Broken time reversal symmetry vestigial state for a two-component superconductor in two spatial dimensions

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We consider the vestigial phase with broken time-reversal symmetry above the superconducting transition temperature of a two-component superconductor in two spatial dimensions. We show that, in contrast to 3D, a vestigial phase is in general allowed within Ginzburg-Landau theory. The vestigial phase occupies an increasing temperature region if the parameters in the Ginzburg-Landau theory gives a larger energy difference between the broken time-reversal symmetry phase and the other ordered phase.

I. INTRODUCTION

Consider a superconductor with order parameter (Φ_1, Φ_2) belonging to a two-dimensional representation, here $\Phi_{1,2}$ are complex fields which transform as $(\Phi_1', \Phi_2') = e^{i\chi}(\Phi_1, \Phi_2)$ under gauge transformation by χ, and among themselves under spatial operations, such as $(\Phi_1', \Phi_2') = (\Phi_1 \cos(\phi) + \Phi_2 \sin(\phi), \Phi_2 \cos(\phi) - \Phi_1 \sin(\phi))$ under rotations by ϕ, where ϕ's are restricted to multiples of $\frac{2\pi}{3}$ ($\frac{2\pi}{6}$) for trigonal (hexagonal) systems. Such mutli-dimensional order parameters have been considered extensively since superfluid ^{3}He [1], heavy fermion superconductors [2, 3], and also more recently in many other systems [4, 5]. In the two-dimensional representation case, below the superconducting transition temperature, depending on details of the free energy, the energy minimum can be achieved by having $(\Phi_1, \Phi_2) \propto (1, 0)$, $(0, 1)$, or $(\Phi_1, \Phi_2) \propto (1, \pm i)$. In the former case, the order parameter breaks gauge invariance and rotational invariance, whereas in the latter, gauge invariance as well as time-reversal invariance. In each case, as oppose to the case of an order parameter Φ belonging to a one-dimensional representation, there is additional symmetry breaking other than the gauge symmetry.

At high temperatures we have a symmetric phase with no broken symmetries. Within mean-field theory, at the superconducting transition temperature, the system goes into a phase with more than one symmetry broken. In principle at least, in a more complex scenario such as when fluctuations are included, these broken symmetries do not have to occur at the same time. In particular, one can have a phase where say the rotational (or time-reversal) symmetry is broken, whereas the gauge symmetry is still intact. In this case, we have the expectation values $\langle \Phi_{1,2} \rangle = 0$, whereas, e.g., $\langle \Phi_1^* \Phi_1 - \Phi_2^* \Phi_2 \rangle \neq 0$ (or $i\langle (\Phi_1^* \Phi_2 - \Phi_2^* \Phi_1) \rangle \neq 0$). In additional to the above scenarios, one can have the possibilities that some other symmetry-violating combinations of $\Phi_{1,2}$ acquiring non-zero expectation values. For example, we can have $\langle \Phi_1^2 + \Phi_2^2 \rangle \neq 0$ even though $\langle \Phi_{1,2} \rangle = 0$. In this latter case, while the order parameter is not preserved under general gauge transformations, it is preserved under special transformation $\chi \to \chi + \pi$, thus describes "4e" pairing [6]. Such phases, often called "vestigial" phases or phase with "composite" or "higher-order" parameters, are gaining attention in the recent literature [7–13], though they have been investigated already in the past in similar [14–17] and related (e.g. [18–22]) context. Besides superconductivity, these exotic phases are also relevant to other, e.g., magnetic, systems [23, 24].

In a previous paper [25], considering three spatial dimensions, we show that, witin a Ginzburg-Landau theory with thermal fluctuations, such a vestigial phase is in general not stable, except for the case of extreme gradient energy terms in the free energy. This is because, when the temperature is lowered so that the completely symmetric phase is no longer the free energy stable minimum, either (i) no saddle point corresponding to the vestigial phase exists, so that one has a direct second order phase transition to the superfluid phase with broken rotational (time-reversal) symmetry as well as gauge symmetry, or (ii) the vestigial phase with composite order parameter is only a saddle point but fails to be a free energy minimum. Instead, the free energy minimum occurs in the region where the expectation value of $\Phi_{1,2}$ is/are finite. The system thus makes a joint first order phase transition into the superconducting case. A similar situation can be shown to occur for a multicomponent Bose gas [26, 27].

In this paper, we consider instead two spatial dimensions. We show that the situation becomes quite different. Case (i) above remains a possibility, but for case (ii), the saddle point does become stable in general for a finite range in temperature, due to a very different free energy landscape. Similar strong dependence on the spatial dimensionality has also been found in e.g., Ref [21, 22].

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We shall mainly be studying the broken time reversal symmetry state. Our calculations will be presented in Sec II. In Sec III, we shall include also a short discussion on the nematic case and the "4e" state as well as conclusions.

II. THEORY FOR VESTIGIAL ORDER

We consider the same Hamiltonian as in ref [25], but only now in two spatial dimensions. We employ the "spin-$\frac{1}{2}$" notation, thus

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_\uparrow \\ \Phi_\downarrow \end{pmatrix} \tag{1}$$

The effective Hamiltonian density

$$\mathcal{H} = \mathcal{H}_K + \mathcal{H}_{int} \tag{2}$$

consists of two parts. For the "kinetic" part, we shall simply take

$$\mathcal{H}_K = \sum_{s=\uparrow,\downarrow} \left[\alpha \Phi_s^* \Phi_s + K \left(\frac{\partial \Phi_s^*}{\partial x_i} \frac{\partial \Phi_s}{\partial x_i} \right) \right] \tag{3}$$

thus ignoring possible more complicated spatial ($x_{1,2}$) derivatives. (We remind the readers that [25] established the absence of the vestigial phase in three spatial dimensions when the gradient energy is of this form). Here $\alpha = \alpha(T)$ is positive (negative) above (below) a mean-field transition temperature which we shall label as T_0, thus $\alpha(T) \approx \alpha'(T - T_0)$ with $\alpha' > 0$. The interacting part reads

$$\mathcal{H}_{int} = \frac{g_1}{2} (|\Phi_\uparrow|^4 + |\Phi_\downarrow|^4) + g_2 (|\Phi_\uparrow|^2 |\Phi_\downarrow|^2) \tag{4}$$

$\Phi_{\uparrow,\downarrow}$ are related to $\Phi_{1,2}$ in the Introduction via $\Phi_{\uparrow,\downarrow} = \frac{1}{\sqrt{2}}[\Phi_1 \pm i\Phi_2]$. See also [25] and [28].

If we simply minimize $\mathcal{H}$, the system is in the completely symmetric (normal) phase if $\alpha > 0$, and for $\alpha < 0$, we have the broken time-reversal symmetry state $\Phi_\uparrow \neq 0$, $\Phi_\downarrow = 0$, $|\Phi_\uparrow|^2 = \frac{|\alpha|}{g_1}$ (or $\uparrow \leftrightarrow \downarrow$). with free energy density $-\frac{\alpha^2}{2g_1}$. The "nematic phase", with $|\Phi_\uparrow| = |\Phi_\downarrow|$, is also a local energy minimum but has free energy $-\frac{\alpha^2}{(g_1+g_2)}$. The stability of these mean-field states require $g_1 > 0$, $g_2 > -g_1$, and we shall restrict ourselves to this region. We shall also mainly deal with $g_2 > g_1 > 0$ in this paper, so that the broken time reversal symmetry state has the lower energy.

At finite temperatures, we need to consider the partition function [29, 30] $Z \equiv \int_{\Phi_s} e^{-\int d^2x \mathcal{H}/T}$, where $\int_{\Phi_s}$ means sum over all configurations of $\Phi_s(\vec{r})$. We employ the Hartree-Fock (HF) approximation. The effective Hamiltonian density becomes

$$\mathcal{H}_{eff} = \mathcal{H}_K - h_\uparrow \Phi_\uparrow^* \Phi_\uparrow - h_\downarrow \Phi_\downarrow^* \Phi_\downarrow \tag{5}$$

where $h_{\uparrow,\downarrow}$ are the self-energies (not to be confused with external magnetic fields), which are to be obtained self-consistently. $h_\uparrow \neq h_\downarrow$ signals that $\langle \Phi_\uparrow^* \Phi_\uparrow \rangle \neq \langle \Phi_\downarrow^* \Phi_\downarrow \rangle$. $h_\uparrow - h_\downarrow$ thus serves as an order parameter for the broken Z_2 symmetry.

After Fourier transform,

$$\mathcal{H}_{eff} = \sum_{s,\vec{k}} \Phi_{\vec{k},s}^* \left(\alpha + Kk^2 - h_s \right) \Phi_{\vec{k},s} \tag{6}$$

where $\vec{k}$ represents the wavevector. We thus have the expectation values

$$\langle \Phi_{\vec{k},s} \Phi_{\vec{k},s'}^* \rangle = T G_s(\vec{k}) \delta_{s,s'} \tag{7}$$

with the "Green's function"

$$G_s(\vec{k}) = \frac{1}{\alpha + Kk^2 - h_s} \tag{8}$$

For the vestigial phase, we must have $\alpha - h_s > 0$.

The free energy density is, within the HF approximation,

$$\begin{aligned} \mathcal{F} \;=\; & \frac{T}{L^2} \sum_{\vec{k},s} \left[\ln(\alpha + Kk^2 - h_s) + h_s G_s(\vec{k}) \right] \\ & + g_1 \left[\left(\frac{T}{L^2} \sum_{\vec{k}} G_\uparrow(\vec{k}) \right)^2 + \left(\frac{T}{L^2} \sum_{\vec{k}} G_\downarrow(\vec{k}) \right)^2 \right] + g_2 \left[\left(\frac{T}{L^2} \sum_{\vec{k}} G_\uparrow(\vec{k}) \right) \times \left(\frac{T}{L^2} \sum_{\vec{k}} G_\downarrow(\vec{k}) \right) \right] \end{aligned} \tag{9}$$

This expression is ultraviolet divergent, both because of the $\ln(\alpha + Kk^2 - h_s)$ and interaction terms. These divergences are also present even for $F \equiv F_0$ where we set h_s to be zero (and thus replace $G_s(\vec{k})$ by $G_0(\vec{k}) \equiv \frac{1}{\alpha + Kk^2}$). However, we note that insertion of the Hartree-Fock self-energies $(2g_1 + g_2) \frac{T}{L^2} \sum_{\vec{k}} G_0(\vec{k})$ to the propagators G_s or G_0 would amount to replacing α by $\alpha + (2g_1 + g_2) \frac{T}{L^2} \sum_{\vec{k}} G_0(\vec{k})$, which can be regarded as a redefinition of α. Using this renormalized $\alpha(T)$, the difference of the free energy density between the phase under consideration and F_0 can then be written as [25]

$$\begin{aligned}\Delta\mathcal{F} &= \frac{T}{L^2}\sum_{\vec{k},s}\left[\ln(\alpha+Kk^2-h_s)-\ln(\alpha+Kk^2)+h_sG_s(\vec{k})\right]\\ &+g_1\left[\left(\frac{T}{L^2}\sum_{\vec{k}}(G_\uparrow(\vec{k})-G_0(\vec{k}))\right)^2+\left(\frac{T}{L^2}\sum_{\vec{k}}(G_\downarrow(\vec{k})-G_0(\vec{k}))\right)^2\right]\\ &+\ g_2\left[\left(\frac{T}{L^2}\sum_{\vec{k}}(G_\uparrow(\vec{k})-G_0(\vec{k}))\right)\times\left(\frac{T}{L^2}\sum_{\vec{k}}(G_\downarrow(\vec{k})-G_0(\vec{k}))\right)\right]\end{aligned} \tag{10}$$

This expression is ultraviolet convergent, and the contributions giving rise to finite $\Delta\mathcal{F}$ arise only for small wavevectors when α and h_s are small, as it should be.

The momentum sums can be easily evaluated. For 3D, we reproduce the result in [25]. In the present case, we get

$$\begin{aligned}\Delta\mathcal{F} &= -\frac{T}{4\pi K}\left\{\left[\alpha\ln(1-\frac{h_\uparrow}{\alpha})+h_\uparrow\right]+\left[\alpha\ln(1-\frac{h_\downarrow}{\alpha})+h_\downarrow\right]\right\}\\ &+\frac{T^2g_1}{(4\pi K)^2}\left[\left(\ln(1-\frac{h_\uparrow}{\alpha})\right)^2+\left(\ln(1-\frac{h_\downarrow}{\alpha})\right)^2\right]+\frac{T^2g_2}{(4\pi K)^2}\left[\ln(1-\frac{h_\uparrow}{\alpha})\ \ln(1-\frac{h_\downarrow}{\alpha})\right]\end{aligned} \tag{11}$$

An important point to note is that, in contrast to the three dimensional case [25], this free energy diverges to $+\infty$ due to the g_1 term (since $g_1>0$) when $h_s\to\alpha_-$. Hence there is no “falling off” to the unphysical ($\alpha-h_s<0$) region, in contrast to [25], and stable non-trivial minima can exist within the physical $h_s<\alpha$ region. See Fig 1.

Expansion of the free energy in terms of $h_z\equiv(h_\uparrow-h_\downarrow)/2$ and $h_0\equiv(h_\uparrow+h_\downarrow)/2$ gives

$$\Delta\mathcal{F}=ah_z^2+bh_z^4+\gamma h_0h_z^2+ch_0^2 \tag{12}$$

where

$$a=TI_2[1+T(2g_1-g_2)I_2] \tag{13}$$

$$b=\frac{3}{2}TI_4+2T^2g_1(I_3^2+2I_2I_4)+T^2g_2(I_3^2-2I_2I_4) \tag{14}$$

$$\gamma=4TI_3+2T^2(6g_1-g_2)I_2I_3 \tag{15}$$

$$c=TI_2[1+T(2g_1+g_2)I_2] \tag{16}$$

Here $I_2=\frac{1}{4\pi K\alpha}$ and generally $I_n=\frac{1}{(n-1)4\pi K\alpha^{n-1}}$ for $n\geq 2$.

The coefficient a changes sign at T at T_2 where

$$0=1+(2g_1-g_2)\frac{T_2}{4\pi K\alpha(T_2)} \tag{17}$$

signalling a phase transition (at T_2 if second order). This transition thus exists only when $g_2-2g_1>0$. Eq (12) implies $h_0=-\frac{\gamma}{2c}h_z^2$. Eliminating h_0, the effective coefficient for h_z^4 becomes $b-\frac{\gamma^2}{4c}$. The value of this coefficient at T_2 is given by $b_{eff}=\frac{T_2}{4\pi K\alpha^3(T_2)}\frac{6g_1-g_2}{24g_2}$ hence positive only when $g_2<6g_1$. Hence transition is second order only when $g_2<6g_1$ [31]. See Appendix for further analysis on this point. Below we shall confine ourselves only to this parameter regime. Since α is rapidly varying with temperature near T_0, eq (17) implies

$$T_2\approx T_0\ [1+\frac{(g_2-2g_1)}{4\pi K\alpha'}] \tag{18}$$

hence a transition temperature increasing from T_0 linearly with g_2-2g_1 when the latter is positive. Below T_2, $h_z^2\approx-\frac{a'}{2b_{eff}}(T-T_2)$, with $a'=-\frac{T_2}{4\pi K\alpha^3(T_2)}\alpha'$.

The above has assumed that the transition is to a state with uniform $h_{0,z}$. One can also consider the free energy F for the case where the self-energies h_s varies with position. If these fields have wavevector $\vec{Q}$, then the free energy has the form

$$\Delta\mathcal{F}=a(Q)h_z(\vec{Q})h_z(-\vec{Q})+... \tag{19}$$

with

$$a(Q)=TI_2(Q)[1+T(2g_1-g_2)I_2(Q)] \tag{20}$$

where

$$I_2(Q)\equiv\frac{1}{L^2}\sum_{\vec{k}}\frac{1}{(\alpha+Kk_+^2)(\alpha+Kk_-^2)} \tag{21}$$

with $\vec{k}_\pm=\vec{k}\pm\frac{\vec{Q}}{2}$. $I_2(Q)=I_2$ if $Q=0$, decreases with increasing Q or α, and is positive definite if $\alpha>0$. Hence

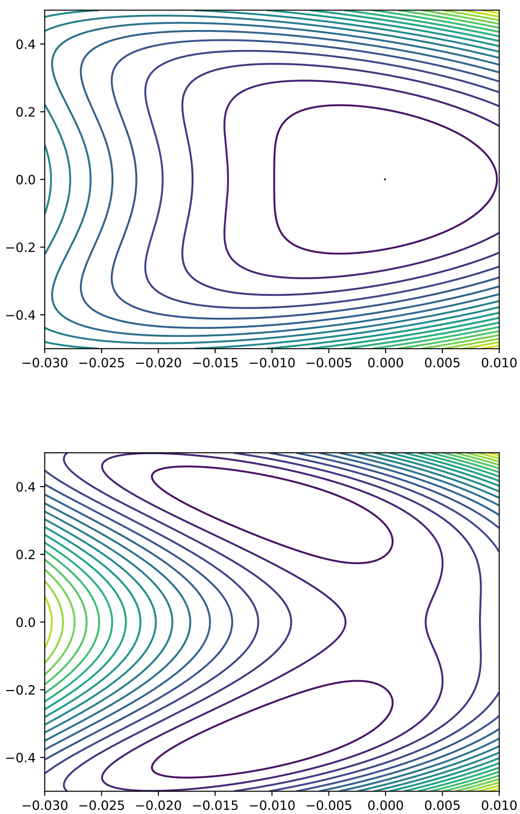

FIG. 1. Example contour plots of the free energy in eq (11). Upper diagram: symmetric phase, lower: broken symmetry phase. Abscissa: $[\ln(1-\frac{h_\uparrow}{\alpha})+\ln(1-\frac{h_\downarrow}{\alpha})]/2$, ordinate: $[\ln(1-\frac{h_\uparrow}{\alpha})-\ln(1-\frac{h_\downarrow}{\alpha})]/2$.

if $2g_1-g_2>0$, $a(Q)$ is positive for any Q and positive α. If $2g_1-g_2<0$, $a(Q)>0$ for all Q's at high temperatures, and at T_2, $a(Q)$ changes sign at $Q=0$ with $a(Q)>0$ at $Q\neq 0$, verifying that the transition is to the uniform state.

The above considerations show that, for long wavelength fluctuations of h_z, the free energy density has the form

$$\Delta\mathcal{F} = a h_z^2 + \tilde{K}(\vec{\nabla}h_z)^2 + b_{eff}h_z^4 \qquad (22)$$

The coefficient $\tilde{K}$ can be obtained from an expansion of $a(Q)$ at small Q. Using $I_2(Q) = I_2 - \alpha\frac{KQ^2}{2}I_4(0)$, we obtain thus

$$\tilde{K} = -\frac{\alpha K}{2}I_4[1+2T(2g_1-g_2)I_2] \qquad (23)$$

Near $a=0$ (T_2 given in eq (17)), $\tilde{K}\approx\frac{T}{12\pi\alpha}>0$ [32]. Eq (22) represents an effective Hamiltonian for a second order Ising transition, with $\tilde{K}>0$ and $b_{eff}>0$ (the latter holds if $g_2<6g_1$, as already mentioned).

The above considerations find the minimum of the free energy in h_z. More precisely, h_z is itself a fluctuating quantity and the free energy density in eq (22) should be regarded as the effective Hamiltonian density for $h_z(\vec{r})$. We thus obtained an effective ϕ^4 theory for the Ising transition where h_z plays the role of the order parameter for the Z_2 transition. The considerations so far thus give, upon lowering of temperature, an Ising transition from a completely symmetric phase to a Z_2 broken symmetry phase ($h_z\neq 0$ yet with $\langle\Phi_s\rangle=0$) at $T_2>T_0$ (thus $\alpha>0$) given by eq (18) if $g_2>2g_1$. This is our vestigial phase. In this region, $h_\uparrow\neq h_\downarrow$, but both $\Phi_\uparrow$ and $\Phi_\downarrow$ have vanishing expectation values. Correlations between Φ_s at different positions decay exponentially in space: $\langle\Phi_s^*(\vec{r})\Phi_s(\vec{r}')\rangle\propto e^{-\frac{|\vec{r}-\vec{r}'|}{\lambda_s}}$. Moreover, due to the finite h_z, $\lambda_\uparrow\neq\lambda_\downarrow$. Upon lowering of the temperature, $h_{z,0}$ both grows in magnitude, whereas α decreases. Within the above considerations, at temperature where $\alpha=h_\uparrow$, the system makes a transition to the state with $\langle\Phi_\uparrow\rangle\neq 0$ but $\langle\Phi_\downarrow\rangle=0$ or vice versa, a state just like $T=0$. At this temperature, $\alpha-h_\downarrow>0$ so that $\langle\Phi_\downarrow^*(\vec{r})\Phi_\downarrow(\vec{r}')\rangle$ still decays exponentially Furthermore, for $g_2<2g_1$, h_s vanishes, the system goes from the symmetric phase to the state $\langle\Phi_\uparrow\rangle\neq 0$ but $\langle\Phi_\downarrow\rangle=0$ or vice versa at T_0, where α vanishes [33].

At finite T, the phase with long range order just described is due to the artifact that phase fluctuation of Φ_s was not considered. Mermin-Wagner theorem states that this long range order is destroyed in 2D. However, quasi-long range order [36] is allowed. For the phase diagram, the simplest possibility is that the above mentioned phase with long range order is instead characterized by power law correlations, thus instead of finite expectation value for $\Phi_\uparrow$, we have simply $\langle\Phi_\uparrow^*(\vec{r})\Phi_\uparrow(\vec{r}')\rangle\propto\frac{1}{|\vec{r}-\vec{r}'|^\eta}$. The resulting phase diagram is as given in Fig 2a.

Another possibility is that, due to thermal fluctuations of the phase, there is always a vestigial Z_2 broken symmetry phase that lies between the completely symmetric phase and the quasi-long range order phase, even for the region $g_2<2g_1$. This possibility has been raised in a few theoretical calculations based on models which are related to though not the same as the one we have in this paper [21, 37–39] (though there are also related studies where such a phase is absent [40]). The resulting vestigial phase again only has short range order, but since Z_2 is broken, the decaying lengths $\lambda_{\uparrow,\downarrow}$ are thus unequal. This phase is indistinguishable from our vestigial phase described by $h_z\neq 0$, though the physical picture giving rise to this broken Z_2 symmetry seems quite different. The resulting phase diagram is sketched qualitatively in Fig 2b. [41] For both Fig 2a and Fig 2b, the phase transition temperatures all vanish at $g_2=g_1$. At this point there the symmetry is enhanced to SO(3), which forbids any order at finite temperature in two spatial dimensions, a fact also pointed out in [39].

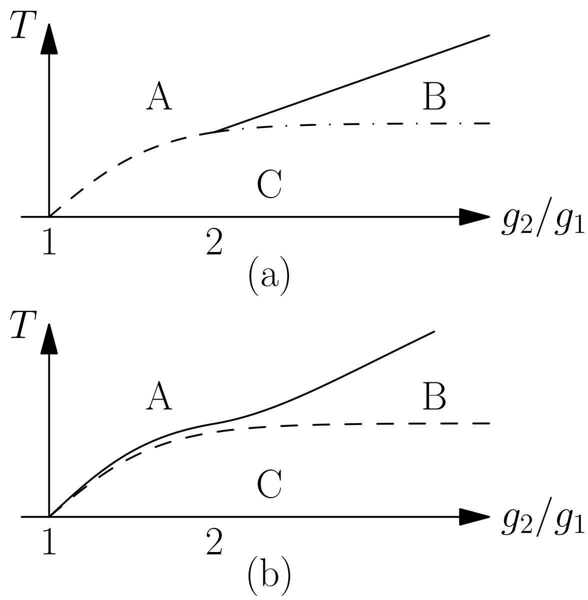

FIG. 2. Possible phase diagrams. Region A is the symmetric phase. Region B is the vestigial phase with broken Z_2 symmetry, but with only short range order for both $\Phi_\uparrow$ and $\Phi_\downarrow$. $\langle \Phi_s^*(\vec{r})\Phi_s(\vec{r}')\rangle \propto e^{-\frac{|\vec{r}-\vec{r}'|}{\lambda_s}}$. but $\lambda_\uparrow \neq \lambda_\downarrow$. In Region C, one of the Φ_s has quasi-long-range order while the other has only short range correlation. Full lines: Ising transitions, dashed or dot-dashed: BKT transitions

III. CONCLUSION

Starting from a Ginzburg-Landau theory for a two-component superconductor, we show that the vestigial broken time reversal symmetry state wtih no superconducting order parameter is possible in two spatial dimensions, provided that the parameters lies in the situable region. This is in strong contrast to the case in three spatial dimensions [25], where such as phase is in general not possible except for some extreme situations. We also obtain the effective ϕ^4 theory for this Ising transition in terms of the parameters entering this Ginsburg-Landau theory.

Similar calculations can be extended also to vestigial nematic order, governed by an order parameter $\vec{h} = (h_x, h_y)$. We have already shown in [25] that the vestigial nematic state is generally unstable in three spatial dimensions. Back to the present case of two spatial dimensions, calculations similar to Sec II can also be carried out. For example, we still have eq. (12) etc if we exchange h_z there by $|\vec{h}|$, provided we also replace $g_{1,2}$ there by $\frac{g_1+g_2}{2}$ and g_1 respectively (*c.f.* also [25]), hence $2g_1 - g_2$ in eq (13) by g_2. A vestigial nematic state thus requires $g_2 < 0$. b_{eff} is now proportional to $\frac{2g_1+3g_2}{g_1}$. The effective gradient energy has the form $\tilde{K}(\partial_i h_j)(\partial_i h_j)$ (note our eq (3) has no "spin-orbit" coupling) with coefficient $\tilde{K}$ given by the same as the expression below eq (23). Instead of an Ising transition, we expect a Kosterlitz-Thouless transition for $\vec{h}$ itself when $g_2 > -\frac{2}{3}g_1$, but a more complicated scenario is feasible if this inequality is not satisfied.

If $g_2 < 0$, we can also have "4e" superconductivity with "pairing" between fields $\Phi_\uparrow$ and $\Phi_\downarrow$. Vestigial "4e" state now corresponds to quasi-long range order of the product $\Phi_\uparrow\Phi_\downarrow$ ($\propto \Phi_1^2 + \Phi_2^2$) but without quasi-long range order of either Φ_s. When the gradient term is simply taken as in (3), the calculations for the effective free energy is entirely parallel to that of the nematic phase, as has already been pointed out in [11–13]. Discussion in the last paragraph also applies in this case with appropriate substitutions.

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Appendix A: Order of phase transition

We analyze this phase transition without expansion in $h_{0,z}$. We define $x_s = -\ln(1 - \frac{h_s}{\alpha})$, where we have chosen the sign so that x_s is an increasing function of h_s. All $h_s < \alpha$, hence $-\infty < x_s < \infty$ are acceptable. Employing $x = (x_\uparrow + x_\downarrow)/2$, and $y = (x_\uparrow - x_\downarrow)/2$, eq (11) can be written as

$$\frac{\Delta\mathcal{F}}{\alpha T/(4\pi K)} = 2\left[x + e^{-x}\cosh(y) - 1\right] + (2\tilde{g}_1 + \tilde{g}_2)x^2 + (2\tilde{g}_1 - \tilde{g}_2)y^2 \tag{A1}$$

where $\tilde{g}_{1,2} = \frac{Tg_{1,2}}{4\pi\alpha K}$.

The stationary point conditions are

$$0 = (1 - e^{-x}\cosh y) + (2\tilde{g}_1 + \tilde{g}_2)x \tag{A2}$$

and

$$0 = e^{-x}\sinh y - (\tilde{g}_2 - 2\tilde{g}_1)y \tag{A3}$$

The second equation is trivially satisfied by $y = 0$. For $y \neq 0$, we solve for x using this second equation and substitute back to the first to yield a single equation for y:

$$G = \frac{\frac{y}{\tanh y} - \frac{\alpha(T)}{\alpha(T_2)}}{\ln\left[\frac{\sinh y}{y}\frac{\alpha(T)}{\alpha(T_2)}\right]} \tag{A4}$$

with $G \equiv \frac{2\tilde{g}_1+\tilde{g}_2}{\tilde{g}_2-2\tilde{g}_1}$. Since we have $0 < 2\tilde{g} < \tilde{g}_2$, G decreases with increasing $\tilde{g}_2/\tilde{g}_1$. For $2\tilde{g}_1 < \tilde{g}_2 < 6\tilde{g}$, G lies between 2 and $+\infty$. For $6\tilde{g}_1 < \tilde{g}_2$, G lies between 1 and 2. Graphical solution shows that for $2 < G < \infty$, y vanishes for $\alpha > \alpha(T_2)$. A non-trivial solution for y starts from zero and grows with decreasing $\alpha < \alpha(T_2)$, thus a typical second order phase transition at $T = T_2$. For $1 < G < 2$, finite y solutions already exist at some $\alpha(T) > \alpha(T_2)$, and with decreasing $\alpha(T)$, one obtains two solutions, one with y decreasing and the other increasing with decrasing α. $\alpha(T_2)$ is the point at which the decreasing solution approaches $y = 0$, which shows a typical first order transition behavior. Note however in contrast to 3D, we have a local free energy minimum, not just a saddle point.

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[28] With $\Phi_{\uparrow,\downarrow} = \frac{1}{\sqrt{2}}(\Phi_1 \pm i\Phi_2)$, then $\mathcal{K} = \left[\alpha\Phi_\nu^*\Phi_\nu + K\left(\frac{\partial\Phi_\nu^*}{\partial x_i}\frac{\partial\Phi_\nu}{\partial x_i}\right)\right]$ while $\mathcal{H}_{int} = \frac{\beta_1}{2}(\Phi_\mu^*\Phi_\mu)(\Phi_\nu^*\Phi_\nu) + \frac{\beta_2}{2}(\Phi_\mu^*\Phi_\mu^*)(\Phi_\nu\Phi_\nu)$ with implicit sum for μ and ν over 1 and 2, and $\beta_1 = g_1$ and $\beta_2 = (g_2 - g_1)/2$. In [25], this was taken to be the model of a superconductor with order parameter (Φ_1, Φ_2) belonging to a two-dimensional representation. This model can also be viewed as describing two superconductors Φ_1 and Φ_2, with the β_2 term contains a contribution representing a pair tunneling term (as well as others).

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[31] The same calculation for 3D, with generally $I_n \equiv \frac{1}{L^d}\sum_{\vec{k}}\frac{1}{(\alpha+Kk^2)^n}$ and $d = 3$ instead of 2, shows that b_{eff} is always negative (given by $-\frac{T}{8\pi\alpha^{5/2}K^{3/2}}\frac{(g_2-2g_1)}{32g_2}$). This result corroborates that in [25], showing that we can only have a joint first order phase transition.

[32] It can be seen from eq (23) that $\tilde{K}$ is negative if $2g_1 > g_2$, and also at sufficiently high temperatures if $g_2 > 2g_1$. However, as explained already below eq (21), this does not lead to transitions to non-uniform states.

[33] This phase diagram cannot be quite right, even within mean-field theory for the Φ_s transitions. It predicts three second order phase transitions line meeting at a point with finite angles with respect to each other, and the finite $\Phi_\uparrow$ phase occupies a region with angle larger than π, violating the phase rules in [34] and [35] respectively. We however would not pursue this question further since the 2D nature requires us to consider phase fluctuations of Φ_s, to be discussed in the next paragraph.

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[41] In the scenarios of [37] and [38], the transition from the symmetric phase to the Z_2 broken phase is via a BKT transition, instead of an Ising transition we deduced for $g_2 > 2g_1$. In that case, for Fig 2b, there must also be an additional multicritical point separating the two regions.