# The pro-Nisnevich topology

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**Abstract**

We construct the pro-Nisnevich topology, an analog of the pro-étale topology. We then show that the Nisnevich $\infty$-topos embeds into the pro-Nisnevich $\infty$-topos, and that the pro-Nisnevich $\infty$-topos is locally of homotopy dimension 0.

## Introduction

In [BS14], Bhatt and Scholze constructed the pro-étale topology, and showed that the étale topos over a scheme embeds into the pro-étale topos.

In this short note, we do a similar construction for the Nisnevich topology. Let $S$ be a qcqs scheme of finite Krull-dimension (e.g. a field), and write $\mathrm{Sm}_S$ for the category of smooth quasi-compact $S$-schemes, equipped with the Nisnevich topology [BH17, Appendix A]. Write $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S)$ for the $\infty$-topos of sheaves on this site. We will construct a site $(\mathrm{ProSm}_S, \mathrm{pronis})$, called the *pro-Nisnevich site*. Write $\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$ for the $\infty$-topos of hypersheaves on this site. This $\infty$-topos is called the *pro-Nisnevich topos*. We prove the following results:

**Theorem A** (Theorem 2.10)**.** *There is a full subcategory $W \subset \mathrm{ProSm}_S$ and an equivalence $\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S) \cong \mathcal{P}_\Sigma(W)$. In particular, the pro-Nisnevich topos is locally of homotopy dimension $0$.*

**Theorem B** (Theorem 2.12)**.** *There is a geometric morphism of $\infty$-topoi*

$$\nu^*\colon \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S) \rightleftarrows \mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)\colon \nu_*,$$

*where $\nu_*$ is given by restriction and $\nu^*$ is fully faithful.*

*Moreover, an $n$-truncated sheaf $F \in \mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$ is in the essential image of $\nu^*$ if and only if for all $U \in \mathrm{ProSm}_S$ and all presentations of $U$ as a cofiltered limit $U \cong \lim_i U_i$ (with the $U_i \in \mathrm{Sm}_S$) the canonical comparison map $\mathrm{colim}_i\, F(U_i) \to F(U)$ is an equivalence.*

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# 1 The Pro-Nisnevich Topology

In this section, we develop an analog of the pro-étale topology from [BS14], adapted for the Nisnevich topology. We show that every affine scheme can be covered in this topology by a cdw-contractible ring, an analog of w-contractible rings from [BS14].

Recall the following definition from [BS14, Section 2.1 and 2.2], especially the alternative characterizations of w-local spaces from [BS14, Lemma 2.1.4]:

**Definition 1.1.** Let $A$ be a ring. We say that $A$ is w-local if the set of closed points $\mathrm{Spec}(A)^c$ is closed in $\mathrm{Spec}(A)$, and the map $\mathrm{Spec}(A)^c \to \mathrm{Spec}(A) \to \pi_0(\mathrm{Spec}(A))$ is bijective.

Recall the following definitions:

**Definition 1.2.** Let $f\colon A \to B$ be a ring map. We say that

1. $f$ is a local isomorphism if for every prime ideal $\mathfrak{q} \subset B$ there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $A \to B_g$ induces an open immersion $\mathrm{Spec}(B_g) \to \mathrm{Spec}(A)$,
2. $f$ is an *ind-Zariski localization* if $f$ can be written as a filtered colimit of local isomorphisms $f_i\colon A \to B_i$, and
3. $f$ is an *ind-Zariski cover* if $f$ is a faithfully flat ind-Zariski localization.

We have the following results about w-local rings:

**Lemma 1.3.** *Let $A$ be a ring. Then there exists an ind-Zariski cover $A \to A^Z$ with $A^Z$ w-local, such that $\mathrm{Spec}(A^Z)^c \to \mathrm{Spec}(A)$ is bijective.*

*Proof.* Combine [BS14, Lemma 2.2.4] and [BS14, Lemma 2.1.10]. ☐

**Lemma 1.4.** *For a ring $A$ and a map $T \to \pi_0(\mathrm{Spec}(A))$ of totally disconnected compact Hausdorff spaces, there is an ind-Zariski localization $A \to B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives rise to the given map $T \to \pi_0(\mathrm{Spec}(A))$ on applying $\pi_0$.*

*Proof.* This is [BS14, Lemma 2.2.8]. ☐

## 1.1 Absolutely flat rings

Recall the following definitions from [BS14]:

**Definition 1.5.** Let $A$ be a ring. We say that $A$ is *absolutely flat* if $A$ is reduced and has dimension 0.

Let $f\colon A \to B$ be a ring map. We say that $f$ is *weakly étale* if $f$ is flat with flat diagonal. We say that $f$ is *ind-étale* if $f$ can be written as a filtered colimit of étale morphisms $f_i\colon A \to B_i$.

**Lemma 1.6.** *Let $f\colon A \to B$ be a morphism of rings with $A$ absolutely flat. Suppose that $f$ is weakly étale. Then $B$ is absolutely flat.*

*Proof.* See [Sta23, Tag 092I]. □

**Lemma 1.7.** *Let $A$ be a ring of dimension* 0 *(e.g. $A$ absolutely flat). Then* $\operatorname{Spec}(A)$ *is totally disconnected compact Hausdorff.*

*Proof.* Clearly $\operatorname{Spec}(A)$ is compact. It is Hausdorff, see e.g. [Sta23, Tag 0CKV], which applies since $\operatorname{Spec}(A)$ is affine and hence separated. Since $\operatorname{Spec}(A)$ is spectral, it has a basis of compact open subsets. In Hausdorff spaces, compact subsets are closed, thus $\operatorname{Spec}(A)$ has a basis of clopen subsets. This implies that $A$ is totally disconnected. □

**Corollary 1.8.** *Let $A$ be absolutely flat. Then $A$ is w-local.*

*Proof.* We know that $\operatorname{Spec}(A)$ is totally disconnected compact Hausdorff from Lemma 1.7. In particular, all points of $\operatorname{Spec}(A)$ are closed and $\operatorname{Spec}(A) \cong \pi_0(\operatorname{Spec}(A))$. □

**Lemma 1.9.** *Let $A = \operatorname{colim}_i A_i$ be a filtered colimit of absolutely flat rings. Then $A$ is absolutely flat.*

*Proof.* Being reduced and of dimension 0 are properties of rings that are stable under filtered colimits: For the first claim, note that a nilpotent element in a ring is the same as a ring map from $\mathbb{Z}[X]/(X^n)$ for some $n \geq 2$. But since $\mathbb{Z}[X]/(X^n)$ is of finite presentation over $\mathbb{Z}$, we see that $\operatorname{Hom}(\mathbb{Z}[X]/(X^n), \operatorname{colim}_i A_i) \cong \operatorname{colim}_i \operatorname{Hom}(\mathbb{Z}[X]/(X^n), A_i) = 0$, since the $A_i$ are reduced. For the second claim, use [Sta23, Tag 01YW and Tag 01YY (2)]. □

**Definition 1.10.** Let $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be a map of affine schemes. We define the *completely decomposed locus* of $f$ as the subset

$$\operatorname{cd}(f) \coloneqq \{\, x \in \operatorname{Spec}(B) \mid k(f(x)) \to k(x) \text{ is an isomorphism} \,\} \subset \operatorname{Spec}(B).$$

We say that $f$ is *completely decomposed* if $f(\operatorname{cd}(f)) = \operatorname{Spec}(A)$, i.e. over every point of $\operatorname{Spec}(A)$ there is a point in $\operatorname{Spec}(B)$ that induces an isomorphism on residue fields.

**Lemma 1.11.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a filtered colimit of rings. Suppose wlog $I$ has an initial element $0$. Denote by $f_i \colon \mathrm{Spec}(A_i) \to \mathrm{Spec}(A_0)$ the map induced by $0 \to i$, and by $f_\infty \colon \mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ the projection.*

*Then $\mathrm{cd}(f_\infty) = \lim_i \mathrm{cd}(f_i)$.*

*Proof.* Let $p_i \colon \mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ be the projections. Let $x_\infty \in \mathrm{Spec}(A)$, and $x_i \coloneqq p_i(x_\infty) \in \mathrm{Spec}(A_i)$.

Suppose $x_\infty \in \mathrm{cd}(f_\infty)$ and thus $k(x_0) \cong k(x_\infty)$. Thus, for all $i$ we have a factorization of this isomorphism $k(x_0) \to k(x_i) \to k(x_\infty)$. Hence $k(x_i) \cong k(x_0)$ and we get $x_i \in \mathrm{cd}(f_i)$. This implies $x_\infty \in \lim_i \mathrm{cd}(f_i)$.

On the other hand, suppose that $x_\infty \in \lim_i \mathrm{cd}(f_i)$. This implies that $x_i \in \mathrm{cd}(f_i)$. By [Sta23, Lemma 0CUG], we know that $k(x_\infty) = \mathrm{colim}_i\, k(x_i)$. But $k(x_0) \cong k(x_i)$ and thus $k(x_\infty) \cong \mathrm{colim}_i\, k(x_0) \cong k(x_0)$. Hence $x_\infty \in \mathrm{cd}(f_\infty)$. $\square$

**Lemma 1.12.** *Let $A$ an absolutely flat ring. Write $A \cong \mathrm{colim}_i A_i$ as a filtered colimit of its finitely generated subrings. Let $x \in \mathrm{Spec}(A)$. For each $i$ write $p_i \colon \mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ for the projection, and $Z_i(x) \subset \mathrm{Spec}(A_i)$ for the connected component of $p_i(x)$. Then $\{x\} \cong \lim_i Z_i(x)$.*

*Proof.* Clearly $x \in \lim_i Z_i(x)$. Suppose on the other hand that $y \in \mathrm{Spec}(A)$, with $y \neq x$. Since $\mathrm{Spec}(A)$ is totally disconnected compact Hausdorff by Lemma 1.7, there exists a clopen subset $U \subset \mathrm{Spec}(A)$ into clopen subsets with $x \in U$ and $y \notin U$. By descent, there is an index $j$, and $U_j \subset \mathrm{Spec}(A_j)$ clopen such that $U = \mathrm{Spec}(A) \times_{\mathrm{Spec}(A_j)} U_j$. But then $Z_i(x) \subseteq \mathrm{Spec}(A_i) \times_{\mathrm{Spec}(A_j)} U_j$ for all $i \geq j$, and hence $\lim_i Z_i(x) \subseteq U$. Since $y \notin U$, we conclude $y \notin \lim_i Z_i(x)$. $\square$

**Lemma 1.13.** *Let $f \colon A \to B$ be an étale ring map, with $A$ absolutely flat. Then $\mathrm{cd}(f) \subset \mathrm{Spec}(B)$ is closed.*

*Proof.* Write $A = \mathrm{colim}_{i \in I} A_i$ as a filtered colimit of its finitely generated subrings. As $f$ is of finite presentation, by descent there is an index $0 \in I$, a ring $B_0$ and an étale morphism $f_0 \colon B_0 \to A_0$ such that $f$ is the basechange of $f_0$. Write $B_i \coloneqq B_0 \otimes_{A_0} A_i$ for each $i \geq 0$, and $f_i \colon A_i \to B_i$. In particular, $\mathrm{Spec}(B) \cong \lim_{i \geq 0} \mathrm{Spec}(B_i)$. Note that $\mathrm{Spec}(B_i)$ is of finite type over $\mathrm{Spec}(\mathbb{Z})$ (as it is étale over $\mathrm{Spec}(A_i)$ which is of finite type over $\mathrm{Spec}(\mathbb{Z})$). In particular, the connected components of $\mathrm{Spec}(B_i)$ are open. Write $W_i \subset \mathrm{Spec}(B_i)$ for the union of those components $W_{i,j} \subset \mathrm{Spec}(B_i)$ such that $f_i|_{W_{i,j}} \colon W_{i,j} \to \mathrm{Spec}(A_i)$ is an isomorphism onto a component of $\mathrm{Spec}(A_i)$. It suffices to prove that $\mathrm{cd}(f) = \lim_i W_i$: Indeed, since $\mathrm{Spec}(B_i)$ has only finitely many components, we see that $W_i$ is closed. Thus, also $\mathrm{cd}(f) = \lim_i W_i$ is closed as a limit of closed subsets.

It is clear that $\lim_i W_i \subseteq \mathrm{cd}(f)$, as $W_i \subseteq \mathrm{cd}(f_i)$. So let $x \in \mathrm{cd}(f)$ and $y \coloneqq f(x)$. We will write $p_i \colon \mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ for the projection, and $Z_i(y) \subset \mathrm{Spec}(A_i)$ for the component of $p_i(y)$. Let $V_i(y) \coloneqq \mathrm{Spec}(B_i) \times_{\mathrm{Spec}(A_i)} Z_i(y) \subseteq \mathrm{Spec}(B_i)$, which is a clopen subset. As the inclusion $\{x\} \hookrightarrow f^{-1}(y)$ is clopen (because $f^{-1}(y)$ is a discrete set of points), we may assume by descent (possibly by replacing 0 by a larger index), that there exists a $T_0 \subseteq V_0(y)$ clopen such that $\{x\} = T_0 \times_{V_0(y)} f^{-1}(y)$, and set $T_i \coloneqq \mathrm{Spec}(B_i) \times_{\mathrm{Spec}(B_0)} T_0$ (which is clopen in $V_i(y)$). Note that $T_i \times_{Z_i(y)} \{y\} \cong T_i \times_{V_i(y)} f^{-1}(y) \cong \{x\}$. As $f$ is completely decomposed at $x$ by definition, it induces an isomorphism $\{x\} \xrightarrow{\cong} \{y\}$. Thus, again by descent and Lemma 1.12, we conclude that $T_i \xrightarrow{\cong} Z_i(y)$ for $i \gg 0$. But $T_i \subseteq V_i(y)$ is clopen, and hence $T_i \subseteq V_i(y) \subseteq \mathrm{Spec}(B_i)$ is also clopen, and maps isomorphically to $Z_i(y)$. Therefore, $T_i \subseteq W_i$ and we conclude that $x \in \lim_i W_i$. □

**Corollary 1.14.** *Let $A$ be absolutely flat and $f\colon A \to B$ an ind-étale algebra. Then $\mathrm{cd}(f)$ is closed.*

*Proof.* As a limit of closed subsets is closed, we can first reduce to the case that $f$ is étale via Lemma 1.11, and then conclude by Lemma 1.13. □

## 1.2 Weakly Contractible Covers

**Definition 1.15.** Let $f\colon A \to B$ be a ring map. We say that $f$ is a *Nisnevich cover* if $f$ is étale and completely decomposed; and that $f$ is an *ind-Nisnevich cover* if $f$ is ind-étale and completely decomposed.

**Definition 1.16.** A ring $A$ is called *cdw-contractible* if every ind-Nisnevich-cover $A \to B$ splits.

**Definition 1.17.** Let $A$ be a ring. We write $\mathrm{Ind}_{et}(A)$ for the category of ind-étale $A$-algebras.

Recall the following definition from [BS14, Definition 2.2.10]:

**Definition 1.18.** Let $A \to B$ be a ring map. We have a base-change functor $\mathrm{Ind}_{et}(A) \to \mathrm{Ind}_{et}(B)$. Define $\mathrm{Hens}_A(-)\colon \mathrm{Ind}_{et}(B) \to \mathrm{Ind}_{et}(A)$ via

$$B' \mapsto \operatorname*{colim}_{A \xrightarrow{et} A' \to B'} A'.$$

This is a right adjoint to the base change functor and called *henselization.*

We say that a ring $A$ is *henselian along an ideal $I$* if $\mathrm{Hens}_A(A/I) \cong A$.

**Lemma 1.19.** *Let $A$ be a ring, and $I \subseteq A$ an ideal contained in the Jacobson radical $I_A$ of $A$. Let $\overline{A}$ be an ind-étale $A/I$-algebra. Then $B \coloneqq \mathrm{Hens}_A(\overline{A})$ is henselian along $IB$ with $B/IB \cong \overline{A}$.*

*Proof.* We first prove the second part. Since $\overline{A}$ is an ind-étale $A/I$-algebra, we know that $\mathrm{Hens}_{A/I}(\overline{A}) \cong \overline{A}$. Note that $B/IB = \operatorname{colim}_{A \xrightarrow{et} A' \to \overline{A}} A'/IA'$. By basechange, $A/I \to A'/IA'$ is an étale morphism for all such $A'$. Thus, it suffices to show that every factorization $A/I \xrightarrow{et} C \to \overline{A}$ is the basechange of a factorization $A \to A' \to \overline{A}$. Note that $A/I \to C$ can be lifted to an étale morphism $A \to A'$ by [Sta23, Tag 04D1]. But since $A'/IA' = C$, this fits into a factorization $A \to A' \to \overline{A}$ as desired.

For the first part we need to show that $B \xrightarrow{\mathrm{id}} B$ is final under étale morphisms $B \to B'$ such that there is a factorization $B \to B' \to B/IB = \overline{A}$. So let

$B \to B'$ such an étale morphism. We need to show that it has a section. But $B = \operatorname{colim}_{A \xrightarrow{et} A' \to \overline{A}} A'$. By finite presentation of $B \to B'$ there is thus a factorization $A \to A' \to \overline{A}$ with $A \to A'$ étale and an étale morphism $A' \to \tilde{B}$ such that $B \otimes_{A'} \tilde{B} \cong B'$. Since $A' \to \tilde{B}$ is étale, also $A \to \tilde{B}$ is étale. Note that there is a factorization $A \to \tilde{B} \to \overline{A}$. We then get a canonical map $\tilde{B} \to B$ by definition of $B$. But then $B' \cong B \otimes_{A'} \tilde{B} \to B \otimes_{A'} B \to B$ gives the desired section. □

**Lemma 1.20.** *Let $A$ be a ring which is henselian along an ideal $I$. Suppose $A/I$ is cdw-contractible. Then $A$ is cdw-contractible.*

*Proof.* Let $f\colon A \to B$ be an ind-Nisnevich cover. Then $f/I\colon A/I \to B/IB$ is an ind-Nisnevich cover and thus has a section. But

$$\operatorname{Hom}_A(B, A) \cong \operatorname{Hom}_A(B, \operatorname{Hens}_A(A/I)) \cong \operatorname{Hom}_{A/I}(B/IB, A/I).$$

This gives us the desired section of $f$. □

**Definition 1.21.** Let $T$ be a totally disconnected compact Hausdorff space. We say that $T$ is *extremally disconnected* if every surjection $S \twoheadrightarrow T$ from a totally disconnected compact Hausdorff space $S$ has a section.

**Lemma 1.22.** *Let $A$ be absolutely flat. Then $A$ is cdw-contractible if and only if $\operatorname{Spec}(A) = \pi_0(\operatorname{Spec}(A))$ is extremally disconnected.*

*Proof.* First we see from Lemma 1.7 that because $A$ is absolutely flat, the topological space $\operatorname{Spec}(A)$ is a totally disconnected compact Hausdorff space, so in particular $\operatorname{Spec}(A) \cong \pi_0(\operatorname{Spec}(A))$.

Suppose that $A$ is cdw-contractible. Let $f_0\colon T \to \operatorname{Spec}(A)$ be a surjective morphism of totally disconnected compact Hausdorff spaces. But $f_0$ is induced by an ind-Zariski cover $A \to B$, see Lemma 1.4, which has a section because $A$ is cdw-contractible and ind-Zariski covers are ind-Nisnevich covers. This gives us the topological section $\operatorname{Spec}(A) \cong \pi_0(\operatorname{Spec}(A)) \to \pi_0(\operatorname{Spec}(B)) \cong T$.

Suppose now that $\operatorname{Spec}(A)$ is extremally disconnected. Let $f\colon A \to B$ be an ind-Nisnevich cover. Thus, $B$ is absolutely flat, see Lemma 1.6 (as ind-étale maps are weakly étale, see [Sta23, Tag 092N]). Using Corollary 1.14, we see that $\operatorname{cd}(\operatorname{Spec}(f)) \subseteq \operatorname{Spec}(B)$ is closed. Since $f$ is completely decomposed, we see that $\operatorname{cd}(\operatorname{Spec}(f)) \to \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is in fact surjective. Let $B'$ be the $B$-algebra realizing the closed subset $\operatorname{cd}(\operatorname{Spec}(f))$ with its reduced scheme structure. Then $B'$ is reduced and thus absolutely flat. By definition of $B'$, $h\colon \operatorname{Spec}(B') \to \operatorname{Spec}(A)$ induces isomorphisms on all residue fields. Now, $h$ has a topological section $s$ because $\operatorname{Spec}(A) = \pi_0(\operatorname{Spec}(A))$ is extremally disconnected, and $\operatorname{Spec}(B')$ is totally disconnected compact Hausdorff by Lemma 1.7. Let $\operatorname{Spec}(B'')$ be the reduced closed subscheme of $\operatorname{Spec}(B')$ realizing the image $s(\operatorname{Spec}(A)) \subset \operatorname{Spec}(B')$ (note that the image of $s$ is closed because $s$ is a map between compact Hausdorff spaces). This maps isomorphically to $\operatorname{Spec}(A)$, which gives us the section

$$\operatorname{Spec}(A) \cong \operatorname{Spec}(B'') \subseteq \operatorname{Spec}(B') \subseteq \operatorname{Spec}(B).$$

$\square$

**Lemma 1.23.** *Let $A$ be a ring. Then there is an ind-Nisnevich cover $A \to \overline{A}$ with $\overline{A}$ a cdw-contractible ring.*

*Proof.* Consider the ind-Zariski cover $A \to A^Z$ by a w-local ring from Lemma 1.3. Note that $A^Z/I_{A^Z}$ is absolutely flat, using [BS14, Lemma 2.2.3]. Let $T$ be an extremally disconnected compact Hausdorff space covering $\mathrm{Spec}(A^Z/I_{A^Z})$. Let $A'$ be an ind-Zariski cover of $A^Z/I_{A^Z}$ such that $\mathrm{Spec}(A') \to \mathrm{Spec}(A^Z/I_{A^Z})$ realizes the map $T \to \mathrm{Spec}(A^Z/I_{A^Z})$, see Lemma 1.4. Then $A'$ is absolutely flat, and cdw-contractible by Lemma 1.22. Define $\overline{A} \coloneqq \mathrm{Hens}_{A^Z}(A')$. Then $\overline{A}$ is henselian along $I_{A^Z}\overline{A}$, with quotient $\overline{A}/I_{A^Z}\overline{A} \cong A'$ (see Lemma 1.19), hence $\overline{A}$ is cdw-contractible using Lemma 1.20. Now note that $A^Z \to \overline{A}$ induces isomorphisms of residue fields at the closed points of $\mathrm{Spec}(A^Z)$. Note that $\mathrm{Spec}(A^Z)^c \to \mathrm{Spec}(A)$ is bijective (see Lemma 1.3) and induces isomorphisms of residue fields. Thus, $\mathrm{Spec}(\overline{A}) \to \mathrm{Spec}(A)$ is completely decomposed. Since $A \to A^Z$ is ind-Zariski, and $A^Z \to \overline{A}$ is ind-étale by construction, we conclude that $A \to \overline{A}$ is an ind-Nisnevich cover. $\square$

# 2 The Pro-Nisnevich $\infty$-Topos

Let $S$ be a qcqs scheme with finite Krull-dimension.

**Definition 2.1.** Write $\mathrm{ProSm}_S$ for the category of pro-smooth schemes over $S$, i.e. morphisms $X \to S$ such that $X$ can be written as a cofiltered limit $X \cong \lim_i X_i$ where $X_i \in \mathrm{Sm}_S$. Write $\mathrm{ProSmAff}_S$ for the full subcategory of pro-smooth schemes over $S$ which are affine.

**Definition 2.2.** Let $\mathcal{U} \coloneqq \{f_i \colon U_i \to U\}_{i \in I}$ be a family of morphisms in $\mathrm{ProSm}_S$. We say that $\mathcal{U}$ is a *pro-Nisnevich cover* if

- $f_i$ is pro-étale for all $i \in I$,
- the morphism $\amalg_i U_i \to U$ is completely decomposed, and
- the $f_i$ form an fpqc-cover [Sta23, Tag 03NW].

Similarly, let $\mathcal{U} \coloneqq \{f_i \colon U_i \to U\}_{i \in I}$ be a family of morphisms in $\mathrm{ProSmAff}_S$. We say that $\mathcal{U}$ is a *pro-Nisnevich cover* if it is a pro-Nisnevich cover in $\mathrm{ProSm}_S$.

*Remark* 2.3. Let $\mathrm{Spec}(f) \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism in $\mathrm{ProSmAff}_S$. Then $\{\mathrm{Spec}(f)\}$ is a pro-Nisnevich cover if and only if $f \colon A \to B$ is an ind-Nisnevich cover.

Recall the definition of a site [Sta23, Tag 00VH] and a morphism of sites [Sta23, Tag 00X1].

**Lemma 2.4.** *There are sites* $(\mathrm{ProSm}_S, \mathrm{pronis})$ *and* $(\mathrm{ProSmAff}_S, \mathrm{pronis})$*, where the covers are given by pro-Nisnevich covers.*

*Proof.* The only nontrivial part is the stability of covers under pullbacks. First note that the pullback of a pro-étale cover exists and is again a pro-étale cover, which follows from [Sta23, Lemma 098L]. Since completely decomposed families of morphisms are stable under pullbacks in the category of all $S$-schemes, the result follows as pullbacks in $\mathrm{ProSm}_S$ along pro-étale morphisms coincide with the underlying pullback in the category of all $S$-schemes (this can be proven analogously to [Sta23, Tag 098M]). $\square$

We can define the following $\infty$-topoi:

**Definition 2.5.** We will write

- $\mathrm{Shv}^{\mathrm{nh}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$ for the $\infty$-topos of sheaves of anima on the site $(\mathrm{ProSm}_S, \mathrm{pronis})$,
- $\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$ for the hypercompletion of $\mathrm{Shv}^{\mathrm{nh}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$,
- $\mathrm{Shv}^{\mathrm{nh}}_{\mathrm{pronis}}(\mathrm{ProSmAff}_S)$ for the $\infty$-topos of sheaves of anima on the site $(\mathrm{ProSmAff}_S, \mathrm{pronis})$, and
- $\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSmAff}_S)$ for the hypercompletion of $\mathrm{Shv}^{\mathrm{nh}}_{\mathrm{pronis}}(\mathrm{ProSmAff}_S)$.

**Lemma 2.6.** *There are equivalences of $\infty$-topoi*

$$\mathrm{Shv}^{\mathrm{nh}}_{\mathrm{pronis}}(\mathrm{ProSmAff}_S) \cong \mathrm{Shv}^{\mathrm{nh}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$$

*and*

$$\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSmAff}_S) \cong \mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$$

*induced by left Kan extension along the inclusion* $\mathrm{ProSmAff}_S \hookrightarrow \mathrm{ProSm}_S$.

*Proof.* Note that the inclusion $u \colon \mathrm{ProSmAff}_S \to \mathrm{ProSm}_S$ preserves covers by definition, and commutes with limits as limits of affine schemes are affine. Thus, the first equivalence is an easy application of [Hoy15, Lemma C.3]. The second equivalence is just the hypercompletion of the first. $\square$

**Definition 2.7.** We write $W$ for the full subcategory of $\mathrm{ProSmAff}_S$ consisting of (spectra of) cdw-contractible rings.

**Lemma 2.8.** *The category $W$ is extensive ([BH17, Definition 2.3]), and the category $\mathcal{P}_\Sigma(W)$ is an $\infty$-topos.*

*Proof.* The category is extensive, as it is a full subcategory of the category of schemes (which is extensive), stable under finite coproducts and summands. The second part is [BH17, Lemma 2.4]. $\square$

Recall the notion of a locally weakly contractible site [Mat24, Definition B.3].

**Lemma 2.9.** *The site* $(\mathrm{ProSmAff}_S, \mathrm{pronis})$ *is locally weakly contractible.*

*Proof.* We have seen that the category $W$ is extensive, see Lemma 2.8, and consists by definition of weakly contractible objects. The pro-Nisnevich topology is a $\Sigma$-topology, because clopen immersions are in particular pro-étale and induce isomorphisms on the residue fields. The pro-Nisnevich topology on $\mathrm{ProSm}_S$ is finitary (cf. [Lur18, Definition A.3.1.1]) by definition, so every object is quasicompact. Note that every object in $\mathrm{ProSmAff}_S$ has a cover by an object in $W$, this is the content of Lemma 1.23. This proves the lemma. □

**Theorem 2.10.** *There is an equivalence of $\infty$-topoi*

$$\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S) \cong \mathcal{P}_{\Sigma}(W).$$

*Thus,* $\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$ *is locally of homotopy dimension* $0$ *and in particular Postnikov-complete.*

*Proof.* There is an equivalence $\mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S) \cong \mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSmAff}_S)$, see Lemma 2.6. Thus, the theorem follows from an application of [Mat24, Lemma B.7], because the site $(\mathrm{ProSmAff}_S, \mathrm{pronis})$ is locally weakly contractible, see Lemma 2.9. See [Mat24, Lemma 4.20] for a proof that $\mathcal{P}_{\Sigma}(W)$ is locally of homotopy dimension 0. The last claim is [Lur09, Proposition 7.2.1.10]. □

**Lemma 2.11.** *The $\infty$-topos* $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S)$ *is Postnikov-complete.*

*Proof.* If $S$ is noetherian, then the proof of this result is completely analogous to the proof of [Mat24, Lemma 5.1]. Here we use that $S$ is qcqs of finite Krull dimension (and that schemes in $\mathrm{Sm}_S$ possess the same property, note that for us, they are by definition quasi-compact). If $S$ is not noetherian, we can still apply the same strategy, but resort to [CM21, Theorem 3.18] instead of [Lur18, Theorem 3.7.7.1]. □

**Theorem 2.12.** *There is a geometric morphism*

$$\nu^*\colon \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S) \rightleftarrows \mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)\colon \nu_*,$$

*where the right adjoint is given by restriction, and the left adjoint is fully faithful.*

*Moreover, an $n$-truncated sheaf $F \in \mathrm{Shv}^{\mathrm{h}}_{\mathrm{pronis}}(\mathrm{ProSm}_S)$ is in the essential image of $\nu^*$ if and only if for all $U \in \mathrm{ProSm}_S$ and all presentations of $U$ as cofiltered limit $U \cong \lim_i U_i$ (with the $U_i \in \mathrm{Sm}_S$) the canonical map $\mathrm{colim}_i F(U_i) \to F(U)$ is an equivalence.*

*Proof.* Both involved $\infty$-topoi are Postnikov-complete, see Theorem 2.10 and Lemma 2.11. We want to apply [Mat24, Proposition B.8] to the natural inclusion of sites

$$(\mathrm{Sm}_S, \mathrm{nis}) \subset (\mathrm{ProSm}_S, \mathrm{pronis}).$$

In the notation of [Mat24, Proposition B.8], it remains to prove that $\iota_h j^* F \cong k^* \iota'_h F$ for every $n$-truncated Nisnevich sheaf $F \in \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S)$, i.e. we have to show that the presheaf $k^* \iota'_h F$ is already a pro-Nisnevich hypersheaf. The proof of this fact is completely analogous to the proof of [Mat24, Theorem B.24]. □

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