GENERALIZED QUANTIFIERS USING TEAM SEMANTICS

FREDRIK ENGSTRÖM

ABSTRACT. Dependence logic provides an elegant approach for introducing dependencies between variables into the object language of first-order logic. In [Eng12] generalized quantifiers were introduced in this context. However, a satisfactory account was only achieved for monotone increasing generalized quantifiers.

In this paper, we modify the fundamental semantical guideline of dependence logic to create a framework that adequately handles both monotone and non-monotone generalized quantifiers. We demonstrate that this new logic can interpret dependence logic and possesses the same expressive power as existential second-order logic (ESO) on the level of formulas. Additionally, we establish truth conditions for generalized quantifiers and prove that the extended logic remains conservative over first-order logic with generalized quantifiers and is able to express the branching of continuous generalized quantifiers.

1. INTRODUCTION

Dependence logic [Vää07] extends first-order logic by dependence atoms of the form

$$D(t_1, \ldots, t_n).$$

These atoms express that the value of the term t_n is functionally determined by the values of $t_1, \ldots, t_{n-1}$.

While in first-order logic, the order of quantifiers alone determines the dependence relations between variables, dependence logic allows for more general and non-linear dependencies between variables. Remarkably, dependence logic is equivalent in expressive power to existential second-order logic (ESO). Historically, it was preceded by the partially ordered quantifiers (known as Henkin quantifiers) introduced by Henkin [Hen61], as well as the Independence-Friendly (IF) logic developed by Hintikka and Sandu [HS89].

The semantical framework of dependence logic, known as team semantics, has proven to be highly flexible. It accommodates not only dependence atoms, but also other intriguing generalizations. For instance, variants of dependence atoms were introduced and explored in works such as [Eng12], [GV12] and [Gal12].

In [Eng12], I investigated extensions of dependence logic using generalized quantifiers. I introduced a general schema for extending dependence logic with these quantifiers. Subsequently, this schema was further examined in subsequent works, including [EK13] and [EKV13]. This extension allows us to express branching behavior of generalized quantifiers–a construct that naturally arises in natural language [Bar79]. Moreover, it provides a means to express different scope readings of generalized quantifiers in a natural and intuitive manner, as exemplified in [AD22].

Key words and phrases. Team semantics, Dependence logic, Generalized quantifiers, Henkin quantifiers.

In the present paper, we take a slightly different approach and redefine the semantics of dependence logic to incorporate non-monotone generalized quantifiers into the logic. The traditional semantics of dependence logic relies on the following guiding principle.

In the present paper we take a slightly different approach and redefine the semantics of dependence logic to be able to incorporate non-monotone generalized quantifiers into the logic. The traditional semantics of dependence logic is based on the following guiding principle, or guideline.

Guideline 1. A formula φ is satisfied by a team X if for every assignment $s : \mathrm{dom}(X) \to M^k$, if $s \in X$ then s satisfies φ.

This principle establishes the semantics for all formulas where it makes sense to say that a single assignment satisfies the formula, i.e., for all formulas that do not contain the dependence atom. From this principle, the semantical clauses of dependence logic naturally follow.

In this paper we will instead base our semantics on the following alternative guideline.

Guideline 2. A formula φ is satisfied by a team X if for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ iff s satisfies φ,

This guideline replaces the implication in Guideline 1 with an equivalence. In the language of first-order logic, nothing exciting happens; it is only when we introduce new atoms, like dependence atoms, or new logical operations that interesting things start to happen.

Dependence logic exhibits several appealing basic properties, including closure under taking subteams and locality (where assignments of variables not occurring in a formula are irrelevant for determining satisfaction). However, this new logic is not closed under subteams and does not strictly adhere to the usual notion of locality. Instead, it possesses a weaker locality-like property, as demonstrated by Proposition 2.4.

In the present section, we introduce the semantics of dependence logic in a slightly non-standard manner. We prove that this alternative semantics is equivalent to the standard one. In the subsequent section, we modify the truth conditions somewhat to create a new logic and prove that this new logic matches the expressive power of existential second-order logic, and consequently, with that of dependence logic. Finally, in the last section, we extend the logic by introducing generalized quantifiers and demonstrate that the resulting system is conservative over first-order logic and prove that it can express branching of continuous generlized quantifiers.

1.1. **Dependence logic.** In this section, we provide a brief introduction to dependence logic, where satisfaction is defined in a non-standard but equivalent manner. For a more detailed account of dependence logic, please refer to [Vää07].

The syntax of dependence logic extends the syntax of first-order logic by introducing new atomic formulas known as dependence atoms. There is one dependence atom for each arity. We denote the dependence atom expressing that the term t_n is uniquely determined by the values of the terms $t_1, \dots, t_{n-1}$ by $D(t_1, \dots, t_n)$.[1] We assume that all formulas of dependence logic are written in negation normal form, meaning that all negations in formulas appear in front of atomic formulas. Given

[1]The dependence atom is often also denoted by $=(t_1, \dots, t_n)$.

a vocabulary τ, $\mathrm{d}[\tau]$ denotes the set of τ-formulas of dependence logic, i.e., $\mathrm{d}[\tau]$ is the set described by

$$\varphi ::= \mathrm{At} \mid \neg\mathrm{At} \mid D(t_1,\dots,t_n) \mid \varphi \mathbin{/\!\!\!\wedge} \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi,$$

where At is an atomic formula in the vocabulary τ.

We denote conjunction and disjunction with $\mathbin{/\!\!\!\wedge}$ and $\mathbin{\vee\!\!\!\vee}$ respectively because we intend to introduce the operators $\wedge$ and $\vee$ with a different meaning later in the paper.

The set of free variables of a formula is defined as in first-order logic by treating the dependence atoms as any other atom. The set of free variables of a formula φ is denoted by $\mathrm{fv}(\varphi)$. We say that a formula is first-order if it contains no dependence atom.

To define a compositional semantics for dependence logic we employ *sets of assignments* known as *teams*, rather than single assignments as in first-order logic. An assignment is a function $s : V \to M$, where V is a finite set of variables and M is the universe under consideration. Given a universe M, a team X over M is a pair of a finite set of variables, denoted $\mathrm{dom}(X)$, and a subset of the function space $\mathrm{dom}(X) \to M$. We will use an abuse of notation and refer to the set of functions as X. Thus, we think of a team as a set of assignments along with a set $\mathrm{dom}(X)$ that specifies the domain of these assignments. Clearly, the set $\mathrm{dom}(X)$ is determined by X in all cases except when X is empty.

If $V = \emptyset$ there is only one assignment: the empty assignment, denoted by ϵ. It is important to observe that the team consisting of the empty assignment $\{\,\epsilon\,\}$ is different from each empty team; remember, there is one empty team for each finite set of variables. We denote an empty team by $\emptyset_V$, where V is the domain of the team.

Given an assignment $s : V \to M$ and $a_1,\dots,a_k \in M$ let

$$s[a_1,\dots,a_k/x_1,\dots,x_k] : V \cup \{\,x_1,\dots,x_k\,\} \to M$$

be the assignment

$$s[a_1,\dots,a_k/x_1,\dots,x_k] : y \mapsto \begin{cases} s(y) & \text{if } y \in V \setminus \{\,x_1,\dots,x_k\,\}, \text{ and} \\ a_i & \text{if } y = x_i. \end{cases}$$

We use the notation $\bar a$ as a shorthand for a finite sequence $a_1,\dots,a_k$, but we also treat $\bar a$ as the finite set of the a_i's when appropriate. The interpretation of the term t in the model $\mathbb{M}$ under the assignment s is denoted by $t^{\mathbb{M},s}$. Furthermore, the tuple obtained by point-wise application of s to the finite sequence $x_1,\dots,x_k$ is denoted by $s(\bar x)$.

For assignments $s : V \to M$ and first-order formulas φ such that $\mathrm{fv}(\varphi) \subseteq V$ we denote the ordinary Tarskian satisfaction by $\mathbb{M}, s \vDash \varphi$. If $\mathrm{fv}(\varphi) \subseteq \{\,\bar x\,\}$ then the semantic value or denotation of φ is given by

$$[\![\varphi]\!]^{\mathbb{M}}_{\bar x} = \{\, s : \{\,\bar x\,\} \to M \mid \mathbb{M}, s \vDash \varphi \,\},$$

a team with domain $\{\,\bar x\,\}$.

Just as union and intersection correspond to disjunction and conjunction, there are operations on denotations that correspond to quantification:

$$\exists x X = \{\, s : \mathrm{dom}(X) \setminus \{\,x\,\} \to M \mid \exists a \in M : s[a/x] \in X \text{ or } s \in X \,\}$$

and

$$\forall x X = \{ s : \mathrm{dom}(X) \setminus \{ x \} \to M \mid \forall a \in M : s[a/x] \in X \text{ or } s \in X \}.$$

Please note that, for these operations to be properly defined, we need to add that

$$\mathrm{dom}(\exists x X) = \mathrm{dom}(\forall x X) = \mathrm{dom}(X) \setminus \{ x \},$$

which is important in the case when $\exists x X = \emptyset_V$. Also, observe that $\exists x X = \forall x X = X$ if $x \notin \mathrm{dom}(X)$.

Lemma 1.1. *Let* $\mathrm{fv}(\varphi) \subseteq \bar{y}$ *and* φ *a first-order formula, then*

- $\exists x \, [\![\varphi]\!]^{\mathbb{M}}_{\bar{y}} = [\![\exists x \varphi]\!]^{\mathbb{M}}_{\bar{y} \setminus \{ x \}}$ *and*
- $\forall x \, [\![\varphi]\!]^{\mathbb{M}}_{\bar{y}} = [\![\forall x \varphi]\!]^{\mathbb{M}}_{\bar{y} \setminus \{ x \}}$.

Proof. Easy to check directly, but note that if $x \notin \bar{y}$, then $[\![\exists x \varphi]\!]^{\mathbb{M}}_{\bar{y}} = [\![\varphi]\!]^{\mathbb{M}}_{\bar{y}} = \exists x \, [\![\varphi]\!]^{\mathbb{M}}_{\bar{y}}$. □

Let us now turn to the satisfaction relation for dependence logic. The following definition, while not the conventional one found in the literature, is equivalent to it, see Proposition 1.3.

Definition 1.2. The satisfaction relation for dependence logic $\mathbb{M}, X \vDash \varphi$, for $\mathrm{fv}(\varphi) \subseteq \mathrm{dom}(X)$ and $\varphi \in D[\tau]$ is defined as follows.

(1) $\mathbb{M}, X \vDash \psi$ iff $\forall s{\in}X : \mathbb{M}, s \vDash \psi$, for first-order atomic or negated atomic formulas ψ.
(2) $\mathbb{M}, X \vDash D(t_1, \ldots, t_{n+1})$ iff $\forall s, s'{\in}X \bigwedge_{1 \leq i \leq n} t_i^{\mathbb{M},s} = t_i^{\mathbb{M},s'} \to t_{n+1}^{\mathbb{M},s} = t_{n+1}^{\mathbb{M},s'}$.
(3) $\mathbb{M}, X \vDash \varphi \mathbin{/\!\!\!\wedge} \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$, and both $\mathbb{M}, Y \vDash \varphi$ and $\mathbb{M}, Z \vDash \psi$.
(4) $\mathbb{M}, X \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$, and both $\mathbb{M}, Y \vDash \varphi$ and $\mathbb{M}, Z \vDash \psi$.
(5) $\mathbb{M}, X \vDash \exists x \varphi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \exists x Y = \exists x X$ and $\mathbb{M}, Y \vDash \varphi$.
(6) $\mathbb{M}, X \vDash \forall x \varphi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \forall x Y = \exists x X$ and $\mathbb{M}, Y \vDash \varphi$.

Note that $X = Y \cup Z$ and $X = Y \cap Z$ implies that $\mathrm{dom}(X) = \mathrm{dom}(Y) = \mathrm{dom}(Z)$.

Also, note that in (5) and (6) we use $\exists x X$ instead of just X, which might seem unconventional. However, this choice is deliberate to handle cases where $x \in \mathrm{dom}(X)$: if $x \notin \mathrm{dom}(X)$ then $\exists x X = X$. The decision to use $\exists x$ as the arity-reducing operation applied to X will become clearer in the argument below, following the proof of Proposition 1.3.

The operations $\exists x$ and $\forall x$ are arity-reducing operations. However, the standard way of dealing with quantifiers in dependence logic involves arity-increasing operations. Let X be a team, then

- $X[M/x]$ is the team $\{ s[a/x] \mid s \in X, a \in M \}$,
- $X[f/x]$ is $\{ s[f(s)/x] \mid s \in X \}$, where $f : X \to M$ and
- $X[F/x]$ is $\{ s[a/x] \mid s \in X, a \in F(s) \}$, where $F : X \to \mathcal{P}(M)$.

Define $X \restriction \bar{x} = \{ s \restriction \bar{x} \mid s \in X \}$ and $\mathrm{dom}(X \restriction \bar{x}) = \mathrm{dom}(X) \cap \{ \bar{x} \}$; $X(\bar{x}) = \{ s(\bar{x}) \mid s \in X \}$, and $\mathrm{rel}(X) = X(\bar{x})$, where $\bar{x}$ is $\mathrm{dom}(X)$ ordered with increasing indices.

Proposition 1.3. *The satisfaction relation* $\mathbb{M}, X \vDash \varphi$ *satisfies:*

(1) *If* $\mathbb{M}, X \vDash \varphi$ *and* $X' \subseteq X$ *then* $\mathbb{M}, X' \vDash \varphi$.
(2) $\mathbb{M}, X \vDash \varphi \mathbin{/\!\!\!\wedge} \psi$ *iff* $\mathbb{M}, X \vDash \varphi$ *and* $\mathbb{M}, X \vDash \psi$.
(3) $\mathbb{M}, X \vDash \exists x \varphi$ *iff* $\exists f : X \to M$ *s.t.* $\mathbb{M}, X[f/x] \vDash \varphi$.

(4) $\mathbb{M}, X \vDash \forall x\varphi$ *iff* $\mathbb{M}, X[M/x] \vDash \varphi$.

Proof. (1) is proved by induction on φ: All cases are trivial except for the quantifer cases; assume $\mathbb{M}, X \vDash \exists x\varphi$ and that $X' \subseteq X$. Let Y be such that $\exists xY = \exists xX$ and $\mathbb{M}, Y \vDash \varphi$. Define

$$Y' = \{ s \in Y \mid s \upharpoonright \bar{y} \in \exists xX' \},$$

where $\bar{y} = \mathrm{dom}(X) \setminus \{ x \}$. For $\mathbb{M}, X' \vDash \exists x\varphi$ we want to prove that $\exists xY' = \exists xX'$, since by the induction hypothesis we have $\mathbb{M}, Y' \vDash \varphi$.

By definition we get that $\exists xY' \subseteq \exists xX'$. For the other inclusion, let $s \in \exists xX'$ which implies that $s \in \exists xX$ and so $s \in \exists xY$. Let $s' = s[a/x] \in Y$ (since $x \in \mathrm{dom}(Y)$). $s' \upharpoonright \bar{y} = s \in \exists xX'$ which proves that $s' \in Y'$ giving us $s \in \exists xY'$. And thus $\exists xX' = \exists xY'$.

For the universal quantifier assume $\mathbb{M}, X \vDash \forall x\varphi$ and that $X' \subseteq X$. Let Y be such that $\forall xY = \exists xX$ and $\mathbb{M}, Y \vDash \varphi$. Define

$$Y' = \{ s \in Y \mid s \upharpoonright \bar{y} \in \exists xX' \},$$

where $\bar{y} = \mathrm{dom}(X) \setminus \{ x \}$. To see that $\mathbb{M}, X' \vDash \forall x\varphi$ we need to prove that $\forall xY' = \exists xX'$, since by the induction hypothesis we have $\mathbb{M}, Y' \vDash \varphi$.

By definition we get that $\forall xY' \subseteq \exists xY' \subseteq \exists xX'$. For the other inclusion, let $s \in \exists xX'$ which implies that $s \in \exists xX$ and so $s \in \forall xY$. Then $s[a/x] \in Y$ for every $a \in M$ (since $x \in \mathrm{dom}(Y)$). Also $s[a/x] \upharpoonright \bar{y} = s \in \exists xX'$ which proves that $s[a/x] \in Y'$ for all $a \in M$; giving us $s \in \forall xY'$. And thus $\forall xY' = \exists xX'$.

(2). Assume $\mathbb{M}, X \vDash \varphi \mathbin{/\!\!\!\wedge} \psi$ and let $Y \cap Z = X$ be such that $\mathbb{M}, Y \vDash \varphi$ and $\mathbb{M}, Z \vDash \psi$. By the induction hypothesis we get that $\mathbb{M}, (Y \cap X) \vDash \varphi$ and $\mathbb{M}, (Z \cap X) \vDash \psi$ and since $(Y \cap X) \cap (Z \cap X) = Y \cap Z \cap X = X \cap X = X$ we have that $\mathbb{M}, X \vDash \varphi \mathbin{/\!\!\!\wedge} \psi$.

For (3) assume first that $\mathbb{M}, X \vDash \exists x\varphi$ and let Y be such that $\mathbb{M}, Y \vDash \varphi$ and $\exists xY = \exists xX$. Define $f' : \exists xX \to M$ in such a way that for all $s \in \exists xX$, $s[f(s)/x] \in Y$. Then let $f : X \to M$ be defined by $f(s) = f'(s \upharpoonright \bar{y})$, where $\bar{y} = \mathrm{dom}(X) \setminus \{ x \}$. The team $X[f/x] \subseteq Y$ and by (1) we then get $\mathbb{M}, X[f/x] \vDash \varphi$.

For the other direction assume that there is an $f : X \to M$ such that $\mathbb{M}, X[f/x] \vDash \varphi$. Let $Y = X[f/x]$, then $\exists xY = \exists xX$.

(4) Assume that $\mathbb{M}, X \vDash \forall x\varphi$ and let Y witness that. Then $\forall xY = \exists xX$ and thus $(\exists xX)[M/x] \subseteq Y$. However $(\exists xX)[M/x] = X[M/x]$. Using (1) we get $\mathbb{M}, X[M/x] \vDash \varphi$. On the other hand if $\mathbb{M}, X[M/x] \vDash \varphi$, we may chose $Y = X[M/x]$. □

For case (3) to be valid, the natural choice of an arity-reducing operation in Definition 1.2 appears to be $\exists x$: Let's assume that another arity-reducing operator was used, denoted by $P(X)$, such that $P(X) \subseteq \exists xX$. Now consider the following scenario: Suppose

$$X = \{ \epsilon \} [A/x][a/y] \cup \{ x \mapsto b, y \mapsto c \},$$

where $A \neq \emptyset$ and $a \neq c$. For case (3) to hold we need

$$\{ y \mapsto a \} \in P(X)$$

. This ensures that X does not satisfy the formula $\exists x(y = c)$. The only reasonable choice for P seems to be $\exists x$. A similar argument demonstrates that $\exists x$ is the only reasonable arity-reducing operation to choose in case (6) of Definition 1.2.

Proposition 1.3 established that the satisfaction relation in this paper is equivalent to the standard one for dependence logic. Consequently, we retain all the familiar properties associated with dependence logic, including locality:

Proposition 1.4. $\mathbb{M}, X \vDash \varphi$ *iff* $\mathbb{M}, X \upharpoonright \mathrm{fv}(\varphi) \vDash \varphi$.

We define $\mathbb{M} \vDash \sigma$ for a sentence σ to hold if $\mathbb{M}, \{\, \epsilon \,\} \vDash \sigma$.

2. A team logic for generalized quantifiers

The reader may wonder why we took the extra effort of defining dependence logic in this slightly roundabout way. The main reason for doing so is to set up the definitions in such a way that only a small alteration gives us a logic which is better suited for generalized quantifiers.

This small alternation is based on Guideline 2 which states that a team satisfies a formula iff the team *is* the semantic value of the formula:

Guideline 2. A formula φ is satisfied by a team X if for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ iff s satisfies φ.

The idea to formulate this "maximal" semantics originates from [Eng12] where a semantics for dependence logic with non-monotone generalized quantifiers is given using a maximality condition on the truth condition for the generalized quantifier. However, the approach in that paper takes us beyond the realm of existential second-order logic. Instead, we propose to base the semantics on the guideline above.

In his PhD thesis [Nur09] Nurmi presents a similar semantical system: 1-semantics for the syntax of dependence logic. In formulating 1-semantics, Nurmi uses the syntax of dependence logic with a dependence atom that is satisfied by a team iff the team is the graph of a total function. He proves that a team satisfies a formula in dependence logic iff there is a superset of the team that satisfies the formula in 1-semantics. In the present paper, instead of using a dependence atom, we opt for an *external* disjunction and conjunction and restricts the use of negation to atoms; but are otherwise using the same basic idea and guiding principle for the semantics as Nurmi. However, we do arrive at slightly different conclusions, and it appears that these seemingly small differences give rise to quite distinct logics.

2.1. mt-logic.

We will define a logic, which we will call *mt-logic* for maximal team logic, using a variation of Definition 1.2 in which clause (1) is replaced by a clause that is directly copied from Guideline 2. Instead of the dependence atoms, we will introduce two new connectives: $\wedge$ and $\vee$, representing the *external* conjunction and disjunction, distinct from the internal operators $\mathbin{/\!\!\wedge}$ and $\mathbin{\vee\!\!\!\vee}$.

Thus, the set of formulas of this logic is

$$\varphi ::= \mathrm{At} \mid \neg\mathrm{At} \mid \varphi \mathbin{/\!\!\wedge} \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi,$$

where At is an atomic formula in the language of some given signature.

We say that a formula is first-order if there is no occurrence of $\wedge$ nor $\vee$.

Definition 2.1. The satisfaction relation for mt-logic $\mathbb{M}, X \vDash \varphi$, where $\mathrm{fv}(\varphi) \subseteq \mathrm{dom}(X)$, is defined as follows.

(1) $\mathbb{M}, X \vDash \psi$ iff $\forall s : \mathrm{dom}(X) \to M (s \in X \text{ iff } \mathbb{M}, s \vDash \psi)$, for literals ψ
(2) $\mathbb{M}, X \vDash \varphi \mathbin{/\!\!\wedge} \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z; \mathbb{M}, Y \vDash \varphi$ and $\mathbb{M}, Z \vDash \psi$
(3) $\mathbb{M}, X \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z; \mathbb{M}, Y \vDash \varphi$ and $\mathbb{M}, Z \vDash \psi$

(4) $\mathbb{M}, X \vDash \varphi \wedge \psi$ iff $\mathbb{M}, X \vDash \varphi$ and $\mathbb{M}, X \vDash \psi$
(5) $\mathbb{M}, X \vDash \varphi \vee \psi$ iff $\mathbb{M}, X \vDash \varphi$ or $\mathbb{M}, X \vDash \psi$
(6) $\mathbb{M}, X \vDash \exists x\varphi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \exists xY = \exists xX$ and $\mathbb{M}, Y \vDash \varphi$
(7) $\mathbb{M}, X \vDash \forall x\varphi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \forall xY = \exists xX$ and $\mathbb{M}, Y \vDash \varphi$

If there is need, we will denote this satisfaction relation by $\vDash_{\mathrm{mt}}$ and the relation for dependence logic by $\vDash_D$. We will also often skip to mention the model $\mathbb{M}$ if that is understood from the context.

Instead of using arity-reducing operations on teams in the quantifier clauses we may use arity-increasing operators as in Proposition 1.3; however, the definition for the universal quantifier then becomes more involved:

Proposition 2.2. *For φ in $T[\tau]$ we have:*

- $\mathbb{M}, X \vDash_{\mathrm{mt}} \exists x\varphi$ *iff there exists a function* $F : \exists xX \to \mathcal{P}(M) \setminus \{ \emptyset \}$ *such that* $(\exists xX)[F/x] \vDash_{\mathrm{mt}} \varphi$.
- $\mathbb{M}, X \vDash_{\mathrm{mt}} \forall x\varphi$ *iff there exists a function* $F : (\exists xX)^c \to \mathcal{P}(M) \setminus \{ M \}$ *such that* $(\exists xX)[M/x] \cup (\exists xX)^c[F/x] \vDash_{\mathrm{mt}} \varphi$.

Proof. Directly from the definitions. □

We note that satisfaction of mt-logic can be expressed in existential second order logic, ESO, i.e., using Σ^1_1-formulas:

Proposition 2.3. *For every formula in mt-logic φ in the signature τ with n free variables there is a Σ^1_1 formula Θ in the language of $\tau \cup \{ R \}$, where R is n-ary, such that for all $\mathbb{M}$ and X:*

$$\mathbb{M}, X \vDash_{\mathrm{mt}} \varphi \textit{ iff } (\mathbb{M}, \mathit{rel}(X)) \vDash \Theta.$$

The proof follows the proof of the similar result regarding dependence logic in [Vää07]. The details are left to the reader.

As mentioned above the basic idea behind this logic is that for first-order formulas φ, $X \vDash \varphi$ iff X *is* the semantic value of φ, $[\![\varphi]\!]^{\mathbb{M}}$. This is indeed true for a large class of formulas, but not true in other cases: Assume $\mathrm{dom}(X) = \{ x \}$ then $\mathbb{M}, X \vDash \exists x(x = x)$ iff there is Y s.t. $x \in \mathrm{dom}(Y)$, $\exists xY = \exists xX$ and $\mathbb{M}, Y \vDash x = x$, i.e., iff $\exists xX = \{ \epsilon \}$. Thus any non-empty X satisfies the sentence.

For this reason we will single out a subclass of formulas that we call *untangled*: A formula is untangled if no quantifier Qx appears in the scope of another quantifier $Q'x$ and no variable is both free and bound.[2]

The next proposition shows that untangled first-order formulas are well-behaved. Here $\mathrm{bv}(\varphi)$ is used to denote the set of bound variables of φ.

Proposition 2.4. *Assume φ is an untangled first-order formula in mt-logic and X a team such that* $\mathrm{dom}(X) \cap \mathrm{bv}(\varphi) = \emptyset$. *Then*

$$X \vDash_{\mathrm{mt}} \varphi \textit{ iff } X = [\![\varphi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)}.$$

Proof. This is proved by a direct induction over formulas. The base case is follows directly from the definitions and the cases for $\mathbin{/\!\!\!\wedge}$ and $\mathbin{\vee\!\!\!\vee}$ follows easily. Let us do the $\exists$ case: If φ is $\exists x\psi$, then $x \notin \mathrm{bv}(\psi)$ and so by the induction hypothesis $Y \vDash \psi$

[2]Here Q and Q' are either $\forall$ or $\exists$.

iff $Y = [\![\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(Y)}$Since $x \notin \mathrm{dom}(X)$ we have that $X \vDash \exists x\psi$ iff there is Y such that $\exists xY = X$ and $Y = [\![\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)\cup\{x\}}$. This is true iff

$$X = \exists x\, [\![\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)\cup\{x\}} = [\![\exists x\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)}. \qquad \square$$

Even though mt-logic does not exhibit full locality, it does have a weaker locality property that is outlined in the next proposition. In fact, in the upcoming section, we will use the absence of complete locality to interpret the independence atom.

Proposition 2.5. *Assume φ is a formula in mt-logic and $x \in \mathrm{dom}(X)$ does not occur in φ, then*

$$X \vDash_{\mathrm{mt}} \varphi \textit{ iff } \exists xX \vDash \varphi \textit{ and } \forall xX = \exists xX.$$

Also, if $\bar{w} = \mathrm{dom}(X) \setminus (\mathrm{fv}(\varphi) \cup \mathrm{bv}(\varphi))$ then

$$X \vDash_{\mathrm{mt}} \forall\bar{w}\varphi \textit{ iff } \exists\bar{w}X \vDash_{\mathrm{mt}} \varphi.$$

Proof. The first statement is proved by induction over formulas. The base case is taken care of by Proposition 2.4 since any formula without quantifiers is untangled. The inductive steps are easy for $\wedge$ and $\vee$. For the other cases:

Assume $X \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$. Then there are $Y \cup Z = X$ such that $Y \vDash \varphi$ and $Z \vDash \psi$, and so by the induction hypothesis $\exists xY = \forall xY$, $\exists xY \vDash \varphi$, $\exists xZ = \forall xZ$, and $\exists xZ \vDash \psi$. Now,

$$\exists xX = \exists x(Y \cup Z) = \exists xY \cup \exists xZ = \forall xY \cup \forall xZ \subseteq \forall x(Y \cup Z) = \forall xX,$$

and thus, since $\forall xX \subseteq \exists xX$ holds in general, $\exists xX = \forall xX$. Since $\exists xX = \exists xY \cup \exists xZ$, $\exists xY \vDash \varphi$ and $\exists xZ \vDash \psi$ we also have $\exists xX \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$.

For the other direction assume $\exists xX \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$ and $\exists xX = \forall xX$. Then there are $Y \cup Z = \exists xX$ such that $Y \vDash \varphi$ and $Z \vDash \psi$. Let $Y' = Y[M/x]$ and $Z' = Z[M/x]$, then $\exists xY' = \forall xY' = Y$ and so $Y' \vDash \varphi$ by the induction hypothesis. Thus $Y' \cup Z' \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$ and $Y' \cup Z' = X$.

The case of $\varphi \mathbin{\wedge\!\!\!\wedge} \psi$ is similar.

Assume $X \vDash \exists y\varphi$. Then there is $Y \vDash \varphi$ such that $\exists yY = X$. By the induction hypothesis we know that $\exists xY \vDash \varphi$ and $\exists xY = \forall xY$. Thus,

$$\exists xX = \exists x\exists yY = \exists y\exists xY = \exists y\forall xY \subseteq \forall x\exists yY = \forall xX. \tag{1}$$

But clearly $\forall xX \subseteq \exists xX$ and so $\forall xX = \exists xX$. Also, $\exists xY \vDash \varphi$ and $\exists y\exists xY = \exists xX$ by (1) and thus, $\exists xX \vDash \exists y\varphi$.

For the other direction assume $\exists xX \vDash \exists y\varphi$ and $\exists xX = \forall xX$. Then there is $Y \vDash \varphi$ such that $\exists yY = \exists xX$. Let $Y' = Y[M/x]$, then $\exists xY' = \forall xY' = Y$ and so, by the induction hypothesis $Y' \vDash \varphi$. Thus, $\exists yY' \vDash \exists y\varphi$ and

$$\exists yY' = \exists y(Y[M/x]) = (\exists yY)[M/x] = (\exists xX)[M/x] = X.$$

Assume $X \vDash \forall y\varphi$. Then there is $Y \vDash \varphi$ such that $\forall yY = X$. By the induction hypothesis we know that $\exists xY \vDash \varphi$ and $\exists xY = \forall xY$. Thus,

$$\exists xX = \exists x\forall yY \subseteq \forall y\exists xY = \forall y\forall xY = \forall x\forall yY = \forall xX. \tag{2}$$

But clearly $\forall xX \subseteq \exists xX$ and so $\forall xX = \exists xX$. Also, $\exists xY \vDash \varphi$ and $\forall y\exists xY = \exists xX$ by (2) and thus, $\exists xX \vDash \forall y\varphi$.

For the other direction assume $\exists x X \vDash \forall y\varphi$ and $\exists x X = \forall x X$. Then there is $Y \vDash \varphi$ such that $\forall y Y = \exists x X$. Let $Y' = Y[M/x]$, then $\exists x Y' = \forall x Y' = Y$ and so, by the induction hypothesis $Y' \vDash \varphi$. Thus, $\forall y Y' \vDash \forall y \varphi$ and

$$\forall y Y' = \forall y(Y[M/x]) = (\forall y Y)[M/x] = (\exists x X)[M/x] = X.$$

For the also-part we note that $X \vDash \forall \bar{w}\varphi$ iff there is Y such that $\forall \bar{w} Y = \exists \bar{w} X$ and $Y \vDash \varphi$. By the previous part of the proof, $Y \vDash \varphi$ iff $\forall \bar{w} Y = \exists \bar{w} Y$ and $\exists \bar{w} Y \vDash \varphi$. Thus, the left to right implication follows directly. For the other, assume $\exists \bar{w} X \vDash \varphi$ and let $Y = (\exists \bar{w} X)[M/\bar{w}]$. Then $Y \vDash \varphi$ and $\forall \bar{w} Y = \exists \bar{w} Y = \exists \bar{w} X$. Thus, $X \vDash \varphi$. ☐

Observe also that for all φ there is a team X such that $\mathbb{M}, X \vDash_{\mathrm{mt}} \varphi$, this follows by an easy inductive argument.

2.2. **Relationship with dependence and independence logic.** Next, we investigate the relationship between mt-logic on the one hand and dependence and independence logic on the other. We will work our way towards interpretations of dependence and independence logic in mt-logic. Let us first see how to deal with atoms of dependence logic that are closed downwards in mt-logic:

Proposition 2.6. *If ψ is a first-order formula then*

$$X \vDash \psi \mathbin{/\!\!\!\wedge} \exists \bar{w}(w_0 = w_0 \vee w_0 \neq w_0) \text{ iff } X \subseteq [\![\psi]\!]_{\bar{w}}\,, \tag{3}$$

where $\bar{w}$ is $\mathrm{dom}(X)$ and $w_0 \in \bar{w}$.

Proof. This is easily seen by first observing that any team Y with domain $\bar{w}$ satisfies $\exists \bar{w}(w_0 = w_0 \vee w_0 \neq w_0)$: If Y is empty then $Z = \emptyset$ satisfies $w_0 = w_0 \vee w_0 \neq w_0$ and $\exists \bar{w} Z = \exists \bar{w} Y = \emptyset$. In the other hand, if Y is non-empty then $Z = \{\epsilon\}[M/\bar{w}]$ satisfies $w_0 = w_0 \vee w_0 \neq w_0$ and $\exists \bar{w} Y = \exists \bar{w} Z = \{\epsilon\}$.

Now, the left hand side of (3) holds iff there is Y such that $X = [\![\psi]\!]_{\bar{w}} \cap Y$, which is equivalent to $X \subseteq [\![\psi]\!]_{\bar{w}}$. ☐

Thus, $X \vDash_{\mathrm{mt}} \psi \mathbin{/\!\!\!\wedge} \top_{\bar{w}}$ iff $X \vDash_{\mathrm{d}} \psi$, where $\top_{\bar{w}}$ denotes the sentence $\exists \bar{w}(w_0 = w_0 \vee w_0 \neq w_0)$; and so we have an interpretation of the literals of dependence logic in mt-logic.

Observe that, in general when $\bar{w} \subseteq \mathrm{dom}(X)$ we have $X \vDash \top_{\bar{w}}$ iff $X = \emptyset$ or $\exists \bar{w} X$ is the full team with domain $\mathrm{dom}(X) \setminus \bar{w}$.

Instead of directly interpreting the dependence atom we turn to the *independence atom* of [GV12][3]: $X \vDash_{\mathrm{d}} \bar{y} \perp_{\bar{x}} \bar{z}$ iff

$$\forall s, s' \in X \exists s_0 \in X\big(s(\bar{x}) = s'(\bar{x}) \to s_0(\bar{x}, \bar{z}) = s(\bar{x}, \bar{z}) \wedge s_0(\bar{x}, \bar{y}) = s'(\bar{x}, \bar{y})\big)$$

We show that this is expressible in mt-logic.

Proposition 2.7. *If $\bar{x}$, $\bar{y}$, and $\bar{z}$ are pair-wise disjoint then*

$$X \vDash_{\mathrm{d}} \bar{y} \perp_{\bar{x}} \bar{z} \text{ iff } X \vDash_{\mathrm{mt}} \forall \bar{w}(\top_{\bar{x},\bar{y}} \mathbin{/\!\!\!\wedge} \top_{\bar{x},\bar{z}}),$$

where $\bar{w}$ is $\mathrm{dom}(X) \setminus \{\, \bar{x}, \bar{y}, \bar{x} \,\}$.

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[3] The independence atom was introduced in [GV12] and is the embedded version of the multi-valued dependence relation used in database theory, see [Eng12].

Proof. Observe first that, by Proposition 2.5:

$$X \vDash_{\mathrm{mt}} \forall \bar{w}(\top_{\bar{x},\bar{y}} \mathbin{/\!\!\wedge} \top_{\bar{x},\bar{z}}) \text{ iff } \exists \bar{w} X \vDash_{\mathrm{mt}} \top_{\bar{x},\bar{y}} \mathbin{/\!\!\wedge} \top_{\bar{x},\bar{z}}.$$

Therefore we may, without loss of generality, assume that $\mathrm{dom}(X) = \{ \bar{x}, \bar{y}, \bar{z} \}$.

By using the fact that $\bar{y}$ and $\bar{z}$ are disjoint we get that $X \vDash_{\mathrm{mt}} \top_{\bar{x},\bar{y}} \mathbin{/\!\!\wedge} \top_{\bar{x},\bar{z}}$ iff

$$X = X_{\bar{z}} \cap X_{\bar{y}}, \tag{4}$$

where $X_{\bar{z}} = (\exists \bar{z} X)[M/\bar{z}]$ and similar for $X_{\bar{y}}$.

Now, assume that $X \vDash \bar{y} \perp_{\bar{x}} \bar{z}$ and prove (4). The left-to-right inclusion is trivial so let us assume that $s_1 \in X_{\bar{z}} \cap X_{\bar{y}}$, i.e., that there are s and s' such that $s(\bar{x}, \bar{y}) = s_1(\bar{x}, \bar{y})$ and $s'(\bar{x}, \bar{z}) = s_1(\bar{x}, \bar{z})$. This tells us that there is $s_0 \in X$ such that $s_0 = s_1$, and thus that $s_1 \in X$.

On the other hand, assume (4) and that $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$. Define s_0 to be such that $s_0(\bar{x}, \bar{z}) = s(\bar{x}, \bar{z})$ and $s_0(\bar{x}, \bar{y}) = s'(\bar{x}, \bar{y})$. From these two equations it follows that $s_0 \in X_{\bar{z}}$ and $s_0 \in X_{\bar{y}}$ and thus, from (4), we get that $s_0 \in X$. Thus, $X \vDash_{\mathrm{d}} \bar{y} \perp_{\bar{x}} \bar{z}$. $\square$

In fact, disjoint independence atoms can express any independence atom by existentially quantifying in new variables. As can be easily checked, $D(\bar{x}, y)$ is equivalent to $y \perp_{\bar{x}} y$ which in Dependence logic is equivalent to $\exists z(y \perp_{\bar{x}} z \wedge y = z)$. Thus, we may express dependence atoms in mt-logic as follows.

Proposition 2.8. $X \vDash_{\mathrm{d}} D(\bar{x}, y)$ *iff*

$$X \vDash_{\mathrm{mt}} \exists z\big(\forall \bar{w}(\top_{\bar{x},y} \mathbin{/\!\!\wedge} \top_{\bar{x},z}) \wedge (y = z \mathbin{/\!\!\wedge} \top_{\bar{x},\bar{w}})\big),$$

where z is not in $\bar{x}, y$ and $\bar{w}$ is $\mathrm{dom}(X) \setminus \{ \bar{x}, y, z \}$.

Proof. Assume that $X \vDash_{\mathrm{d}} D(\bar{x}, y)$ and define $Y = X[f/z]$ where $f(s) = s(y)$. According to Proposition 2.7 Y satisfies the first conjunct of the formula and according to (3) it satisfies the second conjunct.

On the other hand, assume there is Y such that $X = \exists z Y$ and Y satisfies the conjunction. Y satisfies the second conjunct iff $s(z) = s(y)$ for all $s \in Y$. Satisfying the first conjunct implies that for all $s, s' \in Y$ if $s(\bar{x}) = s'(\bar{x})$ then there is $s_0 \in Y$ such that $s_0(\bar{x}, y) = s(\bar{x}, y)$ and $s_0(z) = s'(z)$, i.e., $s(y) = s_0(y) = s_0(z) = s'(z) = s'(y)$. In other words, $X \vDash_{\mathrm{d}} D(\bar{x}, y)$. $\square$

We are now able to define an *almost* compositional translation $^+ : \varphi \mapsto \varphi^+$ of dependence logic into our logic in such a way that

$$\mathbb{M}, X \vDash_{\mathrm{d}} \varphi \text{ iff } \mathbb{M}, X \vDash_{\mathrm{mt}} \varphi^+,$$

for all models $\mathbb{M}$ and teams X with $\mathrm{dom}(X) = \mathrm{fv}(\varphi)$. This is done by replacing atomic formulas ψ by $\psi \mathbin{/\!\!\wedge} \top_{\bar{x}}$ for some suitable choice of variables $\bar{x}$ and using Proposition 2.8.

Definition 2.9. Let $f(\bar{w}, \varphi)$ be defined inductively on the set of dependence logic formulas:

- $f(\bar{w}, D(\bar{x}, y)) = \exists z(\forall \bar{w}'(\top_{\bar{x},y} \mathbin{/\!\!\wedge} \top_{\bar{x},z}) \wedge (y = z \mathbin{/\!\!\wedge} \top_{\bar{x},\bar{w}'})$, where $\bar{w}' = \bar{w} \setminus \{ \bar{x}, y, z \}$ and z is not in $\bar{w}$.
- $f(\bar{w}, \varphi) = \varphi \mathbin{/\!\!\wedge} \top_{\bar{w}}$ if ψ is a literal,
- $f(\bar{w}, \varphi \wedge \psi) = f(\bar{w}, \varphi) \mathbin{/\!\!\wedge} f(\bar{w}, \varphi)$,
- $f(\bar{w}, \varphi \vee \psi) = f(\bar{w}, \varphi) \mathbin{\vee\!\!\!\vee} f(\bar{w}, \varphi)$, and
- $f(\bar{w}, \exists y \varphi) = \exists y f(\bar{w}, y, \varphi)$, and

- $f(\bar{w}, \forall y \varphi) = \forall y f(\bar{w}, y, \varphi)$.

Let φ^+ be the formula $f(\mathrm{fv}(\varphi), \varphi)$.

The translation $\varphi \mapsto \varphi^+$ is not compositional since for example $(x = x \wedge y = y)^+$ is not $(x = x)^+ \mathbin{/\!\!\wedge} (y = y)^+$, and these two formulas are not equivalent.

Proposition 2.10. *For every team X and dependence logic formula φ such that $\mathrm{dom}(X) = \mathrm{fv}(\varphi)$:*

$$X \vDash_{\mathrm{d}} \varphi \textit{ iff } X \vDash_{\mathrm{mt}} \varphi^+.$$

Proof. Observe first that the role of $\bar{w}$ in $f(\bar{w}, \varphi)$ is to "keep track" of $\mathrm{dom}(X)$; this has to be cared for in the induction step and thus we will instead prove the slightly more involved statement that if $\mathrm{fv}(\varphi) \subseteq \mathrm{dom}(X)$ then

$$X \vDash_{\mathrm{d}} \varphi \text{ iff } X \vDash_{\mathrm{mt}} f(\mathrm{dom}(X), \varphi).$$

The two base cases are handled by (3) and Proposition 2.8. For the inductive steps observe that the satisfaction clauses for $\mathbin{\vee\!\!\!\vee}$ and $\mathbin{/\!\!\wedge}$ correspond exactly to the satisfaction clauses in dependence logic for $\wedge$ and $\vee$. Similarly with the two quantifier cases. □

We may, in a similar fashion, give a translation g of independence logic into team logic in such a way that

$$X \vDash_{\mathrm{d}} \varphi \text{ iff } X \vDash_{\mathrm{mt}} g(\varphi, \mathrm{dom}(X)).$$

Note that this translation need to take care of the non-disjoint independence atom by translating it into disjoint independence atom: In independence logic we have that

$$\bar{y} \perp_{\bar{x}} \bar{z} \text{ is equivalent to } \exists \bar{v}, \bar{w}(\bar{v} = \bar{y} \wedge \bar{w} = \bar{z} \wedge \bar{v} \perp_{\bar{x}} \bar{w}).$$

Galliano in [Gal12] proved that every ESO-property can be expressed by an independence formula and thus team logic has the same expressive power:

Theorem 2.11. *The expressive power of team logic is that of existential second-order logic, for both formulas and sentences.*

3. Generalized quantifiers

According to Mostowski [Mos57] and Lindström [Lin66] a *generalized quantifier* is a class (in most cases a *proper class*) of structures in some finite relational signature closed under taking isomorphic images. For example

$$\mathtt{most} = \{ (M, A, B) : |\, A \cap B| \geq |A \setminus B| \}$$

is a generalized quantifier.The truth condition is defined so that

$$\mathbb{M}, s \vDash \mathtt{most}\, x, y\ (\varphi(x), \psi(y)) \quad \text{iff} \quad (\mathbb{M}, [\![\varphi]\!]^s_x, [\![\psi]\!]^s_y) \in \mathtt{most},$$

where

$$[\![\varphi]\!]^s_x = \{ a \in \mathbb{M} \mid \mathbb{M}, s[a/x] \vDash \varphi \}.$$

Thus,

$$\mathbb{M}, s \vDash \mathtt{most}\, x, y\ (\varphi(x), \psi(y)) \text{ iff } |\, [\![\varphi]\!]^s_x \cap [\![\psi]\!]^s_y \,| \geq |\, [\![\varphi]\!]^s_x \setminus [\![\psi]\!]^s_y \,|,$$

which coincides with the intuitive truth condition that most φ's are ψ's.

Given a generalized quantifier Q and a domain M, let the *local* quantifer Q_M be defined as

$$Q_M = \{ \langle A_0, A_1, \ldots, A_k \rangle \mid (M, A_0, A_1, \ldots, A_k) \in Q \}.$$

We say that a generalized quantifier is of *type* $\langle n_1, \dots, n_k \rangle$ if it is a class of structures in the relational signature $\{ R_1, \dots, R_k \}$ where R_i is of arity n_i.

Given a generalized quantifier Q of type $\langle n \rangle$ we may extend mt-logic with it in such a way that (2) in Lemma 2.4 is holds for all FO(Q) formulas. This is done by the following definition:

Definition 3.1. Let Q be of type $\langle n \rangle$ then $\mathbb{M}, X \vDash_{\mathrm{mt}} Q\bar{x}\varphi$ iff there is Y such that $\bar{x} \in \mathrm{dom}(Y)$, $\mathbb{M}, Y \vDash_{\mathrm{mt}} \varphi$ and $\exists\bar{x}X = Q\bar{x}Y$, where

$$Q\bar{x}Y = \{ s : \mathrm{dom}(Y) \setminus \{ \bar{x} \} \to M \mid Y_s(\bar{x}) \in Q_M \} .$$

Remember that $Y_s = \{ s' : \mathrm{dom}(Y) \setminus \mathrm{dom}(s) \to M \mid s \cup s' \in Y \}$. Observe that the clauses for $\forall$ and $\exists$ in Definition 1.2 are special cases of the above definition.

Similar to the cases of $\exists$ and $\forall$ there is an alternative truth condition for generalized quantifiers Q using arity increasing operations:

Lemma 3.2. *$X \vDash_{\mathrm{mt}} Q\bar{x}\varphi$ iff there exists a function*

$$F : \{ \epsilon \} [M/\mathrm{dom}(X) \setminus \{ \bar{x} \}] \to \mathcal{P}(M^k)$$

such that $F(s) \in Q_M$ iff $s \in \exists\bar{x}X$ and $(\exists\bar{x}X)[F/\bar{x}] \vDash_{\mathrm{mt}} \varphi$.

Proof. Directly from the definitions. □

Proposition 3.3. *For every untangled φ formula of $FO(Q)$ and every team X such that $\mathrm{dom}(X) \cap \mathrm{bv}(\varphi) = \emptyset$:*

$$\mathbb{M}, X \vDash_{\mathrm{mt}} \varphi \textit{ iff } X = [\![\varphi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)} .$$

Proof. Induction as in the case with FO-formulas in Lemma 2.4. The only new case is when φ is $Qx\psi$ and $x \notin \mathrm{bv}(\psi)$. By the induction hypothesis $Y \vDash \psi$ iff $Y = [\![\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(Y)}$. Since $x \notin \mathrm{dom}(X)$ we have that $X \vDash Qx\psi$ iff there is Y such that $QxY = X$ and $Y = [\![\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)\cup\{x\}}$. This is true iff

$$X = Qx\, [\![\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)\cup\{x\}} = [\![Qx\psi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)} . \qquad \square$$

The next lemma shows that our definition is true to the truth conditions of monotone increasing generalized quantifiers in Dependence logic introduced in [Eng12]. A generalized quantifier Q of type $\langle n \rangle$ is *monotone increasing* if $R \in Q_M$ and $R \subseteq S$ implies $S \in Q_M$. We remind the reader of the truth condition when Q is monotone increasing: $X \vDash_{\mathrm{d}} Q\bar{x}\varphi$ iff there exists a function $F : \exists\bar{x}X \to Q_M$ such that $(\exists\bar{x}X)[F/\bar{x}] \vDash_{\mathrm{d}} \varphi$.

Lemma 3.4. *If Q is monotone increasing of type $\langle n \rangle$, then $X \vDash_{\mathrm{d}} Q\bar{x}\varphi$ iff there exists Y with $\bar{x} \in \mathrm{dom}(Y)$, $Q\bar{x}Y = \exists\bar{x}X$ and $Y \vDash_{\mathrm{d}} \varphi$.*

Proof. First observe that we may assume $\bar{x} \notin \mathrm{dom}(X)$ to simply notation. Then assume $X \vDash_{\mathrm{d}} Q\bar{x}\varphi$ and let $F : X \to Q_M$ be such that $X[F/\bar{x}] \vDash_{\mathrm{d}} \varphi$. We may now chose $Y = X[F/\bar{x}]$ satisfying $Y \vDash \varphi$ and $Q\bar{x}Y = X$.

For the other direction, assume that there is a $Y \vDash_{\mathrm{d}} \varphi$ such that $Q\bar{x}Y = X$. Define $F : X \to \mathcal{P}(M^k)$ by $F(s) = \{ \bar{a} \mid s[\bar{a}/\bar{x}] \in Y \}$. Clearly $X[F/\bar{x}] \subseteq Y$ proving that $X[F/\bar{x}] \vDash_{\mathrm{d}} \varphi$. Also, if $s \in X = Q\bar{x}Y$, then $F(s) \in Q_M$. □

In [EK13] the strength of $D(Q)$ is characterized for monotone increasing quantifiers Q as the strength of $\mathrm{ESO}(Q)$. Since we may interpret $D(Q)$ in mt-logic extended with Q we see that this logic is at least as strong as $\mathrm{ESO}(Q)$. By a standard argument we can "interpret" mt-logic with Q in $\mathrm{ESO}(Q)$ proving that mt-logic with Q has the same strength as $\mathrm{ESO}(Q)$ for monotone increasing quantifiers Q:

Theorem 3.5. *Mt-logic extended with a monotone increasing Q has the same strength as $D(Q)$, and thus as* $\mathrm{ESO}(Q)$.

As we say in Proposition 3.3 we also have a conservativity result, at least for a large class of $\mathrm{FO}(Q)$-formulas. Before discussing branching quantifiers we also note that our definition of a generalized quantifier respects iterated quantifiers sense below. The iteration of two type $\langle 1\rangle$ quantifiers Q and Q' is defined by

$$(Q \cdot Q')_M = \{ R \subseteq M^2 \mid \{ a \mid R_a \in Q'_M \} \in Q_M \},$$

where $R_a = \{ b \mid \langle a, b\rangle \in R \}$.

Theorem 3.6. *For any two generalized quantifiers Q and Q' of type $\langle 1\rangle$:*

$$\mathbb{M}, X \vDash_{\mathrm{mt}} (Q \cdot Q')xy\, \phi \text{ iff } \mathbb{M}, X \vDash_{\mathrm{mt}} Qx\, Q'x\, \phi.$$

Proof. It is clearly enough to see that $Qx(Q'yY) = (Q \cdot Q')xyY$ for all teams Y including x and y in its domain. But this follows immediate from the definitions. $\square$

4. Branching quantifiers

We conclude this paper by discussing the branching of generalized quantifiers. One of the motivations behind Hintikka and Sandu's introduction of IF-logic [HS89] was to express the branching behavior of quantifiers. In [Eng12], it was demonstrated that there exists a natural way to express the branching of two monotone increasing quantifiers within what is now known as Independence logic. In this section, we establish that mt-logic is sufficiently powerful to capture the branching behavior of a broader class of quantifiers, specifically the continuous ones

Barwise (see [Bar79]), among others, argues that for increasingly monotone quantifiers Q_1 and Q_2 of type $\langle 1\rangle$ the branching of Q_1 and Q_2

$$Q_1 x Q_2 y A(x, y)$$

should be interpreted as

$$\mathrm{Br}(Q_1, Q_2)xy\ A(x, y),$$

where $\mathrm{Br}(Q_1, Q_2)$ is the type $\langle 2\rangle$ quantifier

$$\{ (M, R) \mid \exists A \in Q_1, B \in Q_2, A \times B \subseteq R \}.$$

Westerståhl, in [Wes87], suggests a definition of the branching of a larger class of generalized quantifiers; *continuous* generalized quantifiers. These quantifiers have also been studied under different names: convex quantifiers [Gie+23] and connected quantifiers [CBD19]. For simplicity we only give the definitions here for quantifiers of type $\langle n\rangle$:

Definition 4.1.
- A quantifier Q is *continuous* if for every M and every $R_1 \subseteq R_2 \subseteq R_3$ such that $R_1, R_3 \in Q_M$ we have $R_2 \in Q_M$.
- The branching $\mathrm{Br}(Q_1, Q_2)$ of two continuous quantifiers Q_1 and Q_2 is defined by $R \in \mathrm{Br}(Q_1, Q_2)_M$ iff there exists $S_1, S'_1 \in Q_{1M}$ and $S_2, S'_2 \in Q_{2M}$ such that $S_1 \times S_2 \subseteq R \subseteq S'_1 \times S'_2$.

In mt-logic we are able to express the branching of two continuous quantifiers as the following propositions shows.

Proposition 4.2. *Let Q_1 and Q_2 be continuous quantifiers of type $\langle 1\rangle$ and φ a first-order formula, then $\mathbb{M} \vDash_{\mathrm{mt}} Br(Q_1, Q_2)xy\,\phi$ iff*

$$(5)\quad \mathbb{M} \vDash_{\mathrm{mt}} Q_1x\, Q_2y\, Q_1z\, Q_2w(x \perp y \wedge xy \perp z \wedge xyz \perp w \wedge \\ (\varphi(x,y) \mathbin{/\!\!\wedge} \top_{xyzw}) \wedge (\varphi(z,w) \mathbin{\vee\!\!\vee} \top_{xyzw})).$$

Proof. First note that $X \vDash_{\mathrm{mt}} Q_1x\, Q_2y\, Q_1z\, Q_2w\, (x \perp y \wedge xy \perp z \wedge xyz \perp w)$ iff $X = \{\epsilon\}[A/x][B/y][C/z][D/w]$ where $A, C \in Q_{1M}$ and $B, D \in Q_{2M}$.

Secondly, such an X satisfies $\varphi(x,y) \mathbin{/\!\!\wedge} \top_{xyzw}$ iff $A \times B \subseteq R$ where R is the relation corresponding to the team $[\![\varphi]\!]_{x,y}$. Similarly X satisfies $\varphi(z,w) \mathbin{/\!\!\wedge} \top_{xyzw}$ iff $R \subseteq C \times D$.

Thus (5) holds iff there are $A, C \in Q_{1M}$ and $B, D \in Q_{2M}$ such that $A \times C \subseteq R \subseteq B, D$, i.e., iff $R \in \mathrm{Br}(Q_1, Q_2)$. Which is equivalant to $\mathbb{M} \vDash_{\mathrm{mt}} \mathrm{Br}(Q_1, Q_2)xy\,\phi$. $\square$

Thus, mt-logic is expressible enough to branch these quantifiers, and so should be able to formalize a larger fragment of natural languages than Dependence logic. However, this result is not completely satisfactory as the formula that expresses branching is unnatural and we have not been able to establish a result in which the branching $\mathrm{Br}(Q_1, Q_2)x, y\phi(x,y)$ can be expressed by a formula of the form

$$Q_1xQ_2y(\psi(x,y) \wedge \phi(x,y)).$$

5. Conclusion

The results in this paper demonstrates that it is possible to handle generalized quantifiers within team semantics, given that the flatness principle is relaxed. This opens up for a more general, and more algebraic, definition of team semantics that quantifies over *all* possible "lifts", in the sense that we don't restrict the possible semantic values of atomic formulas. In the context of Dependence logic the semantic values of atomic formulas are confined to principal ideals of teams. In the team semantics used in this paper the restriction is to singleton sets. In a related paper [EO23] we investigate the case of propositional logic when no such restriction is imposed. The resulting logic, known as the logic of teams, proves powerful enough to express propositional dependence logic and all its related variants. Looking ahead, we aim to merge the approach presented in this paper with that of the logic of teams.

Acknowledgments

This paper was written as part of the project: *Foundations for team semantics: Meaning in an enriched framework*, a research project supported by grant 2022-01685 of the Swedish Research Council, Vetenskapsrådet.

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Email address: fredrik.engstrom@gu.se

DEPARTMENT OF PHILOSOPHY, LINGUISTICS AND THEORY OF SCIENCE, UNIVERSITY OF GOTHENBURG, BOX 100, 405 30 GOTHENBURG, SWEDEN