

ON THE GROTHENDIECK DUALITY FOR THE SPACE OF HOLOMORPHIC SOBOLEV FUNCTIONS

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ABSTRACT. We describe the strong dual space $(\mathcal{O}^s(D))^*$ for the space $\mathcal{O}^s(D) = H^s(D) \cap \mathcal{O}(D)$ of holomorphic functions from the Sobolev space $H^s(D)$, $s \in \mathbb{Z}$, over a bounded simply connected plane domain D with infinitely differential boundary ∂D . We identify the dual space with the space of holomorphic functions on $\mathbb{C}^n \setminus \overline{D}$ that belong to $H^{1-s}(G \setminus \overline{D})$ for any bounded domain G , containing the compact \overline{D} , and vanish at the infinity. As a corollary, we obtain a description of the strong dual space $(\mathcal{O}_F(D))^*$ for the space $\mathcal{O}_F(D)$ of holomorphic functions of finite order of growth in D (here, $\mathcal{O}_F(D)$ is endowed with the inductive limit topology with respect to the family of spaces $\mathcal{O}^s(D)$, $s \in \mathbb{Z}$). In this way we extend the classical Grothendieck-Köthe-Sebastião e Silva duality for the space of holomorphic functions.

One of the first dualities in the spaces of holomorphic functions was discovered in 1950-'s independently by A. Grothendieck [1], G. Köthe [2] and J. Sebastião e Silva [3], who described the strong dual $(\mathcal{O}(D))^*$ for the space of holomorphic functions $\mathcal{O}(D)$ (endowed with the standard Frechét topology) in a bounded simply connected domain $D \subset \mathbb{C}$:

$$(0.1) \quad (\mathcal{O}(D))^* \cong \mathcal{O}(\hat{\mathbb{C}} \setminus D),$$

where $\mathcal{O}(\hat{\mathbb{C}} \setminus D)$ is the space of holomorphic functions on neighborhoods of the closed set $\mathbb{C} \setminus D$, vanishing at the infinity, endowed with the standard inductive limit topology of holomorphic functions on closed sets. One of the most general results, describing the duality for the spaces of solutions to elliptic differential operators with the topology of uniform convergence on compact sets, belong to A. Grothendieck, see [4, Theorems 3 and 4]; it is similar in a way to (0.1). Another general scheme of producing dualities for (both determined and overdetermined) elliptic systems was presented in [5]. It involves the concept of Hilbert space with reproducing kernel and the constructed pairings are closely related to the inner products of the used Hilbert spaces. However it works easily for formally self-adjoint strongly elliptic operators, while in general case the application of the scheme depends on the very subtle information regarding the properties of the reproducing kernel that is not always at hands. Actually, similar results (with the use of classical Bergmann reproducing kernel and pairing induced by the inner product of the Lebesgue space $L^2(D)$) was obtained by E. Straube [6, §3] for harmonic and holomorphic functions of finite order of growth of many complex variables. Paper [5] contains also description of a Grothendieck type duality for spaces of solutions of finite order of growth to strongly elliptic systems.

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In the present short note we describe a Grothendick type duality for the spaces $\mathcal{O}^s(D)$ of holomorphic Sobolev functions and $\mathcal{O}_F(D)$ of holomorphic functions of finite order of growth over a bounded simply connected plane domain D with infinitely differential boundary ∂D with the use the pairing induced by the inner product of the Lebesgue space $L^2(\partial D)$.

1. DUALITY FOR THE SPACE OF SOBOLEV HOLOMORPHIC FUNCTIONS

Let $L^2(D)$ be the Lebesgue space and $H^s(D)$, $s \in \mathbb{N}$, be the Sobolev space of functions over plane domain D , endowed with the standard inner products. As it is known the scale extends to all values $s > 0$, as the Sobolev-Slobodetskii scale. We denote by $H^{-s}(D)$, $s > 0$, the strong dual for the space $H_0^s(D)$ (i.e. for the closure of smooth functions with compact support in D in $H_0^s(D)$); the related pairing between elements of $H^{-s}(D)$ and $H^s(D)$ is induced by the inner product in the Lebesgue space $L^2(D)$. Denote by $h(D)$ the space of harmonic functions in D , set $h^s(s) = H^s(D) \cap h(D)$ and, similarly, $\mathcal{O}^s(s) = H^s(D) \cap \mathcal{O}(D)$, $s \in \mathbb{Z}$, where $\mathcal{O}(D)$ the space of holomorphic functions in D . By the standard a priori estimates for harmonic functions, $h^s(s)$ and $\mathcal{O}^s(s)$ are closed subspaces of $H^s(D)$, $s \in \mathbb{Z}$, see, for instance, [6, p. 568]. We note that a holomorphic function is harmonic and then $\mathcal{O}^s(D)$ is a closed subspace in $h^s(D)$. According to [6, Corollary 1.7], any element $u \in h^s(D)$ has a weak boundary value $u|_{\partial D}$ on ∂D belonging to $H^{s-1/2}(\partial D)$, $s \in \mathbb{Z}$. Of course, $u|_{\partial D}$ coincides with the usual trace of u on ∂D if $s \in \mathbb{N}$. It follows immediately from [6, Corollary 1.7] that for each $u \in h^s(D)$ the functional $\|u|_{\partial D}\|_{H^{s-1/2}(\partial D)}$ defines a norm on $h^s(D)$ that is equivalent to the standard one. As $\mathcal{O}^s(D) \subset h^s(D)$, we prefer to endow $\mathcal{O}^s(D)$ with the norm $\|u|_{\partial D}\|_{H^{s-1/2}(\partial D)}$.

In any case, $\mathcal{O}^s(s)$, $s \in \mathbb{Z}$, is a Hilbert space (because $\|\cdot\|_{H^{s-1/2}(\partial D)}$ possesses parallelogram property) and we immediately have the standard Riesz duality with the pairing related to the corresponding inner product:

$$(1.1) \quad (\mathcal{O}^s(D))^* \cong \mathcal{O}^s(D).$$

However we want to produce a Grothendieck type duality for $\mathcal{O}^s(D)$. With this purpose, denote by $\mathcal{O}^s(\hat{\mathbb{C}} \setminus \overline{D})$, $s \in \mathbb{Z}$, the space of holomorphic functions in $\mathbb{C} \setminus \overline{D}$ vanishing at the infinity that belong to $H^s(G \setminus \overline{D})$ for any bounded domain G , containing the compact \overline{D} . By the discussion above, any element $v \in \mathcal{O}^s(\hat{\mathbb{C}} \setminus \overline{D})$ has a weak boundary value $v|_{\partial D}$ on ∂D belonging to $H^{s-1/2}(\partial D)$. Then, taking into the account the connection between the interior and exterior Dirichlet problems for the Laplace operator, for each $v \in \mathcal{O}^s(\hat{\mathbb{C}} \setminus \overline{D})$ functional $\|v|_{\partial D}\|_{H^{s-1/2}(\partial D)}$ defines a norm on $\mathcal{O}^s(\hat{\mathbb{C}} \setminus \overline{D})$ and, by the discussion above, $\mathcal{O}^s(\hat{\mathbb{C}} \setminus \overline{D})$ is Hilbert space.

Theorem 1.1. *Let D be a bounded simply connected domain with C^∞ -smooth boundary. Then for each $s \in \mathbb{Z}$ we have an isomorphism of Banach spaces:*

$$(1.2) \quad (\mathcal{O}^s(D))^* \cong \mathcal{O}^{1-s}(\hat{\mathbb{C}} \setminus \overline{D}).$$

Proof. We begin with the description of the related pairing. First, we note that since ∂D is a compact, then $H^{s'}(\partial D) = H_0^{s'}(\partial D)$ for each $s' \in \mathbb{R}$. Hence there is a natural duality

$$(1.3) \quad H^{-s'}(\partial D) \cong H^{s'}(\partial D), \quad s' \in \mathbb{R},$$

with the pairing

$$\langle \cdot, \cdot \rangle_{\partial D, s'} : H^{-s'}(\partial D) \times H^{s'}(\partial D) \rightarrow \mathbb{C},$$

induced by the inner product in $L^2(\partial D)$. In particular,

$$(1.4) \quad |\langle u, v \rangle_{\partial D, s'}| \leq \|u\|_{H^{s'}(\partial D)} \|v\|_{H^{-s'}(\partial D)} \text{ for all } v \in H^{-s'}(\partial D), u \in H^{s'}(\partial D).$$

For the sake of notations we drop the index s' in the pairing.

Thus, for each $s \in \mathbb{Z}$ we obtain a natural pairing

$$(1.5) \quad \langle u|_{\partial D}, v|_{\partial D} \rangle_{\partial D} : \mathcal{O}^s(D) \times \mathcal{O}^{1-s}(\hat{\mathbb{C}} \setminus \overline{D}) \rightarrow \mathbb{C},$$

inducing a continuous (conjugate-) linear mapping

$$(1.6) \quad \mathcal{O}^{1-s}(\hat{\mathbb{C}} \setminus \overline{D}) \ni v \rightarrow f_v \in (\mathcal{O}^s(D))^*, \quad f_v(u) = \langle u|_{\partial D}, v|_{\partial D} \rangle_{\partial D}.$$

As (1.3) is an isomorphism of normed spaces, we see that

$$\|f_v\|_{(\mathcal{O}^s(D))^*} = \|v|_{\partial D}\|_{H^{s-1/2}(\partial D)}.$$

Now we note that the integral Cauchy formula may be extended to the elements of the space $\mathcal{O}^s(D)$ with the use of the notion of the weak boundary values. Namely, for a distribution $u_0 \in H^{s-1/2}(\partial D)$ denote by Ku_0 its integral Cauchy transform:

$$(Ku_0)(z) = \frac{1}{2\pi\iota} \langle (\cdot - z)^{-1}, \overline{u_0} \rangle_{\partial D}, \quad z \notin \partial D,$$

where ι is the imaginary unit. Of course, $Ku_0(z)$ is just the Cauchy integral for u_0 if $s \in \mathbb{N}$. Then for any $u \in \mathcal{O}^s(D)$ we have

$$(1.7) \quad (Ku|_{\partial D})(z) = \begin{cases} 0, & z \notin \overline{D}, \\ u(z) & z \in D; \end{cases}$$

see, for instance, [7], [8] even for the Martinelli-Bochner integral in \mathbb{C}^n . Similarly, taking into the account the behaviour at the infinity and the orientation of the curve ∂D , for elements $v \in \mathcal{O}^{1-s}(\hat{\mathbb{C}} \setminus \overline{D})$ we have

$$(1.8) \quad -(Kv|_{\partial D})(z) = \begin{cases} 0, & z \in D, \\ v(z) & z \notin \overline{D}. \end{cases}$$

It follows from (1.8) that if $f_v(u) = 0$ for all $u \in \mathcal{O}^s(D)$ then, as the kernel $(\zeta - z)^{-1}$ is holomorphic in D with respect to ζ for all $z \notin \overline{D}$, we conclude that

$$0 = \langle (\cdot - z)^{-1}, v|_{\partial D} \rangle_{\partial D} = 2\pi\iota (Kv|_{\partial D})(z) = 2\pi\iota v(z) \text{ for all } z \notin \overline{D},$$

i.e. mapping (1.6) is injective.

To finish the proof we have to show that mapping (1.6) is surjective. As we noted above, the space $\mathcal{O}^s(D)$ can be treated as a closed subset of the Hilbert space $H^{s-1/2}(\partial D)$. Then, by Hahn-Banach theorem and Riesz theorem on functionals on Hilbert spaces, for any functional $f \in (\mathcal{O}^s(D))^*$ there is a function $v_0 = v_0(f) \in H^{1/2-s}(\partial D)$ such that

$$f(u) = \langle u|_{\partial D}, v_0 \rangle_{\partial D} \text{ for all } u \in \mathcal{O}^s(D).$$

Next, denote by $(Kv_0)^-$ the restriction of the integral Cauchy transform to D and $(Kv_0)^+$ its restriction to $\mathbb{C} \setminus \overline{D}$. Then theorems on the boundedness of potentials, see [9, §2.3.2.5], and the structure of the Cauchy kernel yield $(Kv_0)^- \in \mathcal{O}^{1-s}(D)$, $(Kv_0)^+ \in \mathcal{O}^{1-s}(\hat{\mathbb{C}} \setminus \overline{D})$. Now, by the weak jump theorem of the Cauchy transform, see [8], we have in the sense of weak boundary values:

$$(Kv_0)|_{\partial D}^- - (Kv_0)|_{\partial D}^+ = v_0 \text{ on } \partial D.$$

Clearly, by the definition of the weak boundary values and the classical Cauchy theorem, we have

$$\langle u|_{\partial D}, (Kv_0)|_{\partial D}^- \rangle_{\partial D} = 0 \text{ for all } u \in \mathcal{O}^s(D).$$

Therefore

$$f(u) = -\langle u|_{\partial D}, (Kv_0)|_{\partial D}^+ \rangle_{\partial D} \text{ for all } u \in \mathcal{O}^s(D),$$

and then mapping (1.6) is surjective (i.e. $v = v(f) = -(Kv_0)^+ \in \mathcal{O}^{1-s}(\hat{\mathbb{C}} \setminus \overline{D})$). \square

2. HOLOMORPHIC FUNCTIONS OF FINITE ORDER OF GROWTH

One says that a function $u \in h(D)$ has a finite order of growth near ∂D , if for each point $z_0 \in \partial D$ there are positive numbers γ , C and R such that

$$|u(z)| \leq C|z - z_0|^{-\gamma} \text{ for all } z \in D, |z - z_0| < R.$$

The space of such functions we denote by $h_F(D)$. E. Straube [6] proved that

$$h_F(D) = \cup_{s \in \mathbb{Z}} h^s(D)$$

and hence we may endow the space with the inductive limit topology with respect to the family $\{h^s(D)\}_{s \in \mathbb{Z}}$ of Banach spaces, see, for instance, [10, §6]. Again, as $\mathcal{O}(D) \subset h(D)$, we obtain

$$(2.1) \quad \mathcal{O}_F(D) = \cup_{s \in \mathbb{Z}} \mathcal{O}^s(D);$$

we endow this space of holomorphic functions of finite order of growth near ∂D with the same topology as $h_F(D)$. According to [10, Ch. 4, Exercise 24e], $\mathcal{O}_F(D)$ is a DF-space and then its dual is expected to be a Fréchet space, see [10, Ch. 4, Exercise 24a]. Thus, we denote by $\mathcal{O}(\hat{\mathbb{C}} \setminus \overline{D})$ the space of holomorphic functions in $\mathbb{C} \setminus \overline{D}$ vanishing at the infinity. By the Sobolev Embedding Theorem,

$$(2.2) \quad \mathcal{O}(\hat{\mathbb{C}} \setminus \overline{D}) \cap C^\infty(\mathbb{C} \setminus D) = \cap_{s \in \mathbb{Z}} \mathcal{O}^s(\hat{\mathbb{C}} \setminus \overline{D}).$$

We endow the space with the projective limit topology with respect to the family $\{\mathcal{O}^s(\hat{\mathbb{C}} \setminus \overline{D})\}_{s \in \mathbb{Z}}$ of the Banach spaces, see [6, Ch. I, §5]. Thus, $\mathcal{O}(\hat{\mathbb{C}} \setminus \overline{D})$ is a Fréchet space, see [6, Ch. II, §6, Corollary 1].

Theorem 2.1. *Let D be a bounded simply connected domain with C^∞ -smooth boundary. Then we have a topological isomorphism:*

$$(2.3) \quad (\mathcal{O}_F(D))^* \cong \mathcal{O}(\hat{\mathbb{C}} \setminus \overline{D}) \cap C^\infty(\mathbb{C} \setminus D).$$

Proof. It follows almost immediately from Theorem 2.1. Indeed, as $v|_{\partial D} \in C^\infty(\partial D)$ for each $\mathcal{O}(\hat{\mathbb{C}} \setminus \overline{D}) \cap C^\infty(\mathbb{C} \setminus D)$, formulae (2.1) and (2.2) imply that (1.5) defines a sesquilinear pairing

$$(2.4) \quad \langle u|_{\partial D}, v|_{\partial D} \rangle_{\partial D} : \mathcal{O}_F(D) \times \mathcal{O}(\hat{\mathbb{C}} \setminus \overline{D}) \cap C^\infty(\mathbb{C} \setminus D) \rightarrow \mathbb{C}.$$

Again, taking into the account the topologies of the space and inequality (1.4), we may define continuous mapping

$$(2.5) \quad \mathcal{O}(\hat{\mathbb{C}} \setminus \overline{D}) \cap C^\infty(\mathbb{C} \setminus D) \ni v \rightarrow f_v \in (\mathcal{O}_F(D))^*, \quad f_v(u) = \langle u|_{\partial D}, v|_{\partial D} \rangle_{\partial D}.$$

Its injectivity and surjectivity follow by the same arguments as in the proof of Theorem 2.1. Finally, the continuity of the inverse mapping follows from the Closed Graph Theorem for Fréchet-spaces, see [10, Ch. 3, Theorem 2.3]. \square

Similarly, we obtain the following statement.

Theorem 2.2. *Let D be a bounded simply connected domain with C^∞ -smooth boundary. Then we have a topological isomorphism:*

$$(2.6) \quad (\mathcal{O}(D) \cap C^\infty(\overline{D}))^* \cong \mathcal{O}_F(\hat{\mathbb{C}} \setminus \overline{D}).$$

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