomes gradually larger.

For SF networks adheres to a power-law degree distribution, $P(k) = \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(k_{max}+1)^{1-\lambda} - k_{min}^{1-\lambda}}$, where k_{min} and k_{max} denote the minimum and maximum degree values, λ is the power-law exponent. The generating function of degree distribution $G(x) = \sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(k_{max}+1)^{1-\lambda} - k_{min}^{1-\lambda}} x^k$.

We find that the numerical calculations are in good agreement with the simulation results. From Fig. 2(c), we could observe that the degree distribution of the network obeys a power law distribution at different r when $\alpha > 0$. When $\alpha < 0$ and the r decrease, the degree distribution of the network transforms from obeying a power-law distribution to an approximate Poisson distribution as r decreases as shown in Fig. 2(d). We compared our theory with the simulations also for other values of r and $\langle k(r^\alpha) \rangle$ and the agreement is also excellent. We can find in Fig. 2(d) that the degree distribution of the remaining nodes gradually changes from obeying a power-law distribution to obeying a Poisson distribution as r decreases.

From Fig.3(a) we find that as r decreases gradually, at this point $\langle k(r^\alpha) \rangle$ also becomes smaller. At the same time, for the same average degree k , at $\alpha > 0$, the corresponding $\langle k(r^\alpha) \rangle$ becomes smaller when α is larger. However, for $\alpha < 0$, the corresponding $\langle k(r^\alpha) \rangle$ becomes larger when α is larger, as shown in Fig.3(b). At the same time, as r gets smaller, $\langle k(r^\alpha) \rangle$ will gradually increase. We can find similar results in SF networks as in Figs.3(c) and (d).

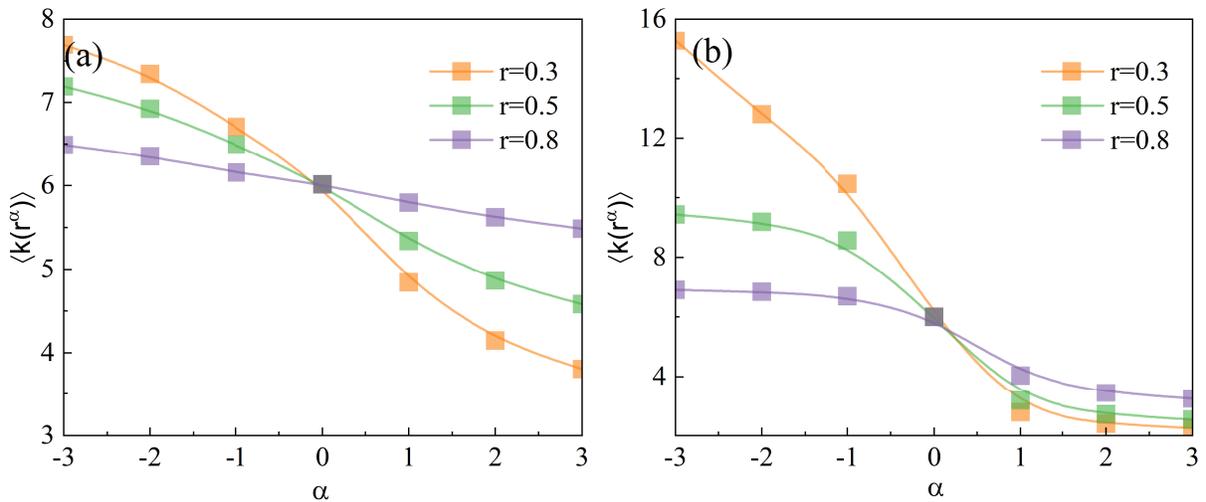


FIG. 4: Comparison between the simulation result and the theoretical prediction for the degree distribution $\langle k(r^\alpha) \rangle$ as a function of α . (a) ER network with $\langle k \rangle = 6$, (b) SF network with $\lambda = 2.5, \langle k \rangle = 6$. Simulation results (symbols) agree well with theoretical results (solid lines).

In Fig.4(a), we also notice that when $\alpha > 0$, the corresponding r is smaller and $\langle k(r^\alpha) \rangle$ is larger. In the opposite case when $\alpha < 0$, the smaller the corresponding r , the smaller the $\langle k(r^\alpha) \rangle$. However, when $\alpha = 0$, then different r corresponds to the same $\langle k(r^\alpha) \rangle$, because the nodes are depending on the degree of the node selected when $\alpha = 0$. In Fig.4(b) we could find the same conclusion as

Fig.4(a).

C. Iterative functional of r_l

In this section, a comprehensive investigation will be undertaken to clarify the recursive relationship between two consecutive shells. Let $S(t)$ be the number of open links at step t , where $S(t) = v(t)N$ and $v(t)$ represents the number of open connections emerging at step t . The number of open links belonging to shell l of the aggregate is defined as $S_l(t)$, where $S_l(t) = v_l(t)N$ and $v_l(t)$ denotes the number of open connections emerging from layer l . Before we begin selecting shell $l+1$, all the open links in the aggregate belong to nodes in shell l , so at $t = t_l$, we have $v_l(t) = v(t)$. And in the process of selecting shell $l+1$, $v_l(t)$ decreases to 0 [12].

At moment t , the links of the remaining r fraction of nodes can be expressed as $rN \langle k[r(t)^\alpha] \rangle$. Then at step t , the probability of connecting to the next shell nodes can be expressed as $\frac{r(t)\langle k[r(t)^\alpha] \rangle}{r(t)\langle k[r(t)^\alpha] \rangle + v[r(t)]}$. Hence in the next moment we get the expression

$$\frac{dr(t)}{dt} = Nr(t+1) - Nr(t) = -\frac{1}{N} \frac{r \langle k[r(t)^\alpha] \rangle}{r \langle k[r(t)^\alpha] \rangle + v[r(t)]}. \quad (6)$$

The probability of connecting to a node at the same shell can be written as $\frac{v[r(t)]}{r(t)\langle k[r(t)^\alpha] \rangle + v[r(t)]}$, thus one can get

$$\frac{dv(t)}{dt} = v(t+1) - v(t) = -\frac{1}{N} + \frac{\tilde{k}[r(t)^\alpha]}{N} \frac{r \langle k[r(t)^\alpha] \rangle}{r(t) \langle k[r(t)^\alpha] \rangle + v[r(t)]} - \frac{1}{N} \frac{v[r(t)]}{r \langle k[r(t)^\alpha] \rangle + v[r(t)]}. \quad (7)$$

By solving Eq. 6 and 7, we obtain the differential equation as a function of r ,

$$\frac{dv(r)}{dr} = -\tilde{k}(r)^\alpha + 1 + \frac{2v(r)}{r \langle k(r^\alpha) \rangle}, \quad (8)$$

where $\tilde{k}(r)^\alpha = \frac{\sum_{k=0}^{\infty} k^2 P_r(k)}{\langle k(r^\alpha) \rangle} - 1$ is the branching factor of a node in the r external E_r [12]. The behaviour of $v_l(t)$ is similar to that of $v(t)$, so it can be written as

$$\frac{dv_l(t)}{dt} = v_l(t+1) - v_l(t) = -\frac{1}{N} - \frac{1}{N} \frac{v_l[r(t)]}{r \langle k[r(t)^\alpha] \rangle + v_l[r(t)]}. \quad (9)$$

By using the formula Eq. 6 and 9, we can derive

$$\frac{dv_l(r)}{dr} = 1 + \frac{v(r)}{r \langle k(r^\alpha) \rangle} + \frac{v_l(r)}{r \langle k(r^\alpha) \rangle}. \quad (10)$$

Substituting $f = G_\alpha^{-1}(r)$, into Eq. 8 , the general solution can be expressed as

$$v(f) = -G'_\alpha(f)f + c_1f^2, \quad (11)$$

where c_1 is an arbitrary constant. In the initial state, with the remaining nodes in the network at $r = 1$, and given that $f = 1$, and $v(1) = 0$, we can deduce that $c_1 = G'_\alpha(1)$. Building upon this outcome, we can further ascertain

$$v_l(f) = G'_\alpha(1)f^2 + c_2f, \quad (12)$$

where c_2 is an arbitrary constant.

When $r = r_l$, at this time $v(r) = v_l(r)$. We define $f_l \equiv G_\alpha^{-1}(r_l)$, $c_2 = -G'_\alpha(f_l)$, Eq.11 and Eq.12 can be rewritten as

$$\begin{cases} v(f) = G'_\alpha(1)f^2 - G'_\alpha(f)f, \\ v_l(f) = G'_\alpha(1)f^2 - G'_\alpha(f_l)f. \end{cases} \quad (13)$$

With the increasing number of selected nodes, the number of open connections emerging v of the remaining shell gradually diminishes until it asymptotically approaches zero, $v(f_\infty) = 0$. We get

$$f_\infty = \frac{G'_\alpha(f_\infty)}{G'_\alpha(1)}. \quad (14)$$

From the formula $f = G_\alpha^{-1}(r)$, one can obtain

$$r_\infty = G_\alpha(f_\infty). \quad (15)$$

It implies the potential existence of a fraction of distant links and nodes that remain unattached to the aggregates upon the completion of the construction process. When $v \rightarrow 0$, the construction of the shell $l + 1$ is complete with $r_l = r_{l+1}$ and $f_l = f_{l+1}$. Subsequently, employing Eq. 15 yields

$$r_{l+1} = G_\alpha(f_{l+1}) = G_\alpha(G'_\alpha(G_\alpha^{-1}(r_l))/G'_\alpha(1)) \quad (16)$$

With Eq. 16 we can get the relationship between the proportion of nodes in shell l and the proportion of nodes in shell $l + 1$ under different point selection strategies i.e., different α . Fig. 5 gets the relationship between the fraction r_l of nodes within shell l and the proportion of nodes in shell r_l . We can find that in ER network, the larger the average degree of the network, the larger the proportion of r_l under the same number of shells. At the same average degree, when $\alpha > 0$, the

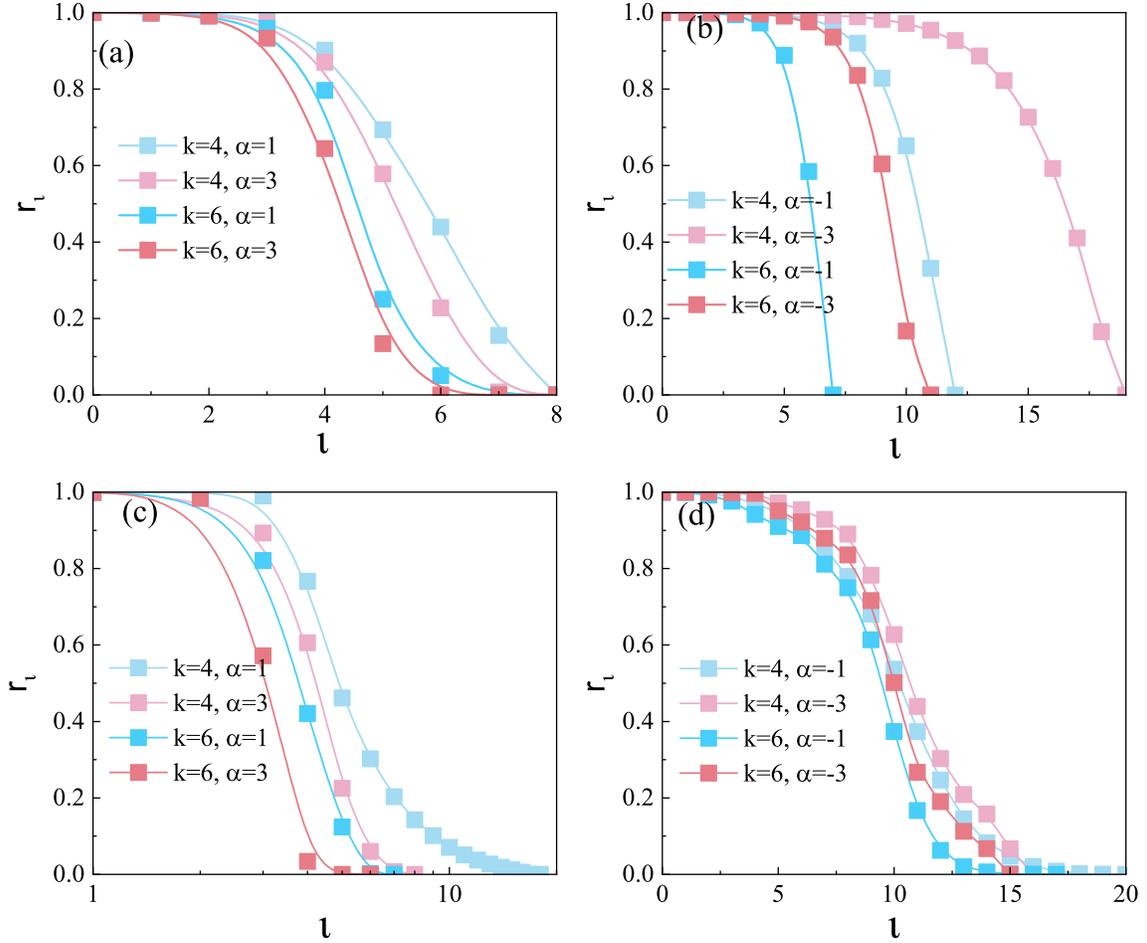


FIG. 5: Fraction of nodes r_l as a function of shell l for for different α and networks. ER network with $k = 4, k = 6$ (a) $\alpha > 0$, (b) $\alpha < 0$; SF network with $\lambda = 2.5, k = 4, k = 6$ (c) $\alpha > 0$, (d) $\alpha < 0$.

fraction of nodes outside the shell l mean that r_l , gradually decreases with the increase of α under the same shell l ; at the same time, we also find that when $\alpha < 0$, the r_l becomes larger with the decrease of α under the same shell l as shown in Fig. 5(a) and (b). Fig. 5(c) and (d) depict the r_l a function of l relationship under the SF network, and we can find that has similar results with the ER network.

4. CONCLUSION

In this paper, we analyse the surrounding neighbourhood of the root node under different α by means of probabilistic formulas as an important method of selecting the root node. We derive analytical relations describing shell properties of a network and study the statistical properties of

the shells of a chosen node that depend on the degree of the node. Meanwhile, we find how the degree distribution is depleted as we approach the boundaries of the network which consist of the r fraction of the most distant node from a root node. Concurrently in the SF network we find that the degree distribution of the remaining part of the nodes gradually changes from a power-law distribution to a Poisson distribution as r and α decreases. It is also found that an explicit analytical expression for the degree distribution as a function of r . One also derive an explicit analytical relation between the values of r for two successive shells l and $l + 1$, as well as relations between the node proportions r_l within the l shell for the function of the l shell.

ACKNOWLEDGMENTS

This research is supported by grants from the National Natural Science Foundation of China (Grant No. 62373169), National Statistical Science Research Project (Grant No. 2022LZ03), Special Project of Emergency Management Institute of Jiangsu University (Grant No. KY-A-08), the National Key Research and Development Program of China (Grant No. 2020YFA0608601), and the Jiangsu Postgraduate Research and Innovation Plan (Grant No. KYCX22_3601).

-
- [1] Xuqing Huang, Jianxi Gao, Sergey V Buldyrev, Shlomo Havlin, and H Eugene Stanley. Robustness of interdependent networks under targeted attack. *Physical Review E*, 83(6):065101, 2011.
 - [2] Ming Li, Run-Ran Liu, Linyuan Lü, Mao-Bin Hu, Shuqi Xu, and Yi-Cheng Zhang. Percolation on complex networks: Theory and application. *Physics Reports*, 907:1–68, 2021.
 - [3] Stefano Boccaletti, Ginestra Bianconi, Regino Criado, Charo I Del Genio, Jesús Gómez-Gardenes, Miguel Romance, Irene Sendina-Nadal, Zhen Wang, and Massimiliano Zanin. The structure and dynamics of multilayer networks. *Physics reports*, 544(1):1–122, 2014.
 - [4] Xueming Liu, Daqing Li, Manqing Ma, Boleslaw K Szymanski, H Eugene Stanley, and Jianxi Gao. Network resilience. *Physics Reports*, 971:1–108, 2022.
 - [5] Amir Bashan, Yehiel Berezin, Sergey V Buldyrev, and Shlomo Havlin. The extreme vulnerability of interdependent spatially embedded networks. *Nature Physics*, 9(10):667–672, 2013.
 - [6] Lucas D Valdez, Louis Shekhtman, Cristian E La Rocca, Xin Zhang, Sergey V Buldyrev, Paul A Trunfio, Lidia A Braunstein, and Shlomo Havlin. Cascading failures in complex networks. *Journal of Complex Networks*, 8(2):cnaa013, 2020.
 - [7] Yangyang Liu, Hillel Sanhedrai, GaoGao Dong, Louis M Shekhtman, Fan Wang, Sergey V Buldyrev, and Shlomo Havlin. Efficient network immunization under limited knowledge. *National Science Review*, 8(1):nwaa229, 2021.
 - [8] Xiangyi Meng, Xinqi Hu, Yu Tian, Gaogao Dong, Renaud Lambiotte, Jianxi Gao, and Shlomo Havlin. Percolation theories for quantum networks. *Entropy*, 25(11):1564, 2023.
 - [9] Yucheng Hao, Limin Jia, and Yanhui Wang. Edge attack strategies in interdependent scale-free networks. *Physica A: Statistical Mechanics and its Applications*, 540:122759, 2020.
 - [10] Gaogao Dong, Yanting Luo, Yangyang Liu, Fan Wang, Huanmei Qin, and André LM Vilela. Percolation behaviors of a network of networks under intentional attack with limited information. *Chaos, Solitons & Fractals*, 159:112147, 2022.

- [11] Hao Peng, Zhen Qian, Zhe Kan, Dandan Zhao, Juan Yu, and Jianmin Han. Cascading failure dynamics against intentional attack for interdependent industrial internet of things. *Complexity*, 2021:1–15, 2021.
- [12] Jia Shao, Sergey V Buldyrev, Lidia A Braunstein, Shlomo Havlin, and H Eugene Stanley. Structure of shells in complex networks. *Physical Review E*, 80(3):036105, 2009.
- [13] Tomer Kalisky, Reuven Cohen, Osnat Mokryn, Danny Dolev, Yuval Shavitt, and Shlomo Havlin. Tomography of scale-free networks and shortest path trees. *Physical Review E*, 74(6):066108, 2006.
- [14] Jia Shao, Sergey V Buldyrev, Reuven Cohen, Maksim Kitsak, Shlomo Havlin, and H Eugene Stanley. Fractal boundaries of complex networks. *Europhysics Letters*, 84(4):48004, 2008.
- [15] Antonios Garas, Frank Schweitzer, and Shlomo Havlin. A k-shell decomposition method for weighted networks. *New Journal of Physics*, 14(8):083030, 2012.
- [16] Reuven Cohen and Shlomo Havlin. Complex networks: structure, robustness and function. 2010.
- [17] A Lachgar and A Achahbar. Shells structure in uncorrelated scale-free networks. *Physica A: Statistical Mechanics and its Applications*, 535:122407, 2019.
- [18] Sergey V Buldyrev, Roni Parshani, Gerald Paul, H Eugene Stanley, and Shlomo Havlin. Catastrophic cascade of failures in interdependent networks. *Nature*, 464(7291):1025–1028, 2010.
- [19] MEJ Newman. Networks: an introduction: Oxford university press.[google scholar]. 2010.
- [20] Federico Battiston, Giulia Cencetti, Iacopo Iacopini, Vito Latora, Maxime Lucas, Alice Patania, Jean-Gabriel Young, and Giovanni Petri. Networks beyond pairwise interactions: Structure and dynamics. *Physics Reports*, 874:1–92, 2020.
- [21] Leto Peel, Tiago P Peixoto, and Manlio De Domenico. Statistical inference links data and theory in network science. *Nature Communications*, 13(1):6794, 2022.
- [22] Scott Freitas, Diyi Yang, Srijan Kumar, Hanghang Tong, and Duen Horng Chau. Graph vulnerability and robustness: A survey. *IEEE Transactions on Knowledge and Data Engineering*, 35(6):5915–5934, 2022.
- [23] Wei Wang, Wenyao Li, Tao Lin, Tao Wu, Liming Pan, and Yanbing Liu. Generalized k-core percolation on higher-order dependent networks. *Applied Mathematics and Computation*, 420:126793, 2022.
- [24] Sibel A. Alumur, James F. Campbell, Ivan Contreras, Bahar Y. Kara, and Morton E. O’Kelly. Perspectives on modeling hub location problems. *European Journal of Operational Research*, 291(1):1–17, 2021.
- [25] Morton E. O’Kelly. Network hub structure and resilience. *Networks and Spatial Economics*, 15:235–251, 2015.
- [26] Emre Guney, J?Rg Menche, Marc Vidal, and Albert-László Barábasi. Network-based in silico drug efficacy screening. *Nature Communications*, 7:10331, 2016.
- [27] Gaogao Dong, Nan Wang, Fan Wang, Ting Qing, Yangyang Liu, and André LM Vilela. Network resilience of non-hub nodes failure under memory and non-memory based attacks with limited information. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 32(6):063110, 2022.
- [28] Weixia. Xu, Yunfeng. Dong, Jihong. Guan, and Shuigeng. Zhou. Identifying essential proteins from protein-protein interaction networks based on influence maximization. *BMC bioinformatics*, 23:339, 2022.
- [29] Sybil Derrible and Kennedy Christopher. The complexity and robustness of metro networks. *Physica A: Statistical Mechanics and its Applications*, 389:3678–3691, 2010.
- [30] Marco Bardoscia, Stefano Battiston, Fabio Caccioli, and Guido Caldarelli. Pathways towards instability in financial networks. *Nature Communications*, 8(1):1–9, 2016.
- [31] Run-Ran Liu, Chun-Xiao Jia, Ming Li, and Fanyuan Meng. A threshold model of cascading failure on random hypergraphs. *Chaos, Solitons & Fractals*, 173:113746, 2023.
- [32] Haggai Bonneau, Aviv Hassid, Ofer Biham, Reimer Kühn, and Eytan Katzav. Distribution of shortest cycle lengths in random networks. *Physical Review E*, 96(6):062307, 2017.