

FILTERED BOOLEAN POWERS OF FINITE SIMPLE NON-ABELIAN MAL'CEV ALGEBRAS

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ABSTRACT. Let $\mathbf{A}$ be a finite simple non-abelian Mal'cev algebra (e.g. a group, loop, ring). We investigate the Boolean power $\mathbf{D}$ of $\mathbf{A}$ by the countable atomless Boolean algebra $\mathbf{B}$ filtered at some idempotents $e_1, \dots, e_n$ of $\mathbf{A}$. When $e_1, \dots, e_n$ are *all* idempotents of $\mathbf{A}$ we establish two concrete representations of $\mathbf{D}$: as the Fraïssé limit of the class of finite direct powers of $\mathbf{A}$, and as congruence classes of the countable free algebra in the variety generated by $\mathbf{A}$. Further, for arbitrary $e_1, \dots, e_n$, we show that $\mathbf{D}$ is ω -categorical and that its automorphism group has the small index property, strong uncountable cofinality and the Bergman property. As necessary background we establish some general properties of congruences and automorphisms of filtered Boolean powers of $\mathbf{A}$ by any Boolean algebra $\mathbf{B}$, including a semidirect decomposition for their automorphism groups.

1. INTRODUCTION, PRELIMINARIES AND STATEMENTS OF THE MAIN RESULTS

The purpose of this paper is to investigate filtered Boolean powers $\mathbf{D}$ of a finite simple non-abelian Mal'cev algebra $\mathbf{A}$ by a Boolean algebra $\mathbf{B}$. On a general level, we establish some properties of congruences and automorphisms of $\mathbf{D}$. When $\mathbf{B}$ is the countable atomless Boolean algebra, we investigate the role $\mathbf{D}$ plays in the variety generated by $\mathbf{A}$, and prove that it has a number of combinatorial properties: it is ω -categorical and its automorphism group has the small index property, strong uncountable cofinality and the Bergman property.

In this section we present the background and motivation for our work, introduce the concepts and notation that will be used throughout and give statements of the main results.

A *variety* is a class of algebraic structures (*algebras* for short) of the same type that is defined by equations. By Birkhoff's Theorem the

Date: April 29, 2024.

2020 *Mathematics Subject Classification.* 03C05 (20B27, 22A05, 08A05, 06E15, 54H10).

Key words and phrases. Fraïssé limit, direct powers, free algebra of countable rank, countable atomless Boolean algebra, Stone space, Cantor space, small index property, uncountable cofinality, Bergman property.

Supported by the Engineering and Physical Sciences Research Council EP/V003224/1 and by the Heilbronn Institute for Mathematical Research.

variety generated by a class of algebras C is the class of all homomorphic images of subalgebras of products of elements in C . We refer to [7] for background on general algebra.

Varieties generated by a finite simple group have been studied by Neumann [24], Apps [2] and others. Many of the techniques carry over from groups to Mal'cev algebras in general. Here an algebra $\mathbf{A}$ is *Mal'cev* if it has a ternary term operation m satisfying $m(x, x, y) = y = m(y, x, x)$. Note that groups, quasigroups, loops, rings as well as their expansions with additional operations are all Mal'cev.

The generalization of commutator theory from groups to algebras as described in [12] allows us to talk about (non)-abelian algebras. A Mal'cev algebra is *abelian* if its basic operations can be represented as affine functions over a module; otherwise we call it *non-abelian*. This is consistent with the classical notions for groups and loops. A ring is abelian if and only if its multiplication is constant 0.

Unless otherwise indicated, $\mathbf{A}$ will stand for a finite simple non-abelian Mal'cev algebra throughout. Throughout, we will use bold letters to denote algebras and other structures, and matching standard letters will be used for the underlying universes. Thus, here, A would be the universe of $\mathbf{A}$. Let V denote the variety generated by $\mathbf{A}$, let V_{fin} be the class of its finite members, and let W be the variety generated by all proper subalgebras of $\mathbf{A}$. Then W is the unique maximal subvariety of V . Every element in V_{fin} is isomorphic to some $\mathbf{A}^k \times \mathbf{B}$ for an integer $k \geq 0$ and $\mathbf{B} \in W_{\text{fin}}$ (cf. [24] for groups, [11] for primal algebras). This indicates the distinguished role the class $K := \{\mathbf{A}^k : k \geq 1\}$ of finite powers of $\mathbf{A}$ plays in the structure of the finite members of V . It turns that this class has some nice model-theoretic properties, leading to existence of a Fraïssé limit.

In order to be able to state this, we briefly review the standard Fraïssé theory following [16]. Let K be any collection of finitely generated structures of the same type. Consider the following three properties that such a collection may have:

Hereditary property (HP): Every finitely generated substructure of a member of K is isomorphic to a member of K .

Joint embedding property (JEP): For any $\mathbf{A}, \mathbf{B} \in K$ there exists $\mathbf{C} \in K$ into which both $\mathbf{A}$ and $\mathbf{B}$ embed.

Amalgamation property (AP): For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in K$ with embeddings $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ and $\psi: \mathbf{A} \rightarrow \mathbf{C}$ there exists $\mathbf{D} \in K$ and embeddings $\mu: \mathbf{B} \rightarrow \mathbf{D}$ and $\nu: \mathbf{C} \rightarrow \mathbf{D}$ such that $\mu\varphi = \nu\psi$.

Note that V_{fin} has HP but in general not JEP or AP. On the other hand the subclass $K = \{\mathbf{A}^k : k \geq 1\}$ we are looking at has JEP and AP but in general not HP. (The assumption $k \neq 0$ is necessary for AP only if the algebra $\mathbf{A}$ has more than one trivial subalgebra.)

By a generalization of Fraïssé's Theorem [16, Theorem 7.1.2] there exists a unique (up to isomorphism) countable algebra $\mathbf{D}$ such that (i) every finitely generated subalgebra of $\mathbf{D}$ embeds into some element of K , (ii) $\mathbf{D}$ is a direct limit of algebras in K , and (iii) every isomorphism between subalgebras of $\mathbf{D}$ that are isomorphic to some element in K extends to an automorphism of $\mathbf{D}$ (i.e., $\mathbf{D}$ is *K-homogeneous*). We call such a $\mathbf{D}$ the *Fraïssé limit* of K .

This Fraïssé limit can be explicitly described as a filtered Boolean power of $\mathbf{A}$, as defined by Arens and Kaplansky [3] (see also [10]). We briefly review this construction.

Let $\mathbf{B}$ be a Boolean algebra. The *Stone space* of $\mathbf{B}$ is the set X of ultrafilters on $\mathbf{B}$ with the topology whose basic open sets are $\{x \in X : b \in x\}$ for $b \in B$. The original Boolean algebra $\mathbf{B}$ is isomorphic to the Boolean algebra of clopen sets of X . An arbitrary topological space X is homeomorphic to the Stone space of a Boolean algebra if and only if it is compact and totally separated. For basics on Boolean algebras we refer the reader to [23, Chapter 1], and for Stone duality to [23, Chapter 3].

Recall that there is a unique (up to isomorphism) countable atomless Boolean algebra $\mathbf{B}$. It is in fact the Fraïssé limit of all finite Boolean algebras, and will play a particularly important role in this paper. Its Stone space is the *Cantor space* X , i.e. the unique space up to homeomorphism which is compact, Hausdorff, has no isolated points and has a countable basis of clopen sets. A concrete representation of X is as $A^{\mathbb{N}}$ under the product topology, where A is any finite set endowed with the discrete topology.

Let $\mathbf{B}$ be a Boolean algebra with Stone space X , and let $\mathbf{A}$ be an algebra. The *Boolean power* $\mathbf{A}^{\mathbf{B}}$ is the subalgebra of $\mathbf{A}^X$ with universe

$$A^{\mathbf{B}} := \{f : X \rightarrow A : f \text{ is continuous}\},$$

where A is endowed with the discrete topology.

Borrowing the concept from semigroups, we call an element e of an algebra $\mathbf{A}$ an *idempotent* if $\{e\}$ forms a trivial subalgebra of $\mathbf{A}$. For pairwise distinct $x_1, \dots, x_n \in X$ and idempotents $e_1, \dots, e_n$ of $\mathbf{A}$, the subalgebra $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ of $\mathbf{A}^{\mathbf{B}}$ with universe

$$\{f \in A^{\mathbf{B}} : f(x_1) = e_1, \dots, f(x_n) = e_n\}$$

is a *filtered Boolean power* of $\mathbf{A}$.

We can now state our first main result.

{thm:Flim}

Theorem 1.1. *Let $\mathbf{A}$ be a finite simple non-abelian Mal'cev algebra, let $e_1, \dots, e_n$ be the idempotents of $\mathbf{A}$, let $\mathbf{B}$ be the countable atomless Boolean algebra, let X be the Stone space of $\mathbf{B}$, and let $x_1, \dots, x_n \in X$ be distinct.*

{it:Flim}

- (1) *The Fraïssé limit of $\{\mathbf{A}^k : k \geq 1\}$ exists and is isomorphic to $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$.*

`{it:free}`

- (2) Let $\mathbf{F}$ be the free algebra of countable rank in the variety V generated by $\mathbf{A}$, and let θ be the smallest congruence such that $\mathbf{F}/\theta$ is in the variety W generated by all proper subalgebras of $\mathbf{A}$. Then every θ -class which is a subalgebra of $\mathbf{F}$ is isomorphic to $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$.

In the theorem above it is in fact sufficient to take $e_1, \dots, e_n$ to be a set of orbit representatives under $\text{Aut} \mathbf{A}$ for the idempotents in $\mathbf{A}$ (see Corollary 2.8).

Part (1) of Theorem 1.1 is proved in Section 3 and part (2) in Section 4.

Note that when $\mathbf{A}$ is a group, the filtered Boolean power arising in Theorem 1.1(2) is of the form $(\mathbf{A}^{\mathbf{B}})_e^x$, where e is the identity of $\mathbf{A}$. This is in turn easily seen to be isomorphic to $\mathbf{A}^{\mathbf{R}}$ where $\mathbf{R}$ is the countable atomless Boolean ring without identity. (For Boolean powers over Boolean rings see [2, 15].) Theorem 1.1(2) in this case gives that the kernel of the homomorphism from $\mathbf{F}$ to the free group of countable rank in W is isomorphic to $(\mathbf{A}^{\mathbf{B}})_e^x$. This observation was made in [4, p. 367-8] and [5, p. 201]. As a consequence of Theorem 1.1(2), for groups and loops and rings we obtain the stronger result that every countable algebra $\mathbf{C}$ in V is a split extension of a Boolean power of $\mathbf{A}$ by an algebra in W (Corollary 4.1).

We also provide an illustration of Theorem 1.1 for a specific algebra with more than one idempotent.

`{exa:twoids}`

Example 1.2. Let $\mathbf{A} := (\mathbb{Z}_2, x - y + z, \cdot)$ be the idempotent reduct of the field of size 2. The only proper subalgebras of $\mathbf{A}$ are the trivial ones $\{0\}$ and $\{1\}$, and $\text{Aut} \mathbf{A}$ is trivial as well. It turns out that the variety V generated by $\mathbf{A}$ is the class of (isomorphic copies of) filtered Boolean powers of $\mathbf{A}$.

Every finite algebra in V is isomorphic to $\mathbf{A}^k$ for some integer $k \geq 0$. Let $\mathbf{B}$ be the countable atomless Boolean algebra. Then the Fraïssé limit of $\{\mathbf{A}^k : k \geq 1\}$ is isomorphic to the free algebra of countable rank in V and to

$$(\mathbf{A}^{\mathbf{B}})_{0,1}^{x_0,x_1} \cong (\mathbf{A}^{\mathbf{B}})_0^{x_0} \times (\mathbf{A}^{\mathbf{B}})_1^{x_1}.$$

We now investigate automorphism groups of filtered Boolean powers. For a group G acting on a set X and $Y \subseteq X$, we use G_Y to denote the pointwise stabilizer $\{g \in G : g(x) = x \text{ for all } x \in Y\}$ of Y . Similarly for a set Z of subsets of X , we let G_Z denote the setwise stabilizer $\{g \in G : g(Y) = Y \text{ for all } Y \in Z\}$ of all sets in Z .

It is well-known that for any countably infinite first order structure M , the group of automorphisms $G := \text{Aut} M$ affords a natural *topology of pointwise convergence* with basic open sets the cosets of pointwise stabilizers G_Y of finite subsets Y of M . This makes G a topological (in fact, Polish) group.

The group G has the *small index property* (SIP) if any subgroup of index less than $2^{\aleph_0}$ is open, that is, contains the pointwise stabilizer of some finite subset of M .

Note that if $\text{Aut}M$ has the SIP, then its topological structure is completely determined by its abstract algebraic structure. Specifically, if N is another countable structure and $\text{Aut}M \cong \text{Aut}N$ as groups, then in fact $\text{Aut}M$ and $\text{Aut}N$ are isomorphic as topological groups [21, Proposition 5.2.2]. In particular $\text{Aut}M$ has a unique Polish topology.

The *cofinality* of a group G is the smallest cardinality κ such that G is the union of a chain of length κ of proper subgroups.

The *strong cofinality* of G is the smallest cardinality κ such that G is the union of a chain $(U_i)_{i < \kappa}$ of proper subsets of G such that for all $i < j$ we have $U_i \subseteq U_j$, $U_i = U_i^{-1}$ and $U_i U_i \subseteq U_k$ for some $k \in \kappa$.

The group G has the *Bergman property* if for each subset E of G such that $1 \in E = E^{-1}$ and E generates G there exists $k \in \mathbb{N}$ such that $E^k = G$.

Finally G has *ample generics* if for any $k \in \mathbb{N}$ the diagonal conjugacy action of G on G^k has a comeager orbit, that is, an orbit containing the intersection of countably many dense open subsets of G^k .

These properties are related as follows. A group G has uncountable strong cofinality if and only if G has uncountable cofinality and the Bergman property (cf. [9, Theorem 2.2]). The existence of ample generics implies SIP as shown by Hodges, Hodkinson, Lascar and Shelah [17]. Further, if M is ω -categorical and $\text{Aut}M$ has ample generics, then $\text{Aut}M$ has uncountable strong cofinality by Kechris and Rosendal [18].

The automorphism group G of the countable atomless Boolean algebra was among the first for which these properties were established. Truss [26] showed that it has SIP. Droste and Göbel [8] proved uncountable strong cofinality. Later Kwiatkowska [19] subsumed these results by showing that G actually has ample generics.

As our second main result we show that SIP and uncountable strong cofinality carry over from the countable atomless Boolean algebra $\mathbf{B}$ to filtered Boolean powers of a finite simple non-abelian Mal'cev algebra by $\mathbf{B}$:

{thm:SIP}

Theorem 1.3. *Let $\mathbf{A}$ be a finite simple non-abelian Mal'cev algebra, let $e_1, \dots, e_n$ be the idempotents of $\mathbf{A}$, let $\mathbf{B}$ be the countable atomless Boolean algebra, let X be its Stone space, and let $x_1, \dots, x_n \in X$ be distinct. Then*

{it:omegacat}

{it:SIP}

{it:Bergman}

- (1) $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ is ω -categorical;
- (2) $\text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ has the SIP;
- (3) $\text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ has uncountable strong cofinality, in particular, strong cofinality and the Bergman property.

Theorem 1.3(1) will follow from results of Macintyre and Rosenstein [20] in Section 2.5. We will prove items (2),(3) of Theorem 1.3 in

Sections 5, 6 following the strategies for the countable atomless Boolean algebra by Truss [26] and Droste and Göbel [8], respectively. It remains open whether $\text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ has ample generics. In particular, it is unclear whether Kwiatkowska's approach for the countable atomless Boolean algebra from [19] can be extended to filtered Boolean powers.

Throughout the paper $\mathbb{N}$ stands for the set of natural numbers $\{1, 2, \dots\}$. For $n \in \mathbb{N}$ we write $[n] := \{1, \dots, n\}$.

2. FILTERED BOOLEAN POWERS

We collect the structural information on filtered Boolean powers and their automorphism groups that we need for proving our main results. Throughout this section we use the following notation unless specified otherwise:

- $\mathbf{A}$ is a finite simple non-abelian Mal'cev algebra;
- $n \in \mathbb{N} \cup \{0\}$;
- $e_1, \dots, e_n$ are idempotents of $\mathbf{A}$; they do not need to be distinct or to include all the idempotents;
- $\mathbf{B}$ is a Boolean algebra with Stone space X ; the latter will be considered to consist of ultrafilters on $\mathbf{B}$;
- $x_1, \dots, x_n$ are pairwise distinct points in X , and $X^\circ := X \setminus \{x_1, \dots, x_n\}$ is equipped with the subspace topology from X .

2.1. Filtered Boolean powers represented on arbitrary sets.

Filtered Boolean powers can be defined using an arbitrary representation of a Boolean algebra as field of sets, not only via its Stone space.

Lemma 2.1. *Let $\mathbf{B}$ be a subalgebra of the Boolean algebra of subsets of a set Y and let X be the Stone space of $\mathbf{B}$.*

- (1) *For $y \in Y$, $y' := \{b \in B : y \in b\}$ is an ultrafilter on $\mathbf{B}$.*
- (2) *Let $\mathbf{A}$ be a finite algebra with idempotents $e_1, \dots, e_n$, and let $y_1, \dots, y_n \in Y$ be distinct. Then the set*

$$D_{e_1, \dots, e_n}^{y_1, \dots, y_n} := \{f \in A^Y : f^{-1}(a) \in B \text{ for } a \in A, f(y_i) = e_i \text{ for } i \in [n]\}$$

is the universe of a subalgebra $\mathbf{D}_{e_1, \dots, e_n}^{y_1, \dots, y_n}$ of $\mathbf{A}^Y$ isomorphic to the filtered Boolean power $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{y'_1, \dots, y'_n}$.

Proof. (1) is clear.

(2) For $b \in B$, let $b^{**} := \{x \in X : b \in x\}$. Let $B^{**} := \{c \subseteq X : c \text{ clopen}\}$. By Stone's Representation Theorem

$$\varphi: \mathbf{B} \rightarrow \mathbf{B}^{**}, b \mapsto b^{**},$$

is an isomorphism. Now

$$D := \{f \in A^Y : f^{-1}(a) \in B \text{ for all } a \in A\}$$

is the universe of a subalgebra $\mathbf{D}$ of $\mathbf{A}^Y$. For $f \in D$ define $f': X \rightarrow A$ by

$$(f')^{-1}(a) = (f^{-1}(a))^{**} \text{ for all } a \in A.$$

Equivalently, if $Y = b_1 \sqcup \cdots \sqcup b_k$ (disjoint union) and $f(b_j) = \{a_j\}$ for $b_j \in B, j \in [k]$, then $X = b_1^{**} \sqcup \cdots \sqcup b_k^{**}$ and $f'(b_j^{**}) = \{a_j\}$ for $j \in [k]$. Then

$$\varphi': \mathbf{D} \rightarrow \mathbf{A}^{\mathbf{B}}, f \mapsto f',$$

is an isomorphism. We claim that φ' restricts to the required isomorphism between $\mathbf{D}_{e_1, \dots, e_n}^{y_1, \dots, y_n}$ and $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{y'_1, \dots, y'_n}$. To see that the domain and codomain are correct, note that for $f \in D$ and $i \in [n]$,

$$\begin{aligned} f(y_i) = e_i &\Leftrightarrow y_i \in f^{-1}(e_i) \Leftrightarrow f^{-1}(e_i) \in y'_i \\ &\Leftrightarrow y'_i \in (f^{-1}(e_i))^{**} = (f')^{-1}(e_i) \Leftrightarrow f'(y'_i) = e_i. \end{aligned}$$

Hence

$$\varphi'(\mathbf{D}_{e_1, \dots, e_n}^{y_1, \dots, y_n}) = (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{y'_1, \dots, y'_n},$$

completing the proof. $\square$

2.2. Congruences. Our description of congruences of filtered Boolean powers extends those of Boolean powers of simple algebras by Burris [6, Theorem 3.5, Corollary 3.6].

For $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ and $Y \subseteq X$ let

$$\theta_Y := \{(f, g) \in D^2 : f|_Y = g|_Y\}$$

denote the kernel of the projection of $\mathbf{D}$ to Y .

Lemma 2.2. [6, cf. Theorem 3.5] *Let $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$.*

- (1) *The principal congruence θ of $\mathbf{D}$ generated by $(f, g) \in D^2$ is the kernel of the projection of $\mathbf{D}$ onto the clopen set*

$$b := \{x \in X : f(x) = g(x)\}.$$

- (2) *Every congruence θ of $\mathbf{D}$ is the kernel of the projection of $\mathbf{D}$ onto*

$$Y := \bigcap_{(f, g) \in \theta} \{x \in X : f(x) = g(x)\}.$$

Proof. (1) Note that every finite subalgebra of $\mathbf{D}$ is contained in a filtered Boolean power $\mathbf{D} \cap \mathbf{A}^{\mathbf{C}}$ for some finite subalgebra $\mathbf{C}$ of $\mathbf{B}$. Here we view C as a finite set of clopens on X and $\mathbf{A}^{\mathbf{C}}$ as the subalgebra of $\mathbf{A}^{\mathbf{B}}$ with the universe

$$A^{\mathbf{C}} = \{h: X \rightarrow A : h^{-1}(a) \in C \text{ for all } a \in A\}.$$

Now suppose that the subalgebra $\mathbf{C}$ is such that $f, g \in A^{\mathbf{C}}$. For all $x \in b$ the projection of $\theta \cap (D \cap A^{\mathbf{C}})^2$ onto x is equality on A ; else if $f(x) \neq g(x)$, the projection of $\theta \cap (D \cap A^{\mathbf{C}})^2$ onto x yields the total congruence on $\mathbf{A}$, as $\mathbf{A}$ is simple. It is folklore that every congruence of

a finite direct power of a simple non-abelian Mal'cev algebra is a product congruence (cf. [13, Lemma 9.69, Corollary 9.66]). In particular each congruence of the finite algebra $\mathbf{D} \cap \mathbf{A}^{\mathbf{C}}$ is a product congruence and uniquely determined by its projections onto the points $x \in X$. It follows that

$$\theta \cap (D \cap A^{\mathbf{C}})^2 = \{(u, v) \in (D \cap A^{\mathbf{C}})^2 : u|_b = v|_b\}.$$

Hence the restrictions of θ and θ_b onto any finite $\mathbf{D} \cap \mathbf{A}^{\mathbf{C}}$ containing f and g coincide. This proves (1).

(2) The inclusion $\theta \subseteq \theta_Y$ is clear. For the converse let $(f', g') \in \theta_Y$. Then $b' := \{x \in X : f'(x) = g'(x)\}$ contains Y . By the definition of Y we have $(f, g) \in \theta$ with $b := \{x \in X : f(x) = g(x)\}$ such that $Y \subseteq b \subseteq b'$. So $(f', g') \in \theta_b$ by construction and $\theta_b \leq \theta$ by (1). Thus $\theta_Y \subseteq \theta$ and (2) is proved. $\square$

For a subset Y of X , the quotient $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n} / \theta_Y$ is isomorphic to the restriction of $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ to Y . The latter is again isomorphic to a filtered Boolean power by Lemma 2.1. Thus every homomorphic image of a filtered Boolean power is a filtered Boolean power again.

Lemma 2.2 actually yields a bijection between the congruences θ of the filtered Boolean power $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ and the filters $\bigcap Y$ contained in $\bigcap_{i=1}^n x_i$ on the Boolean algebra $\mathbf{B}$.

{sec:autos}

2.3. Automorphisms. In this subsection we give a detailed analysis of the automorphisms of $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$, in terms of automorphisms of $\mathbf{A}$ and of $\mathbf{B}$. With some additional mild assumptions regarding $e_1, \dots, e_n$ we obtain a semidirect decomposition of $\text{Aut} \mathbf{D}$, and describe the kernel as the closure of a filtered Boolean power of $\text{Aut} \mathbf{A}$ by $\mathbf{B}$. Our results generalize those for Boolean powers of groups by Boolean rings by Apps [2, Theorem C].

Automorphisms of $\mathbf{B}$ act naturally on the ultrafilters of $\mathbf{B}$, i.e. the points in the Stone space X of $\mathbf{B}$. For a topological space X , let $\text{Homeo} X$ denote the group of all homeomorphism of X . For $x_1, \dots, x_n \in X$, let

$$(\text{Homeo} X)_{\{x_1, \dots, x_n\}} := \{\psi \in \text{Homeo} X : \psi(x_i) = x_i \text{ for all } i \leq n\}.$$

For $\psi \in (\text{Homeo} X)_{\{x_1, \dots, x_n\}}$ define

$$\psi^{\mathbf{D}}: D \rightarrow D, f \mapsto f\psi^{-1}.$$

Equivalently $[\psi^{\mathbf{D}}(f)]^{-1}(a) = \psi f^{-1}(a)$ for all $f \in D, a \in A$. It is easy to check that $\psi^{\mathbf{D}}$ is an automorphism of $\mathbf{D}$, and that

$$(2.1) \quad g: (\text{Homeo} X)_{\{x_1, \dots, x_n\}} \rightarrow \text{Aut} \mathbf{D}, \psi \mapsto \psi^{\mathbf{D}},$$

is a group homomorphism (injective if $e_1, \dots, e_n$ are pairwise distinct).

In the following lemma we show that, conversely, automorphisms of $\mathbf{D}$ induce homeomorphisms on X° , using the fact that every automorphism of $\mathbf{D}$ induces an automorphism on the congruence lattice of $\mathbf{D}$.

{lem:hphi}

Lemma 2.3. *For $x \in X$, let $\pi_x: (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n} \rightarrow \mathbf{A}$, $f \mapsto f(x)$, be the projection at x , and let $\theta_x := \ker \pi_x$ be its kernel. For any $\varphi \in \text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ the following hold:*

{it:phicirc1}

(1) $\varphi^\circ: X^\circ \rightarrow X^\circ$ defined by

$$\varphi(\theta_x) = \theta_{\varphi^\circ(x)}$$

{it:phix}

is a homeomorphism.

{it:phicirc2}

(2) $\varphi_x := \pi_{\varphi^\circ(x)} \varphi \pi_x^{-1}$ is an automorphism of $\mathbf{A}$ for all $x \in X^\circ$.

(3) For any convergent net $(y_i)_{i \in I}$ in X° with limit x_k for $k \in [n]$, all cluster points of $(\varphi^\circ(y_i))_{i \in I}$ are in $\{x_\ell : \ell \in [n], e_\ell \in (\text{Aut } \mathbf{A})(e_k)\}$.

Proof. Let $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$.

(1) To see that φ° is well-defined, let $x \in X^\circ$. Since $\mathbf{A}$ is simple, θ_x is a maximal congruence of $\mathbf{D}$, which φ maps to another maximal congruence

$$\varphi(\theta_x) = \{(\varphi(f), \varphi(g)) \in D^2 : f(x) = g(x)\}.$$

By Lemma 2.2 we have a unique point $\varphi^\circ(x)$ in X° such that $\varphi(\theta_x) = \theta_{\varphi^\circ(x)}$ as required.

Next note that φ° is bijective with inverse $(\varphi^{-1})^\circ$ since by definition

$$\theta_x = \varphi^{-1}(\theta_{\varphi^\circ(x)}).$$

For continuity, it suffices to show that φ° maps any clopen subset b of X that is contained in X° to such a clopen subset again. Let $f, g \in D$ be such that

$$\{x \in X : f(x) = g(x)\} = X \setminus b.$$

By Lemma 2.2(1) the congruence of $\mathbf{D}$ generated by (f, g) is the kernel of the projection to $X \setminus b$. It follows that the congruence generated by $(\varphi(f), \varphi(g))$ is the kernel of the projection to the clopen

$$\{x \in X : \varphi(f)(x) = \varphi(g)(x)\} = X \setminus \varphi^\circ(b).$$

Hence $\varphi^\circ(b)$ is clopen in X and disjoint from $\{x_1, \dots, x_n\}$.

(2) Let $x \in X^\circ$. To see that φ_x is well-defined, let $f, g \in D$ with $f(x) = g(x)$, that is, $(f, g) \in \theta_x$. Then $(\varphi(f), \varphi(g))$ is in $\varphi(\theta_x) = \theta_{\varphi^\circ(x)}$. Hence

$$\varphi_x(f(x)) = \pi_{\varphi^\circ(x)}(\varphi(f)) = \pi_{\varphi^\circ(x)}(\varphi(g)) = \varphi_x(g(x))$$

and φ_x is well-defined.

To check that φ_x is a homomorphism on $\mathbf{A}$, let t be a k -ary operation in the signature of $\mathbf{A}$ and let $f_1, \dots, f_k \in D$. Then

$$\begin{aligned} \varphi_x(t^{\mathbf{A}}(f_1(x), \dots, f_k(x))) &= \pi_{\varphi^\circ(x)} \varphi \pi_x^{-1}(t^{\mathbf{A}}(f_1(x), \dots, f_k(x))) \\ &= \pi_{\varphi^\circ(x)} \varphi(t^{\mathbf{D}}(f_1, \dots, f_k)) \\ &= t^{\mathbf{A}}(\pi_{\varphi^\circ(x)} \varphi(f_1), \dots, \pi_{\varphi^\circ(x)} \varphi(f_k)) \\ &= t^{\mathbf{A}}(\pi_{\varphi^\circ(x)} \varphi \pi_x^{-1}(f_1(x)), \dots, \pi_{\varphi^\circ(x)} \varphi \pi_x^{-1}(f_k(x))) \\ &= t^{\mathbf{A}}(\varphi_x(f_1(x)), \dots, \varphi_x(f_k(x))). \end{aligned}$$

Finally for the bijectivity of φ_x , recall that $(\varphi^{-1})^\circ = (\varphi^\circ)^{-1}$. Hence

$$(\varphi^{-1})_{\varphi^\circ(x)} = \pi_x \varphi^{-1} \pi_{\varphi^\circ(x)}^{-1}$$

is the inverse of φ_x . Thus (2) is proved.

(3) By (1) the mapping $(\varphi^{-1})^\circ$ is continuous on X° . It follows that all cluster points of $(\varphi^\circ(y_i))_{i \in I}$ are in $\{x_1, \dots, x_n\}$.

Next from the definition of φ_x we see that

$$\varphi_{y_i}(f(y_i)) = \varphi(f)(\varphi^\circ(y_i)) \quad \text{for all } f \in D, i \in I.$$

Since $\lim y_i = x_k$, we have

$$\forall f \in D \exists j \in I \forall i \geq j : y_i \in f^{-1}(e_k).$$

Note that $y_i \in f^{-1}(e_k)$ implies $\varphi_{y_i}(f(y_i)) \in (\text{Aut } \mathbf{A})(e_k)$ and hence

$$\forall f \in D \exists j \in I \forall i \geq j : \varphi(f)(\varphi^\circ(y_i)) \in (\text{Aut } \mathbf{A})(e_k).$$

Since φ is an automorphism of D , this can be written as

$$\forall g \in D \exists j \in I \forall i \geq j : g(\varphi^\circ(y_i)) \in (\text{Aut } \mathbf{A})(e_k).$$

It follows that if x_ℓ is a cluster point of $(\varphi^\circ(y_i))_{i \in I}$, then $g(x_\ell) = e_\ell$ is in $(\text{Aut } \mathbf{A})(e_k)$. $\square$

In general φ° from the previous lemma does not have a continuous extension on X as shown in the following.

Example 2.4. Let $\mathbf{A}$ be a finite non-abelian simple Mal'cev algebra with an idempotent e , let $\mathbf{B}$ be the countable atomless Boolean algebra, let X be its Stone space, and let $x_0, x_1 \in X$ be distinct. We will construct an automorphism φ of the Boolean power $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})_{e,e}^{x_0,x_1}$ such that the homeomorphism φ° of X° cannot be extended to a homeomorphism of X .

To this end, we identify X with 2^ω , the set of all infinite binary sequences. Let $x_0 := 00\dots$ and $x_1 := 11\dots$. Consider the following clopen sets in X :

$$\begin{aligned} b_i &:= \{\sigma \in X : \sigma_1 = i\} \quad \text{for } i \in \{0, 1\}, \\ b_{ij} &:= \{\sigma \in X : \sigma_l = i \text{ for } l \in [j], \sigma_{j+1} = 1 - i\} \quad \text{for } i \in \{0, 1\}, j \geq 1. \end{aligned}$$

Note that

$$\left(\bigcup_{j \geq 1} b_{ij}\right) \cup \{x_i\} = b_i \quad \text{for } i \in \{0, 1\}.$$

Define $\psi : X^\circ \rightarrow X^\circ$ by

$$\begin{aligned} \psi : (\underbrace{i, \dots, i}_j, 1-i, \sigma_{j+2}, \sigma_{j+3}, \dots) \\ \mapsto \begin{cases} (\underbrace{i, \dots, i}_j, 1-i, \sigma_{j+2}, \sigma_{j+3}, \dots) & \text{if } j \text{ is odd,} \\ (\underbrace{1-i, \dots, 1-i}_j, i, \sigma_{j+2}, \sigma_{j+3}, \dots) & \text{if } j \text{ is even.} \end{cases} \end{aligned}$$

In particular,

$$\psi(b_{ij}) = \begin{cases} b_{ij} & \text{if } j \text{ is odd,} \\ b_{1-i,j} & \text{if } j \text{ is even.} \end{cases}$$

It is clear that ψ is indeed a homeomorphism of X° . But ψ cannot be extended to a homeomorphism of X . Indeed, if U_i is any neighbourhood of x_i for $i \in \{0, 1\}$, then $\psi(U_0)$ has a non-empty intersection with each of U_1 and U_2 .

Now define $\varphi : D \rightarrow D$ by

$$(\varphi(f))(x) := \begin{cases} f\psi^{-1}(x) & \text{if } x \in X^\circ, \\ e & \text{if } x \in \{x_0, x_1\}. \end{cases}$$

To verify that φ is well-defined, we show that $\varphi(f)$ is continuous on X . It is certainly continuous on X° as a composition of two continuous maps. We now claim that $\varphi(f)$ is also continuous at x_i for $i \in \{0, 1\}$. Since $f \in D$, the set $c := f^{-1}(e)$ is clopen in X and contains x_0, x_1 . Therefore there exists $k \in \mathbb{N}$ such that $d := \bigcup_{j \geq k} (b_{0j} \cup b_{1j}) \subseteq c$. Since $\psi(d) = d$, it follows that $d \subseteq (\varphi(f))^{-1}(e)$. Hence

$$(\varphi(f))^{-1}(e) = \{x_0, x_1\} \cup d \cup \bigcup_{i=0}^1 \bigcup_{j=1}^{k-1} (b_{ij} \cap (\varphi(f))^{-1}(e)).$$

Note that

$$\{x_0, x_1\} \cup d = X \setminus \left(\bigcup_{i=0}^1 \bigcup_{j=1}^{k-1} b_{ij}\right)$$

is clopen. Also each

$$b_{ij} \cap (\varphi(f))^{-1}(e) = \begin{cases} b_{ij} \cap f^{-1}(e) & \text{if } j \text{ is odd,} \\ \psi(b_{1-i,j} \cap f^{-1}(e)) & \text{if } j \text{ is even} \end{cases}$$

is clopen. Therefore $(\varphi(f))^{-1}(e)$ is clopen and so $\varphi(f) \in D$ as required. It is now straightforward to show that φ is bijective and also that it is a homomorphism because of the componentwise definition of operations

in $\mathbf{D}$. Finally we have $\varphi^\circ = \psi$, a homeomorphism of X° that cannot be extended to a homeomorphism of X .

In contrast to the previous example we show in the following theorem that φ° can be extended to X for automorphisms φ of $(\mathbf{A}^\mathbf{B})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ if $e_1, \dots, e_n$ are in distinct orbits under $\text{Aut} \mathbf{A}$. Under this condition we can even give a semidirect decomposition of the automorphism group of $(\mathbf{A}^\mathbf{B})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ and identify the kernel as the closure in $\text{Aut} \mathbf{D}$ of a filtered Boolean power of $\text{Aut} \mathbf{A}$.

{thm:AutABxe}

Theorem 2.5. *Let $\mathbf{A}$ be a finite simple Mal'cev algebra with idempotents $e_1, \dots, e_n$ in distinct $\text{Aut} \mathbf{A}$ -orbits, let $\mathbf{B}$ be a Boolean algebra with Stone space X , and let $x_1, \dots, x_n \in X$ be distinct. For $\varphi \in \text{Aut}(\mathbf{A}^\mathbf{B})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ and $x \in X$, define $\overline{\varphi}(x) := \begin{cases} x & \text{if } x \in \{x_1, \dots, x_n\}, \\ \varphi^\circ(x) & \text{else.} \end{cases}$*

{it:AAB1}

(1) *The mapping*

$$h: \text{Aut}(\mathbf{A}^\mathbf{B})_{e_1, \dots, e_n}^{x_1, \dots, x_n} \rightarrow (\text{Homeo} X)_{\{x_1, \dots, x_n\}}, \quad \varphi \mapsto \overline{\varphi},$$

is a group epimorphism and

$$\text{Aut}(\mathbf{A}^\mathbf{B})_{e_1, \dots, e_n}^{x_1, \dots, x_n} \cong \ker h \rtimes (\text{Homeo} X)_{\{x_1, \dots, x_n\}}.$$

{it:AAB2}

(2) *Let K be the set of all continuous maps $\psi: X^\circ \rightarrow \text{Aut} \mathbf{A}$ such that $\psi^{-1}((\text{Aut} \mathbf{A})_{e_i}) \cup \{x_i\}$ is open in X for each $i \in [n]$. Then $\ker h \cong K$ via*

$$p: \varphi \mapsto [\varphi_*: X^\circ \rightarrow \text{Aut} \mathbf{A}, \quad x \mapsto \varphi_x].$$

Proof. (1) By Lemma 2.3(1) $\overline{\varphi}$ is a homeomorphism on X° . Let $(y_i)_{i \in I}$ be a convergent net in X° with limit x_k for $k \in [n]$. Then x_k is the unique cluster point (i.e. the limit) of $(\varphi^\circ(y_i))_{i \in I}$ by Lemma 2.3(3) and the assumption that $e_1, \dots, e_n$ are in distinct $\text{Aut} \mathbf{A}$ -orbits. Hence $\overline{\varphi}$ is continuous on X . Further $\overline{\varphi} \in (\text{Homeo} X)_{\{x_1, \dots, x_n\}}$.

That h is a group homomorphism is straightforward. That h is surjective and that $\ker h$ has a complement follows from the claim

$$(2.2) \quad hg \text{ is the identity on } (\text{Homeo} X)_{\{x_1, \dots, x_n\}},$$

with g defined in (2.1). To see this let $\mathbf{D} := (\mathbf{A}^\mathbf{B})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ and let $\psi \in (\text{Homeo} X)_{\{x_1, \dots, x_n\}}$. Then $hg(\psi) = \overline{\psi^\mathbf{D}}$. Clearly $\overline{\psi^\mathbf{D}}(x_i) = x_i$ for all $i \in [n]$. Else for $x \in X^\circ$ we note that

$$\begin{aligned} \theta_{\overline{\psi^\mathbf{D}}(x)} &= \psi^\mathbf{D}(\theta_x) \\ &= \psi^\mathbf{D}(\{(f, g) \in D^2 : f(x) = g(x)\}) \\ &= \{(f\psi^{-1}, g\psi^{-1}) \in D^2 : f(x) = g(x)\} \\ &= \{(f', g') \in D^2 : f'\psi(x) = g'\psi(x)\} \\ &= \theta_{\psi(x)}. \end{aligned}$$

Thus $\overline{\psi^\mathbf{D}} = \psi$ and (2.2) is proved.

(2) First we show that p maps $\ker h$ into K . Let $\varphi \in \ker h$, i.e. $\varphi \in \text{Aut} \mathbf{D}$ with $\varphi^\circ = \text{id}_{X^\circ}$. We claim that $\varphi_*: X^\circ \rightarrow \text{Aut} \mathbf{A}$, $x \mapsto \varphi_x$, is continuous. Note that for each clopen b in X that is contained in X° and each $a \in A$, we have some $f_{a,b} \in D$ such that $f_{a,b}(b) = a$. Let $\alpha \in \text{Aut} \mathbf{A}$. For $x \in b$ we have $\varphi_x = \alpha$ if and only if for all $a \in A$

$$\alpha(a) = \varphi_x(a) = \pi_x \varphi \pi_x^{-1}(a) = \pi_x \varphi(f_{a,b}).$$

Hence

$$b \cap \varphi_*^{-1}(\alpha) = \bigcap_{a \in A} \{x \in b : \pi_x \varphi(f_{a,b}) = \alpha(a)\}.$$

Since $\varphi(f_{a,b}): X \rightarrow A$ is continuous and b clopen, every set $\{x \in b : \pi_x \varphi(f_{a,b}) = \alpha(a)\}$ on the right hand side is clopen. Thus their intersection $b \cap \varphi_*^{-1}(\alpha)$ is clopen in X as well. As a union of open sets

$$\varphi_*^{-1}(\alpha) = \bigcup \{b \cap \varphi_*^{-1}(\alpha) : b \text{ clopen in } X, b \subseteq X^\circ\}$$

is open for every $\alpha \in \text{Aut} \mathbf{A}$. (Since $X^\circ = \bigcup_{\alpha \in \text{Aut} \mathbf{A}} \varphi_*^{-1}(\alpha)$ is a finite union, each $\varphi_*^{-1}(\alpha)$ for $\alpha \in \text{Aut} \mathbf{A}$ is actually clopen in X° .) So $\varphi_*: X^\circ \rightarrow \text{Aut} \mathbf{A}$ is continuous.

Next we show that $\varphi_*^{-1}((\text{Aut} \mathbf{A})_{e_i}) \cup \{x_i\}$ is also open in X for all $i \in [n]$. Note that for each clopen b in X such that $b \cap \{x_1, \dots, x_n\} = x_i$ we have some $f_b \in D$ such that $f_b(b) = e_i$. For $x \in b \setminus \{x_i\}$ we have $\varphi_x \in (\text{Aut} \mathbf{A})_{e_i}$ if and only if $\pi_x \varphi(f_b) = e_i$. Hence

$$[\varphi_*^{-1}((\text{Aut} \mathbf{A})_{e_i}) \cup \{x_i\}] \cap b = \{x \in b : \pi_x \varphi(f_b) = e_i\}$$

is clopen in X . As a union of open sets

$$\begin{aligned} & \varphi_*^{-1}((\text{Aut} \mathbf{A})_{e_i}) \cup \{x_i\} = \\ & \bigcup \{[\varphi_*^{-1}((\text{Aut} \mathbf{A})_{e_i}) \cup \{x_i\}] \cap b : b \text{ clopen in } X, b \cap \{x_1, \dots, x_n\} = \{x_i\}\} \end{aligned}$$

is open in X for each $i \in [n]$. This completes the proof that $p(\ker h) \subseteq K$.

That p is a group homomorphism follows from

$$\varphi_x \psi_x = \pi_x \varphi \pi_x^{-1} \pi_x \psi \pi_x^{-1} = (\varphi \psi)_x$$

for all $\varphi, \psi \in \ker h$ and $x \in X^\circ$.

To see that p is injective consider any $\varphi \in \ker p$. This means that $\varphi^\circ = \text{id}_X$ and $\varphi_x = \text{id}_A$ for all $x \in X^\circ$. Now, for all $f \in D$, $x \in X^\circ$, we have

$$f(x) = \varphi_x(f(x)) = \pi_x \varphi \pi_x^{-1}(f(x)) = \pi_x \varphi(f) = \varphi(f)(x).$$

Hence $\varphi = \text{id}_D$.

Finally, we verify that p is surjective. Let $\psi \in K$. Define

$$\psi': D \rightarrow D, f \mapsto \begin{cases} X \rightarrow A, x \mapsto [\psi(x)](f(x)) & \text{if } x \in X^\circ, \\ f(x) & \text{else.} \end{cases}$$

Let $f \in D$. Then clearly $[\psi'(f)](x_i) = f(x_i) = e_i$ for all $i \in [n]$. To show that $\psi'(f)$ is continuous, let $a \in A$ and consider

$$X^\circ \cap [\psi'(f)]^{-1}(a) = \bigcup_{\alpha \in \text{Aut } \mathbf{A}} \{x \in X^\circ : \psi(x) = \alpha, f(x) = \alpha^{-1}(a)\}.$$

Since ψ and f are continuous on X° and X , respectively, and X° is open in X , we obtain that

$$(2.3) \quad X^\circ \cap [\psi'(f)]^{-1}(a) \text{ is open in } X \text{ for all } a \in A.$$

In particular, if $a \notin \{e_1, \dots, e_n\}$, then $[\psi'(f)]^{-1}(a) = X^\circ \cap [\psi'(f)]^{-1}(a)$ is open in X .

On the other hand, for $i \in [n]$, the set $b := f^{-1}(e_i)$ is open in X and contains x_i . By assumption $c := \psi^{-1}((\text{Aut } \mathbf{A})_{e_i}) \cup \{x_i\}$ is open in X as well. Hence $b \cap c$ is open and contains x_i . Since $\psi'(f)(b \cap c) = \{e_i\}$, we obtain with (2.3) that

$$[\psi'(f)]^{-1}(e_i) = [X^\circ \cap [\psi'(f)]^{-1}(e_i)] \cup [b \cap c]$$

is open in X . Thus $\psi'(f)$ is continuous and consequently in D .

From the pointwise definition of ψ' it is straightforward that ψ' is an automorphism of $\mathbf{D}$, $\psi' \in \ker h$ and $p(\psi') = \psi$. Thus p is surjective. $\square$

2.4. Isomorphisms. In this subsection we discuss isomorphisms between filtered Boolean powers, in the case where $\mathbf{B}$ is the countable atomless Boolean algebra. The aim is to show that one can add or remove filtering idempotents which belong to the orbits of the remaining ones under $\text{Aut } \mathbf{A}$ without changing the isomorphism type of the power.

For $b \subseteq X$ let $(\mathbf{A}^{\mathbf{B}})^{x_1, \dots, x_n}_{e_1, \dots, e_n}|_b$ denote the restriction of the functions in $(\mathbf{A}^{\mathbf{B}})^{x_1, \dots, x_n}_{e_1, \dots, e_n}$ to b . We characterize when such restrictions are isomorphic.

Lemma 2.6. *Let b_1, b_2 be disjoint clopen subsets of the Stone space X such that $b_i \cap \{x_1, \dots, x_n\} = \{x_i\}$ for $i = 1, 2$. Then the following are equivalent:*

{it:iso1}
{it:iso2}
{it:iso3}

- (1) $(\mathbf{A}^{\mathbf{B}})^{x_1, \dots, x_n}_{e_1, \dots, e_n}|_{b_1} \cong (\mathbf{A}^{\mathbf{B}})^{x_1, \dots, x_n}_{e_1, \dots, e_n}|_{b_2}$;
- (2) $\varphi^\circ(b_1 \setminus \{x_1\}) = b_2 \setminus \{x_2\}$ for some $\varphi \in \text{Aut}(\mathbf{A}^{\mathbf{B}})^{x_1, \dots, x_n}_{e_1, \dots, e_n}$;
- (3) There exist $\alpha \in \text{Aut } \mathbf{A}$ such that $\alpha(e_1) = e_2$ and a homeomorphism $\psi: b_1 \rightarrow b_2$ such that $\psi(x_1) = x_2$.

Proof. Let $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})^{x_1, \dots, x_n}_{e_1, \dots, e_n}$.

(1) $\Rightarrow$ (2): For $\psi: \mathbf{D}|_{b_1} \rightarrow \mathbf{D}|_{b_2}$ an isomorphism, note that

$$\varphi := \psi \cup \psi^{-1} \cup \text{id}_{\mathbf{D}|_{X \setminus (b_1 \cup b_2)}}$$

is an automorphism of $\mathbf{D}$ with the required properties.

(2) $\Rightarrow$ (3): We claim that the mapping

$$\psi: b_1 \rightarrow b_2, \quad x \mapsto \begin{cases} \varphi^\circ(x) & \text{if } x \neq x_1, \\ x_2 & \text{if } x = x_1, \end{cases}$$

is a homeomorphism. It is clearly a bijection and is continuous on $b_1 \setminus \{x_1\}$ by Lemma 2.3(1). To show that ψ is continuous on x_1 , let $(y_i)_{i \in I}$ be a net in $b_1 \setminus \{x_1\}$ with limit x_1 . Then $(\varphi^\circ(y_i))_{i \in I}$ is a net in $b_2 \setminus \{x_2\}$. By Lemma 2.3(3) all its cluster points are in $\{x_1, \dots, x_n\}$. Note that $X \setminus b_2$ is a neighborhood of $\{x_1, x_3, \dots, x_n\}$ that contains no points of $\varphi^\circ(y_i)$. Hence x_2 remains as the unique cluster point of that net, hence its limit point. This completes the proof that ψ is a homeomorphism. Furthermore, by Lemma 2.3(3), we also have $e_2 = \alpha(e_1)$ for some $\alpha \in \text{Aut} \mathbf{A}$.

(3) $\Rightarrow$ (1): Note that $\varphi: D|_{b_1} \rightarrow D|_{b_2}, f \mapsto \alpha f \psi^{-1}$, is the required isomorphism. $\square$

As a stepping stone towards the main result of this section, the following lemma establishes one important special case when the number of filter points in a filtered Boolean power can be reduced.

{lem:isoproduct}

Lemma 2.7. *Let $\mathbf{A}$ be an algebra with idempotent e , let $\mathbf{B}$ be the countable atomless Boolean algebra, let X be the Stone space of $\mathbf{B}$, and let $x_0 \in X$.*

Then $(\mathbf{A}^{\mathbf{B}})_e^{x_0} \times (\mathbf{A}^{\mathbf{B}})_e^{x_0} \cong (\mathbf{A}^{\mathbf{B}})_e^{x_0}$.

Proof. Let X_1, X_2 be two disjoint copies of the Cantor space X and fix $x_i \in X_i, i \in [2]$. We will represent the two copies of $(\mathbf{A}^{\mathbf{B}})_e^{x_0}$ as

$$D_i := \{f: X_i \rightarrow A : f \text{ is continuous and } f(x_i) = e\} \quad \text{for } i \in [2].$$

Consider now the space $X_1 \cup X_2$ with the union topology; it is homeomorphic to X . Let $\sim$ be the equivalence relation on $X_1 \cup X_2$ with the equivalence classes $\{x_1, x_2\}$ and $\{x\}$ for $x \in (X_1 \cup X_2) \setminus \{x_1, x_2\}$. The quotient space $(X_1 \cup X_2)/\sim$ is again homeomorphic to X . This is easy to show directly by verifying that it is Hausdorff, compact, has no isolated points and has a countable basis of clopen sets. Alternatively, one can note that $\sim$ is a Boolean equivalence in the sense of [23], and that the resulting quotient $(X_1 \cup X_2)/\sim$ has a countable basis of clopen sets and no isolated points. Henceforth we will identify X with $(X_1 \cup X_2)/\sim$. We will take its points to be $(X_1 \setminus \{x_1\}) \cup (X_2 \setminus \{x_2\}) \cup \{x_{12}\}$ and note that there are two types of open sets:

- $U_1 \cup U_2$ where U_i is open in X_i and $x_i \notin U_i$ for $i \in [2]$;
- $(U_1 \setminus \{x_1\}) \cup (U_2 \setminus \{x_2\}) \cup \{x_{12}\}$ where U_i is open in X_i and $x_i \in U_i$ for $i \in [2]$.

Let

$$D := \{f: X \rightarrow X : f \text{ is continuous and } f(x_{12}) = e\}$$

be yet another copy of $(\mathbf{A}^{\mathbf{B}})_e^{x_0}$. We will prove the lemma by showing that $\psi: D_1 \times D_2 \rightarrow D$ defined by

$$\psi(f_1, f_2)(x) := \begin{cases} f_1(x) & \text{if } x \in X_1 \setminus \{x_1\}, \\ f_2(x) & \text{if } x \in X_2 \setminus \{x_2\}, \\ e & \text{if } x = x_{12}, \end{cases}$$

is an isomorphism.

We prove that ψ is well-defined, i.e. that $\psi(f_1, f_2) \in D$ for all $f_i \in D_i$, $i \in [n]$. Let $a \in A$. If $a \neq e$, then

$$(\psi(f_1, f_2))^{-1}(a) = f_1^{-1}(a) \cup f_2^{-1}(a),$$

where each $f_i^{-1}(a)$ is a clopen in X_i not containing x_i . Hence $(\psi(f_1, f_2))^{-1}(a)$ is clopen in X . Similarly, for $a = e$ we have

$$(\psi(f_1, f_2))^{-1}(a) = (f_1^{-1}(e) \setminus \{x_1\}) \cup (f_2^{-1}(e) \setminus \{x_2\}) \cup \{x_{12}\},$$

again a clopen in X . Hence indeed $\psi(f_1, f_2) \in D$. It is clear from the definition that ψ is bijective. It is a homomorphism by the componentwise definition of operations in D_1, D_2 and D . $\square$

From Lemmas 2.6 and 2.7 it follows that any filtered Boolean power of $\mathbf{A}$ by the countable atomless Boolean $\mathbf{B}$ is isomorphic to $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ with $e_1, \dots, e_n$ in distinct orbits under $\text{Aut} \mathbf{A}$.

{cor:isoreduced}

Corollary 2.8. *Let $\mathbf{A}$ be a finite simple non-abelian Mal'cev algebra, $n \leq m$ and let $e_1, \dots, e_n$ be orbit representatives of idempotent elements $e_1, \dots, e_m$ in $\mathbf{A}$ under $\text{Aut} \mathbf{A}$. Let $\mathbf{B}$ be the countable atomless Boolean algebra, let X be the Stone space of $\mathbf{B}$, and let $x_1, \dots, x_m \in X$ be distinct.*

Then $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_m}^{x_1, \dots, x_m}$ is isomorphic to $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$.

Proof. By partitioning X into disjoint clopens $b_1, \dots, b_m$ with $x_i \in b_i$, and recalling that all b_i are homeomorphic to the Cantor space X , we obtain

$$(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_m}^{x_1, \dots, x_m} \cong \prod_{i=1}^m (\mathbf{A}^{\mathbf{B}})_{e_i}^{x_i}.$$

Now we can use Lemmas 2.6 and 2.7 to eliminate the factors $(\mathbf{A}^{\mathbf{B}})_{e_i}^{x_i}$ for $i > n$, leaving us with

$$(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_m}^{x_1, \dots, x_m} \cong \prod_{i=1}^n (\mathbf{A}^{\mathbf{B}})_{e_i}^{x_i} \cong (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n},$$

as required. $\square$

{sec:omegacat}

2.5. ω -categoricity.

Proof of Theorem 1.3(1). We apply a description of ω -categorical filtered Boolean powers due to Macintyre and Rosenstein [20]. They use Stone spaces of maximal ideals instead of ultrafilters. So we change to that dual perspective in the following. Note that $x'_i := B \setminus x_i$ is a maximal ideal of $\mathbf{B}$ for $i \in [n]$. The Heyting algebra $\mathbf{H}_0$ of ideals of $\mathbf{B}$ generated by $x'_1, \dots, x'_n, 0, B$ has universe $\{x'_{i_1} \wedge \dots \wedge x'_{i_k} : k, i_1, \dots, i_k \in [n]\} \cup \{0\}$ since $(x'_1 \wedge \dots \wedge x'_n) \rightarrow 0 = 0$. In particular $\mathbf{H}_0$ is finite and $\mathbf{B}/J$ has finitely many atoms for any ideal J in $\mathbf{H}_0$. Now the augmented Boolean algebra $(\mathbf{B}, x'_1, \dots, x'_n)$ is a first order structure with operations from the Boolean algebra and unary relations that are ideals $x'_1, \dots, x'_n$

(see [20, p. 137]). Then $(\mathbf{B}, x'_1, \dots, x'_n)$ is ω -categorical by [20, Theorem 7]. Hence $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ is ω -categorical by [20, Theorem 5]. $\square$

{sec:Flim}

3. THE FRAÏSSÉ LIMIT

We apply standard model theoretic arguments to show that the (generalized) Fraïssé limit of $\{\mathbf{A}^k : k \geq 1\}$ exists and is isomorphic to a filtered Boolean power.

Proof of Theorem 1.1(1). Let $\mathbf{A}$ be a finite simple non-abelian Mal'cev algebra. We prove that the Fraïssé limit of the family $K := \{\mathbf{A}^k : k \geq 1\}$ exists by using a modification of Fraïssé's Theorem [16, Theorem 7.1.2] given in [16, Exercise 7.1.11]. For this we need to check that all $\mathbf{A}^k$ are finitely generated, and that K has JEP and AP. The first two statements are clear.

To see that K has AP, consider $u, v, w \in \mathbb{N}$ and embeddings $\varphi: \mathbf{A}^u \rightarrow \mathbf{A}^v$, $\psi: \mathbf{A}^u \rightarrow \mathbf{A}^w$. Since $\mathbf{A}$ is simple, we can identify the endomorphisms of $\mathbf{A}$ that are not automorphisms with the set of idempotents $E := \{e_1, \dots, e_n\}$ of $\mathbf{A}$. From the Foster–Pixley Theorem [7, Corollary IV 10.2] it follows that for every $i \in [v]$ there exist $\alpha_i \in \text{Aut} \mathbf{A} \cup E$ and $j_i \in [u]$ such that

$$\varphi(a_1, \dots, a_u) = (\alpha_1(a_{j_1}), \dots, \alpha_v(a_{j_v})) \text{ for all } a_1, \dots, a_u \in A.$$

Further $\{j_i : \alpha_i \in \text{Aut} \mathbf{A}\} = [u]$ since φ is an embedding. By permuting the copies of $\mathbf{A}$ and renaming their elements we may assume that

$$\varphi(a_1, \dots, a_u) = (\underbrace{a_1, \dots, a_1}_{p_1 \text{ times}}, \dots, \underbrace{a_u, \dots, a_u}_{p_u \text{ times}}, \underbrace{e_1, \dots, e_1}_{q_1 \text{ times}}, \dots, \underbrace{e_n, \dots, e_n}_{q_n \text{ times}}),$$

with multiplicities $p_1, \dots, p_u \geq 1, q_1, \dots, q_n \geq 0$. Analogously,

$$\psi(a_1, \dots, a_u) = (\underbrace{a_1, \dots, a_1}_{r_1 \text{ times}}, \dots, \underbrace{a_u, \dots, a_u}_{r_u \text{ times}}, \underbrace{e_1, \dots, e_1}_{s_1 \text{ times}}, \dots, \underbrace{e_n, \dots, e_n}_{s_n \text{ times}}),$$

where $r_1, \dots, r_u \geq 1, s_1, \dots, s_n \geq 0$. Letting $v_i := \max(p_i, r_i)$ for $i \in [u]$, $w_i := \max(q_i, s_i)$ for $i \in [n]$ and $m := \sum_{i=1}^u v_i + \sum_{i=1}^n w_i$ it is now straightforward to define embeddings $\varphi': \mathbf{A}^v \rightarrow \mathbf{A}^m$ and $\psi': \mathbf{A}^w \rightarrow \mathbf{A}^m$ in the same form as φ and ψ above, taking the sequences of multiplicities to be the concatenations of:

$$(v_i - p_i + 1, \underbrace{1, \dots, 1}_{p_i - 1}) \text{ (} i \in [u] \text{) and } (w_i - q_i + 1, \underbrace{1, \dots, 1}_{q_i - 1}) \text{ (} i \in [n] \text{) for } \varphi',$$

$$(v_i - r_i + 1, \underbrace{1, \dots, 1}_{r_i - 1}) \text{ (} i \in [u] \text{) and } (w_i - s_i + 1, \underbrace{1, \dots, 1}_{s_i - 1}) \text{ (} i \in [n] \text{) for } \psi',$$

and to show that they satisfy $\varphi'\varphi = \psi'\psi$.

Since K has JEP and AP, by [16, Exercise 7.1.11] there exists a unique (up to isomorphism) countable algebra $\mathbf{C}$ such that

- (i) every finitely generated subalgebra of $\mathbf{C}$ embeds into some element of K ,

- (ii) $\mathbf{C}$ is a direct limit of algebras in K , and
- (iii) every isomorphism between subalgebras of $\mathbf{C}$ that are isomorphic to some element in K extends to an automorphism of $\mathbf{C}$ (i.e., $\mathbf{C}$ is K -homogenous).

Let $\mathbf{B}$ be the countable atomless Boolean algebra with Stone space, and let $x_1, \dots, x_n \in X$ be distinct. To see that the filtered Boolean power $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ is isomorphic to the Fraïssé limit $\mathbf{C}$ it suffices to show that $\mathbf{D}$ satisfies (i), (ii), (iii). Properties (i) and (ii) follow from the fact that $\mathbf{D}$ is locally finite and that every finite subalgebra of $\mathbf{D}$ is contained in a subalgebra isomorphic to some finite direct power of $\mathbf{A}$, as in the proof of Lemma 2.2. To prove (iii) we emulate [16, Lemma 7.1.4], and show that $\mathbf{D}$ is *weakly K -homogeneous*, in the sense that for any $u \leq v$ and any embeddings $\varphi: \mathbf{A}^u \rightarrow \mathbf{A}^v$ and $\psi: \mathbf{A}^u \rightarrow \mathbf{D}$ there exists an embedding $\psi': \mathbf{A}^v \rightarrow \mathbf{D}$ such that $\psi'\varphi = \psi$. That the weak K -homogeneity implies K -homogeneity is straightforward using a standard back-and-forth argument as in [16, cf. Lemma 7.1.4].

To verify weak K -homogeneity, as above we may assume that

$$\varphi(a_1, \dots, a_u) = (\underbrace{a_1, \dots, a_1}_{p_1 \text{ times}}, \dots, \underbrace{a_u, \dots, a_u}_{p_u \text{ times}}, \underbrace{e_1, \dots, e_1}_{q_1 \text{ times}}, \dots, \underbrace{e_n, \dots, e_n}_{q_n \text{ times}})$$

with multiplicities $p_1, \dots, p_u \geq 1, q_1, \dots, q_n \geq 0$ for all $a_1, \dots, a_u \in A$. For $\psi: \mathbf{A}^u \rightarrow \mathbf{D}$ there exist $m_i \in \mathbb{N}$ ($i \in [u]$), automorphisms α_{ij} ($i \in [u], j \in [m_i]$), and a partition of X into clopens c_{ij} ($i \in [u], j \in [m_i]$) and b_i ($i \in [n]$) such that

$$\psi(a_1, \dots, a_u)(x) = \begin{cases} \alpha_{ij}(a_i) & \text{if } x \in c_{ij} \text{ for } i \in [u], j \in [m_i], \\ e_i & \text{if } x \in b_i \text{ for } i \in [n]. \end{cases}$$

Clearly we may assume that $m_i \geq p_i$ for all $i \in [u]$. Now define an embedding $\psi': \mathbf{A}^v \rightarrow \mathbf{D}$ as follows. Subdivide each b_i into $q_i + 1$ clopens b_{ij} ($0 \leq j \leq q_i$) with $x_i \in b_{i0}$. Set

$$p_{ij} := \begin{cases} \sum_{k=1}^{i-1} p_k + j & \text{if } i \in [u], j \in [p_i], \\ \sum_{k=1}^i p_k & \text{if } i \in [u], j \in [m_i] \setminus [p_i], \end{cases}$$

and

$$q_{ij} := \sum_{k=1}^u p_k + \sum_{k=1}^{i-1} q_k + j \text{ if } i \in [n], j \in [q_i].$$

Then define

$$\psi'(a_1, \dots, a_v)(x) = \begin{cases} \alpha_{ij}(a_{p_{ij}}) & \text{if } x \in c_{ij} \text{ for } i \in [u], j \in [m_i], \\ a_{q_{ij}} & \text{if } x \in b_{ij} \text{ for } i \in [n], j \in [q_i], \\ e_i & \text{if } x \in b_{i0} \text{ for } i \in [n]. \end{cases}$$

It is straightforward to verify that ψ' is indeed an embedding and that $\psi'\varphi = \psi$.

This completes the proof of weak homogeneity of $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$, and hence of the theorem. $\square$

{sec:variety}

4. THE VARIETY GENERATED BY A SIMPLE ALGEBRA

We investigate the countable free algebra in the variety V generated by a finite simple non-abelian Mal'cev algebra $\mathbf{A}$.

Proof of Theorem 1.1(2). First we build explicit representations of finite and countable free algebras in V . For $i \in \mathbb{N}$ let $x_i: A^{\mathbb{N}} \rightarrow A$, $a \mapsto a(i)$. Then the universe of $\mathbf{F} := \langle x_i : i \in \mathbb{N} \rangle \leq \mathbf{A}^{A^{\mathbb{N}}}$ forms the clone of term operations on $\mathbf{A}$ (see [22, Section 4.1]). Hence $\mathbf{F}$ is free in V over $\{x_i : i \in \mathbb{N}\}$ (see [22, Section 4.11]). Note that $A^{\mathbb{N}}$ with the product topology is homeomorphic to the Cantor space. Since x_i for $i \in \mathbb{N}$ are continuous functions from $A^{\mathbb{N}}$ to A , so are all the elements of $\mathbf{F}$. Hence $\mathbf{F}$ is contained in the Boolean power $\mathbf{A}^{\mathbf{B}}$.

Consider the lexicographic ordering on $A^{\mathbb{N}}$, where A is linearly ordered in an arbitrary way. Let R be the transversal of $\text{Aut} \mathbf{A}$ orbits on $A^{\mathbb{N}}$ consisting of lexicographically minimal elements. It is easy to see that:

- For every $k \in \mathbb{N}$ the restriction $R|_{[k]}$ is a set of orbit representatives of A^k under $\text{Aut} \mathbf{A}$.
- The restriction $\mathbf{F}|_R$ is isomorphic to $\mathbf{F}$.

We will also need the following explicit description of the free algebra $\mathbf{F}_k = \langle x_1, \dots, x_k \rangle$ of rank k as a subalgebra of $\mathbf{A}^{\mathbf{B}}$, as well as the descriptions of the corresponding free algebras in the variety W generated by the proper subalgebras of $\mathbf{A}$. Let

$$S_k := \{a \in R : \langle a(1), \dots, a(k) \rangle \neq A\}.$$

Then $\mathbf{F}_k|_{S_k}$ is free in W over the restrictions of $x_1, \dots, x_k$ to S_k . For $p \in R|_{[k]}$ let $c_p := \{r \in R : r|_{[k]} = p\}$ be a basic clopen in R , and note that every element of $\mathbf{F}_k$ is constant on every c_p . So $\mathbf{F}_k|_{R \setminus S_k}$ is a finite subdirect power of $\mathbf{A}$, hence isomorphic to a direct power of $\mathbf{A}$ by the Foster–Pixley Theorem [7, Corollary IV.10.2]. Clearly $\mathbf{F}_k$ is a subdirect product of its projections $\mathbf{F}_k|_{R \setminus S_k}$ and $\mathbf{F}_k|_{S_k}$. Since every quotient of the former is isomorphic to a direct power of $\mathbf{A}$ again (cf. Lemma 2.2), $\mathbf{F}_k|_{R \setminus S_k}$ and $\mathbf{F}_k|_{S_k}$ have no non-trivial common homomorphic image. So Fleischer's Lemma [7, Lemma IV.10.1] yields that $\mathbf{F}_k$ is the direct product of its projections to $R \setminus S_k$ and S_k respectively. This in turn implies

$$(4.1) \quad F_k = \left\{ f \in A^R : \begin{array}{l} f|_{S_k} \in F_k|_{S_k}, f \text{ is constant on } c_p \\ \text{for all } p \in (R \setminus S_k)|_{[k]} \end{array} \right\}.$$

For

$$S := \bigcap_{k \in \mathbb{N}} S_k = \{a \in R : F|_a \neq A\}$$

the restriction $\mathbf{F}|_S$ is free in W over $\{x_i|_S : i \in \mathbb{N}\}$. Note that for every proper subalgebra $\mathbf{A}' \leq \mathbf{A}$ there is a sequence $a \in S$ such that $\langle a \rangle = \mathbf{A}'$. It follows that $\mathbf{F}$ restricted to individual elements of S gives the collection of all proper subalgebras of $\mathbf{A}$, from which it readily follows that the kernel of the restriction map to S is the smallest congruence θ such that $\mathbf{F}/\theta \in W$.

So let $e \in F$ such that $e|_S$ forms a trivial subalgebra of $\mathbf{F}|_S$. Then $e(S) = \{e_1, \dots, e_n\}$ is a set of idempotents of $\mathbf{A}$ containing at least one idempotent from each $\text{Aut} \mathbf{A}$ -orbit. The sets $s_j := e^{-1}(e_j) \cap S$ for $j \in [n]$ partition S . Also let

$$B := \{b \text{ clopen in } R : b \cap s_j \in \{\emptyset, s_j\} \text{ for all } j \in [n]\},$$

which is the universe of a Boolean algebra $\mathbf{B}$.

We claim that $\mathbf{B}$ is the countable atomless Boolean algebra. To see this consider an arbitrary basic clopen $c_p \in B$ where $p \in R|_{[k]}$. Let $l > k$ and $q \in R|_{[l]}$ be such that p is a subsequence of q and $\langle q \rangle = \mathbf{A}$. Then $c_q \cap S = \emptyset$, and hence $c_q \in B$. Also $c_q \subsetneq c_p$, and so c_p is not an atom. Thus, $\mathbf{B}$ is atomless, and since it is clearly countable, the claim follows.

We may view $\mathbf{B}$ as a subalgebra of the Boolean algebra of subsets of $Y := (R \setminus S) \cup \{s_1, \dots, s_n\}$. Hence

$$D := \{f \in A^R : f^{-1}(a) \in B \text{ for } a \in A, f(s_j) = \{e_j\} \text{ for } j \in [n]\}$$

forms a subalgebra of $\mathbf{A}^R$ which is naturally isomorphic to $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{s'_1, \dots, s'_n}$ by Lemma 2.1.

We claim that

$$(4.2) \quad e/\theta = D.$$

The inclusion $\subseteq$ is immediate from the definitions and the fact that every element in $F|_R$ is continuous.

For the converse, let $f \in D$. Let $k \in \mathbb{N}$ be large enough so that the following hold:

$\{\text{it:fc}\}$
 $\{\text{it:efK}\}$

- f is constant on the basic clopens c_p for all $p \in R|_{[k]}$,
- $e \in F_k$,
- $f|_{s_k} = e|_{s_k}$.

Then $f \in F_k$ by (4.1). Hence (4.2) is proved.

Summing up, any subalgebra e/θ of $\mathbf{F}$ is isomorphic to $(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{s'_1, \dots, s'_n}$ by (4.2) and Lemma 2.1. Here $\mathbf{B}$ is a countable atomless Boolean algebra, and the set of idempotents $e_1, \dots, e_n$ contains at least one representative from each $\text{Aut} \mathbf{A}$ -orbit. By Corollary 2.8 we may assume that $e_1, \dots, e_n$ is the set of all idempotents of $\mathbf{A}$. Thus Theorem 1.1(2) is proved. $\square$

For loops (in particular for groups) we can sharpen Theorem 1.1(2) as follows. (We note that an equivalent result holds for rings with the obvious notational changes by the same proof.)

{cor:group}

Corollary 4.1. *Let $\mathbf{A}$ be a finite simple non-abelian loop with identity e , let V be the variety generated by $\mathbf{A}$, and let W be the variety generated by all proper subloops of $\mathbf{A}$.*

{it:countable}

- (1) *The countable free loop $\mathbf{F}$ in V has a normal subloop N and a subloop H such that:*

{it:kernel}

- (a) *$N \cong (\mathbf{A}^{\mathbf{B}})_e^x$ where $\mathbf{B}$ is the countable atomless Boolean algebra and x is an ultrafilter on $\mathbf{B}$,*

{it:complement}

- (b) *H is isomorphic to the countable free loop in W ,*

{it:semidirect}

- (c) *$N \cap H = \{e\}$ and $F = NH$.*

{it:V}

- (2) *Every countable loop $\mathbf{C}$ in V is a split extension of a filtered Boolean power $(\mathbf{A}^{\mathbf{B}})_e^x$ for some Boolean algebra $\mathbf{B}$ by some loop in W .*

Proof. For (1) we reuse the explicit representation $\mathbf{F} = \langle x_i : i \in \mathbb{N} \rangle \leq \mathbf{A}^R$ from the proof of Theorem 1.1(2) above.

(1a) Again let $S := \{a \in R : F|_a \neq A\}$. Then

$$N := \{f \in F : f(S) = e\}$$

is the kernel of the projection π_S from $\mathbf{F}$ to $\mathbf{F}|_S$, with the latter isomorphic to the free loop in W over $\{\pi_S(x_i) : i \in \mathbb{N}\}$. Further N is isomorphic to $(\mathbf{A}^{\mathbf{B}})_e^x$ by Theorem 1.1(2).

(1b) Let $k \in \mathbb{N}$ and $S_k := \{a \in R : \langle a(1), \dots, a(k) \rangle \neq A\}$. Consider $\mathbf{F}_k := \langle x_1, \dots, x_k \rangle$. As in the proof of Theorem 1.1(2) above, we have $F_k = K_k \times H_k$, where $K_k := F_k|_{R \setminus S_k}$ and $H_k := F_k|_{S_k}$. Further K_k is isomorphic to a finite direct power of $\mathbf{A}$, while H_k is isomorphic to the free loop of rank k in W . The constant function $\bar{e}_k : R \setminus S_k \rightarrow \{e\}$ forms the unique trivial subloop of K_k . So $\{\bar{e}_k\} \times H_k$ is a subloop of $\mathbf{F}_k$, which is isomorphic to H_k . In particular

$$y_k : R \rightarrow A, a \mapsto \begin{cases} a_k & \text{if } a \in S_k \\ e & \text{else,} \end{cases}$$

is in F_k , hence in F .

Then

$$H := \langle y_i : i \in \mathbb{N} \rangle \leq \mathbf{F}$$

and $H \in W$ since $H|_a \in W$ for all $a \in R$. Moreover, since $\bigcap_{i \in \mathbb{N}} S_i = S$, it follows that H is isomorphic to $\mathbf{F}|_S$, which in turn is isomorphic to the free loop in W over $\{y_i : i \in \mathbb{N}\}$.

(1c) Since $y_k \in H$ and $x_k y_k^{-1} \in N$, we have $x_k \in NH$ for all $k \in \mathbb{N}$. Thus $F = NH$.

For proving that $N \cap H$ is trivial, let $t(y_1, \dots, y_k) \in H$ for some term t and $k \in \mathbb{N}$ such that $t(y_1, \dots, y_k)|_S$ is constant e . By the latter every proper subalgebra of $\mathbf{A}$ satisfies the identity $t \approx e$. Since $\mathbf{H}$ is in W , it follows that $t(y_1, \dots, y_k) = \bar{e}$, the constant function with value e on R . Thus $N \cap H = \{\bar{e}\}$.

For (2), let $\mathbf{C} \in V$ be isomorphic to $\mathbf{F}/K$ for the countable free loop $\mathbf{F}$ in V with some normal K . Let N, H be as in (1). Then $\mathbf{F}/K$ is a split extension of NK/K and HK/K . The former is isomorphic to $(\mathbf{A}^{\mathbf{B}/\theta})_e^x$ for a congruence θ of $\mathbf{B}$ by Lemma 2.2. The latter is clearly in W . $\square$

5. SMALL INDEX PROPERTY

{sec:SIP}

5.1. SIP for the punctured Cantor space. The proof of Theorem 1.3(2) requires the following generalization of Truss' result that the automorphism group of the countable atomless Boolean algebra has SIP [26, Theorem 3.7].

{thm:Truss}

Theorem 5.1. *Let $n \in \mathbb{N} \cup \{0\}$, let X be the Cantor space with distinct points $x_1, \dots, x_n \in X$, and let $G := (\text{Homeo}X)_{\{x_1, \dots, x_n\}}$.*

Then for every subgroup $H \leq G$ with $|G : H| < 2^{\aleph_0}$ there exists a partition of X into clopen sets $b_1, \dots, b_m$ such that $G_{\{b_1, \dots, b_m\}} \leq H$.

The proof makes up the rest of this subsection. It heavily relies on the corresponding results from [26] concerning the Cantor space itself (Theorem 3.7) and $\mathbb{Q}$ with a point removed (Theorem 2.7). We do not reproduce the arguments that are essentially repetitions of those from [26] but just point out the necessary adaptations. The arguments that are new to our situation are proved in full.

For the rest of this section we use the notation and assumptions of Theorem 5.1, assume $n \geq 1$ and let $X^\circ := X \setminus \{x_1, \dots, x_n\}$. Following Truss [25, 26] we say that a subset b of X *abuts* an element x of X if $x \notin b$ and $b \cup \{x\}$ is closed in X .

{1a:Xo3Tk3}

Lemma 5.2. *Let $(b_i)_{i \in \mathbb{N}}$ be a sequence of pairwise disjoint clopen subsets of X° that converges to x_1 such that either (a) each b_i is clopen in X or (b) each b_i abuts x_1 .*

Then $G_{X \setminus b_i} \leq H$ for some $i \in \mathbb{N}$.

Proof. (a) This is identical to the argument in the first paragraph of the proof of [26, Theorem 3.7]. Here we can use that $G_{X \setminus b_i} \cong \text{Homeo}X$ for all $i \in \mathbb{N}$, which is simple by a result of Anderson [1].

(b) This follows [26, Lemma 2.5]. This time $G_{X \setminus b_i} \cong (\text{Homeo}X)_{x_1}$ for all $i \in \mathbb{N}$, which is not simple. Instead, by [25, Theorem 3.13], the latter has precisely one non-trivial proper normal subgroup

$$N := \{g \in \text{Homeo}X : g \text{ fixes pointwise a neighborhood of } x_1\}.$$

However, this normal subgroup has uncountable index in $(\text{Homeo}X)_{x_1}$, which can be proved by following [26, p. 499, par. 3]. So as in [26, Lemma 2.5] we can show that $H \cap G_{X \setminus b_i} = G_{X \setminus b_i}$ for some $i \in \mathbb{N}$. $\square$

{1a:Xo5Tk3}

Lemma 5.3. *There exists a clopen set b of X such that $b \cap \{x_1, \dots, x_n\} = \{x_1\}$ and for every clopen set $c \subseteq b$ of X not containing x_1 we have $G_{X \setminus c} \leq H$.*

Proof. Suppose the statement is not true. Then we can inductively define two sequences of clopen sets $(b_i)_{i \in \mathbb{N}}$ and $(c_i)_{i \in \mathbb{N}}$ of X as follows. For $i \in \mathbb{N}$ let b_i be a clopen satisfying the following properties:

- $b_i \cap \{x_1, \dots, x_n\} = \{x_1\}$,
- b_i is contained in the ball of radius $1/i$ around x_1 ,
- b_i is disjoint from each of $c_1, \dots, c_{i-1}$.

By our assumption we have a clopen $c_i \subseteq b_i$ not containing x_1 such that $G_{X \setminus c_i} \not\leq H$. The sequence $(c_i)_{i \in \mathbb{N}}$ consists of pairwise disjoint clopen sets not containing x_1 , which converges to x_1 , and such that $G_{X \setminus c_i} \not\leq H$ for all $i \in \mathbb{N}$. This contradicts Lemma 5.2 (a). $\square$

{1a:Xo6Tk3}

Lemma 5.4. *There exists a clopen set b of X such that $b \cap \{x_1, \dots, x_n\} = \{x_1\}$ and $G_{X \setminus b} \leq H$.*

Proof. Let b be the clopen set of X guaranteed by Lemma 5.3: thus $b \cap \{x_1, \dots, x_n\} = \{x_1\}$ and for every clopen $c \subseteq b \setminus \{x_1\} =: b^\circ$ of X we have $G_{X \setminus c} \leq H$. Let

$$\Gamma := \{d : d \subseteq b \text{ is clopen in } X^\circ \text{ abutting } x_1 \text{ and } G|_{X \setminus d} \leq H\}.$$

We will show that $b^\circ \in \Gamma$ via a sequence of claims.

{c1:Xo62Tk3}

Claim 5.5. *If c and d are clopen sets of X° such that c , d and $c \cap d$ abut x_1 , then $G_{X \setminus (c \cup d)} = \langle G_{X \setminus c}, G_{X \setminus d} \rangle$.*

Proof. This is identical to [26, Lemma 2.6]. $\square$

{c1:Xo63Tk3}

Claim 5.6. *If $c, d \in \Gamma$, then $c \cup d \in \Gamma$.*

Proof. This can be proved by following the paragraph preceding [26, Lemma 2.9] using Lemma 5.2 (b) and Claim 5.5. $\square$

{c1:Xo64Tk3}

Claim 5.7. *If $d \in \Gamma$ and $c \subseteq b^\circ$ clopen in X , then $c \cup d \in \Gamma$.*

Proof. This follows [26, Lemma 2.9]. $\square$

The proof of Lemma 5.3 is now completed by following the Conclusion of the proof of Theorem 2.7 in [26, p. 501, 502]. $\square$

Proof of Theorem 5.1. We use induction on n . The base case for $n = 0$ is Truss' result [26, Theorem 3.7]. So assume $n \geq 1$ in the following. By Lemma 5.4 there exists a clopen b_1 in X such that $b_1 \cap \{x_1, \dots, x_n\} = \{x_1\}$ and $G_{X \setminus b_1} \leq H$. If $b_1 = X$, then $G = H$ and there is nothing left to prove. So suppose $b_1 \neq X$. Since $H \cap G_{b_1}$ has index $< 2^{\aleph_0}$ in $G_{b_1} \cong (\text{Homeo}X)_{x_2, \dots, x_n}$, we have a partition of $X \setminus b_1$ into clopen sets $b_2, \dots, b_m$ such that $(G_{b_1})_{\{b_2, \dots, b_m\}} \leq H \cap G_{b_1}$ by the induction assumption. But then H contains the direct product

$$G_{X \setminus b_1} (G_{b_1})_{\{b_2, \dots, b_m\}} = G_{\{b_1, b_2, \dots, b_m\}}$$

as required. $\square$

5.2. Orbits of clopens in the punctured Cantor space. Let X be the Cantor space, $x_1, \dots, x_n \in X$ and $X^\circ := \{x_1, \dots, x_n\}$. For a clopen b in X° let

$$I := \{i \in [n] : x_i \text{ is a limit point of } b\},$$

$$I' := \{i \in [n] : x_i \text{ is a limit point of } X^\circ \setminus b\}.$$

We call the pair (I, I') the *type* of b . Note that $I \cup I' = [n]$ but that union is not necessarily disjoint. Note that the following hold:

- $I = \emptyset$ if and only if b is clopen in X ;
- $I' = \emptyset$ if and only if $X \setminus b$ is clopen in X ;
- $I, I' \neq \emptyset$ if and only if there exist pairwise disjoint clopens b_{ij} ($i \in I, j \in \mathbb{N}$) and b'_{ij} ($i \in I', j \in \mathbb{N}$) such that

$$(5.1) \quad \begin{aligned} b &= \bigcup_{i \in I, j \in \mathbb{N}} b_{ij}, & \lim_{j \rightarrow \infty} b_{ij} &= x_i \ (i \in I), \\ X^\circ \setminus b &= \bigcup_{i \in I', j \in \mathbb{N}} b'_{ij}, & \lim_{j \rightarrow \infty} b'_{ij} &= x_i \ (i \in I'). \end{aligned}$$

We can characterize the orbits of clopens in X° under $(\text{Homeo}X)_{\{x_1, \dots, x_n\}}$ as follows.

{lem:types}

Lemma 5.8. *For $I, I' \subseteq [n]$ with $I \cup I' = [n]$, the set of clopens in X° of type (I, I') is an orbit under $(\text{Homeo}X)_{\{x_1, \dots, x_n\}}$.*

Proof. Homeomorphisms from $(\text{Homeo}X)_{\{x_1, \dots, x_n\}}$ preserve types of clopens because they fix $x_1, \dots, x_n$. Conversely, consider two clopens b, c of the same type (I, I') . If $I = \emptyset$ then taking the union of a homeomorphism $b \rightarrow c$ and a homeomorphism $X \setminus b \rightarrow X \setminus c$ that fixes $x_1, \dots, x_n$ yields a homeomorphism $\psi \in (\text{Homeo}X)_{\{x_1, \dots, x_n\}}$ with $\psi(b) = c$. An analogous construction deals with the case $I' = \emptyset$. Consider now the case where $I, I' \neq \emptyset$. Decompose each of b and c according to (5.1); let b_{ij} ($i \in I, j \in \mathbb{N}$), b'_{ij} ($i \in I', j \in \mathbb{N}$) be the clopens associated with b , and let c_{ij} ($i \in I, j \in \mathbb{N}$), c'_{ij} ($i \in I', j \in \mathbb{N}$) be those for c . Take arbitrary homeomorphisms $\psi_{ij}: b_{ij} \rightarrow c_{ij}$ ($i \in I, j \in \mathbb{N}$) and $\psi'_{ij}: b'_{ij} \rightarrow c'_{ij}$ ($i \in I', j \in \mathbb{N}$). Their union ψ extended by $\psi(x_i) = x_i$ ($i \in [n]$) is a homeomorphism of X because of the limit conditions in (5.1), and clearly $\psi(b) = c$, completing the proof. $\square$

5.3. SIP for the filtered Boolean power. We need one more auxiliary result for the proof of Theorem 1.3(2).

{lem:invariant}

Lemma 5.9. *Let $\mathbf{A}$ be a finite simple Mal'cev algebra with an idempotent e_1 , let $\mathbf{B}$ be the countable atomless Boolean algebra with Cantor space X , let $x_1 \in X$, and let $h: \text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1}^{x_1} \rightarrow (\text{Homeo}X)_{x_1}$ be as in Theorem 2.5(1).*

Let L be subgroup of $K := \ker h$ with $|K : L| < 2^{\aleph_0}$ that is normal in $\text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1}^{x_1}$. Then L contains

$$K' := \{\varphi \in K : \varphi_x \in (\text{Aut} \mathbf{A})_{e_1} \text{ for all } x \in X^\circ\}.$$

Proof. If $(\text{Aut}\mathbf{A})_{e_1}$ is trivial, then so is K' and the result clearly holds. So we assume that $(\text{Aut}\mathbf{A})_{e_1}$ is non-trivial for the rest of the proof.

For $\varphi \in K'$, let $\varphi_*: X^\circ \rightarrow (\text{Aut}\mathbf{A})_{e_1}$, $x \mapsto \varphi_x$, be as in Theorem 2.5(2). Then $\varphi_*^{-1}(\alpha)$ is clopen in X° for any $\alpha \in (\text{Aut}\mathbf{A})_{e_1}$ by that lemma.

We will prove the result by showing that

$$W := \{\varphi \in K' : x_1 \text{ is a limit point of } \varphi_*^{-1}(\alpha) \text{ for all } \alpha \in (\text{Aut}\mathbf{A})_{e_1}\}$$

is contained in L and generates K' . Note that $\varphi_*^{-1}(\alpha) \neq \emptyset$ for any $\varphi \in W$ and $\alpha \in (\text{Aut}\mathbf{A})_{e_1}$, in particular, $\varphi_*^{-1}(\alpha)$ is clopen of type $([1], [1])$.

First we claim that

$$(5.2) \quad L \cap W \neq \emptyset.$$

To see this, partition X° into a sequence of clopens in X converging to x_1 . Then subdivide each of the clopens into $|(\text{Aut}\mathbf{A})_{e_1}|$ non-empty clopens to obtain a partition of X° into $|(\text{Aut}\mathbf{A})_{e_1}|$ disjoint sequences of clopens $b_{i\alpha}$ for $i \in \mathbb{N}, \alpha \in (\text{Aut}\mathbf{A})_{e_1}$ that converge to x_1 . For any subset $T \subseteq \mathbb{N}$ define $\varphi^T \in K'$ such that

$$\varphi_x^T := \begin{cases} \alpha & \text{if } x \in b_{i\alpha} \text{ and } i \in T, \\ \text{id}_A & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}$ be an uncountable family of subsets of $\mathbb{N}$ such that $T_1 \triangle T_2$ (the symmetric difference) is infinite for any two distinct $T_1, T_2 \in \mathcal{T}$. Since $|K : L| < 2^{\aleph_0}$, there exist $T_1, T_2 \in \mathcal{T}$, $T_1 \neq T_2$ with $\varphi^{T_1}(\varphi^{T_2})^{-1} \in L$. Since $T_1 \triangle T_2$ is infinite, we see that $\varphi^{T_1}(\varphi^{T_2})^{-1} \in W$ and (5.2) is proved.

Recall the definition of g from (2.1). We show that

$$(5.3) \quad C := g((\text{Homeo}X)_{x_1}) \text{ acts transitively on } W \text{ by conjugation.}$$

Let $\sigma, \tau \in W$. For each $\alpha \in (\text{Aut}\mathbf{A})_{e_1}$ the clopens $\sigma_*^{-1}(\alpha)$ and $\tau_*^{-1}(\alpha)$ have type $([1], [1])$. By Lemma 5.8 there exists $\psi \in (\text{Homeo}X)_{x_1}$ such that $\psi(\sigma_*^{-1}(\alpha)) = \tau_*^{-1}(\alpha)$ for all $\alpha \in (\text{Aut}\mathbf{A})_{e_1}$. For $\mathbf{D} := \text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1}^{x_1}$, recall that $g(\psi) = \psi^{\mathbf{D}} \in C$ and $(\tau^{\psi^{\mathbf{D}}})_x = \tau_{\psi(x)}$. It follows that $\tau^{\psi^{\mathbf{D}}} = \sigma$ and (5.3) is proved.

Since L is normal in $\text{Aut}\mathbf{D}$ by assumption, claims (5.2) and (5.3) imply $W \subseteq L$. It remains to show that

$$(5.4) \quad \langle W \rangle = K'.$$

For $\alpha \in (\text{Aut}\mathbf{A})_{e_1}$ and c clopen in X° , define $\chi^{c,\alpha} \in K$ by

$$\chi_x^{c,\alpha} := \begin{cases} \alpha & \text{if } x \in c, \\ \text{id}_A & \text{else.} \end{cases}$$

To show that $\chi^{c,\alpha}$ is generated by W we distinguish two cases, depending on the type of c .

Case 1: c is clopen in X . As in the proof of Claim (5.2) we partition $X^\circ \setminus c$ into $|(\text{Aut } \mathbf{A})_{e_1}|$ sequences of clopens converging to x_1 . Use these to define $\sigma, \tau \in W$ with $c \subseteq \sigma_*^{-1}(\alpha)$, $c \subseteq \tau_*^{-1}(\text{id})$ and $\sigma_x = \tau_x$ for all $x \in X^\circ \setminus c$. Then $\chi^{c,\alpha} = \sigma\tau^{-1} \in \langle W \rangle$.

Case 2: $X \setminus c$ is clopen in X . This is analogous to Case 1.

Case 3: c is a disjoint union of clopens in X converging to x_1 . This time we partition c itself into $|(\text{Aut } \mathbf{A})_{e_1}|$ disjoint sequences of clopens $b_{i\beta}$ on X for $i \in \mathbb{N}, \beta \in (\text{Aut } \mathbf{A})_{e_1}$ that converge to x_1 . Define $\sigma, \tau \in W$ by

$$\sigma_x := \begin{cases} \alpha\beta & \text{if } x \in b_{i\beta}, \\ \text{id}_A & \text{else,} \end{cases} \quad \tau_x := \begin{cases} \beta & \text{if } x \in b_{i\beta}, \\ \text{id}_A & \text{else.} \end{cases}$$

Then $\chi^{c,\alpha} = \sigma\tau^{-1} \in \langle W \rangle$.

Since every element in K' is a product of elements $\chi^{c,\alpha}$ for c clopen in X° and $\alpha \in (\text{Aut } \mathbf{A})_{e_1}$, it follows that K' is generated by W . This proves (5.4) and the lemma. $\square$

Finally we are ready to prove the main result of this section.

Proof of Theorem 1.3(2). Let $\mathbf{D} := (\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ and let H be a subgroup of $\text{Aut } \mathbf{D}$ of index $< 2^{\aleph_0}$. Let $G := (\text{Homeo } X)_{\{x_1, \dots, x_n\}}$. By Corollary 2.8 we may assume that $e_1, \dots, e_n$ are in distinct $\text{Aut } \mathbf{A}$ -orbits. So, by Theorem 2.5(1) and (2.1), $\text{Aut } \mathbf{D}$ is a semidirect product of $K := \ker h$ and $C := g(G)$.

Then $|C : H \cap C| < 2^{\aleph_0}$. By Theorem 5.1 we have a partition of X into clopens $b_1, \dots, b_m$ such that

$$C' := \{\psi^{\mathbf{D}} : \psi \in G, \psi(b_i) = b_i \text{ for } i \in [m]\} \leq H \cap C.$$

Furthermore, by refining the b_i and reordering, we may assume that $m \geq n$ and $x_i \in b_i$ for all $i \in [n]$. For $i \in [n]$, let

$$K_i := \{\varphi \in K : \varphi_x = \text{id}_A \text{ for } x \in X^\circ \setminus b_i\}.$$

Since $H \cap K$ is invariant under conjugation by C' , it follows that $H \cap K_i$ is invariant under the natural action of $(\text{Homeo } b_i)_{x_i}$. Further $|K_i : H \cap K_i| < 2^{\aleph_0}$. Hence Lemma 5.9 yields that $H \cap K_i$ contains

$$K'_i := \{\varphi \in K : \varphi_x \in (\text{Aut } \mathbf{A})_{e_i} \text{ for } x \in b_i \setminus \{x_i\}, \varphi_x = \text{id}_A \text{ for } x \in X^\circ \setminus b_i\}.$$

It follows that

$$K' := K'_1 \dots K'_n \leq H \cap K.$$

We complete the proof by showing that $K'C' \leq H$ contains the stabilizer of finitely many elements in D . Specifically, for a in $T := \prod_{i=1}^n \{e_i\} \times A^{m-n}$, we define

$$f_a : X \rightarrow A, \quad x \mapsto a_i \text{ if } x \in b_i, i \in [m],$$

and claim that

$$(5.5) \quad (\text{Aut } \mathbf{D})_{\{f_a : a \in T\}} \leq K'C' \leq H.$$

Consider an element in the above stabilizer decomposed as $\varphi\psi^{\mathbf{D}}$ for $\varphi \in K$, $\psi \in G$. We first show that ψ stabilizes each set $b_1, \dots, b_m$, so $\psi^{\mathbf{D}} \in C'$. Let $a \in T$. Then, using Theorem 2.5(1),

$$f_a = \varphi\psi^{\mathbf{D}}(f_a) = \varphi(f_a\psi^{-1}).$$

Combining this with Lemma 2.3(2) gives

$$f_a(\psi(x)) = \varphi_{\psi(x)}(f_a(x))$$

for $x \in X^\circ$. If $x \in b_i$ and $\psi(x) \in b_j$ for $i, j \in [m]$, this yields

$$(5.6) \quad a_j = \varphi_{\psi(x)}(a_i).$$

So, for all $a \in T$, a_j is uniquely determined by a_i and conversely. Suppose that $i \neq j$. Since $|A| > 1$, it follows that $i, j \in [n]$. But then (5.6) yields that e_i, e_j are in the same orbit under $\text{Aut}\mathbf{A}$, contradicting our assumption. Thus $i = j$ and $\psi^{\mathbf{D}} \in C'$.

Now (5.6) simplifies to

$$a_i = \varphi_x(a_i) \text{ for all } x \in b_i \cap X^\circ, i \in [m].$$

If $i \in [n]$, we obtain $\varphi_x(e_i) = e_i$ for $x \in b_i$. If $i \in [m] \setminus [n]$, we obtain $\varphi_x(a_i) = a_i$ for all $a_i \in A$, i.e., $\varphi_x = \text{id}_A$ for all $x \in b_i$. Thus $\varphi \in K'$ and (5.5) is proved, completing the proof that $\text{Aut}\mathbf{D}$ has SIP. $\square$

6. COFINALITY AND BERGMAN PROPERTY

{sec:Bergman}

Throughout this section X is the Cantor space, $x_1, \dots, x_n \in X$ are distinct, and $X^\circ := X \setminus \{x_1, \dots, x_n\}$.

To establish uncountable cofinality and the Bergman property for filtered Boolean powers we first need some auxiliary results on $G := (\text{Homeo}X)_{\{x_1, \dots, x_n\}}$.

For $\sigma \in G$ let

$$\text{supp}\sigma := \{x \in X : \sigma(x) \neq x\}.$$

{lem:boundconj}

Lemma 6.1. *Let $\sigma, \tau \in (\text{Homeo}X)_{\{x_1, \dots, x_n\}}$ be such that $x_1, \dots, x_n$ are limit points of $\text{supp}\sigma$. Then τ can be written as a product of $8n$ conjugates of $\sigma^{\pm 1}$ by elements in $(\text{Homeo}X)_{\{x_1, \dots, x_n\}}$.*

Proof. This is a small modification of the final paragraph of the proof of [25, Theorem 3.17]. Use [25, Lemma 3.16] to write $\tau = \tau_n \dots \tau_1$, where τ_i fixes pointwise a clopen b_i containing $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$. We show that every τ_i can be written as a product of 8 conjugates of $\sigma^{\pm 1}$. For ease of notation, we take $i = 1$. By [25, Lemma 3.15], there exist a clopen set b of X which does not contain $x_2, \dots, x_n$, and $\mu \in (\text{Homeo}X)_{\{x_1, \dots, x_n\}}$ such that $\text{supp}(\sigma\mu\sigma^{-1}\mu^{-1})$ has x_1 as limit point and is contained within b . Let $c := b \cup (X \setminus b_1)$. This again is a clopen that does not contain $x_2, \dots, x_n$, and contains $\text{supp}(\sigma\mu\sigma^{-1}\mu^{-1})$; furthermore $\text{supp}\tau_1 \subseteq c$. Both τ_1 and $\sigma\mu\sigma^{-1}\mu^{-1}$ restrict to homeomorphisms of c since they fix the complement of c in X . Since $\sigma\mu\sigma^{-1}\mu^{-1}$ does not fix any clopen around x_1 pointwise, it follows by [25, Corollary 3.14]

that τ_1 can be written as a product of 4 conjugates of $\sigma\mu\sigma^{-1}\mu^{-1}$, i.e. a product of 8 conjugates of $\sigma^{\pm 1}$. $\square$

{1e:Xucsc}

Lemma 6.2. *Let X be the Cantor space and $x_1, \dots, x_n \in X$ distinct. Then $(\text{Homeo}X)_{\{x_1, \dots, x_n\}}$ has uncountable strong cofinality.*

Proof. For $n = 0$ this is part of [8, Theorem 3.3].

Assume $n \geq 1$. Let us call clopens of X° of type $([n], [n])$ *good*. Recall that $c \subseteq X^\circ$ is a good clopen if and only if both c and $X^\circ \setminus c$ can be partitioned into n sequences of clopens of X converging to $x_1, \dots, x_n$ respectively. We will prove the lemma by applying [8, Theorem 2.8] with:

- $G := (\text{Homeo}X)_{\{x_1, \dots, x_n\}}$;
- $\Omega := X^\circ := X \setminus \{x_1, \dots, x_n\}$;
- $\mathfrak{K}$ the set of all good clopens;
- $F := \{b_1, \dots, b_{n+2}\}$, a set of good clopens that partition X° .

We will follow the path offered by part (I) of [8, Theorem 2.8] and verify that

- the ‘piecewise-patching’ properties (1), (2), (5), (6), (7), (8) of [8, p. 337] hold;
- $G = E^3$, where $E := \bigcup_{b \in F} G_{\{b\}}$ and $G_{\{b\}}$ denotes the setwise stabiliser of b in G .

(1) $\emptyset, X^\circ \notin \mathfrak{K}$ and $\mathfrak{K}$ is closed under complements: This follows from the definition.

(2) Every good set c splits into a disjoint union of two good sets: Take the partition of c into n sequences of disjoint clopens converging to $x_1, \dots, x_n$, respectively, and split each of these clopens into two non-empty clopens. This yields $2n$ sequences of disjoint clopens with two of them converging to x_i for $i \in [n]$. The unions of n of these sequences converging to $x_1, \dots, x_n$, respectively, yields a good clopen again. So c can be split into two good clopens.

(5) G is transitive on $\mathfrak{K}$: This follows from Lemma 5.8.

(6) Let c, d_1, d_2, e_1, e_2 be good such that $c = d_1 \dot{\cup} d_2 = e_1 \dot{\cup} e_2$ and $f_i \in G$ with $f_i(d_i) = e_i$ for $i \in [2]$. Then $f_1|_{d_1} \cup f_2|_{d_2} \cup \text{id}_{X \setminus c}$ is in G : These partial homeomorphisms can be glued together since d_1, d_2 have $x_1, \dots, x_n$ as limit points.

(7) There are disjoint good c_i for $i \in \mathbb{N}$ such that for all $f_i \in G_{\{c_i\}}$, we have $\bigcup_{i \in \mathbb{N}} f_i|_{c_i} \cup \text{id}_{X \setminus \bigcup_{i \in \mathbb{N}} c_i}$ in G : Partition X° into n sequences of clopens of X converging to $x_1, \dots, x_n$, respectively. Next split each of these sequences into countably many subsequences. Taking unions of these sequences yields our countably many good clopens c_i .

(8) There exists $m \in \mathbb{N}$ such that for every good clopen c , and any $f, g \in G$ with $\text{supp} f, \text{supp} g \subseteq c$ and $\text{supp} g$ containing some good clopen, we have

$$f = (g'_1)^{h_1} \dots (g'_m)^{h_m}$$

for some $h_i \in G$ with $\text{supp} h_i \subseteq c$ and $g'_i \in \{g, g^{-1}\}$ for all $i \in [m]$: This follows from Lemma 6.1 with $m = 8n$ since the union of any good clopen c with $\{x_1, \dots, x_n\}$ is homeomorphic to X .

To check that $G = E^3$ we proceed via a series of claims, motivated by [14, Lemma 2.1].

$\{c1:GbGcGb\}$

Claim 6.3. *Let b, c, d be pairwise disjoint good clopens such that $X^\circ = b \cup c \cup d$ and let $\sigma \in G$ such that $d \setminus \sigma^{-1}(b)$ is good. Then $\sigma \in G_b G_c G_b$ (pointwise stabilizers).*

Proof. We aim to define three homeomorphisms $\sigma_1, \sigma_3 \in G_b$, $\sigma_2 \in G_c$ such that $\sigma = \sigma_3 \sigma_2 \sigma_1$,

Since $(c \cup d) \cap \sigma^{-1}(b)$ is clopen in X° and d is good, we have $\tau_1 \in G$ such that

$$(6.1) \quad f := \tau_1((c \cup d) \cap \sigma^{-1}(b)) \text{ is contained in } d \text{ and } d \setminus f \text{ is good.}$$

By assumption, the clopen $d \setminus \sigma^{-1}(b)$ is good; in particular, $x_1, \dots, x_n$ are among its limit points. Hence the set $(c \cup d) \setminus \sigma^{-1}(b)$ also has all x_i as limit points. The complement of this set contains the good clopen b , and hence also has $x_1, \dots, x_n$ among its limit points. We conclude that $(c \cup d) \setminus \sigma^{-1}(b)$ is good. An analogous argument shows that $(c \cup d) \setminus f \in \mathfrak{K}$. By Lemma 5.8 there exists a homeomorphism

$$\tau_2: (c \cup d) \setminus \sigma^{-1}(b) \rightarrow (c \cup d) \setminus f.$$

We can now define the first of our three target homeomorphisms on X :

$$(6.2) \quad \sigma_1(x) := \begin{cases} x & \text{if } x \in b \cup \{x_1, \dots, x_n\}, \\ \tau_1(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(b), \\ \tau_2(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(c \cup d). \end{cases}$$

To see that $\sigma \in G$, observe that it is defined piecewise by homeomorphisms on three clopens that partition X° , and that the images of those three clopens, namely b , f and $(c \cup d) \setminus f$, also partition X° . That $\sigma \in G_b$ is immediate from (6.2).

Next the clopen

$$(b \cap \sigma^{-1}(c \cup d)) \cup (d \setminus f)$$

is good: indeed it has $x_1, \dots, x_n$ as limit points because $d \setminus f$ is good by (6.1), while its complement also has $x_1, \dots, x_n$ as limit points because it contains $c \in \mathfrak{K}$. Pick a homeomorphism

$$\tau_3: (b \cap \sigma^{-1}(c \cup d)) \cup (d \setminus f) \rightarrow d.$$

Define our second target homeomorphism by

$$(6.3) \quad \sigma_2(x) := \begin{cases} x & \text{if } x \in c \cup \{x_1, \dots, x_n\}, \\ \sigma \tau_1^{-1}(x) & \text{if } x \in f, \\ \sigma(x) & \text{if } x \in b \cap \sigma^{-1}(b), \\ \tau_3(x) & \text{if } x \in (b \cap \sigma^{-1}(c \cup d)) \cup (d \setminus f). \end{cases}$$

Then $\sigma_2 \in G_c$ since it is defined piecewise by homeomorphisms on clopens partitioning X° with images $c, \sigma(c \cup d) \cap b, \sigma(b) \cap b, d$ also partitioning X° .

Finally, define

$$(6.4) \quad \sigma_3(x) := \begin{cases} x & \text{if } x \in b \cup \{x_1, \dots, x_n\}, \\ \sigma\tau_2^{-1}(x) & \text{if } x \in c, \\ \sigma\tau_3^{-1}(x) & \text{if } x \in \tau_3(b \cap \sigma^{-1}(c \cup d)), \\ \sigma\tau_2^{-1}\tau_3^{-1}(x) & \text{if } x \in \tau_3(d \setminus f). \end{cases}$$

That $\sigma_3 \in G_b$ follows as usual since the images of its constituting partial homeomorphisms $b, \sigma\tau_2^{-1}(c), \sigma(b) \cap (c \cup d), \sigma\tau_2^{-1}(d \setminus f)$ partition X° .

Now we verify that $\sigma = \sigma_3\sigma_2\sigma_1$ using the definitions (6.2), (6.3), (6.4) of $\sigma_1, \sigma_2, \sigma_3$ respectively. For $x \in X^\circ$

$$\begin{aligned} & \sigma_3\sigma_2\sigma_1(x) \\ &= \sigma_3\sigma_2 \begin{pmatrix} x & \text{if } x \in b \\ \tau_1(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(b) \\ \tau_2(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(c \cup d) \end{pmatrix} \\ &= \sigma_3 \begin{pmatrix} \sigma(x) & \text{if } x \in b \cap \sigma^{-1}(b) \\ \tau_3(x) & \text{if } x \in b \cap \sigma^{-1}(c \cup d) \\ \sigma(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(b) \\ \tau_2(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(c \cup d) \cap \tau_2^{-1}(c) \\ \tau_3\tau_2(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(c \cup d) \cap \tau_2^{-1}(d \setminus f) \end{pmatrix} \\ &= \begin{cases} \sigma(x) & \text{if } x \in b \cap \sigma^{-1}(b) \\ \sigma\tau_3^{-1}\tau_3(x) & \text{if } x \in b \cap \sigma^{-1}(c \cup d) \\ \sigma(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(b) \\ \sigma\tau_2^{-1}\tau_2(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(c \cup d) \cap \tau_2^{-1}(c) \\ \sigma\tau_2^{-1}\tau_3^{-1}\tau_3\tau_2(x) & \text{if } x \in (c \cup d) \cap \sigma^{-1}(c \cup d) \cap \tau_2^{-1}(d \setminus f) \end{cases} \\ &= \sigma(x). \end{aligned}$$

Thus Claim 6.3 is proved. $\square$

{cl:abutone}

Claim 6.4. *Let b, c, d be clopens in X° and $k \in [n]$ such that $b \cap c = \emptyset$ and x_k is a limit point of d . If x_k is not a limit point of $d \setminus b$, then it is a limit point of $d \setminus c$.*

Proof. The assumption that x_k is not a limit point of $d \setminus b$ means that there exists a clopen neighbourhood f of x_k in X such that $f \setminus \{x_k\}$ is contained in the complement of $d \setminus b$. Since $f \cap d$ is clopen in X° and is contained in d , this implies $f \cap d \subseteq b$. Since $b \cap c = \emptyset$, it follows that $(f \cap d) \cap c = \emptyset$. Hence $f \cap d \subseteq d \setminus c$ and x_k is a limit point of $d \setminus c$ as required. $\square$

Recall that $F = \{b_1, \dots, b_{n+2}\}$ is a set of disjoint good clopens that partition X° .

{cl:pigeon}

Claim 6.5. *For every $\sigma \in G$ there exist $i, j \in [n+2]$ such that $\sigma \in G_{b_i}G_{b_j}G_{b_i}$.*

Proof. Let $k \in [n]$. By Claim 6.4 x_k is a limit point of $b_{n+2} \setminus \sigma^{-1}(b_i)$ for all but possibly one $i \in [n+1]$. So, by the pigeonhole principle, there exists $i \in [n+1]$ such that $b_{n+2} \setminus \sigma^{-1}(b_i)$ has all of $x_1, \dots, x_n$ as limit points. Let $b := b_i$, $c := \bigcup_{j \in [n+1] \setminus \{i\}} b_j$ and $d := b_{n+2}$. Then $d \setminus \sigma^{-1}(b)$ is good and $\sigma \in G_b G_c G_b$ by Claim 6.3. For any $j \in [n+1] \setminus \{i\}$ we have $G_c \subseteq G_{b_j}$ and hence $\sigma \in G_{b_i} G_{b_j} G_{b_i}$ as required. $\square$

To complete the proof of Lemma 6.2, recall that $E = \bigcup_{b \in F} G_{\{b\}}$. Using Claim 6.5, for any $\sigma \in G$ there exist $b_i, b_j \in F$ such that $\sigma \in G_{b_i} G_{b_j} G_{b_i} \subseteq G_{\{b_i\}} G_{\{b_j\}} G_{\{b_i\}} \subseteq E^3$.

Since $G = E^3$ and all assumptions of case (I) of [8, Theorem 2.8] are satisfied, G has uncountable strong cofinality. $\square$

Using the semidirect decomposition of $\text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ from Theorem 2.5 and the previous lemma, it is not hard to show our last main result.

Proof of Theorem 1.3(3). By Corollary 2.8 we may assume that $e_1, \dots, e_n$ are in distinct $\text{Aut} \mathbf{A}$ -orbits. So, by Theorem 2.5(1) and (2.1), $G := \text{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1, \dots, e_n}^{x_1, \dots, x_n}$ is a semidirect product of the groups $K := \ker h$ and $C := g((\text{Homeo} X)_{\{x_1, \dots, x_n\}})$.

Seeking a contradiction suppose that G has countable strong cofinality. That is, there exists a sequence

$$H_1 \subseteq H_2 \subseteq \dots$$

of proper subsets of G such that $H_i^{-1} = H_i$, $H_i H_i \subseteq H_{i+1}$ for all $i \in \mathbb{N}$ and

$$\bigcup_{i \in \mathbb{N}} H_i = G.$$

Since C has uncountable strong cofinality by Lemma 6.2, we may assume that $C \subseteq H_1$.

For $\alpha \in \text{Aut} \mathbf{A}$ and c clopen in X° of type (I, I') , assume that there exists $\chi^{c, \alpha} \in K$ such that

$$\chi_x^{c, \alpha} := \begin{cases} \alpha & \text{if } x \in c, \\ \text{id}_A & \text{else.} \end{cases}$$

Then $\alpha(e_i) = e_i$ for all $i \in I$ by Theorem 2.5(2) and $I \cup I' = [n]$. We call such a triple (α, I, I') *legal* and call functions of the form $\chi^{c, \alpha}$ *characteristic*. Note that conversely for every legal triple (α, I, I') there exists a clopen c in X° of type (I, I') and a characteristic $\chi^{c, \alpha}$ in K .

Since there are only finitely many legal triples, we may assume that H_1 contains at least one characteristic function $\chi^{c, \alpha}$ for any legal triple. By Lemma 5.8 any two characteristic functions corresponding to the

same legal triple are conjugate under C . Hence H_3 contains all characteristic functions from K . Since every element of K is a product of $|\text{Aut}\mathbf{A}|$ many characteristic functions from K , it follows that $K \subseteq H_{|\text{Aut}\mathbf{A}|+3}$. Then $G = H_{|\text{Aut}\mathbf{A}|+4}$ contradicting our assumption. Thus G has uncountable strong cofinality. $\square$

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