

ON THE LOCALITY OF FORMAL DISTRIBUTIONS OVER PRE-LIE AND NOVIKOV ALGEBRAS

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ABSTRACT. The Dong Lemma in the theory of vertex algebras states that the locality property of formal distributions over a Lie algebra is preserved under the action of a vertex operator. A similar statement is known for associative algebras. We study local formal distributions over pre-Lie (right-symmetric), pre-associative (dendriform), and Novikov algebras to show that the analogue of the Dong Lemma holds for Novikov algebras but does not hold for pre-Lie and pre-associative ones.

INTRODUCTION

Conformal algebras were introduced in [16] as a tool for the theory of vertex algebras in mathematical physics and representation theory. A keystone in the construction of a vertex algebra is the Dong Lemma stating that, given a system of pairwise mutually local formal distributions over a Lie algebra, all conformal n -products of these distributions are also mutually local. The Dong Lemma is also essential for constructing free (associative and Lie) conformal algebras since it is enough to fix locality on the generators to define a universal structure [27].

Associative and Lie conformal algebras, as well as Jordan conformal algebras are well studied in a series of papers (see, e.g., [6, 9, 17, 19, 29]). From the categorial point of view, if a class \mathfrak{M} of “ordinary” algebras over a field \mathbb{k} is presented by morphisms of operads $\mathcal{O}_{\mathfrak{M}} \rightarrow \text{Vec}_{\mathbb{k}}$, where $\mathcal{O}_{\mathfrak{M}}$ is the operad corresponding to the class \mathfrak{M} and $\text{Vec}_{\mathbb{k}}$ is the multi-category of linear spaces over \mathbb{k} , then \mathfrak{M} -conformal algebras are presented by morphisms $\mathcal{O}_{\mathfrak{M}} \rightarrow \mathbb{k}[\partial]\text{-mod}$, where $\mathbb{k}[\partial]\text{-mod}$ is the multi-category (pseudo-tensor category) of modules over the polynomial Hopf algebra (see [2] for details).

In this way, other varieties of conformal algebras appear naturally, and the study of their structure and relations between different varieties is an interesting algebraic problem. The class of Poisson conformal algebras was studied in [21], right-symmetric (pre-Lie) conformal algebras were introduced in [15] along with Novikov conformal algebras. The latter class turns to be closely related with Novikov–Poisson algebras, and the study of Novikov conformal algebras helps solving problems in this area [22]. Dendriform (pre-associative) conformal algebras appeared in [14, 28], they are related to deformation theory problems for conformal algebras.

Recall that the class of right-symmetric algebras consists of all pairs (V, \circ) , where V is a linear space and \circ is a bilinear operation on V , $(x, y) \mapsto x \circ y$, such that

$$(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y) \quad (1)$$

for all $x, y, z \in V$. If, in addition, the identity

$$x \circ (y \circ z) = y \circ (x \circ z) \quad (2)$$

holds on V then (V, \circ) is said to be a Novikov algebra. This notion emerged in [11] and [3], where the identities (1), (2) (or their opposite versions) were used to describe relations on the coordinates of a rank 3 tensor appeared in different studies in functional analysis and partial differential equations.

For example, if A is a commutative (and associative) algebra with a derivation $d : A \rightarrow A$ then the same space A equipped with the new operation $(x, y) \mapsto x \circ y = d(x)y$ is a Novikov

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algebra denoted $A^{(d)}$. It was shown in [8] and [5] that for every Novikov algebra V there exists a commutative algebra A with a derivation d such that V is isomorphic to a subalgebra of $A^{(d)}$. A similar statement was derived in [22] for quadratic Novikov conformal algebras, those related with Novikov–Poisson algebras. In order to study this embedding problem more precisely and generally, the notion of a free Novikov conformal algebra is needed. To that end, an analogue of the Dong Lemma for formal distributions over Novikov algebras is crucially important.

The purpose of this paper is to establish an analogue of the Dong Lemma for Novikov algebras and show that this statement does not hold for pre-Lie and pre-associative algebras.

Throughout the paper, \mathbb{k} is a field of characteristic zero and \mathbb{Z}_+ stands for the set of non-negative integers.

1. FORMAL DISTRIBUTIONS, LOCALITY, AND THE OPE FORMULA

Let A be an algebra, i.e., a linear space over a field \mathbb{k} , $\text{char } \mathbb{k} = 0$, equipped with a bilinear operation (multiplication) $\mu : A \times A \rightarrow A$, $\mu(a, b) = ab$, for $a, b \in A$. We do not assume the operation μ is associative or commutative.

A formal distribution over A is a two-side infinite formal power series

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a_n \in A, \quad (3)$$

where z is a formal variable. The space of all formal distributions over A is denoted $A[[z, z^{-1}]]$. In contrast to ordinary power series $A[[z]]$ or Lawrent series over A , it is in general impossible to multiply formal distributions due to potentially infinite sums emerging at coefficients. However, given two formal distributions $a(z), b(z) \in A[[z, z^{-1}]]$ as above, one may easily define the product

$$a(w)b(z) = \sum_{n, m \in \mathbb{Z}} a_n b_m w^{-n-1} z^{-m-1} \in A[[w, w^{-1}, z, z^{-1}]].$$

A pair $(a(z), b(z))$ of formal distributions over an algebra A is said to be *local* if there exists $N \in \mathbb{Z}_+$ such that

$$a(w)b(z)(w - z)^N = 0. \quad (4)$$

The minimal such N is said to be the *locality value* of the pair $(a(z), b(z))$.

Two formal distributions $a(z), b(z) \in A[[z, z^{-1}]]$ are said to be mutually local if $(a(z), b(z))$ and $(b(z), a(z))$ are local pairs.

Remark 1. Let ∂ be the formal derivation of formal distributions, i.e., if $a(z)$ is given by (3) then

$$(\partial a)(z) = \sum_{n \in \mathbb{Z}} (-n - 1) a_n z^{-n-2} = -n \sum_{n \in \mathbb{Z}} a_{n-1} z^{-n-1}.$$

If $a(w)b(z)(w - z)^N = 0$ then $(\partial a)(w)b(z)(w - z)^{N+1} = 0$ as well as $a(w)(\partial b)(z)(w - z)^{N+1} = 0$.

Example 1. Let A be the associative algebra generated by q, t, t^{-1} such that $qt - tq = 1$. (This is the algebra of differential operators on the algebra of Lawrent polynomials $\mathbb{k}[t, t^{-1}]$.) Then the formal distributions

$$q^{(m)}(z) = \sum_{n \in \mathbb{Z}} \frac{1}{m!} t^n q^m z^{-n-1} \in A[[z, z^{-1}]], \quad m \in \mathbb{Z}_+,$$

are pairwise mutually local and the locality value for $q^{(m)}(z), q^{(k)}(z)$ is equal to $N = m + 1$.

Example 2. Let V be a Novikov algebra with an operation $(a, b) \mapsto a \circ b$, $a, b \in V$. Then the space of Lawrent polynomials $V[t, t^{-1}]$ equipped with a new operation

$$[at^n, bt^m] = (n(b \circ a) - m(a \circ b))t^{m+n-1}, \quad a, b \in V, n, m \in \mathbb{Z},$$

is a Lie algebra, and the series of the form

$$a(z) = \sum_{n \in \mathbb{Z}} at^n z^{-n-1}, \quad a \in V,$$

are pairwise local with $N \leq 2$.

In particular, for 1-dimensional commutative (hence, Novikov) algebra $V = \mathbb{k}$ such a series generates the Virasoro conformal Lie algebra [6].

It follows by direct computations [18] that a pair of formal distributions $(a(z), b(z))$ over an algebra A is local if and only if there exists $N \in \mathbb{Z}_+$ such that

$$\sum_{s \in \mathbb{Z}_+} (-1)^s \binom{N}{s} a_{n-s} b_{m+s} = 0 \quad (5)$$

for all $n, m \in \mathbb{Z}$.

Given a local pair of formal distributions $a(z), b(z) \in A[[z, z^{-1}]]$, their λ -product is a polynomial in λ with coefficients in $A[[z, z^{-1}]]$ defined as follows:

$$(a(z) \underset{\lambda}{(}\ b(z)) = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} c^{(n)}(z),$$

where

$$c^{(n)}(z) = \underset{w=0}{\text{Res}} a(w)b(z)(w-z)^n, \quad n \in \mathbb{Z}_+. \quad (6)$$

Here $\underset{w=0}{\text{Res}}$ denotes the residue of a formal distribution, i.e., the coefficient at w^{-1} . The coefficients $c^{(n)}(z)$ are called n -products of $a(z)$ and $b(z)$, denoted $(a \underset{(n)}{(} b))(z)$. It is easy to compute (see also [18, 27]) that

$$c^{(n)}(z) = \sum_{m \in \mathbb{Z}} (a \underset{(n)}{(} b)_m z^{-m-1}, \quad (a \underset{(n)}{(} b)_m = \sum_{s \in \mathbb{Z}_+} (-1)^s \binom{n}{s} a_{n-s} b_{m+s}.$$

The importance of n -products becomes clear from the following observation. Suppose we are given a local pair $(a(z), b(z))$ of formal distributions. It follows from (5) that every element in A of the form $a_k b_m$, $k, m \in \mathbb{Z}$, may be expressed as a linear combination of

$$a_0 b_{k+m}, a_1 b_{k+m-1}, \dots, a_{N-1} b_{k+m-N+1}.$$

The explicit form of such an expression is known as the operator product expansion (OPE) formula:

$$a(w)b(z) = \sum_{s=0}^{N-1} \frac{1}{s!} (a \underset{(s)}{(} b)(z) \frac{\partial^s}{\partial z^s} \delta(w-z),$$

i.e., the product of two distributions in different variables expands into a finite sum with respect to derivatives of the formal delta-function

$$\delta(w-z) = \sum_{r+s=-1} w^r z^s \in \mathbb{k}[[w, w^{-1}, z, z^{-1}]].$$

Theorem 1 (Dong Lemma, see, e.g., [10]). *Let A be a Lie algebra, and let $a(z), b(z), c(z) \in A[[z, z^{-1}]]$ be pairwise mutually local distributions. Then for every $n \in \mathbb{Z}_+$ the pairs of formal distributions $(a \underset{(n)}{(} b))(z), c(z)$ and $a(z), (b \underset{(n)}{(} c))(z)$ are local.*

For the case of formal distributions over an associative algebra A , the Dong Lemma is also true (see, e.g., [27]). The aim of this paper is to study related varieties: pre-Lie (right-symmetric), Novikov, and pre-associative (dendriform) algebras.

The latter class of algebras is defined via two bilinear products, so the locality statement (similar to that of Theorem 1) should be checked for eight pairs of formal distributions. We will also mention the cases of Leibniz algebras and associative dialgebras.

2. LOCALITY OF FORMAL DISTRIBUTIONS OVER PRE-ASSOCIATIVE ALGEBRAS

The notion of a pre-associative algebra (under the term “dendriform algebra”) was proposed by J.-L. Loday in [23]. The original definition involves two bilinear operations $(x, y) \mapsto x \prec y$ and $(x, y) \mapsto x \succ y$ satisfying some identities of degree 3. These identities may be obtained from associativity by means of a general procedure (dendriform splitting, [1]). Note that the same procedure transforms Lie algebra identities into pre-Lie ones. This is why we prefer the term “pre-associative” rather than “dendriform”.

It is convenient to define pre-associative algebras in terms of other two bilinear operations.

Definition 1. A pre-associative algebra is a linear space A equipped with two bilinear operations $(x, y) \mapsto x * y$ and $(x, y) \mapsto xy$ satisfying the following axioms:

$$(x * y) * z = x * (y * z), \quad (7)$$

$$(x * y)z = x(yz), \quad (8)$$

$$x(y * z) = (xy) * z + x(yz) - (xy)z. \quad (9)$$

These new operations have the following relation to the original ones: $x * y = x \prec y + x \succ y$, and $xy = x \succ y$.

Suppose A is a pre-associative algebra and $a(z), b(z) \in A[[z, z^{-1}]]$ are two formal distributions over A . Since A has two operations, the definition of locality for the pair $(a(z), b(z))$ includes two conditions. Namely, if there exists $N \in \mathbb{Z}_+$ such that $a(w) * b(z)(w - z)^N = 0$ then we say that $(a(z), b(z))$ is a $*$ -local pair; if $a(w)b(z)(w - z)^N = 0$ for some $N \in \mathbb{Z}_+$ then the pair $(a(z), b(z))$ is said to be \succ -local.

Given $n \in \mathbb{Z}_+$, denote

$$(a \cdot_{(n)} b)(z) = \underset{w=0}{\operatorname{Res}} a(w)b(z)(w - z)^n, \quad (a *_{(n)} b)(z) = \underset{w=0}{\operatorname{Res}} a(w) * b(z)(w - z)^n,$$

as for “ordinary” algebras.

The purpose of this section is to show that the Dong Lemma does not hold “in full” for pre-associative algebras. First, let us prove positive parts of the statement.

Theorem 2. Let A be a pre-associative algebra. Suppose $a(z), b(z), c(z) \in A[[z, z^{-1}]]$ are pairwise mutually $*$ -local and \succ -local. Then for every $n \in \mathbb{Z}_+$

- the pair $a(z), (b \cdot_{(n)} c)(z)$ is \succ -local;
- the pair $a(z), (b *_{(n)} c)(z)$ is $*$ -local and \succ -local;
- the pair $(a *_{(n)} b)(z), c(z)$ is $*$ -local and \succ -local.

Proof. All these statements follow from the embedding of a pre-associative (dendriform) algebra A into an associative (Rota–Baxter) algebra \hat{A} described in [13]. Let us recall the essential part of the construction.

As a linear space, \hat{A} is a direct sum of two copies of the space A :

$$\hat{A} = A \dot{+} \bar{A}.$$

A multiplication $(x, y) \mapsto x \cdot y$, for $x, y \in \hat{A}$, is defined by

$$\begin{aligned} u \cdot v &= u * v, \quad u, v \in A; \\ \bar{u} \cdot v &= \overline{u * v} - \overline{uv}, \quad \bar{u} \in \bar{A}, v \in A; \\ u \cdot \bar{v} &= \overline{uv}, \quad u \in A, \bar{v} \in \bar{A}; \\ \bar{u} \cdot \bar{v} &= 0, \quad \bar{u}, \bar{v} \in \bar{A}. \end{aligned}$$

Then \hat{A} with the operation \cdot is an associative algebra. For $x \in \{a, b, c\}$ consider $x(z)$ and $\bar{x}(z)$ as formal distributions over \hat{A} : if $x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1}$, $x_n \in A$, then

$$\bar{x}(z) = \sum_{n \in \mathbb{Z}} \bar{x}_n z^{-n-1}, \quad \bar{x}_n \in \bar{A} \subset \hat{A}.$$

By the hypothesis, $a(z)$, $b(z)$, $c(z)$, $\bar{a}(z)$, $\bar{b}(z)$, $\bar{c}(z)$ are pairwise mutually local formal distributions from $\hat{A}[[z, z^{-1}]]$.

In particular, consider $a(z)$, $b(z)$, and $\bar{c}(z)$. By the Dong Lemma for associative algebras, the pair $(a(z), \overline{(b \cdot c)}(z))$ of formal distributions over \hat{A} is local. Since $(b \cdot \bar{c})(z) = \overline{(b \cdot c)}(z)$ by the definition of operation in \hat{A} , the pair $(a(z), (b \cdot c)(z))$ is \succ -local.

Similarly, since $\overline{(b * c)}(z) = (\bar{b} \cdot c)(z) + (b \cdot \bar{c})(z)$, the pair $(a(z), (b * c)(z))$ is \succ -local in $A[[z, z^{-1}]]$. The $*$ -locality of the same pair is an immediate corollary of the Dong Lemma for associative algebras. The remaining statements are proved in a similar way. \square

Consider the remaining locality statements on formal distributions over a pre-associative algebra: $*$ -locality for the pairs $(a(z), (b \cdot c)(z))$, $((a \cdot b)(z), c(z))$, and \succ -locality for $((a \cdot b)(z), c(z))$. Let us show that these statements do not hold in general. We will construct an appropriate pre-associative algebra A by means of generators and defining relations.

Proposition 1. *There exists a pre-associative algebra A and a formal distribution $a(z) \in A[[z, z^{-1}]]$ such that the pair $(a(z), a(z))$ is both $*$ -local and \succ -local, but $(a(z), (a \cdot a)(z))$ is not $*$ -local and $((a \cdot a)(z), a(z))$ is neither $*$ -local, nor \succ -local.*

Proof. Let $X = \{a_n \mid n \in \mathbb{Z}\}$ be a countable set indexed by integers and let $F(X)$ stand for the free pre-associative algebra generated by X (we will recall its structure below). Denote by I the ideal of $F(X)$ generated by the elements

$$a_n * a_m - a_0 * a_{n+m}, \quad a_n a_m - a_0 a_{n+m},$$

for all $n, m \in \mathbb{Z}$, $n \neq 0$. Consider the quotient pre-associative algebra $A = F(X)/I$. Then the formal distribution

$$a(z) = \sum_{n \in \mathbb{Z}} (a_n + I) z^{-n-1} \in A[[z, z^{-1}]]$$

has the following properties:

$$(a(w) * a(z))(w - z) = 0, \quad (a(w)a(z))(w - z) = 0.$$

In other words, the pair $(a(z), a(z))$ is both $*$ -local and \succ -local with locality level $N = 1$.

In order to show that the pair of formal distributions $((a \cdot a)(z), a(z))$ is not \succ -local, consider the elements

$$f_{n,m}^{(N)} = \sum_{s=0}^N (-1)^s \binom{N}{s} (a_0 a_{n-s}) a_{m+s} \in F(X), \quad n, m \in \mathbb{Z}, \quad N \in \mathbb{Z}_+. \quad (10)$$

It is enough to prove that for every $N \in \mathbb{Z}_+$ there exist $n, m \in \mathbb{Z}$ such that $f_{n,m}^{(N)} \notin I$.

For the latter, we may use the homogeneity of defining relations which allows us to compute explicitly all polynomials of degree 3 from I . One may organize the computation better by making use of the Gröbner–Shirshov bases (GSB) method for pre-associative (dendriform) algebras proposed in [20], but it is not necessary to apply this technique.

It is easy to see that the identities (7)–(9) allow one to rewrite an arbitrary term in the free pre-associative algebra $F(X)$ generated by a set X as a linear combination of monomials

$$u_1 * u_2 * \cdots * u_k, \quad k \geq 1, \quad (11)$$

where each u_i is a nonassociative (magmatic) word in X involving only the second operation $(x, y) \mapsto xy$. It was shown in [20] that the monomials (11) are linearly independent, i.e., they form a linear basis of $F(X)$. For the set

$$S = \{a_n a_m - a_0 a_{n+m}, a_n * a_m - a_0 * a_{n+m} \mid n, m \in \mathbb{Z}, n \neq 0\},$$

$S \subseteq F(X)$, consider the space I_3 of all elements of degree 3 in the ideal I generated by S : they are spanned by $sa_k, a_k s, s * a_k, a_k * s$, where $s \in S$. Hence, the space $I_3 \subset I$ is spanned by

$$(a_n a_m) a_k - (a_0 a_{n+m}) a_k, \quad a_k (a_n a_m) - a_k (a_0 a_{n+m}), \quad (12)$$

$$(a_n a_m) * a_k - (a_0 a_{n+m}) * a_k, \quad a_k * (a_n a_m) - a_k * (a_0 a_{n+m}), \quad (13)$$

$$(a_n * a_m) * a_k - (a_0 * a_{n+m}) * a_k, \quad a_k * (a_n * a_m) - a_k * (a_0 * a_{n+m}), \quad (14)$$

$$(a_n * a_m - a_0 * a_{n+m}) a_k = a_n (a_m a_k) - a_0 (a_{n+m} a_k), \quad (15)$$

$$a_k (a_n * a_m - a_0 * a_{n+m}) = (a_k a_n) * a_m - (a_k, a_n, a_m) - (a_k a_0) * a_{n+m} + (a_k, a_0, a_{n+m}), \quad (16)$$

where $n, m, k \in \mathbb{Z}$, $n \neq 0$. By means of linear reduction, (15) and (16) may be replaced with

$$a_n (a_0 a_k) - a_0 (a_0 a_{n+k}), \quad (17)$$

$$(a_0 a_n) * a_m - (a_0 a_n) a_m - (a_0 a_0) * a_{n+m} + (a_0 a_0) a_{n+m}, \quad (18)$$

for $n, m, k \in \mathbb{Z}$, $n \neq 0$.

Since all monomials of the form $(a_0 a_{n-s}) a_{m+s}$ that appear in $f_{n,m}^{(N)}$ are linearly independent modulo (12)–(14), (17), (18), we may conclude $f_{n,m}^{(N)} \notin I$, as desired.

In a similar way, if the pair $((a \circledast a)(z), a(z))$ is $*$ -local then there exists $N \in \mathbb{Z}_+$ such that

$$g_{n,m}^{(N)} = \sum_{s \in \mathbb{Z}_+} (-1)^s \binom{N}{s} (a_0 a_{n-s}) * a_{m+s} \in I$$

for all $n, m \in \mathbb{Z}$. This is impossible since

$$I \ni \sum_{s \in \mathbb{Z}_+} (-1)^s \binom{N}{s} ((a_0 a_{n-s}) * a_{m+s} - (a_0 a_{n-s}) a_{m+s} - (a_0 a_0) * a_{n+m} + (a_0 a_0) a_{n+m}) = g_{n,m}^{(N)} - f_{n,m}^{(N)}$$

for $N > 0$ (c.f. (18)), but we know $f_{n,m}^{(N)} \notin I$.

Finally, if the pair $(a(z), (a \circledast a)(z))$ is $*$ -local then there exists $N \in \mathbb{Z}_+$ such that

$$h_{n,m}^{(N)} = \sum_{s \in \mathbb{Z}_+} (-1)^s \binom{N}{s} a_{n-s} * (a_0 a_{m+s}) \in I$$

for all $n, m \in \mathbb{Z}$. The only option is to present $h_{n,m}^{(N)}$ as a linear combination of elements (13), but this is obviously impossible. \square

3. PRE-LIE ALGEBRAS AND THE DONG LEMMA FOR NOVIKOV ALGEBRAS

Given a pre-associative algebra A as above, its “commutator” algebra $A^{(-)}$ is a right-symmetric (pre-Lie) algebra: this is the same space A equipped with bilinear operation

$$x \circ y = x * y - xy - yx, \quad x, y \in A.$$

In terms of “original” operations \prec and \succ we have $x \circ y = x \prec y - y \succ x$. It was proved in [12] that every right-symmetric algebra V may be embedded into its universal enveloping pre-associative algebra $U(V)$, and an analogue of the Poincaré–Birkhoff–Witt Theorem holds (see also [7]).

Theorem 3. Let V be a right-symmetric (pre-Lie) algebra with an operation $(x, y) \mapsto x \circ y$, $x, y \in V$. Assume $a(z), b(z), c(z) \in V[[z, z^{-1}]]$ are pairwise mutually local formal distributions over V . Then $((a_{(n)} b)(z), c(z))$ is a local pair for every $n \in \mathbb{Z}_+$, but $(a(z), (b_{(n)} c)(z))$ is not necessarily local.

Proof. Note that if two formal distributions over V form a local pair $(a(z), b(z))$ then we cannot say that the same series are local if considered as formal distributions over $U(V)$. This is why we need to recall a relation between pre-Lie and Lie algebras [13].

Given a right-symmetric algebra V , consider the “ordinary” algebra \hat{V} constructed as follows. As a linear space, \hat{V} is a direct sum of two copies of the space V :

$$\hat{V} = V \dot{+} \bar{V}.$$

A multiplication $(x, y) \mapsto [xy]$, for $x, y \in \hat{V}$, is defined by

$$\begin{aligned} [uv] &= u \circ v - v \circ u, \quad u, v \in V; \\ [\bar{u}\bar{v}] &= \overline{u \circ v}, \quad \bar{u} \in \bar{V}, \quad v \in V; \\ [u\bar{v}] &= -\overline{v \circ u}, \quad u \in V, \quad \bar{v} \in \bar{V}; \\ [\bar{u}\bar{v}] &= 0, \quad \bar{u}, \bar{v} \in \bar{V}. \end{aligned}$$

Then \hat{V} with the operation $[..]$ is a Lie algebra.

Assume

$$x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1}, \quad x_n \in V, \quad x \in \{a, b, c\}.$$

Since $V \subset \hat{V}$, we may consider these distributions as elements of $\hat{V}[[z, z^{-1}]]$. The distributions with coefficients \bar{x}_n , $x \in \{a, b, c\}$, belong to the same space of formal distributions over \hat{V} :

$$\bar{x}(z) = \sum_{n \in \mathbb{Z}} \bar{x}_n z^{-n-1}, \quad \bar{x}_n \in \bar{V} \subset \hat{V}.$$

According to the hypothesis, the pairs of distributions $(\bar{x}(z), y(z))$ are local in $\hat{V}[[z, z^{-1}]]$ for all $x, y \in \{a, b, c\}$. It follows immediately from the construction that

$$[\bar{a}_{(n)} b](z) = \sum_{s \in \mathbb{Z}_+} (-1)^s \binom{n}{s} [\bar{a}_{n-s} b_{m+s}] = \sum_{s \in \mathbb{Z}_+} (-1)^s \binom{n}{s} \overline{\bar{a}_{n-s} \circ b_{m+s}} = \overline{(a_{(n)} b)}(z)$$

for every $n \in \mathbb{Z}_+$. Hence, by the Dong Lemma for Lie algebras, $\overline{(a_{(n)} b)}(z)$ and $c(z)$ form a local pair of formal distributions over \hat{V} . The latter means

$$0 = [\overline{(a_{(n)} b)}(w)c(z)](w-z)^N = \overline{(a_{(n)} b)(w) \circ c(z)}(w-z)^N$$

for sufficiently large $N \in \mathbb{Z}_+$. Since \bar{V} is an isomorphic copy of V , the first claim follows: the pair $((a_{(n)} b)(z), c(z))$ is local in $V[[z, z^{-1}]]$.

In order to construct a counterexample to approve the second statement, let us apply Gröbner–Shirshov bases method for right-symmetric algebras proposed in [4].

Suppose $X = \{a_n \mid n \in \mathbb{Z}\}$ is a set of generators which is well-ordered in the following way:

$$a_0 < a_{-1} < a_1 < a_{-2} < a_2 < \dots$$

As in [4], denote by $\text{RS}(X)$ the free right-symmetric algebra generated by the set X . Let I be the ideal of $\text{RS}(X)$ generated by the set

$$S = \{h_{n,m} = a_n \circ a_m - a_0 \circ a_{n+m} \mid n, m \in \mathbb{Z}, \quad n \neq 0\},$$

and let V stand for the quotient $\text{RS}(X)/I$. Then the formal distribution

$$a(z) = \sum_{n \in \mathbb{Z}} (a_n + I) z^{-n-1} \in V[[z, z^{-1}]]$$

is local to itself with locality level $N = 1$.

Note that it follows from right symmetry that

$$(a_0 \circ a_k) \circ a_m - (a_0 \circ a_m) \circ a_k = a_0 \circ (a_k \circ a_m) - a_0 \circ (a_m \circ a_k) \in a_0 \circ (a_0 \circ a_{m+k}) - a_0 \circ (a_0 \circ a_{k+m}) + I$$

for all $n, m \in \mathbb{Z}$. Hence, $(a_0 \circ a_k) \circ a_m - (a_0 \circ a_m) \circ a_k \in I$ for all $k, m \in \mathbb{Z}$.

Next, consider

$$\begin{aligned} I \ni h_{1,m} \circ a_k &= a_1 \circ (a_m \circ a_k) + (a_1 \circ a_k) \circ a_m - a_1 \circ (a_k \circ a_m) - (a_0 \circ a_{m+1}) \circ a_k \\ &= a_1 \circ h_{m,k} + a_1 \circ (a_0 \circ a_{m+k}) + h_{1,k} \circ a_m + (a_0 \circ a_{k+1}) \circ a_m - a_1 \circ h_{k,m} - a_1 \circ (a_0 \circ a_{k+m}) - (a_0 \circ a_{m+1}) \circ a_k \\ &\quad \in (a_0 \circ a_{k+1}) \circ a_m - (a_0 \circ a_k) \circ a_{m+1} + I. \end{aligned}$$

Therefore,

$$g_{m,k} = (a_0 \circ a_{k+1}) \circ a_m - (a_0 \circ a_k) \circ a_{m+1} \in I$$

for all $k, m \in \mathbb{Z}$.

Since the generators S of I are homogeneous, it is straightforward to find all elements of degree 3 in the Gröbner–Shirshov bases of I . In fact, according to the scheme from [4] we only need to add all compositions of right multiplication

$$h_{n,m} \circ a_k, \quad a_k < a_m. \quad (19)$$

It is easy to see that all these compositions are trivial modulo the set of relations the set

$$S' = \{h_{n,m}, g_{m,k} \mid n, m, k \in \mathbb{Z}, n \neq 0\}.$$

Namely, the principal monomial of $h_{n,m}$ (according to the same order as in [4]) is $a_n \circ a_m$, $(a_n \circ a_m) \circ a_k$ is not a “good” word, and

$$\begin{aligned} h_{n,m} \circ a_k &= a_n \circ (a_m \circ a_k) + (a_n \circ a_k) \circ a_m - a_n \circ (a_k \circ a_m) - (a_0 \circ a_{n+m}) \circ a_k \\ &= a_n \circ h_{m,k} + a_n \circ (a_0 \circ a_{m+k}) + h_{n,k} \circ a_m + (a_0 \circ a_{n+k}) \circ a_m - a_n \circ h_{k,m} - a_n \circ (a_0 \circ a_{k+m}) - (a_0 \circ a_{n+m}) \circ a_k \\ &= a_n \circ h_{m,k} + h_{n,k} \circ a_m - a_n \circ h_{k,m} + (a_0 \circ a_{n+k}) \circ a_m - (a_0 \circ a_{n+m}) \circ a_k \\ &= a_n \circ h_{m,k} + h_{n,k} \circ a_m - a_n \circ h_{k,m} + \sum_{s=0}^{k-m} g_{m+s,n+k-s}. \end{aligned}$$

Principal monomials of all summands in the right-hand side are smaller than $(a_n \circ a_m) \circ a_k$, as needed for triviality of a composition.

Hence, S' is the degree 3 component of a Gröbner–Shirshov basis of the ideal I .

Return to the formal distribution $a(z) \in A[[z, z^{-1}]]$ and assume $(a(z), (a \circ a)(z))$ is a local pair. Then there exists $N \in \mathbb{Z}_+$ such that

$$\sum_{s \in \mathbb{Z}_+} (-1)^s \binom{N}{s} a_{n-s} \circ (a_0 \circ a_{m+s}) \in I$$

for all $n, m \in \mathbb{Z}$. Since all words in this combination are S' -reduced, they are linearly independent modulo I , a contradiction. \square

Now we turn to the particular case when V is a Novikov algebra, i.e., satisfies the identities (1) and (2).

Corollary 1. *In the conditions of Theorem 3, assume V is a Novikov algebra. Then the formal distributions $a(z)$, $(b \circ a)(z)$ form a local pair in $V[[z, z^{-1}]]$.*

Proof. Suppose $(a(w) \circ c(z))(w - z)^N = 0$. Then for the same $N \in \mathbb{Z}_+$ we have

$$\begin{aligned} (a(w) \circ (b_{(n)} c)(z))(w - z)^N &= (a(w) \circ \text{Res}_{y=0}(b(y) \circ c(z))(y - z)^n)(w - z)^N \\ &= \text{Res}_{y=0}((a(w) \circ (b(y) \circ c(z)))(w - z)^N)(y - z)^n \\ &= \text{Res}_{y=0}(b(y) \circ (a(w) \circ c(z))(w - z)^N)(y - z)^n = 0 \end{aligned}$$

due to the left commutativity (2). \square

As a result, Theorem 3 and Corollary 1 imply that the Dong Lemma holds for Novikov algebras.

Remark 2. The locality estimates from the Dong Lemma for Lie and associative algebras provides us the corresponding estimates for Novikov algebras. Suppose V is a Novikov algebra with a binary operation $(x, y) \mapsto x \circ y$, and let $a(z), b(z), c(z) \in V[[z, z^{-1}]]$ be three pairwise mutually local formal distributions over V :

$$a(w) \circ b(z)(w - z)^{N(a,b)} = a(w) \circ c(z)(w - z)^{N(a,c)} = b(w) \circ c(z)(w - z)^{N(b,c)} = 0$$

for some $N(a, b), N(a, c), N(b, c) \in \mathbb{Z}_+$. Then by Corollary 1 $N(a, b_{(n)} c) \leq N(a, c)$, and it follows from the proof of Theorem 3 that over the Lie algebra \hat{V} the distributions $\bar{a}(z), b(z), c(z) \in \hat{V}[[z, z^{-1}]]$ are also pairwise mutually local with $N(\bar{a}, x) = N(a, x)$, $x \in \{b, c\}$. Hence,

$$N(a_{(n)} b, c) = N([\bar{a}_{(n)} b], c) \leq N(a, b) + N(a, c) + N(b, c) - n$$

(see, e.g., [27]).

4. ON THE FORMAL DISTRIBUTIONS OVER REPLICATED ALGEBRAS

The class of right-symmetric algebras includes the variety of associative algebras. On the other hand, as mentioned above, the notion of a right-symmetric algebra appears as a result of the dendriform splitting procedure (see [1]) applied to the defining identities of Lie algebras. It is reasonable to consider the result of a dual procedure (called “replication” in [26]) applied to the same variety of Lie algebras. This is a well-known class of Leibniz algebras which is often considered as a “non-commutative” version of Lie algebras [24].

Recall that a linear space L equipped with a bilinear operation $(x, y) \mapsto [xy]$ is said to be a *Leibniz algebra* if

$$[x[yz]] - [y[xz]] = [[xy]z]$$

for all $x, y, z \in L$.

Note that for every Leibniz algebra L the subspace $I \subset L$ spanned by the set $\{[ab] + [ba] \mid a, b \in L\}$ is an ideal of L , and the quotient $\bar{L} = L/I$ is a Lie algebra. Define an algebra \hat{L} in the following way. As a linear space, \hat{L} is a direct sum of \bar{L} and L . Assuming $\bar{a} = a + I \in \bar{L}$ for $a \in L$, define a product $(x, y) \mapsto [x, y]$ on $\hat{L} = \bar{L} + L$ by the rule

$$\begin{aligned} [u, v] &= 0, \quad u, v \in L; \\ [\bar{u}, v] &= [uv], \quad \bar{u} \in \bar{L}, v \in L; \\ [u, \bar{v}] &= -[vu], \quad u \in L, \bar{v} \in \bar{L}; \\ [\bar{u}, \bar{v}] &= \overline{[uv]}, \quad \bar{u}, \bar{v} \in \bar{L}. \end{aligned}$$

Then \hat{L} is a Lie algebra relative to the operation $[\cdot, \cdot]$.

Proposition 2. *Let L be a Leibniz algebra, and let $a(z), b(z), c(z)$ be pairwise local formal distributions over L . Then the pairs of formal distributions $(a_{(n)} b)(z), c(z)$ and $a(z), (b_{(n)} c)(z)$ are local for every $n \in \mathbb{Z}_+$.*

Proof. Since the map $\tau : L \rightarrow \bar{L}$, $\tau(x) = \bar{x}$, is a homomorphism from the Leibniz algebra L onto the Lie algebra \bar{L} , the locality of a pair $a(z), b(z) \in L[[z, z^{-1}]]$ implies that the pairs $(\bar{x}(z), y(z)), (x(z), \bar{y}(z)), (\bar{x}(z), \bar{y}(z))$ are local in $\hat{L}[[z, z^{-1}]]$ for all $x, y \in \{a, b, c\}$. By the Dong Lemma applied to the Lie algebra \hat{L} , we obtain the pairs $([\bar{a}_{(n)} \bar{b}](z), c(z))$ and $(\bar{a}(z), [\bar{b}_{(n)} c](z))$ are local in $\hat{L}[[z, z^{-1}]]$. By the definition of \hat{L} , the latter implies the required locality. \square

Remark 3. In the very similar way, the Dong Lemma holds for formal distributions over *associative dialgebras* [25], i.e., algebras with two bilinear operations $(x, y) \mapsto x \vdash y$ and $(x, y) \mapsto (x \dashv y)$ satisfying the axioms

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \quad x \dashv (y \vdash z) = x \dashv (y \dashv z)$$

along with replicated associativity

$$(x \vdash y) \vdash z - x \vdash (y \vdash z) = (x \vdash y) \dashv z - x \vdash (y \dashv z) = (x \dashv y) \dashv z - x \dashv (y \dashv z) = 0.$$

In this case, as for pre-associative algebras, we have eight locality statements to be checked similarly to what is done in Proposition 2.

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