# Fractional Order Sunflower Equation: Stability, Bifurcation and Chaos

Deepa Gupta[1] and Sachin Bhalekar[2*]

[1,2]School of Mathematics and Statistics, University of Hyderabad, Hyderabad, 500046, Telangana, India.

*Corresponding author(s). E-mail(s): sachinbhalekar@uohyd.ac.in; Contributing authors: deepanew96@gmail.com;

**Abstract**

The sunflower equation describes the motion of the tip of a plant due to the auxin transportation under the influence of gravity. This work proposes the fractional-order generalization to this delay differential equation. The equation contains two fractional orders and infinitely many equilibrium points. The coefficients in the linearized equation near the equilibrium points are delay-dependent. We provide a detailed stability analysis of each equilibrium point. We observed the following bifurcation phenomena: stable for all the delay values, a single stable region in the delayed interval, and a stability switch. We also observed a multi-scroll chaotic attractor for some values of the parameters.

**Keywords:** Delay, Stability, Fractional Derivative, Chaos

## 1 Introduction

Mathematical analysis is the fundamental tool used to study complex Biological systems [1, 2]. These systems can be modeled using differential or difference equations. If the "time" variable in these systems is continuous, then one can use the ordinary or partial differential equation [3, 4]. Such equations may be improved by including "time-delay," which gives a better fit for the "system-memory" [5, 6]. The obvious nonlocality of the Biological systems can be captured in the model with the help of "fractional order derivative (FOD)" [7–9]. If the order of the derivative is a non-integer (e.g., a positive real number or a complex number with a positive real part), then it is termed as FOD [10]. Nowadays, Mathematicians are working on derivatives whose order depends on time or is distributed over some interval [11, 12].

In 1967, Israelsson and Johnsson [13] proposed a model explaining the helical movements (circumnutations) of the apex of sunflower plants. The theory is about the interplay between gravity and the growth hormone (auxin). The time delay arises because the hormone takes some time to spread in the plant body (geotropic reaction time for the hypocotyls). Somolinos [14] in 1978 carried out the rigorous mathematical analysis of the sunflower equation. Oscillations in this equation are studied by Kulenović and Ladas [15]. In this work, we generalize the sunflower equation to include the fractional order derivatives. The system possesses infinitely many equilibrium points. We provide the stability analysis of all these equilibrium points and discuss the possible types of bifurcations in detail. Furthermore, we present the chaotic solutions of this system.

The rest of the paper is organized as below: Section 2 gives the details of the sunflower equation and its fractional-order counterpart. In Section (3), we describe the stability and bifurcation analysis of equilibrium points of the sunflower equation. Chaos in the proposed model is studied in Section (4). Validation of results is done in Section (5). Section (6) presents the conclusions.

# 2 The Sunflower Equation

The sunflower equation [13] described by (1) is a modeling nonlinear delay differential equation that defines the helical movement of the tip of a growing plant which accumulates growth hormone (auxin).

$$\frac{\tau}{l}\ddot{x}(t) + \dot{x}(t) = \frac{-m}{l}\sin(x(t-\tau)) \tag{1}$$

where $m$, $l$ and delay $\tau$ are positive numbers. Now, we generalize it to the fractional order case as:

$$\frac{\tau}{l}D^{2\alpha}x(t) + D^{\alpha}x(t) = \frac{-m}{l}\sin(x(t-\tau)), \quad 0 < \alpha \le 1. \tag{2}$$

Here, $D^{\alpha}$ and $D^{2\alpha}$ represent Caputo Fractional Derivatives [10, 16–19].

Note that $x^*$ is an equilibrium point of equation (2) if and only if $\sin(x^*) = 0$ as we have $D^{\alpha}x^* = 0$ and $D^{2\alpha}x^* = 0$. So, equation (2) has infinitely many equilibrium points given by $x^*_{1,n} = 2n\pi$ and $x^*_{2,n} = (2n+1)\pi$, $n \in \mathbb{Z}$.

By taking a small perturbation near the equilibrium point and using Taylor's approximations we get the local linearization of equation (2) as

$$\frac{\tau}{l}D^{2\alpha}x(t) + D^{\alpha}x(t) = \frac{-m}{l}x(t-\tau), \tag{3}$$

near the equilibrium point $x^*_{1,n}$ and

$$\frac{\tau}{l}D^{2\alpha}x(t) + D^{\alpha}x(t) = \frac{m}{l}x(t-\tau), \tag{4}$$

near the equilibrium point $x^*_{2,n}$.

# 3 Stability Analysis

If we consider the non-delayed equation (3) then we get $D^\alpha x(t) = \frac{-m}{l}x(t)$ which implies that the equilibrium point $x^*_{1,n}$ is stable at $\tau = 0$ [20]. Similarly, the equilibrium point $x^*_{2,n}$ is unstable at $\tau = 0$ as the equation (4) gets reduced to $D^\alpha x(t) = \frac{m}{l}x(t)$.

## 3.1 Stability results for the equilibrium points $x^*_{1,n} = 2n\pi$

Now, let us consider the equilibrium points $x^*_{1,n}$ and $\tau > 0$.
By using Laplace transform, the characteristic equation of (3) is:

$$\frac{\tau}{l}\lambda^{2\alpha} + \lambda^\alpha + \frac{m}{l}\exp(-\lambda\tau) = 0. \tag{5}$$

We have change in stability only when the root $\lambda = u + iv$ of equation (5) crosses the imaginary axis.
Therefore, by putting $\lambda = iv$, $v > 0$ in the equation (5), we get the boundary of the stable region i.e. $\frac{\tau}{l}(iv)^{2\alpha} + (iv)^\alpha + \frac{m}{l}\exp(-iv\tau) = 0$.
Separating the real and imaginary parts we get,

$$\frac{\tau}{l}v^{2\alpha}\cos(\alpha\pi) + v^\alpha\cos\left(\frac{\alpha\pi}{2}\right) = \frac{-m}{l}\cos(v\tau) \tag{6}$$

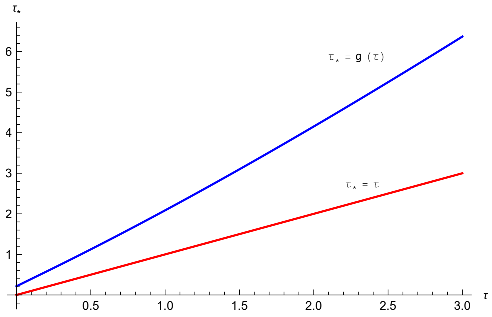
and

$$\frac{\tau}{l}v^{2\alpha}\sin(\alpha\pi) + v^\alpha\sin\left(\frac{\alpha\pi}{2}\right) = \frac{m}{l}\sin(v\tau). \tag{7}$$

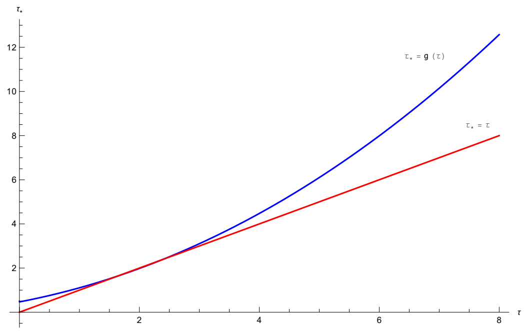
Now, by squaring and adding equations (6) and (7), we get

$$l^2v^{2\alpha} + \tau^2v^{4\alpha} + 2lv^{3\alpha}\tau\cos(\frac{\alpha\pi}{2}) - m^2 = 0 \tag{8}$$

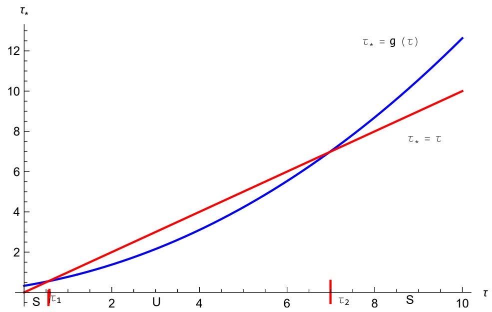
Since, $l$ and $m$ are positive numbers, we get only one positive root $v^\alpha$ of equation (8) given in the Data Set 1 available at https://drive.google.com/drive/folders/1jOuemmKoSxZfzFSRlotf94YYp5nJI-iy?usp=sharing. By putting this value of $v$ in the equation (6), we get a critical value of delay $\tau_*$ where we have change in stability for the equilibrium point $x^*_{1,n}$. By Section (6) in [21], $\exists$ only one critical value of delay $\tau_*$. Since the coefficient in equation (5) depends on $\tau$, the expression for $\tau_*$ also depends on $\tau$, say $\tau_* = g(\tau)$ and is given in the above Data Set 1.
Now, note that if the curve $g(\tau)$ does not meet the line $\tau_* = \tau$ in the $\tau\tau_*$plane then $x^*_{1,n}$ is asymptotically stable (cf. Figure (1)(a)). If it meets twice then $x^*_{1,n}$ will generate stability switch (SS) as shown in Figure (1)(c). So, there exists $\tau_1$ and $\tau_2$ where $g(\tau_1) = \tau_1$ and $g(\tau_2) = \tau_2$ such that when $\tau \in [0, \tau_1)$ then $x^*_{1,n}$ is asymptotically stable, if $\tau \in (\tau_1, \tau_2)$ then $x^*_{1,n}$ is unstable and if $\tau > \tau_2$ again we get $x^*_{1,n}$ asymptotically



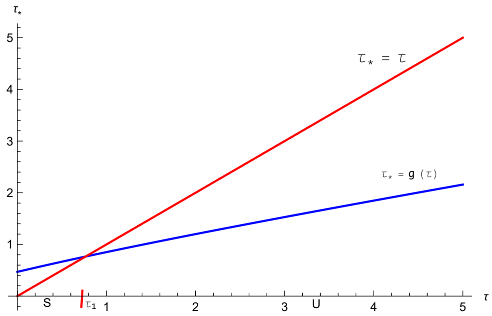

(a) $x^*_{1,n}$ is stable when there is no intersection between $g(\tau)$ and $\tau$




(b) Curve that bifurcates the stable region from the stability switch




(c) Two intersections between $g(\tau)$ and $\tau$ results the stability switch S-U-S




(d) Single stable region since we have only one intersection between $g(\tau)$ and $\tau$

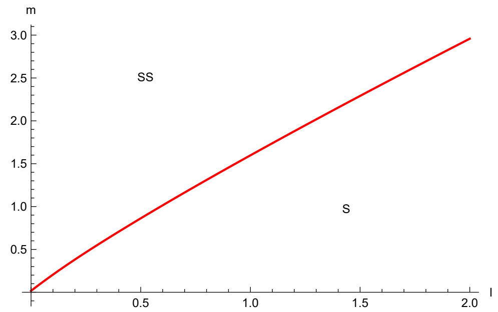
**Fig. 1**: Different behaviors of the critical curve $\tau_* = g(\tau)$ in the $\tau\tau_*$ plane

stable (cf. Figure (1)(c)). If $g(\tau)$ cuts only once the line $\tau = \tau_*$ then we have a single stable region (SSR) as given in Figure (1)(d). So, if $\tau_1$ is the only intersection point where $g(\tau_1) = \tau_1$ then then $x^*_{1,n}$ is asymptotically stable when $\tau < g(\tau_1)$ whereas $\tau > g(\tau_1)$ implies that $x^*_{1,n}$ is unstable.

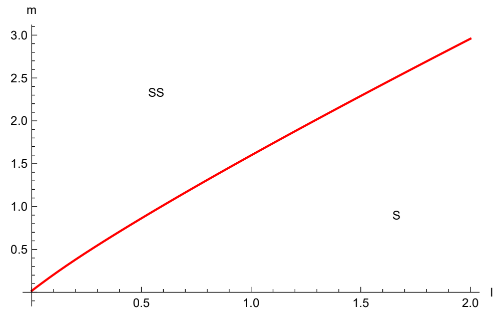
## 3.2 Bifurcation analysis of $x^*_{1,n}$

- For $\alpha = 0.1$, 0.2 and 0.3, we observed only two behaviors of $x^*_{1,n}$ viz. Stable (Figure (1)(a)) and stability switch (SS) (Figure (1)(c)). At the bifurcation of these two behaviors, the curve $\tau_* = \tau$ becomes tangent to the curve $\tau_* = g(\tau)$ (cf. Figure (1)(b)). The values of $l$ and $m$ (cf. Figure (2)) at such tangent form a curve in $lm-$plane that bifurcates stable region with the stability switch.

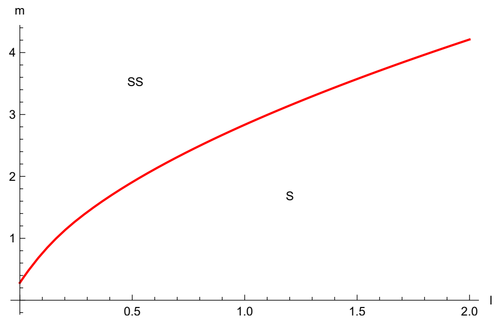
- All the stability behaviors described in Figure (1) was given by $\alpha = 0.4$.
  In Figure (2)(d), the curve $m = h_2(l)$ separating the stable region from the stability switch (SS) is obtained as described above.



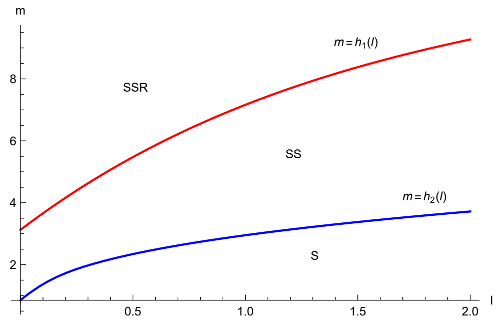

(a) Bifurcation curve in the $lm-$plane for $\alpha = 0.1$




(b) Bifurcation curve in the $lm-$plane for $\alpha = 0.2$




(c) Bifurcation curve in the $lm-$plane for $\alpha = 0.3$

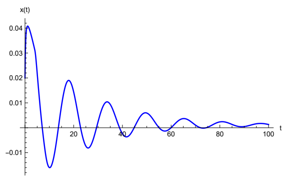



(d) Bifurcation curve in the $lm-$plane for $\alpha = 0.4$

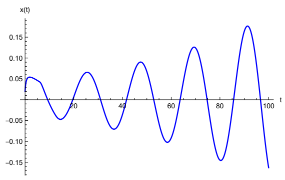
**Fig. 2**: Bifurcation curves in the $lm-$plane for different values of $\alpha$ separating the stable region (S), the stability switch (SS) and the single stable region (SSR)

If we take $m > h_2(l)$ then there are two intersections between $g(\tau)$ and $\tau$ which results in SS region. The second intersection point goes away from the first as we increase $m$ further. At the another bifurcation $m = h_1(l)$ (see Figure (2)(d)), the second intersection point $\to \infty$ and vanishes.
Therefore, for $m > h_1(l)$, there is only one intersection between $g(\tau)$ and $\tau$ and we get the SSR. This gives another bifurcation curve $m = h_1(l)$ separating SS from SSR in the $lm-$plane.

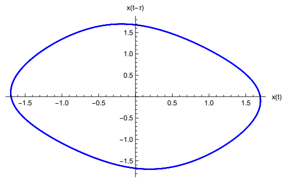
- For $1/2 \leq \alpha < 1$, we observed that the curve $\tau_* = g(\tau)$ and $\tau_* = \tau$ have only one intersection (Figure (1)(d)). Thus, there is SSR for $x^*_{1,n}$ and no bifurcation is observed.



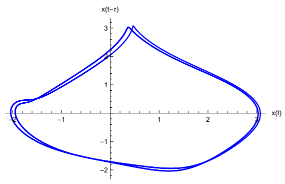

(a) Stable solution for $\tau = 4$



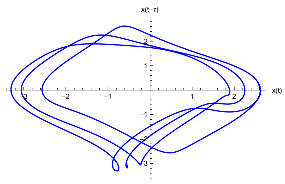

(b) Unstable solution for $\tau = 6$



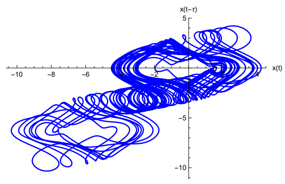

(c) Periodic solution for $\tau = 8$




(d) Two closed cycle for $\tau = 14$




(e) Four closed cycle for $\tau = 15$




(f) Chaotic attractor for $\tau = 20$

**Fig. 3**: Period doubling route to chaos for $l = 14$, $m = 5.6$ and $\alpha = 0.85$

### 3.3 Stability results for the equilibrium points $x^*_{2,n} = (2n+1)\pi$

Note that the equilibrium points $x^*_{2,n}$ are unstable at $\tau = 0$. If we compare equation (8) given in the paper [21] with the equation (4) we get $a = 0$, $b = \dfrac{m}{l}$ so $a_1 = \dfrac{m}{l}$ and $c = \dfrac{\tau}{l}$. Therefore, for any $\tau$, $l$ and $m$ positive, we have $a_1 > 0$ and $c > 0$. So, by the Figure (17) and Figure (19) in [21], the equilibrium points $x^*_{2,n} = (2n+1)\pi$ are unstable for all $\tau \geq 0$ and for any $0 < \alpha \leq 1$.

## 4 Chaos in the Sunflower equation

If we take $l = 14$, $m = 5.6$ and $\alpha = 0.85$ then we get the critical value $\tau_1 = 5.16433$ (Figure (1)(d)) where we have $g(\tau) = \tau$. So, if we take $\tau < \tau_1$ we get stable solution near $x^*_{1,n}$. The stable solution for $\tau = 4$ with initial data $x(t) = 6.9$ and $\dot{x}(t) = 2.5$, $-\tau < t \leq 0$ is shown in Figure (3)(a). We get the unstable solution for $\tau > \tau_1$ as shown in Figure (3)(b) with $\tau = 6$.

If we further increase the delay $\tau$ e.g. $\tau = 8$, we get a periodic solution with one closed orbit (cf. Figure (3)(c)). For $\tau = 14$ and $\tau = 15$, we get two-cycle and four-cycle, respectively, as shown in Figures (3)(d) and (3)(e). The period doubling leads to chaos (3)(f). The infinite scroll chaotic attractor is given in Figure (4) for $\alpha = 1$, $l = 14$, $m = 5.6$ and $\tau = 20$.
Note that, the fractional order systems do not have exactly periodic orbits [22]. However, we can have asymptotic-periodic orbits or limit cycles as observed in this work.

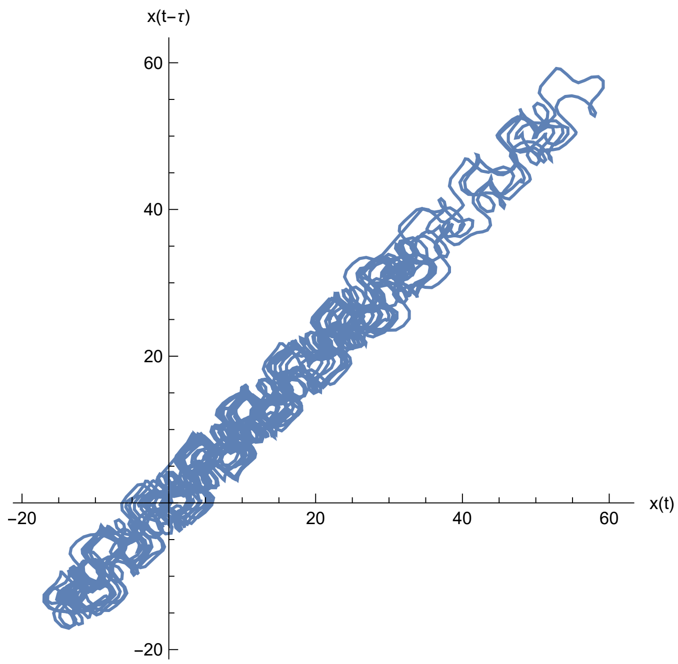



**Fig. 4**: Infinite scroll chaotic attractor

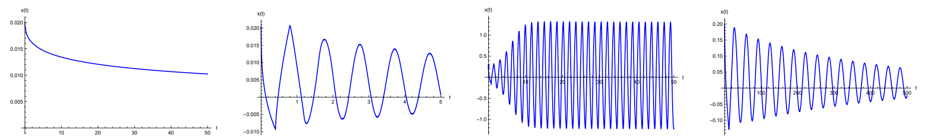
# 5 Examples

**Example 5.1.** *Figure (2) shows that there is only one bifurcation curve in the $lm-$plane separating the region $S$ from $SS$ for $\alpha = 0.1,\ 0.2,$ and $0.3$.*

*So, if we take $\alpha = 0.3$ and $l = 3$ with the initial data $x(t) = 0.02$ for $t \in (-\tau, 0]$ near $x^*_{1,0} = 0$ then along the bifurcation curve we get the critical value $m = 5.3092$. If we take $m = 1 < 5.3092$ then the equilibrium point $0$ is stable $\forall \tau \geq 0$. Figure (5)(a) shows the stable solution for $\tau = 4$.*
*Now, if we take $m = 6 > 5.3092$ then we are in the stability switch region. The two critical values of $\tau$ are $\tau_1 = 0.567501$ and $\tau_2 = 10.133$ (see Figure (1)(c)) where $g(\tau) = \tau$. If we take $\tau < 0.567501$, we get stable solution near $0$ (cf. Figure (5)(b) for $\tau = 0.4$), if we take $0.567501 < \tau < 10.133$ we get unstable solution near $0$ (cf. Figure (5)(c) for $\tau = 0.7$) and if we take $\tau > 10.133$ (cf. Figure (5)(d) for $\tau = 12$) again we get stable solution near $0$.*

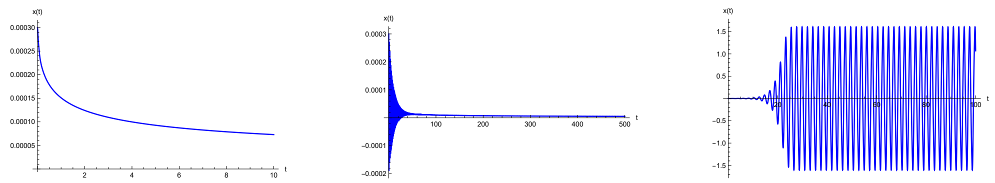
**Example 5.2.** *Figure (2)(d) shows that there are all the three types of behaviors viz. $S$, $SS$ and $SSR$ for $\alpha = 0.4$ in a neighborhood of the equilibrium points $x^*_{1,n}$.*

*So, if we fix $l = 1$ then along the curve $m = h_2(l)$ we get $m = 2.95108$ and $m = 7.16$ along $m = h_1(l)$. We take initial data as $x(t) = 0.0003$ $\forall t \in (-\tau, 0]$ near $x^*_{1,0} = 0$.*
*Hence, if we take $m = 1 < 2.95108$, we get a stable solution near $0$ for all $\tau > 0$. Figure (6)(a) shows stable solution near $0$ for $\tau = 0.08$ .*

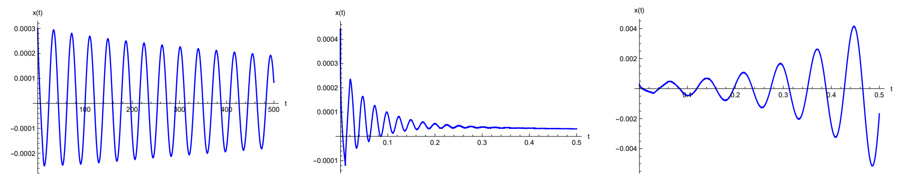


(a) Stable solution for $\tau = 4$ and $m = 1$ (b) Stable solution for $\tau = 0.4$ and $m = 6$ (c) Unstable solution for $\tau = 0.7$ and $m = 6$ (d) Stable solution for $\tau = 12$ and $m = 6$

**Fig. 5**: Figures of Example (5.1)



(a) Stable solution for $\tau = 0.08$ (b) Stable solution for $\tau = 0.4$ (c) Unstable solution for $\tau = 0.8$



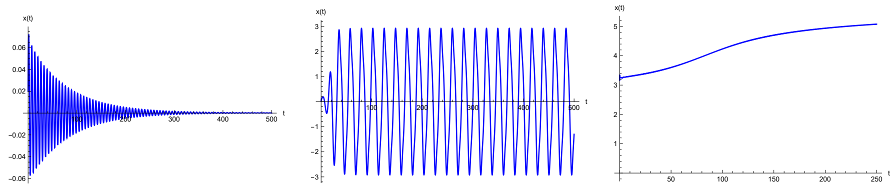
(d) Stable solution for $\tau = 12$ (e) Stable solution for $\tau = 0.01$ (f) Unstable solution for $\tau = 0.03$

**Fig. 6**: Figures of Example (5.2)

*Now, if we take $m = 3.2 \in (2.95108, 7.16)$, then it is in the stability switch region, and we get $\tau_1 = 0.616608$ and $\tau_2 = 10.733$ given in the Figure (1)(c). So, we get stable solution for $\tau \in [0, 0.616608)$, unstable solution for $\tau \in (0.616608, 10.733)$ and again stable solution for $\tau > 10.733$ near the equilibrium points $x^*_{1,n}$. The stable solution near $x^*_{1,0}$ for $\tau = 0.4$ is given in Figure (6)(b), unstable solution for $\tau = 0.8$ in Figure (6)(c) and stable solution for $\tau = 12$ in Figure (6)(d).*

*If $m = 8 > 7.16$ then we are in the SSR region from Figure (2)(d). So, we get $\tau_1 = 0.0173043$ from Figure (1)(d) where $g(\tau) = \tau$. So, for $\tau < 0.0173043$ we get stable solutions (cf. Figure (6)(e) with $\tau = 0.01$) and for $\tau > 0.0173043$ we get unstable solutions (cf. Figure (6)(f) with $\tau = 0.03$) near the equilibrium points $x^*_{1,n}$.*

**Example 5.3.** *Let us now consider the case for $1/2 \le \alpha < 1$. From Subsection (3.2), there is no any bifurcation for this case. We get only one critical value $\tau_1$ (see Figure (1)(d)) where $g(\tau) = \tau$ such that $0 < \tau < \tau_1$ gives stability of $x^*_{1,n}$.*
*So, if we fix $\alpha = 0.9$, $l = 1$ and $m = 1.5$ with the initial data $x(t) = 0.02$, $\dot{x}(t) = 0.1$*



(a) Stable solution for $\tau = 1$ (b) Stable solution for $\tau = 3$ (c) Untable solution for $\tau = 2.8$ near the equilibrium point $\pi$

**Fig. 7**: Figures of Example (5.3) and Example (5.4)

*for* $-\tau < t \leq 0$ *then we get a critical value* $\tau_1 = 1.03915$ *such that for* $\tau < \tau_1$ *we get stable solution near* $x^*_{1,n}$ *(cf. Figure* (7)*(a) for* $\tau = 1$*). For* $\tau > \tau_1$*, we get unstable solution (cf. Figure* (7)*(b) for* $\tau = 3$*).*

**Example 5.4.** *The equilibrium points* $x^*_{2,n}$ *are unstable for all* $\tau \geq 0$ *and* $0 < \alpha < 1$ *(Section* (3.3)*). So, if we fix* $l = 5$*,* $m = 2$*,* $\tau = 2.8$ *and* $\alpha = 0.3$ *with the initial condition* $x(t) = 3.17$ *for* $2.8 < t \leq 0$ *then the equilibrium point* $\pi$ *is unstable as shown in Figure* (7)*(c).*

# 6 Conclusions

Fractional order generalizations of the classical equations are useful in improving the models. The resulting models are more realistic than their classical counterparts. We generalized the sunflower equation to include two fractional order derivatives. The stability analysis of the equilibrium points is provided by linearizing the equations near the respective equilibrium and using the theory developed in the literature. For the fractional order $0 < \alpha < 0.4$, we observed the stable solutions for all the delay parameters and the stability switches for some parameters $l$ and $m$. For $1/2 \leq \alpha \leq 1$, we observed a single stable region where the existence of the critical value $\tau_1$ of the delay bifurcates the stable behavior from the unstable one. The fractional order $\alpha = 0.4$ shows richer dynamics. We observe all the three bifurcation behaviors described above. The interesting observation is chaos at some parameter sets. The chaotic attractor generated has many scrolls because of the involvement of the sine function in the model.

# Acknowledgment

S. Bhalekar acknowledges the University of Hyderabad for Institute of Eminence-Professional Development Fund (IoE-PDF) by MHRD (F11/9/2019-U3(A)). D. Gupta thanks University Grants Commission for financial support (No.F.82-44/2020(SA-III)).

# Declarations

- **Competing Interests:** The authors declare that there are no competing interests regarding the publication of this research article.
- **Data availability:** Data generated in this work is made available at https://drive.google.com/drive/folders/1jOuemmKoSxZfzFSRlotf94YYp5nJI-iy?usp=sharing.
- **Authors' contributions:** S.B. designed the problem. D.G. conducted the research, and both authors collaborated in the discussion. D.G. took the lead in writing the paper, and S.B. provided proofreading and editing. S.B. provided research supervision.

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