OMEGA THEORMES FOR LOGARITHMIC DERIVATIVES OF ZETA AND L-FUNCTIONS NEAR THE 1-LINE

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Abstract

We establish an omega theorem for logarithmic derivative of the Riemann zeta function near the 1-line by resonance method. We show that the inequality $\left|\zeta'\left(\sigma_A+it\right)/\zeta\left(\sigma_A+it\right)\right| \geqslant \left(\left(e^A-1\right)/A\right)\log_2 T + O\left(\log_2 T/\log_3 T\right)$ has a solution $t\in[T^{\beta},T]$ for all sufficiently large T, where $\sigma_A = 1 - A/\log_2 T$. Furthermore, we give a conditional lower bound for the measure of the set of t for which the logarithmic derivative of the Riemann zeta function is large. Moreover, similar results can be generalized to Dirichlet L-functions.

Keywords. Omega theorems, Riemann zeta function, Dirichlet L-functions, Resonance methods.

1 Introduction

Let $\zeta(s)$ be the Riemann zeta function and $s=\sigma+it$, where σ and t are real numbers. The logarithmic derivative of the Riemann zeta function is one of the significant objects in the analytic number theory, as it plays a key role in the proof of the prime number theorem. The study of it can be traced back to the 1910s. In 1911, Landau [14] proved that there exists a positive constant k_1 such that $\left|\zeta'(s)/\zeta(s)\right| \geqslant k_1\log_2 t$ has a solution $s=\sigma+it$ in $\{\sigma+it : \sigma>1, t>\tau\}$ for any given $\tau>0$, where $\log_2 t := \log\log t$. In 1913, this result had been improved by Bohr and Landau [4]. They proved that there exists a positive constant k_2 such

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that $|\zeta'(s)/\zeta(s)| \geqslant (\log t)^{k_2\theta}$ has a solution $s=\sigma+it$ in $\{\sigma+it:\sigma\geqslant 1-\theta, t\geqslant\tau\}$ for any given $\theta\in\left(0,\frac{1}{2}\right)$ and for any given $\tau>0$.

Assuming the Riemann hypothesis(RH), Littlewood [15] showed the classical upper bound

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \leqslant \sum_{n\leqslant(\log t)^2}\frac{\Lambda(n)}{n^\sigma}+O\left((\log t)^{2-2\sigma}\right)$$

holds uniformly for $\frac{1}{2}+1/\log_2 t\leqslant\sigma\leqslant\frac{3}{2}$ and $t\geqslant 10$. Until now, this bound has not been improved significantly, especially its main terms. When assuming RH, Chirre, Valås Hagen and Simonič [7] gave the following conditional upper bound on the 1-line

$$\left|\frac{\zeta'}{\zeta}(1+it)\right| \leqslant 2\log_2 t-0.4989+5.35\frac{(\log_2 t)^2}{\log t},\ \forall t\geqslant 10^{30}.$$

In 2006, Ihara [8] focused on the study of the Euler-Kronecker constants of the cyclotomic fields. Values of $L'(1,\chi)/L(1,\chi)$ appear naturally in this study, and there is a certain correlation between them. This provides new motivation to study the logarithmic derivative of Dirichlet L-functions.

Assuming the Generalized Riemann hypothesis(GRH), Ihara, Kumar Murty and Mahoro Shimura [10] obtained the following conditional upper bound

$$\left|\frac{L'}{L}(1,\chi)\right| \leqslant 2\log_2 q+2(1-\log 2)+O\left(\frac{\log_2 q}{\log q}\right)$$

for all non-principal primitive characters $\chi\,(\mathrm{mod}\ q)$. Later, their result has been optimized greatly. For instance, assuming GRH, Chirre, Valås Hagen and Simonič [7] established the conditional upper bound as follows

$$\left|\frac{L'}{L}(1,\chi)\right| \leqslant 2\log_2 q-0.4989+5.91\frac{(\log_2 q)^2}{\log q},\ \forall q\geqslant 10^{30},$$

where χ is a primitive character modulo q. Note that $-0.4989<2(1-\log 2)=0.6137\cdots$. In fact, in the past years, there were many results on upper bounds of logarithmic derivatives of Dirichlet L-functions. For instance, we can see [19, 21, 23].

There are much fewer omega results, in other words, lower bounds of maximal values of logarithmic derivatives of $\zeta(s)$ and $L(s,\chi)$. Mourtada and Kumar Murty [18] proved that there are infinitely many fundamental discriminants D, regardless of whether they are positive or negative, such that $L'(1,\chi_D)/L(1,\chi_D)\geqslant\log_2|D|+O(1)$, where χ_D is the quadratic Dirichlet character of conductor D. The key point in proving this result is making use of the explicit formula in [9]. Furthermore, assuming GRH, they also proved that for x large enough, there are $\gg x^{\frac{1}{2}}$ primes $q\leqslant x$ such that $-L'(1,\chi_q)/L(1,\chi_q)\geqslant\log_2 x+\log_3 x+O(1)$, where $\log_3 x:=\log\log\log x$.

In 2023, Paul [20] gave that if $f\in S_k(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized Hecke eigenform, then there are infinitely many fundamental discriminants D and quadratic Dirichlet characters $\chi\,(\mathrm{mod}\ D)$ such that $L'/L(1,f,\chi)\geqslant\frac{1}{8}\log_2(Dk)+O(1)$, where we denote $L(s,f,\chi)$ as Dirichlet L-functions associated by f and $\chi\,(\mathrm{mod}\ D)$.

Yang [26] established several omega theorems for any $\sigma \in (\frac{1}{2}, 1]$, not just limited to the case $\sigma = 1$. Fix $\beta \in (0,1)$, he showed that

$$\max_{T^\beta \leqslant t \leqslant T} -\operatorname{Re}\frac{\zeta'}{\zeta}(1+it) \geqslant \log_2 T + \log_3 T + C_1 - \epsilon,$$

where $\epsilon \in (0,1)$ and C_1 is an explicit constant. In addition, when $\sigma \in \left(\frac{1}{2}, 1\right)$, he showed that

$$\max_{T^\beta \leqslant t \leqslant T} -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it) \geqslant C_3(\sigma)(\log T)^{1-\sigma}(\log_2 T)^{1-\sigma},$$

where $C_3(\sigma)$ is a constant. Similar results on Dirichlet L-functions had been given as well.

We take inspiration from Yang's exceptional work and harvest some similar results. We focus on omega theorems for logarithmic derivatives of $\zeta(s)$ and $L(s,\chi)$ near the 1-line. Specifically, we set $\sigma_A = 1 - A/\log_2 T$ for all sufficiently large T when we concentrate on the Riemann zeta function. Naturally, we set $\sigma_A = 1 - A/\log_2 q$ as $q \to \infty$ when we consider Dirichlet L-functions. To be honest, we can shift the proof of the theorems about $\zeta(s)$ to the conclusions on $L(s,\chi)$, as the methods of getting them are roughly same.

We have the following result for the logarithmic derivative of the Riemann zeta function near the 1-line.

Theorem 1.1. *Fix $\beta \in (0,1)$. Let A be any positive real number, $\sigma_A := 1 - A/\log_2 T$. Then for all sufficiently large T, we have*

$$\max_{T^\beta \leqslant t \leqslant T} -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma_A+it) \geqslant \frac{e^A-1}{A}\log_2 T + O\left(\frac{\log_2 T}{\log_3 T}\right).$$

In particular, for all sufficiently large T, we have

$$\max_{T^\beta \leqslant t \leqslant T} \left|\frac{\zeta'}{\zeta}(\sigma_A+it)\right| \geqslant \frac{e^A-1}{A}\log_2 T + O\left(\frac{\log_2 T}{\log_3 T}\right).$$

With the help of the above theorem and methods of [22, 26], we obtain the next theorem, which gives a conditional lower bound for the measure of the set $\mathcal{M}(T,x)$. By observing this set, we can show the distribution of values of logarithmic derivatives of the Riemann zeta function.

Theorem 1.2. *Assuming RH. Fix $\beta \in (0,1)$. Let $x > 0$ be given. Define the set $\mathcal{M}(T,x)$ as*

$$\mathcal{M}(T,x) = \left\{t \in [T^\beta, T] : -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma_A+it) \geqslant \frac{\log_2 T}{\log_3 T}\left(\frac{e^A-1}{A}\log_3 T - x\right)\right\}.$$

Then we have

$$\operatorname{meas}(\mathcal{M}(T,x)) \geqslant T^{1-(1-\beta)e^{-x}+o(1)}, \quad \textit{as} \quad T \to \infty.$$

Generalize the above results to Dirichlet L-functions, we obtain two similar theorems as follows.

Theorem 1.3. *Let A be any positive real number, $\sigma_A := 1 - A/\log_2 q$. Then for all sufficiently large prime q, we have*

$$\max_{\substack{\chi \neq \chi_0 \\ \chi (\bmod\ q)}} -\operatorname{Re}\frac{L'}{L}(\sigma_A,\chi) \geqslant \frac{e^A-1}{A}\log_2 q + O\left(\frac{\log_2 q}{\log_3 q}\right),$$

where χ_0 *is the principal character modulo* q*, and the above maximal is taken over all non-principal characters* χ *which modulo* q*. In particular, for all sufficiently large prime* q*, we have*

$$\max_{\substack{\chi\neq\chi_0\\ \chi (\mathrm{mod}\ \ q)}}\left|\frac{L'}{L}\left(\sigma_A,\chi\right)\right|\geqslant\frac{e^A-1}{A}\log_2 q+O\left(\frac{\log_2 q}{\log_3 q}\right).$$

Theorem 1.4. *Assuming GRH. Let* $x>0$ *be given. Define the set* $\mathcal{N}\left(q,x\right)$ *as*

$$\mathcal{N}\left(q,x\right)=\left\{\chi\left(\mathrm{mod}\ \ q\right):-\mathrm{Re}\,\frac{L'}{L}\left(\sigma_A,\chi\right)\geqslant\frac{\log_2 q}{\log_3 q}\left(\frac{e^A-1}{A}\log_3 q-x\right)\right\}.$$

Then for prime $q\to\infty$*, we have*

$$\#\mathcal{N}\left(q,x\right)\geqslant q^{1-e^{-x}+o(1)}.$$

We will make use of resonance methods to obtain our theorems in this paper. In 1988, the resonance method was first invented by Voronin [24]. This method is an effective and ingenious way to get the lower bounds of the complex functions by taking the help of some "resonators". But unfortunately, his work was not taken seriously at that time. About twenty years later, Soundararajan [22] introduced a broader and more practical method based on Voronin's work, which called short resonator methods. Later, Aistleitner [1] gave another long resonator methods, developing the resonance methods. We recommend readers to see [2, 3, 6, 22, 24] to get more information on resonance methods.

In this paper, we will employ long resonator methods. The main idea is to find a Dirichlet series $R\left(t\right)=\sum_{n\in\mathbb{N}}r\left(n\right)n^{-it}$, named resonator, to resonate with logarithmic derivatives of the Riemann zeta function and pick out its large values.

The outline of this paper is as follows. In Section 2 we present some preliminary lemmas. In Section 3 we prove Theorem 1.1 by the resonance method. In Section 4 we prove Theorem 1.2. In fact, the proof is a slightly modification of the proof of Theorem 1.1. The proof of Theorems 1.3 and 1.4 is provided in Sections 5 and 6, respectively.

Notation. In this paper, we denote p and q as prime numbers. We denote χ as Dirichlet characters and χ_0 as the principal character. We denote $\Lambda\left(n\right)$ as the Mangoldt function, and $\vartheta\left(x\right)=\sum_{p\leqslant x}\log p$ as the Chebyshev's function. In addition, we use the short-hand notations, $\log_2 T:=\log\log T$, $\log_3 T:=\log\log\log T$ and so on.

2 Preliminary Lemmas

First, we need an approximate formula for $\zeta'\left(\sigma_A+it\right)/\zeta\left(\sigma_A+it\right)$. We establish the following lemma, which is useful when combining with zero-density result of $\zeta\left(s\right)$. This lemma is similar to [26, Lemma 1].

Lemma 2.1. *Let* A *be any positive real number,* T *be sufficiently large real number. Set* $\sigma_A:=1-A/\log_2 T$*. Let* $Y\geqslant 3, t\geqslant Y+3$ *and* $\frac{1}{2}\leqslant\sigma_0\leqslant 1$*. Suppose that the rectangle* $\{s:\sigma_0<\mathrm{Re}\left(s\right)\leqslant 1,\left|\mathrm{Im}\left(s\right)-t\right|\leqslant Y+2\}$ *is free of zeros of* $\zeta\left(s\right)$*. Then for* $\sigma_A\left(\leqslant 3\right)$ *and any* $\xi\in\left[t-Y,t+Y\right]$*, we have*

$$\left|\frac{\zeta'}{\zeta}\left(\sigma_A+i\xi\right)\right|\ll\frac{\log t}{\sigma_A-\sigma_0}.$$

Further, for $\sigma_1 \in (\sigma_0, \sigma_A)$, *we have*

$$-\frac{\zeta'}{\zeta}(\sigma_A + it) = \sum_{n \leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A + it}} + O\left(\frac{\log t}{\sigma_1 - \sigma_0} Y^{\sigma_1 - \sigma_A} \log \frac{Y}{\sigma_A - \sigma_1}\right).$$

Proof. We write $\rho = \alpha + i\gamma, 0 < \alpha < 1$ as the non-trivial zeros of $\zeta(s)$. By [13, Lemma 8.2], we have

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_{|\gamma - t| \leqslant 1} \frac{1}{s - \rho} + O(\log|s| + 2).$$

Then we have

$$\left|\frac{\zeta'}{\zeta}(\sigma_A + i\xi)\right| \leqslant \sum_{|\gamma - \xi| \leqslant 1} \frac{1}{\sigma_A - \alpha} + O(\log \xi) \ll \frac{\log t}{\sigma_A - \sigma_0}$$

because of our assumption on the rectangle. Now we obtain the first claim.

Taking $c = -\sigma_A + 1 + 1/\log Y$ in Perron's formula of [13], we find the following asymptotic formula

$$\sum_{n \leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A + it}} = \frac{1}{2\pi i} \int_{c - iY}^{c + iY} -\frac{\zeta'(\sigma_A + it + s)}{\zeta(\sigma_A + it + s)} \frac{Y^s}{s} \mathrm{d}s + O\left(Y^{-\sigma_A} (\log Y)^2\right). \tag{2.1}$$

Note that if $\sigma_1 - \sigma_A \leqslant \mathrm{Re}(s) \leqslant c$ and $|\mathrm{Im}(s)| \leqslant Y$, then $\sigma_1 \leqslant \mathrm{Re}(\sigma_A + it + s) \leqslant 1 + 1/\log Y$ and $|\mathrm{Im}(\sigma_A + it + s) - t| \leqslant Y$. As the rectangle is a zero-free region of $\zeta(s)$, by the residue theorem, we can move the line of integration from the line $\mathrm{Re}(s) = c$ to the line $\mathrm{Re}(s) = \sigma_1 - \sigma_A$ to obtain

$$\begin{aligned}\frac{1}{2\pi i} \int_{c - iY}^{c + iY} -\frac{\zeta'(\sigma_A + it + s)}{\zeta(\sigma_A + it + s)} \frac{Y^s}{s} \mathrm{d}s = &-\frac{\zeta'(\sigma_A + it)}{\zeta(\sigma_A + it)} \\ &+ \frac{1}{2\pi i}\left(\int_{c - iY}^{\sigma_1 - \sigma_A - iY} + \int_{\sigma_1 - \sigma_A - iY}^{\sigma_1 - \sigma_A + iY} + \int_{\sigma_1 - \sigma_A + iY}^{c + iY}\right).\end{aligned} \tag{2.2}$$

For simplicity, we omitted integrands on the right-hand side of (2.2) without causing confusion.

According to the first claim of this lemma, the first and third integrals on the right-hand side of (2.2) are bounded by

$$\ll \int_{\sigma_1 - \sigma_A}^{1 - \sigma_A + \frac{1}{\log Y}} \frac{\log t}{\sigma_A + x - \sigma_0} \frac{Y^x}{Y} \mathrm{d}x \ll \frac{\log t}{(\sigma_1 - \sigma_0)\log Y} Y^{-\sigma_A}, \tag{2.3}$$

meanwhile, the second integral on the right-hand of (2.2) is bounded by

$$\ll \int_{-Y}^{Y} \frac{\log t}{\sigma_1 - \sigma_0} \frac{Y^{\sigma_1 - \sigma_A}}{\sqrt{(\sigma_1 - \sigma_A)^2 + \xi^2}} \mathrm{d}\xi \ll \frac{\log t}{\sigma_1 - \sigma_0} Y^{\sigma_1 - \sigma_A} \log \frac{Y}{\sigma_A - \sigma_1}. \tag{2.4}$$

To compute (2.4), we have divided the integral into two integrals , which are $|\xi| \leqslant 1$ and $1 < |\xi| \leqslant Y$. Combining (2.1), (2.2), (2.3) and (2.4), we obtain the second claim as well. $\square$

Similar to Lemma 2.1, we have the following lemma on Dirichlet L-functions. In fact, this lemma is similar to [26, Lemma 2].

Lemma 2.2. *Let A be any positive real number, $q \geqslant 3$ be a prime number, $\sigma_A := 1 - A/\log_2 q$. Let $Y \geqslant 3, -3q \leqslant t \leqslant 3q$ and $\frac{1}{2} \leqslant \sigma_0 < 1$. Suppose that the rectangle $\{s : \sigma_0 < \mathrm{Re}(s) \leqslant 1, |\mathrm{Im}(s) - t| \leqslant Y + 2\}$ is free of zeros of $L(s,\chi)$, where χ is a non-principal character modulo q. Then for $\sigma_A (\leqslant 3)$ and any $\xi \in [t - Y, t + Y]$, we have*

$$\left|\frac{L'}{L}(\sigma_A + i\xi, \chi)\right| \ll \frac{\log q}{\sigma_A - \sigma_0}.$$

Further, for $\sigma_1 \in (\sigma_0, \sigma_A)$, we have

$$-\frac{L'}{L}(\sigma_A + it, \chi) = \sum_{n \leqslant Y} \frac{\Lambda(n)\chi(n)}{n^{\sigma_A + it}} + O\left(\frac{\log q}{\sigma_1 - \sigma_0} Y^{\sigma_1 - \sigma_A} \log \frac{Y}{\sigma_A - \sigma_1}\right).$$

Proof. We write $\rho' = \alpha' + i\gamma', 0 < \alpha' < 1$ as the non-trivial zeros of $L(s,\chi)$. By [13, Lemma 11.4], we have

$$\left|\frac{L'}{L}(\sigma_A + i\xi, \chi)\right| \leqslant \sum_{|\gamma' - \xi| \leqslant 1} \frac{1}{\sigma_A - \alpha'} + O(\log q) \ll \frac{\log q}{\sigma_A - \sigma_0}.$$

The following steps are the same as in the proof of Lemma 2.1, so we omitted them. □

3 Proof of Theorem 1.1

We still need the zero-density result of $\zeta(s)$. Let $N(\sigma, T)$ denote the number of zeros of $\zeta(s)$ inside the rectangle $\{s : \mathrm{Re}(s) \geqslant \sigma, 0 < \mathrm{Im}(s) \leqslant T\}$. For $\frac{1}{2} \leqslant \sigma_0 \leqslant 1$ and $T \geqslant 2$, Ingham [11] obtained a classical result

$$N(\sigma_0, T) \ll T^{\frac{3(1-\sigma_0)}{2-\sigma_0}} \log^5 T.$$

Fix $\epsilon \in \left(0, \sigma_A - \frac{1}{2}\right)$. We set $\sigma_0 = \sigma_A - \epsilon$, $\sigma_1 = \sigma_0 + 1/\log Y$, and $Y = (\log T)^{20/\epsilon}$. Combining with Ingham's zero-density result of $\zeta(s)$ we establish the following approximate formula

$$-\frac{\zeta'}{\zeta}(\sigma_A + it) = \sum_{n \leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A + it}} + O\left((\log T)^{-18}\right), \ \forall t \in [T^\beta, T] \setminus \mathfrak{B}(T), \tag{3.1}$$

where $\mathfrak{B}(T)$ is a "bad" set satisfying

$$\mathrm{meas}(\mathfrak{B}(T)) \ll T^{\frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon}} (\log T)^{5+\frac{20}{\epsilon}}. \tag{3.2}$$

Next, we take $X = \kappa \log T \log_2 T$. Define $\Phi(t) = e^{-t^2/2}$ as in [5], we note that for any $\xi \in \mathbb{R}$, the Fourier transform of Φ is

$$\hat{\Phi}(\xi) := \int_{-\infty}^{\infty} \Phi(x) e^{-ix\xi} \mathrm{d}x = \sqrt{2\pi}\Phi(\xi),$$

which is always positive. Let $r(n)$ be a completely multiplicative function with values at primes as

$$r(p) = \begin{cases} 1 - X^{\sigma_A - 1}, & \text{if } p \leqslant X, \\ 0, & \text{if } p > X. \end{cases} \tag{3.3}$$

Define

$$R(t)=\prod_{p}\frac{1}{1-r(p)\,p^{-it}}=\sum_{n\in\mathbb{N}}r(n)\,n^{-it}.$$

Notice that

$$\log|R(t)|\leqslant\sum_{p\leqslant X}\log\left|\frac{1}{1-r(p)}\right|=\sum_{p\leqslant X}(1-\sigma_A)\log X,$$

where the first inequality holds by the fact that $\left|p^{it}-r(p)\right|\geqslant\left|\left|p^{it}\right|-|r(p)|\right|=|1-r(p)|$. By the prime number theorem, we obtain that

$$|R(t)|^2\leqslant T^{2A\kappa+o(1)}. \tag{3.4}$$

Define $M_2(R,T)$, $M_1(R,T)$, $I_2(R,T)$ and $I_1(R,T)$ as follows.

$$\begin{aligned}
M_2(R,T)&=\int_{T^\beta}^{T}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right)|R(t)|^2\,\Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t,\\
M_1(R,T)&=\int_{T^\beta}^{T}|R(t)|^2\,\Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t,\\
I_2(R,T)&=\int_{-\infty}^{+\infty}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right)|R(t)|^2\,\Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t,\\
I_1(R,T)&=\int_{-\infty}^{+\infty}|R(t)|^2\,\Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t.
\end{aligned}$$

Plainly, we have

$$\max_{T^\beta\leqslant t\leqslant T}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right)\geqslant\frac{M_2(R,T)}{M_1(R,T)}. \tag{3.5}$$

Clearly, we have the following crude bound by the definition of Y

$$\left|\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right)\right|\leqslant\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A}}\ll Y\ll T^{o(1)}. \tag{3.6}$$

By (3.4), (3.6) and the definition of the function Φ, we immediately obtain two upper bounds on partial integrals of $I_2(R,T)$

$$\begin{aligned}
\left|\int_{|t|\leqslant T^\beta}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right)|R(t)|^2\,\Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t\right|&\ll T^{\beta+2A\kappa+o(1)},\\
\left|\int_{|t|\geqslant T}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right)|R(t)|^2\,\Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t\right|&\ll 1.
\end{aligned}$$

As a result, we find the following approximate formula

$$2M_2(R,T)=I_2(R,T)+O\left(T^{\beta+2A\kappa+o(1)}\right). \tag{3.7}$$

Similarly,

$$2M_1(R,T)=I_1(R,T)+O\left(T^{\beta+2A\kappa+o(1)}\right). \tag{3.8}$$

To give a lower bound for $I_1(R,T)$, we expand the product of $R(t)$. Since the Fourier transform $\hat{\Phi}$ is always positive, we can only consider the contributions from the diagonal terms and drop off-diagonal terms:

$$I_1(R,T) = \int_{-\infty}^{+\infty} \sum_{n,m\in\mathbb{N}} r(n)\, r(m) \left(\frac{n}{m}\right)^{it} \Phi\left(\frac{t\log T}{T}\right) \mathrm{d}t \geqslant \sqrt{2\pi}\frac{T}{\log T}\sum_{n\in\mathbb{N}} r^2(n).$$

By properties of the multiplicative function $r(n)$, we have $r(1)=1$. Dropping all terms except $n=1$ in the summation, we immediately get the following crude lower bound

$$I_1(R,T) \geqslant \sqrt{2\pi}\frac{T}{\log T}\sum_{n\in\mathbb{N}} r^2(n) \geqslant \sqrt{2\pi}\frac{T}{\log T} \gg T^{1+o(1)}.$$

To make the error terms small, we require κ is a positive number satisfying

$$\beta + 2A\kappa < 1. \tag{3.9}$$

By (3.7), (3.8), $I_1(R,T) \gg T^{1+o(1)}$ and the trivial relation that $M_1(R,T) \leqslant I_1(R,T)$, we obtain

$$\frac{M_2(R,T)}{M_1(R,T)} \geqslant \frac{I_2(R,T)}{I_1(R,T)} + O\left(T^{\beta+2A\kappa-1+o(1)}\right). \tag{3.10}$$

Now we get

$$\max_{T^\beta\leqslant t\leqslant T} \operatorname{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right) \geqslant \frac{I_2(R,T)}{I_1(R,T)} + O\left(T^{\beta+2A\kappa-1+o(1)}\right) \tag{3.11}$$

by (3.5) and (3.10).

We still want to obtain the lower bound for $I_2(R,T)$, so we compute $I_2(R,T)$ directly

$$\begin{aligned}
I_2(R,T) &= \int_{-\infty}^{+\infty} \operatorname{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)}{n^{\sigma_A+it}}\right) |R(t)|^2\, \Phi\left(\frac{t\log T}{T}\right) \mathrm{d}t \\
&= \sum_{p^v\leqslant Y}\frac{\log p}{p^{v\sigma_A}} \operatorname{Re}\left(\int_{-\infty}^{+\infty} \sum_{n,m\in\mathbb{N}} r(n)\, r(m)\left(\frac{n}{p^v m}\right)^{it} \Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t\right) \\
&\geqslant \sum_{p\leqslant X}\frac{\log p}{p^{\sigma_A}} \int_{-\infty}^{+\infty} \sum_{\substack{k,m\in\mathbb{N}\\ n=pk}} r(n)\, r(m)\left(\frac{n}{pm}\right)^{it} \Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t \\
&= \left(\sum_{p\leqslant X}\frac{\log p}{p^{\sigma_A}} r(p)\right) \int_{-\infty}^{+\infty} \sum_{k,m\in\mathbb{N}} r(k)\, r(m)\left(\frac{k}{m}\right)^{it} \Phi\left(\frac{t\log T}{T}\right)\mathrm{d}t \\
&= \left(\sum_{p\leqslant X}\frac{\log p}{p^{\sigma_A}} r(p)\right) I_1(R,T)
\end{aligned}$$

by the definitions of $\Lambda(n)$ and $I_1(R,T)$. Thus, we have

$$\frac{I_2(R,T)}{I_1(R,T)} \geqslant \sum_{p\leqslant X}\frac{\log p}{p^{\sigma_A}} r(p). \tag{3.12}$$

Now we compute the summation on the right-hand side of the above inequality by the prime number theorem and partial summation as $T \to \infty$

$$\begin{aligned}\sum_{p \leqslant X} \frac{\log p}{p^{\sigma_A}} &= \vartheta(X) X^{-\sigma_A} - \int_2^X \vartheta(t) \,\mathrm{d}\left(t^{-\sigma_A}\right) + O(1) \\ &= X^{1-\sigma_A} + \sigma_A \int_2^X t^{-\sigma_A} \mathrm{d}t + O\left(\frac{X^{1-\sigma_A}}{\log X}\right) + O\left(\sigma_A \int_2^X \frac{t^{-\sigma_A}}{\log t} \mathrm{d}t\right) \\ &= \frac{1}{1-\sigma_A} X^{1-\sigma_A} + O\left(\frac{X^{1-\sigma_A}}{\log X}\right) + O\left(\frac{\sigma_A}{1-\sigma_A} \frac{X^{1-\sigma_A}}{\log X}\right) \\ &= \frac{1}{1-\sigma_A} X^{1-\sigma_A} + O\Big(\frac{1}{1-\sigma_A} \frac{X^{1-\sigma_A}}{\log X}\Big).\end{aligned}$$

To optimize the error terms, we use integration by parts. Furthermore, when $p \leqslant X$, the value of $r(p)$ is independent of p. So we have the following approximate formula

$$\sum_{p \leqslant X} \frac{\log p}{p^{\sigma_A}} r(p) = \left(\frac{1}{1-\sigma_A} X^{1-\sigma_A} + O\left(\frac{1}{1-\sigma_A} \frac{X^{1-\sigma_A}}{\log X}\right)\right)\left(1 - X^{\sigma_A - 1}\right).$$

Recall that $X = \kappa \log T \log_2 T$, for all sufficiently large T, we obtain

$$\sum_{p \leqslant X} \frac{\log p}{p^{\sigma_A}} r(p) = \frac{e^A - 1}{A} \log_2 T + O\left(\frac{\log_2 T}{\log_3 T}\right)$$

from the above asymptotic formula by directly calculation. By (3.12), we have that

$$\frac{I_2(R,T)}{I_1(R,T)} \geqslant \frac{e^A - 1}{A} \log_2 T + O\left(\frac{\log_2 T}{\log_3 T}\right). \tag{3.13}$$

By (3.2), (3.4) and (3.6), for the integrand of $M_2(R,T)$, the integration over $\mathcal{B}(T)$ is at most

$$\int_{\mathcal{B}(T)} \operatorname{Re}\left(\sum_{n \leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A + it}}\right) |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) \mathrm{d}t \ll T^{2A\kappa + \frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon} + o(1)}.$$

If we require

$$2A\kappa + \frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon} < 1, \tag{3.14}$$

this contribution will be negligible.

Let κ satisfies (3.9) and (3.14), combining (3.1), (3.11) and (3.13), we have

$$\max_{T^\beta \leqslant t \leqslant T} -\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma_A + it) \geqslant \frac{e^A - 1}{A} \log_2 T + O\left(\frac{\log_2 T}{\log_3 T}\right).$$

Now the proof of Theorem 1.1 is completed.

Remark 3.1. *In view of the definition of σ_A, we find that σ_A is close to the 1-line very much. Specifically, $|\sigma_A - 1| \ll 1/\log_2 T$ as $T \to \infty$. So we can use other more precise zero-density results. For instance, by* [12, Theorem 11.4]

$$N(\sigma, T) \ll T^{\frac{6(1-\sigma)}{5\sigma-1}}, \quad \frac{13}{17} \leqslant \sigma \leqslant 1$$

and

$$N(\sigma, T) \ll T^{2-2\sigma}, \ \frac{11}{14} \leqslant \sigma \leqslant 1.$$

Using the above results, we can optimize the upper bound for the measure of $\mathfrak{B}(T)$*, but there is no effect on our Theorem 1.1.*

4 Proof of Theorem 1.2

In fact, the proof is a slightly modification of the proof of Theorem 1.1. Set $Y = (\log T)^{20/\epsilon}$ and $X = \kappa \log T \log_2 T$ as in Section 3. Assuming RH, we can neglect the contribution from $\mathcal{B}(T)$, and immediately establish the following approximate formula from (3.1)

$$-\frac{\zeta'}{\zeta}(\sigma_A + it) = \sum_{n \leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A + it}} + O\left((\log T)^{-18}\right), \ \forall t \in [T^\beta, T]. \tag{4.1}$$

Set

$$\kappa = \frac{1-\beta}{2A} \exp\left(-x + (\log T)^{-E} + f(T)\right),$$

where $f(T) = \exp(-(\log_2 T))$, and E is slightly smaller than 18. Obviously, for all sufficiently large T, we obtain $\beta - 1 + 2A\kappa < \beta - 1 + (1-\beta)\exp\left(-\frac{1}{2}x\right) < 0$.

Define J_x and $\widetilde{J_x}$ as follows

$$J_x = \frac{e^A - 1}{A} \log_3 T - x + (\log T)^{-E} + \frac{1}{2} f(T),$$
$$\widetilde{J_x} = \frac{e^A - 1}{A} \log_3 T - x + (\log T)^{-E}.$$

Then by (3.10), (3.12) and (3.13), we have

$$\frac{M_2(R,T)}{M_1(R,T)} \geqslant \frac{\log_2 T}{\log_3 T} \cdot J_x \tag{4.2}$$

for all sufficiently large T.

Define V_x, W_x and Z_x as follows

$$V_x = \left\{ t \in [T^\beta, T] : \mathrm{Re}\left(\sum_{n \leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A + it}}\right) \leqslant \frac{\log_2 T}{\log_3 T} \cdot \widetilde{J_x} \right\},$$
$$W_x = \left\{ t \in [T^\beta, T] : \mathrm{Re}\left(\sum_{n \leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A + it}}\right) > \frac{\log_2 T}{\log_3 T} \cdot \widetilde{J_x} \right\},$$
$$Z_x = \left\{ t \in [T^\beta, T] : -\mathrm{Re}\, \frac{\zeta'}{\zeta}(\sigma_A + it) > \frac{\log_2 T}{\log_3 T} \cdot \left(\widetilde{J_x} - (\log T)^{-E}\right) \right\}.$$

By the definitions, we have $V_x \cap W_x = \varnothing$ and $V_x \cup W_x = [T^\beta, T]$. Recall the definition of

$M_2(R,T)$, we have

$$\begin{aligned} M_2(R,T) &= \int_{T^\beta}^{T} \mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A+it}}\right) |R(t)|^2 \Phi\left(\frac{t\log T}{T}\right) \mathrm{d}t \\ &= \left(\int_{V_x} + \int_{W_x}\right) \mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A+it}}\right) |R(t)|^2 \Phi\left(\frac{t\log T}{T}\right) \mathrm{d}t \\ &\leqslant \frac{\log_2 T}{\log_3 T} \cdot \widetilde{J_x} \cdot M_1(R,T) + \int_{W_x} \mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A+it}}\right) |R(t)|^2 \Phi\left(\frac{t\log T}{T}\right) \mathrm{d}t. \end{aligned}$$

Combining the above computation with (4.2), we have

$$\frac{\log_2 T}{2\log_3 T} f(T) \cdot M_1(R,T) \leqslant \int_{W_x} \mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A+it}}\right) |R(t)|^2 \Phi\left(\frac{t\log T}{T}\right) \mathrm{d}t. \tag{4.3}$$

We have already obtained $|R(t)|^2 \leqslant T^{2A\kappa+o(1)}$ from (3.4). Meanwhile, by the prime number theorem, we have

$$\left|\mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A+it}}\right)\right| \leqslant \sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A}} \ll \log Y \ll \log_2 T.$$

By above estimates and (4.1), we get

$$\int_{W_x} \mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A+it}}\right) |R(t)|^2 \Phi\left(\frac{t\log T}{T}\right) \mathrm{d}t \leqslant (\log_2 T)\, T^{2A\kappa+o(1)} \,\mathrm{meas}(W_x). \tag{4.4}$$

Combining (3.8) and $I_1(R,T) \geqslant T^{1+o(1)}$, we obtain $M_1(R,T) \geqslant T^{1+o(1)}$ because κ satisfies $\beta - 1 + 2A\kappa < 0$. Furthermore, by (4.3) and (4.4), we have

$$\mathrm{meas}(W_x) \geqslant \frac{\frac{1}{2} f(T) \log_2 T \cdot M_1(R,T)}{(\log_2 T \log_3 T)\, T^{2A\kappa+o(1)}} \geqslant T^{(1+o(1))(1-2A\kappa)}.$$

By the definition of κ, we have $\mathrm{meas}(W_x) \geqslant T^{1-(1-\beta)e^{-x}+o(1)}$. As W_x is a subset of Z_x, we get $\mathrm{meas}(Z_x) \geqslant \mathrm{meas}(W_x)$. So we show that Theorem 1.2 holds.

5 Proof of Theorem 1.3

We denote by G_q the set of Dirichlet characters $\chi \pmod q$. Long before, Gronwall and Titchmarsh (for instance, see [17, Theorem 11.3]) had given a classical result, which is about the zero-free region on Dirichlet L-functions. They proved that there exists a constant $C > 0$, such that for any $\chi \in G_q$, the region

$$\left\{ s = \sigma + it : \sigma > 1 - \frac{C}{\log(q(|t|+2))} \right\}$$

is free of zeros of $L(s,\chi)$, unless $\chi = \chi_e$ is a quadratic character. In this case, $L(s,\chi)$ has at most one zero in this region. We define $G_q^* = G_q \setminus \{\chi_0, \chi_e\}$.

Similar to Section 3, we need the zero-density result of $L(s,\chi)$. Let $N(\sigma,T,\chi)$ denote the number of zeros of $L(s,\chi)$ inside the rectangle $\{s:\mathrm{Re}(s)\geqslant\sigma, |\mathrm{Im}(s)|\leqslant T\}$. For $\frac{1}{2}\leqslant\sigma_0\leqslant 1$ and $T\geqslant 2$, Montgomery [16] obtained a similar zero-density result to Ingham's result on Dirichlet L-functions as follows

$$\sum_{\chi\in G_q} N(\sigma_0,T,\chi)\ll (qT)^{\frac{3(1-\sigma_0)}{2-\sigma_0}}\log^{14}(qT).$$

Fix $\epsilon\in\left(0,\sigma_A-\frac{1}{2}\right)$. We set $t=0,\ \sigma_0=\sigma_A-\epsilon/2,\ \sigma_1=\sigma_0+1/\log Y$ and $Y=(\log q)^{20/\epsilon}$ in Lemma 2.2. Taking $T=Y+2$ and combining Montgomery's zero-density result, we establish the following approximate formula

$$-\frac{L'}{L}(\sigma_A,\chi)=\sum_{n\leqslant Y}\frac{\Lambda(n)\chi(n)}{n^{\sigma_A}}+O\left((\log q)^{-18}\right),\ \forall\chi\in G_q^*\setminus B_q, \tag{5.1}$$

where B_q is a set of "bad" characters with cardinality satisfying

$$\#B_q\ll q^{\frac{3\left(1-\sigma_A+\frac{1}{2}\epsilon\right)}{2-\sigma_A+\frac{1}{2}\epsilon}}(\log q)^{O(1)}. \tag{5.2}$$

Let $X=\kappa\log q\log_2 q$. And we let $r(n)$ be the same function as in Section 3. Meanwhile, we define $R(\chi)=\sum_{n\in\mathbb{N}}r(n)\chi(n)$. By the prime number theorem, we can obtain the following bound similar to Section 3

$$|R(\chi)|^2\leqslant q^{2A\kappa+o(1)}. \tag{5.3}$$

Define $S_2(R,q)$, $S_1(R,q)$, $S_2^*(R,q)$ and $S_1^*(R,q)$ as follows

$$\begin{aligned}
S_2(R,q)&=\sum_{\chi\in G_q}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)\chi(n)}{n^{\sigma_A}}\right)|R(\chi)|^2,\\
S_1(R,q)&=\sum_{\chi\in G_q}|R(\chi)|^2,\\
S_2^*(R,q)&=\sum_{\chi\in G_q^*\setminus B_q}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)\chi(n)}{n^{\sigma_A}}\right)|R(\chi)|^2,\\
S_1^*(R,q)&=\sum_{\chi\in G_q^*\setminus B_q}|R(\chi)|^2.
\end{aligned}$$

Plainly, we have

$$\max_{\chi\in G_q^*\setminus B_q}\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)\chi(n)}{n^{\sigma_A}}\right)\geqslant\frac{S_2^*(R,q)}{S_1^*(R,q)}. \tag{5.4}$$

Meanwhile, we have

$$\mathrm{Re}\left(\sum_{n\leqslant Y}\frac{\Lambda(n)\chi(n)}{n^{\sigma_A}}\right)\ll q^{o(1)}. \tag{5.5}$$

By (5.2), (5.3) and (5.5), we obtain the following approximate formulas

$$\begin{aligned}
S_2(R,q)&=S_2^*(R,q)+O\left(q^{2A\kappa+\frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon}+o(1)}\right),\\
S_1(R,q)&=S_1^*(R,q)+O\left(q^{2A\kappa+\frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon}+o(1)}\right).
\end{aligned}$$

To give a lower bound for $S_1(R,q)$, we make use of the orthogonality of Dirichlet characters and the fact that $r(1)=1$:

$$\begin{aligned} S_1(R,q) &= \sum_{\chi\in G_q} |R(\chi)|^2 \\ &= \sum_{n,k\in\mathbb{N}} r(n)\,r(k) \sum_{\chi\in G_q} \chi(n)\,\bar{\chi}(k) \\ &= \phi(q) \sum_{\substack{n,k\in\mathbb{N} \\ n\equiv k (\bmod\ q) \\ \gcd(k,q)=1}} r(n)\,r(k) \\ &\geqslant (q-1)\sum_{n\in\mathbb{N}} r^2(n) \\ &\gg q^{1+o(1)}. \end{aligned}$$

Combining the above approximate formulas and the trivial relation $S_1^*(R,q) \leqslant S_1(R,q)$, we get

$$\frac{S_2^*(R,q)}{S_1^*(R,q)} \geqslant \frac{S_2(R,q)}{S_1(R,q)} + O\left(q^{2A\kappa+\frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon}-1+o(1)}\right). \tag{5.6}$$

To make the error terms small, we require

$$2A\kappa + \frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon} - 1 < 0. \tag{5.7}$$

By (5.4) and (5.6), we get

$$\max_{\chi\in G_q^*\setminus B_q} \operatorname{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)\,\chi(n)}{n^{\sigma_A}}\right) \geqslant \frac{S_2(R,q)}{S_1(R,q)} + O\left(q^{2A\kappa+\frac{3(1-\sigma_A+\epsilon)}{2-\sigma_A+\epsilon}-1+o(1)}\right). \tag{5.8}$$

We still want to obtain the lower bound for $S_2(R,q)$. We compute $S_2(R,q)$ directly

$$\begin{aligned} S_2(R,q) &= \sum_{\chi\in G_q} \operatorname{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)\,\chi(n)}{n^{\sigma_A}}\right) |R(\chi)|^2 \\ &= \sum_{p^v\leqslant Y} \frac{\log p}{p^{v\sigma_A}} \operatorname{Re}\left(\sum_{n,m\in\mathbb{N}} r(n)\,r(m) \sum_{\chi\in G_q} \chi(p^v n)\,\bar{\chi}(m)\right) \\ &\geqslant \sum_{p\leqslant Y} \frac{\log p}{p^{\sigma_A}} \phi(q) \sum_{\substack{n,m\in\mathbb{N} \\ pn\equiv m (\bmod\ q) \\ \gcd(m,q)=1}} r(n)\,r(m) \\ &\geqslant \left(\sum_{p\leqslant X} \frac{\log p}{p^{\sigma_A}} r(p)\right) \phi(q) \sum_{\substack{n,k\in\mathbb{N} \\ pn\equiv pk (\bmod\ q) \\ \gcd(pk,q)=1}} r(n)\,r(k). \end{aligned}$$

Since q is a prime, by [25] we have

$$\phi(q) \sum_{\substack{n,k\in\mathbb{N} \\ pn\equiv pk (\bmod\ q) \\ \gcd(pk,q)=1}} r(n)\,r(k) = \phi(q) \sum_{\substack{n,k\in\mathbb{N} \\ n\equiv k (\bmod\ q) \\ \gcd(k,q)=1}} r(n)\,r(k) = S_1(R,q).$$

So we obtain
$$\frac{S_2(R,q)}{S_1(R,q)} \geqslant \sum_{p\leqslant Y} \frac{\log p}{p^{\sigma_A}} r(p). \tag{5.9}$$

Recall $X = \kappa \log q \log_2 q$, as $q \to \infty$, we immediately get
$$\frac{S_2(R,q)}{S_1(R,q)} \geqslant \frac{e^A - 1}{A} \log_2 q + O\left(\frac{\log_2 q}{\log_3 q}\right). \tag{5.10}$$

Let κ satisfies (5.7), by (5.1), (5.8) and (5.10) we have
$$\max_{\substack{\chi\neq\chi_0 \\ \chi (\mathrm{mod}\ q)}} -\mathrm{Re}\,\frac{L'}{L}(\sigma_A,\chi) \geqslant \frac{e^A - 1}{A} \log_2 q + O\left(\frac{\log_2 q}{\log_3 q}\right).$$

Now the proof of Theorem 1.3 is completed.

6 Proof of Theorem 1.4

Again, the proof is a slightly modification of the proof of Theorem 1.3 and is similar as the proof of Theorem 1.2. Set $Y = (\log q)^{20/\epsilon}$ and $X = \kappa \log q \log_2 q$ as in Section 5. Assuming GRH, we can neglect the contribution from B_q, so we immediately establish the following approximate formula from (5.1)
$$-\frac{L'}{L}(\sigma_A,\chi) = \sum_{n\leqslant Y} \frac{\Lambda(n)\chi(n)}{n^{\sigma_A}} + O\left((\log q)^{-18}\right), \ \forall \chi \in G_q^*. \tag{6.1}$$

Meanwhile, we can rewrite the definitions of $S_2^*(R,q)$ and $S_1^*(R,q)$ as follows
$$S_2^*(R,q) = \sum_{\chi\in G_q^*} \mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)\chi(n)}{n^{\sigma_A}}\right) |R(\chi)|^2,$$
$$S_1^*(R,q) = \sum_{\chi\in G_q^*} |R(\chi)|^2.$$

We derive $|R(\chi)|^2 \leqslant q^{2A\kappa+o(1)}$ from (5.3), and
$$\mathrm{Re}\left(\sum_{n\leqslant Y} \frac{\Lambda(n)\chi(n)}{n^{\sigma_A}}\right) \leqslant \sum_{n\leqslant Y} \frac{\Lambda(n)}{n^{\sigma_A}} \ll \log Y \ll \log_2 q \tag{6.2}$$
by the prime number theorem. Naturally, we find that $S_1(R,q) = S_1^*(R,q) + O\left(q^{2A\kappa+o(1)}\right)$ and $S_1(R,q) \geqslant q^{1+o(1)}$.

We set
$$\kappa = \frac{1}{2A} \exp\left(-x + (\log q)^{-E} + f(q)\right),$$
where $f(q) = \exp(-(\log_2 q))$, and E is slightly smaller than 18. Obviously, as $q\to\infty$, we have $-1 + 2A\kappa < -1 + \exp\left(-\frac{1}{2}x\right) < 0$.

Define J_x and $\widetilde{J_x}$ as follows

$$J_x = \frac{e^A - 1}{A} \log_3 q - x + (\log q)^{-E} + \frac{1}{2} f(q),$$
$$\widetilde{J_x} = \frac{e^A - 1}{A} \log_3 q - x + (\log q)^{-E}.$$

Then by the new definitions of $S_2^*(R,q)$, $S_1^*(R,q)$, (5.9) and (5.10), we obtain that

$$\frac{S_2^*(R,q)}{S_1^*(R,q)} \geqslant \frac{\log_2 q}{\log_3 q} \cdot J_x \tag{6.3}$$

as $q \to \infty$.

Define V_x, W_x and Z_x as follows

$$V_x = \left\{ \chi \in G_q^* : \mathrm{Re} \left(\sum_{n \leqslant Y} \frac{\Lambda(n) \chi(n)}{n^{\sigma_A}} \right) \leqslant \frac{\log_2 q}{\log_3 q} \cdot \widetilde{J_x} \right\},$$
$$W_x = \left\{ \chi \in G_q^* : \mathrm{Re} \left(\sum_{n \leqslant Y} \frac{\Lambda(n) \chi(n)}{n^{\sigma_A}} \right) > \frac{\log_2 q}{\log_3 q} \cdot \widetilde{J_x} \right\},$$
$$Z_x = \left\{ \chi \in G_q^* : - \mathrm{Re} \frac{L'}{L}(\sigma_A, \chi) > \frac{\log_2 q}{\log_3 q} \cdot \left(\widetilde{J_x} - (\log T)^{-E} \right) \right\}.$$

Then we have

$$S_2^*(R,q) = \left(\sum_{\chi \in V_x} + \sum_{\chi \in W_x} \right) \mathrm{Re} \left(\sum_{n \leqslant Y} \frac{\Lambda(n) \chi(n)}{n^{\sigma_A}} \right) |R(\chi)|^2$$
$$\leqslant \frac{\log_2 q}{\log_3 q} \cdot \widetilde{J_x} \cdot S_1^*(R,q) + \sum_{\chi \in W_x} \mathrm{Re} \left(\sum_{n \leqslant Y} \frac{\Lambda(n) \chi(n)}{n^{\sigma_A}} \right) |R(\chi)|^2.$$

Combining the above computation with (6.3), we have

$$\frac{\log_2 q}{2 \log_3 q} f(q) \cdot S_1^*(R,q) \leqslant \sum_{\chi \in W_x} \mathrm{Re} \left(\sum_{n \leqslant Y} \frac{\Lambda(n) \chi(n)}{n^{\sigma_A}} \right) |R(\chi)|^2. \tag{6.4}$$

By (5.3), (6.1) and (6.2), we obtain

$$\sum_{\chi \in W_x} \mathrm{Re} \left(\sum_{n \leqslant Y} \frac{\Lambda(n) \chi(n)}{n^{\sigma_A}} \right) |R(\chi)|^2 \leqslant (\log_2 q)\, q^{2A\kappa + o(1)} \# (W_x). \tag{6.5}$$

Combining $S_1(R,q) = S_1^*(R,q) + O\left(q^{2A\kappa + o(1)}\right)$ and $S_1(R,q) \geqslant q^{1+o(1)}$, we obtain $S_1^*(R,q) \geqslant q^{1+o(1)}$ because κ satisfies $-1 + 2A\kappa < 0$. Furthermore, by (6.4) and (6.5), we get

$$\# W_x \geqslant \frac{\frac{1}{2} f(q) \log_2 q \cdot S_1^*(R,q)}{(\log_2 q \log_3 q)\, q^{2A\kappa + o(1)}} \geqslant q^{(1+o(1))(1-2A\kappa)}.$$

By the definition of κ, we know $\# W_x \geqslant q^{1-(1-\beta)e^{-x}+o(1)}$. As W_x is a subset of Z_x, we get $\# Z_x \geqslant \# W_x$. So we show that Theorem 1.4 holds.

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