Perturbations in dense matter relativistic stars induced by internal sources

Seema Satin
Dept. of Physical Sciences, Indian Institute for Science Education and Research, Kolkata India*

Einstein's field equations with a background source term that induces perturbations and the applications of this new formalism to a compact dense matter relativistic star are presented. We introduce a new response kernel in the field equations between the metric and fluid perturbations. A source term which drives or induces the sub-hydro mesoscopic scales perturbations in the astrophysical system, is of importance here. Deterministic as well as stochastic perturbations for the radial case are worked out as solutions of field equations. We also touch upon polar perturbations that are deterministic with oscillatory parts. The stochastic perturbation are of significance in terms of two point or point separated correlations which form the building blocks for studying equilibrium and non-equilibrium statistical mechanics for the system. Our main aim is to build a theory for intermediate scale physics, for dense exotic matter and investigate structure of the compact astrophysical objects. Specifically, turbulence which connects various scales in superfluid matter in the dense stars is an area gaining importance. The work presented here is the starting point for new theoretical frame work that touches upon various yet unexplored scales where mechanical and dynamical effects interior to the matter of the star are of significance. Thus there is scope for extending studies in asteroseismology to mesoscopic effects at the new intermediate scales in the cold dense matter fluid . This is expected to enable us to probe astrophysical features at refined scales through theoretical formulations as well as for observational consequences. We provide a first principles approach with our formulations to study the new mesoscopic scales which are yet unexplored in asteroseismology and will form a bridge between the macroscopic scales defined by the hydrodynamics and microscopic details characterized by nuclear physics and quantum aspects in the exotic fluid.

* seemasatin@iiserkol.ac.in

I. INTRODUCTION

Exploring dense matter in compact objects is an active topic of research at present [1–6] and of prime importance in asteroseismology . Several ongoing efforts to understand the nature of dense matter focus on microscopic nuclear physics details. There are ongoing efforts to connect the macroscopic picture with the microscopic details [7, 8] in order to find out and refine the equation of state which is still unknown for the interiors and core of the relativistic stars. While such studies are based on nuclear physics and thermodynamics, there are new avenues in the direction of turbulence and large eddies in rotating configurations which have come under survey where magnetohydrodynamics is considered as a rich topic for new investigations [9, 10]. Mesoscopic studies with emphasis on relativistic fluids that the exotic objects are composed of, have just begun [11, 12], while fluctuations in relativistic fluids is another focus area with good scope for new developments [13]. While the large eddy simulations focus on the hydrodynamics and macro scales, an initial attempt for mesoscopic studies taking into consideration quantum vortices has just begun [11]. We raise an interesting new query towards sub-hydro mesoscopic scales, which lie a little below the macroscopic hydro scales but much above the microscopic and quantum vortex scales. The purpose of doing so, is to attempt to establish an intermediate scale theory for dense compact matter, where effects of strong gravity can be captured for non-local properties. For such a study, Einstein's equations in a perturbative approach are important from first principles. With such theoretical developments, it will be possible to address long range (or point separated) structure inside the dense stars and touch upon the scales at which curvature of spacetime starts showing effect on the physical variables. With such a purpose, recently, a classical Einstein-Langevin formalism has been proposed as a new theoretical formulation [14, 15]. This has wide scope for theoretical modelling at basic level and exploring connections with realistic models of compact stars. Such an effort is based on foundational level new constructs in general relativity, where a stochastic noise term has been added to the linearized Einstein's field equations making them inhomogeneous in nature. The distinct feature of such a new construct are the noise term and a reponse kernel, which relates fluid (matter) perturbations to the spacetime metric perturbations . While the idea has been borrowed from the semiclassical Einstein Langevin equation [16], where noise is due to quantum fluctuations, the classical counterpart has very different construct and applications. A linear reponse kernel explicity calls for attention towards the new formulation and needs further elaborate considerations. In the present article we further modify our earlier attempts, mathematically, to better suit the astrophysical conditions that we intend to investigate. In previous work [17] a response function has been taken as a bi-scalar, while here we consider a bi-tensor as a response kernel that looks more appropriate, mathematically, as well as for physical considerations in the Einstein's equations where component-wise segregation is meaningful with a metric structure. A consistent noise model that drives the perturbations is also important here. Thus we re-build our proposed formalism with a tensor resonse kernel and a slightly different driving source term. From now on we use the term linearized Einstein's equation with a background source instead of the Einstien-Langevin equation which is restricted to stochastic noise only. In this article we give a model for deterministic source in the background spacetime which drives the perturbations, as well as consider a stochastic model separately on the lines of our previous work. Theoretically these are two are separate cases, giving rise to different contexts for the perturbations of relativistic stars. The simple models of non-rotating configuration are worked out in this article, with analytical results obtained for radial perturbations and the semi-analytical equations for the polar perturbations. We do not however carry out full solutions for polar perturbations here, as that needs seperate rigorous methods to be developed as part of further research with numerical modelling for stochasticity in the Einstein's equations. This will also give scope for new avenues to explore in numerical relativity .

II. PERTURBED EINSTEIN'S EQUATIONS WITH A SOURCE TERM

A linear response relation between metric and fluid perturbations has been defined in [15], we modify it slightly in the present article with a new form of response kernel, which is more suitable for our considerations. The fundamental elements of perturbation are h_{ab} for metric, and ξ^a the Lagrangian displacement vector for the matter. We propose in this article a linear response relation between the two which is of the form

$$F^{ab}[h,x) = \int \mathcal{K}^{ab}{}_{cd}(x-x')M^{cd}[\xi,x')dx' \tag{1}$$

where $F^{ab}[h]$ represents tensors with metric perturbations h and $M^{cd}[\xi]$ denotes tensors with fluid perturbations ξ, while the bitensor $\mathcal{K}^{ab}{}_{cd}(x-x')$ is the response kernel giving a new physically relevant quantity "susceptibility" of spacetime to get perturbed. We add to this form, an internal background source which induces or drives perturbations in the gravitating configuration. The above relation splits terms of Einstein's equations into parts with h_{ab} and parts with ξ^a which can be written phenomenologically, as below.

This form of the linearized Einstein's equations with an added source as the inhomogeneous part, reads

$$\delta G^{ab}[h,x] - 8\pi\delta T^{ab}[h,x] - 8\pi \int \mathcal{K}^{ab}{}_{cd}(x-x')\delta T^{cd}[\xi,x']dx' = \tau^{ab}[g,x] \tag{2}$$

where $\tau^{ab}[g,x]$ is the source of perturbations defined on the background g_{ab} and satisfies $\nabla_a\tau^{ab}$ w.r.t the background unperturbed metric . If the driving source $\tau^{ab}(x) = 0$, we anticipate no induced perturbations in the configuration. Hence this formalism is based on the inhomogeneous source term.

In this article, we propose two cases for $\tau_{ab}(x)$,

- $\tau^{ab}(x)$ defines a deterministic source in the background fluid given by $\delta_s T^{ab}(\vec{x})e^{i\omega t}$, δ_s denotes source, as opposed to δ used in equation (2) for the perturbed quantities and
- $\tau^{ab}(x)$ as stochastic or a Langevin source, which is similar to the previous formulations in [14, 15], with a slight change. We model stochasticity with $\delta_s T^{ab}(\vec{x},t)e^{i\omega t}$ here δ_s denotes a stochastic source. Thus the amplitude $\delta_s T^{ab}(\vec{x},t)$ in general is random in space as well as in time, with the oscillatory part given by $e^{i\omega t}$.

In general equation (2) can be seen as a sourced inhomogeneous Einstein's perturbed equation rather than the Einstein-Langevin equation as proposed in our previous work. The response kernel bi-tensor with indices is the basic modification to our earlier proposed linear response kernel for gravitating systems where it was defined as a bi-scalar. In this article we work out a specific model for such a response kernel which looks reasonable for our configuration. This response kernel gives us a susceptibility tensor after taking a Laplace transform as shown later in the expressions and results.

In the following section, we will give solutions of the equation (2) for radial and polar perturbations of a non-rotating, spherically symmetric relativistic star. This is different from our earlier work [14, 15, 17], in that, the perturbations are deterministic in one case, while we also consider generalized stochastic nature as the second case. We have used a more consistent source and noise model in this article compared to previous articles.

For simplicity of analytical solutions, in the spherically symmetric configuration, we assume for the response kernel the form,

$$\mathcal{K}^{ab}{}_{cd}(x-x') = \mathcal{K}^{ab}{}_{cd}(t-t')\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \tag{3}$$

One can certainly have other models, where the response is felt over spatial separations intervals also. But that increases the difficulty to get analytical solutions for the equations and one will have to resort to numerical solutions.

With the simple model that we propose here, we intend to present the basic form of final solutions in closed analytical form which can be taken as a physically meaningful toy model. We use in the expressions that follow, the response kernel $\mathcal{K}^{ab}{}_c{}^d(x-x')$, with 3 indices up and one down for convenience such that the corresponding term in the sourced Einstein's equation, can be evaluated using the form $\int \mathcal{K}^{ab}{}_c{}^d(x-x')\delta T^c{}_d[\xi,x']dx'$.

In this article, for a perfect fluid matter we use the following model with non-zero components of the response kernel given by,

$$\begin{aligned}&\mathcal{K}^{11}{}_1{}^1(x-x') = \mathcal{K}^{22}{}_2{}^2(x-x') = \mathcal{K}^{33}{}_3{}^3(x-x') = \bar{\mathcal{K}}_1(x-x')\\ &\mathcal{K}^{00}{}_0{}^0(x-x') = \bar{\mathcal{K}}_0(x-x'), \mathcal{K}^{01}{}_0{}^1(x-x') = \bar{\mathcal{K}}(x-x')\end{aligned} \tag{4}$$

Such a model of response kernel is suitable for perfect fluid perturbations keeping the response kernel components same for $\delta T^1{}_1, \delta T^2{}_2, \delta T^3{}_3$ due to pressure perturbations while $\delta T^0{}_0$ and $\delta T^0{}_1$ for energy density and velocity variables as different, in general, from that of components for pressure variables.

In the following section, we will solve the sourced Einstein's linearized equation (2) for perturbations (radial and polar) induced in the spherically symmetric geometry of a relativistic star, composed of cold dense fluid.

III. SOLUTIONS FOR RADIAL AND NON-RADIAL POLAR PERTURBATIONS

A spherically symmetric relativistic star has line element,

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5}$$

The cold dense matter being modelled as perfect fluid with

$$T^{ab}(x) = u^a u^b(\epsilon + p) + g^{ab} p \tag{6}$$

where $u^a u_a = -1$ and $u^a = e^{-\nu}\{1, 0, 0, 0\}$ for the static equilibirum case. At a later time a radial velocity v can be introduced in the fluid, $v = e^{\lambda-\nu}\dot{r}$ such that four-velocity has components, $u^a \equiv \frac{1}{\sqrt{1-v^2}}(e^{-\nu}, e^{-\lambda}v, 0, 0)$.

Accordingly, the components of field equations are given as

$$\begin{aligned}
G^t_t = 8\pi T^t_t \quad &: \\
e^{-2\lambda}(\frac{1}{r^2} - \frac{2}{r}\lambda') \; - \; \frac{1}{r^2} &= -8\pi\epsilon\frac{1}{1-v^2} && (7) \\
G^r_r = 8\pi T^r_r \quad &: \\
e^{-2\lambda}(\frac{1}{r^2} + \frac{2}{r}\nu') \; - \; \frac{1}{r^2} &= 8\pi(\epsilon\frac{v^2}{1-v^2} + p) && (8) \\
G^t_r = 8\pi T^t_r \quad &: \\
-\frac{2}{r}e^{-2\nu}\dot{\lambda} \; &= \; 8\pi e^{\lambda-\nu}(\epsilon + p)\frac{v}{1-v^2} && (9) \\
e^{2\lambda}G^\theta_\theta = 8\pi e^{2\lambda}T^\theta_\theta \quad &: \\
\nu'' + \nu'^2 \; - \; \nu'\lambda' + \frac{1}{r}(\nu' - \lambda') &= 8\pi e^{2\lambda}p && (10)
\end{aligned}$$

Also we can easily obtain the relation,

$$\nu' + \lambda' = 4\pi(\epsilon + p)e^{2\lambda}p \tag{11}$$

from the above Einstein's equations. Its easy to see , if we put $v = 0$, in the above, we get equations for the static configuration. The radial velocity has been introduced to get the correct form of perturbed field equations in a heuristic way , such that after perturbing the equations correctly, one can put $v = 0$ back in the unperturbed equations. The perturbations in the fluid can be introduced using the radial velocity such that,

$$\delta v = e^{\lambda-\nu}\dot{\xi} \tag{12}$$

where $\xi \equiv \xi_r$ is the only non-zero fluid displacement vector.

A. Radial perturbations induced by the oscillating sources

In this section we work out solutions for radial perturbations near the end stages of collapse of a relativistic star. Ideally during a core collapse, one gets a rotating configuration, which will need numerical solutions for perturbed field equations. Our purpose is to lay foundations for the study of perturbations which are induced due to internal sources present in the fluid at mesoscopic scales. The collapse mechanism itself can give rise to mesoscopic oscillations inside the core of dense matter, where the degeneracy pressure and energy density fluctuations act as seeds that can grow for different modes of oscillations. One such internal effect is the turbulence in the interior of the cold dense matter fluid, which has not yet been modelled so far from the first principles. For perturbations induced by such effects we restrict the present model in order to account for analytical expressions and hence consider a non-rotating model. The analytical form give us better insight to further implement numerical solutions for more involved and realistic cases . We expect to carry out this in future work. It is important to formulate the new theoretical base with simpler models first.

1. Models of source term in cold dense matter

The cold dense matter is modelled by a perfect fluid. In this article we consider two models, (i) deterministic oscillations $\delta_s T^{ab}(\vec{x})e^{i\omega t}$ of the stress tensor, and (ii) stochastic fluctuations of the stress tensor $\delta T^{ab}(\vec{x}, t)e^{i\omega t}$.

(*i*) Thus $\tau^{ab}(x)$ has the following non-zero components for the deterministic case.

$$\begin{aligned}
\tau^0{}_0(r,t) &= \delta_s\epsilon(r)e^{i\omega t} \\
\tau^1{}_0(r,t) &= (\epsilon(r)+p(r))\delta_s v(r)e^{i\omega t} \\
\tau^1{}_1(r,t) &= \delta_s p(r)e^{i\omega t} \\
\tau^2{}_2(r,t) &= \delta_s p(r)e^{i\omega t} \\
\tau^3{}_3(r,t) &= \delta_s p(r)e^{i\omega t}
\end{aligned} \tag{13}$$

Here 's' in δ_s denotes source, and the radial amplitudes oscillate with frequency ω. Such oscillations can be taken due to the collapse mechanism itself which leaves the fluid oscillating in the interiors. Due to internal dynamics of the configuration then, these can grow and affect the linear perturbations in many ways. As we have mentioned earlier such end of collapse states are dynamical and rotating, it leaves substantial scope for dynamical effects to take over at mesoscopic scales and affect the perturbations of the gravitating system. We touch upon with our formulations the coarse grained effects.

(ii) The stochastic fluctuations of $\tau^{ab}(x)$ have the following non-zero components,

$$\begin{aligned}
\tau^0{}_0(r,t) &= \delta_s\epsilon(r,t)e^{i\omega t} \\
\tau^1{}_0(r,t) &= (\epsilon(r)+p(r))\delta_s v(r,t)e^{i\omega t} \\
\tau^1{}_1(r,t) &= \delta_s p(r,t)e^{i\omega t} \\
\tau^2{}_2(r,t) &= \delta_s p(r,t)e^{i\omega t} \\
\tau^3{}_3(r,t) &= \delta_s p(r,t)e^{i\omega t}
\end{aligned} \tag{14}$$

Here $\delta_s v, \delta_s p, \delta_s\epsilon$ are defined stochastically and hence have meaning only as statistical averages given by $\langle\delta_s v\rangle, \langle\delta_s\epsilon\rangle, \langle\delta_s p\rangle$. The expectation is taken with respect to both space and time variables. However we see that the amplitudes of oscillations in the stochastic model are functions of the temporal variable also, though we have emphasized more on the generalized stochastic models in our earlier work [15].

In general, the sources are defined such that , for example, $\delta_s p(r,t) = p(r,t) - \langle p(r,t)\rangle$, where $\langle p(r,t)\rangle$ is the expectation or average of the pressure in the background unperturbed configuration. For a Langevin noise $\langle\tau^{ab}\rangle = 0$ hence one needs to obtain the rms value or two point correlations for a meaningful analysis.

2. The radial perturbations

There should be no confusion between the sources and the perturbations in the system. While sources are the fluctuations in the background and denoted by δ_s, perturbations are defined as shift in the trajectories and the metric given by h_{ab} and ξ^a.

For near equilibrium configuration, consider $\delta v_r(r,t), \delta p(r,t), \delta\epsilon(r,t)$ as radial perturbations in the fluid where $\xi \equiv \xi_r$ is the only non-zero component of the Lagrangian displacement vector, then,

$$\delta p[\xi] = -\Gamma_1 p\frac{e^{-\lambda}}{r^2}[e^{\lambda}r^2\xi]' - \xi p' \tag{15}$$

$$\delta\epsilon[\xi] = -(p+\epsilon)\frac{e^{-\lambda}}{r^2}[e^{\lambda}r^2\xi]' - \xi\epsilon' \tag{16}$$

$$\delta u^r[\xi] = \mathcal{L}_u\xi^r = e^{-\nu}\dot{\xi} \tag{17}$$

The remaining part $\delta p[h] = -\Gamma_1 p\delta\lambda$ and $\delta\epsilon[h] = -(p+\epsilon)\delta\lambda$, while $\delta u^r[h] = 0$. We assume for a near-equilibrium spherically symmetric configuration, the non-zero components (2) to read,

$$\delta G^0{}_0[h;x] - 8\pi\delta T^0{}_0[h;x] - 8\pi\int\bar{\mathcal{K}}_0(x-x')\delta T^0{}_0[\xi,x')dx' = \tau^0{}_0[g;x) \tag{18}$$

$$\delta G^0{}_1[h;x) - 8\pi\delta T^0_1[h;x) - 8\pi\int\bar{\mathcal{K}}(x-x')\delta T^0{}_1[\xi,x')dx' = \tau^0{}_1[g;x) \tag{19}$$

$$\delta G^1{}_1[h;x) - 8\pi\delta T^1{}_1[h;x) - 8\pi\int\bar{\mathcal{K}}_1(x-x')\delta T^1{}_1[\xi,x')dx' = \tau^1{}_1[g;x) \tag{20}$$

$$\delta G^2{}_2[h;x) - 8\pi\delta T^2{}_2[h;x) - 8\pi\int\bar{\mathcal{K}}_1(x-x')\delta T^2{}_2[\xi,x')dx' = \tau^2{}_2[g;x) \tag{21}$$

$$\delta G^3{}_3[h;x) - 8\pi\delta T^3{}_3[h;x) - 8\pi\int\bar{\mathcal{K}}_1(x-x')\delta T^3{}_3[\xi,x')dx' = \tau^3{}_3[g;x) \tag{22}$$

where we have used the forms of response kernel tensor given in equation (4).

3. *Solutions for deterministic induced perturbations in cold dense matter*

To obtain solutions from equations (18), (19), (20) we assume the form of $\xi(r,t) \equiv \xi(r)e^{\gamma_r t}$. Then, from equation (19) and the source term with $\delta_s v(r,t) = \delta_s v(r)e^{i\omega t}$ along with the response kernel from (3) and (4), we have,

$$\delta\lambda(r,t) = (\nu' + \lambda')\bar{\mathcal{K}}(\gamma_r)\xi(r,t) + ie^{(\nu-\lambda)}(\nu'+\lambda')\frac{\delta_s v(r)e^{i\omega t}}{\omega} \tag{23}$$

where Laplace transform for the response kernel is given by $\bar{\mathcal{K}}(\gamma_r) = \int \mathcal{K}_0(\tau)e^{-\gamma_r\tau}d\tau$, $\tau = t - t'$ has been used.
Substituting this in equation (18),

$$g(r)\xi'(r,t) + f(r)\xi(r,t) = -j(r)i\delta_s v(r)\frac{e^{i\omega t}}{\omega} - 8\pi\delta_s\epsilon(r,t) \tag{24}$$

where

$$g(r) = -\frac{(\nu'+\lambda')}{r}[2\bar{\mathcal{K}}(\gamma_r) - \bar{\mathcal{K}}_0(\gamma_r)] \tag{25}$$

$$f(r) = -[\frac{2}{r}\{(\nu'+\lambda')\bar{\mathcal{K}}(\gamma_r)\}' - (\frac{3\lambda'}{r} - \frac{2}{r^2} - \frac{\nu'}{r})(\nu'+\lambda')\bar{\mathcal{K}}(\gamma_r) - \frac{(\nu'+\lambda')}{2}(\lambda' + \frac{2}{r}) + 8\pi\epsilon' e^{2\lambda}] \tag{26}$$

$$j(r) = (\frac{3\lambda'}{r} - \frac{2}{r^2} - \frac{\nu'}{r})e^{\nu-\lambda}(\nu'+\lambda') \tag{27}$$

Solution of equation (24) gives,

$$\xi(r,t) = e^{\int\frac{f(r')}{g(r')}dr'}\int e^{-\int\frac{f(r'')}{g(r'')}dr''}(j(r')i\frac{\delta_s v(r')e^{i\omega t}}{\omega} - 8\pi\delta_s\epsilon(r',t))dr' \tag{28}$$

thus from (23) we get

$$\begin{aligned}\delta\lambda(r,t) = {} & (\nu\ +\lambda')\bar{\mathcal{K}}(\gamma_r)e^{\int\frac{f(r')}{g(r')}dr'}\int e^{-\int\frac{f(r'')}{g(r'')}dr''}[\{j(r') - e^{\nu(r')-\lambda(r')}(\nu'(r)+\lambda'(r))\delta(r-r')\}\\ & (i\frac{\delta_s v(r')e^{i\omega t}}{\omega}) - 8\pi\delta\epsilon_s(r',t)]dr'\end{aligned} \tag{29}$$

From equation (20),

$$\begin{aligned}\delta\nu(r,t) = {} & \int\int\{[\mathcal{L}_1(r'')g_1(r') + \mathcal{L}_2(r'')g_2(r') + \mathcal{L}_3(r')g_3(r')](i\frac{\delta_s v(r')e^{i\omega t}}{\omega})\\ & -8\pi\bar{\mathcal{K}}(\gamma_r)e^{m_1(r')}[(\mathcal{L}_4(r'') + \mathcal{L}_6(r'') + \mathcal{L}_5(r'')(1+\delta(r-r')]\delta_s\epsilon(r',t)\}dr'dr''\\ & +8\pi\int\delta_s p(r',t)dr'\end{aligned} \tag{30}$$

where

$$
\begin{aligned}
m_1(r) &= \int f(r')/g(r')dr' \\
\mathcal{L}_1(r) &= \{2e^{-2\lambda}(\frac{1}{r^2}+\frac{2}{r}\nu') - 8\pi\Gamma_1 p\}\{(\nu'+\lambda')\bar{K}(\gamma_r)e^{-m_1(r)}\} \\
g_1(r) &= (j(r) - e^{\nu(r)-\lambda(r)}(\nu'(r')-\lambda'(r'))\delta(r-r'))e^{m_1(r)} \\
\mathcal{L}_2(r) &= 8\pi\bar{\mathcal{K}}_1(\gamma_r)\Gamma_1 p m_1(r)e^{-m_1(r)} \\
g_2(r) &= e^{m_1(r)}(1+\delta(r-r'))j(r') \\
\mathcal{L}_3(r) &= \bar{\mathcal{K}}(\gamma_r)(\frac{\gamma_1 p e^{-\lambda}}{r^2}(e^{\lambda}r^2)' + p')e^{-m_1(r)} \\
g_3(r) &= e^{m_1(r)}j(r) \\
\mathcal{L}_4(r) &= (\nu'+\lambda')\{2e^{-2\lambda}(\frac{1}{r^2}+\frac{2}{r}\nu') - 8\pi\Gamma_1 p\}e^{-m_1(r)} \\
\mathcal{L}_5(r) &= 8\pi\bar{\mathcal{K}}(\gamma_r)m_1'(r)e^{-m_1(r)} \\
\mathcal{L}_6(r) &= (\Gamma_1 p e^{-\lambda}(e^{\lambda}r^2)' + p')e^{-m_1(r)}
\end{aligned}
\tag{31}
$$

Equations (28), (29) , (30) are the main results for the deterministic perturbations. One can see that the rhs of these equations consists of the integrated effect of the sources $\delta_s v(r,t), \delta_s\epsilon(r,t), \delta_s p(r,t)$ over some radial depth. It is hence important to take care of the the causal effects, for the limits of integration in the expressions, while doing numerics and evaluating the terms explicitly. For example a relevant condition can be put as the radial depth $r - r' \leq c_s(t-t')$ where c_s is the velocity of sound in the interior of the fluid. Interestingly we see $\delta\lambda, \delta\nu, \xi_r$ have explicit imaginary parts and are complex in nature also for the amplitudes of the oscillations. we will do further analysis of these perturbations in the work to follow in future article where we will build up more on mesoscopic theory for the exotic matter. The expressions presented here will be used as building blocks.

B. Stochastic perturbations in cold dense matter

For modeling stochastic effects, we consider the noise model given by equation (14) in the earlier subsection. The stochastic perturbations are of the form $\xi(r,t) \equiv \tilde{\xi}(r,t)e^{\gamma_r t}$ where γ_r is complex in general and the amplitude $\tilde{\xi}(r,t)$ is a random function of t. For a generalized stochastic nature as defined in [15](randomness w.r.t spatial as well as temporal coordinates for a spacetime structure), we also assume the randomness w.r.t 'r' . Here we consider the stochastic noise model (ii) for τ^{ab}.

Then from (19) we obtain,

$$
\dot{\delta\lambda} + (\nu'+\lambda')\int \bar{\mathcal{K}}(t-t')\dot{\xi}(r,t')dt' = -e^{\lambda-\nu}(\lambda'+\nu')\delta_s v(r,t) \tag{32}
$$

The term with integral over the response kernel can be written as,

$$
\int \bar{\mathcal{K}}(\tau)\dot{\xi}(r,t-\tau)d\tau
$$

using Taylor expansion for $\dot{\xi}(r,t-\tau)$ we obtain,

$$
\delta\lambda = (\nu'+\lambda')(\tilde{\xi}(r,t)\bar{\mathcal{K}}(\gamma) - \dot{\tilde{\xi}}(r,t)\tilde{\mathcal{K}}(\gamma))e^{\gamma_r t} - e^{\lambda-\nu}(\lambda'+\nu')\int \delta_s v(r,t')dt' \tag{33}
$$

where $\bar{\mathcal{K}}(\gamma_r) = \int \bar{\mathcal{K}}(\tau)e^{-\gamma_r\tau}d\tau$ with $\tau = t-t'$, and $\tilde{\mathcal{K}}(\tau) = \tau\bar{\mathcal{K}}(\tau)$ such that $\tilde{\mathcal{K}}(\gamma_r) = \int \tilde{\mathcal{K}}(\tau)d\tau$. Similar relations hold for $\bar{\mathcal{K}}_0(\gamma_r), \bar{\mathcal{K}}_1(\gamma_r), \tilde{\mathcal{K}}_0(\gamma_r)$ and $\tilde{\mathcal{K}}_1(\gamma_r)$. From equation (18),one gets

$$
\begin{aligned}
&-2e^{-2\lambda}\frac{\delta\lambda}{r}(\frac{1}{r}\nu'-\lambda') - \frac{2}{r}\delta\lambda' e^{-2\lambda}8\pi\int\{\bar{\mathcal{K}}_0(t-t')\{(\epsilon+p)\xi'(r,t') \\
&+(\lambda'+\frac{2}{r}+\epsilon'(r))\xi(r,t')\}dt' = 8\pi\delta_s\epsilon(r,t)
\end{aligned}
\tag{34}
$$

From equation (20), the following can be obtained,

$$2e^{-2\lambda}[\frac{\delta\nu'}{r}+(\frac{1}{r^2}+\frac{2\nu'}{r})\delta\lambda]+8\pi\Gamma_1 p\delta\lambda-8\pi\int\bar{\mathcal{K}}_1(t-t')\Gamma_1 p[\xi'(r,t')+(\frac{2}{r}+\lambda'+\frac{\epsilon'}{\epsilon+p})\xi(r,t')]=\delta_s p(r,t) \quad (35)$$

We also use the covariant conservations of the stress tensor and perturb it

$$\delta\nabla_a T^a{}_b(x)=0 \quad (36)$$

The component $\delta\nabla_a T^a{}_0(x)=0$ gives,

$$-2(\epsilon+p)\delta\lambda+\xi[-(\epsilon+p)(\lambda'+\frac{2}{r})-(p'+\lambda'+\nu'+\frac{1}{2r})]=0 \quad (37)$$

From (33) and (37),

$$\tilde{\xi}(r,t)e^{\gamma_r t} = e^{(\gamma_r-X_1(r))t}\int e^{(X_1(r)-\gamma_r)t''}x_1(r)(\int\delta_s v(r,t')dt')dt'' \quad (38)$$

where

$$X_1(r)=\frac{1}{\tilde{\mathcal{K}}(\gamma_r)}[(\nu'+\lambda')\bar{\mathcal{K}}(\gamma_r)+\frac{1}{2}(\lambda'+\frac{2}{r})+\frac{1}{2}\frac{(p'+\lambda'+\nu'+1/(2r))}{(\epsilon+p)}] \quad (39)$$

and

$$x_1(r)=-\frac{e^{\lambda-\nu}}{\tilde{\mathcal{K}}(\gamma_r)}(\lambda'+\nu') \quad (40)$$

$$\begin{aligned}\delta\lambda &= (\nu'+\lambda')[(\bar{\mathcal{K}}(\gamma_r)-X_1(r))e^{(\gamma_r-X_1(r))t}\int e^{(X_1(r)-\gamma_r)t''}x_1(r)\int\delta_s v(r,t')dt'dt''] \\ &\quad -(\nu'+\lambda')(\tilde{\mathcal{K}}(\gamma_r)x_1(r)+e^{\lambda-\nu})(\int\delta_s v(r,t')dt') \end{aligned} \quad (41)$$

Substituting

$$\delta\nu(r,t) = \int\mathcal{W}(r',t,t'')\delta_s v(r',t')dt'dt''dr'-8\pi\int\delta_s p(r',t)dr' \quad (42)$$

where

$$\begin{aligned}\mathcal{W}(r',t,t'') &= S_1(r')[x_1(r')(1-X_1(r'))e^{(\gamma_r-X-1(r'))t+(X_1(r')-\gamma_r)t''}]+S_2(r')[\{x_1'(r')(1+X_1(r'))+ \\ &\quad x_1(r')(X_1(r')X_1'(r')(t-t'')-X_1'(r'))\}e^{(\gamma_r-X_1(r))+(X_1(r')-\gamma_r)t''}] \\ &\quad -S_3(r')[(X_1'(r')t+x_1'(r')-x_1(r')(1+X_1'(r')t''))e^{-X_1(r')t+(X_1(r')-\gamma_r)t''}]+ \\ &\quad S_4(r')[e^{-X_1(r')t+(X_1(r')-\gamma_r)t''}x_1(r')]+S_5(r')\end{aligned} \quad (43)$$

where

$$\begin{aligned} S_1(r) &= \frac{e^{2\lambda}}{2}r[-\tilde{\mathcal{K}}(\gamma_r)(\nu'+\lambda')\{8\pi\Gamma_1 p-2e^{-2\lambda}(\frac{1}{r^2}+\frac{2}{r}\nu')\}-8\pi(\Gamma_1 p(\lambda'+\frac{2}{r})+p')\tilde{\mathcal{K}}_1(\gamma_r)] \\ S_2(r) &= e^{2\lambda}4\pi rp\tilde{\mathcal{K}}_0(\gamma_r) \\ S_3(r) &= 4\pi\Gamma_1 pr\tilde{\mathcal{K}}_1 e^{2\lambda} \\ S_4(r) &= (\nu''+\lambda'')\{8\pi\Gamma_1 p-2e^{-2\lambda}(\frac{1}{r^2}+\frac{2}{r}\nu')\}\tilde{\mathcal{K}}(r)\frac{r}{2}e^{2\lambda}+4\pi re^{-2\lambda}(\Gamma_1 p(\lambda'+\frac{2}{r})+p')\tilde{\mathcal{K}}_1(r) \\ S_5(r) &= e^{3\lambda-\nu}r(\nu'+\lambda')\{4\pi\Gamma_1 p-e^{-2\lambda}(\frac{1}{r^2}+\frac{2}{r}\nu')\}\end{aligned} \quad (44)$$

In the expressions obtained above $\delta_s v(r,t), \delta_s \epsilon(r,t)$ and $\delta_s p(r,t)$ are of the form $\delta_s s(r,t) \equiv \delta_s \tilde{s}(r,t) e^{i\omega t}$ where the amplitude $\tilde{s}(r,t)$ is stochastic in nature and with oscillatory part $e^{i\omega t}$. Thus the above equations are stochastic in nature, this only the expectation value is meaningful. We assume Langevin type noise, hence $\langle s(r,t)\rangle = 0$ while the two point correlations are given by $\langle s(r_1,t_1)s(r_2,t_2)\rangle$. Thus, it can be easily seen that $\langle \xi(r,t)\rangle = 0, \langle \delta\lambda(r,t)\rangle = 0, \langle \delta\nu(r,t)\rangle = 0$, while the two point correlations take the form

$$\begin{aligned}\langle \xi^*(r_1,t_1)\xi(r_2,t_2)\rangle &= e^{(\gamma^*_{r_1} - X^*_1(r_1))t_1 + (\gamma_{r_2} - X_1(r_2))t_2} t_1 t_2 \int e^{(X^*_1(r_1) - \gamma^*_{r_1})t''_1 + (X_1(r_2) - \gamma_{r_2})t''_2} x^*_1(r_1) x_1(r_2) \\ &\quad \langle \delta_s v^*(r_1,t'_1)\delta_s v(r_2,t'_2)\rangle dt'_1 dt'_2 dt''_1 dt''_2 \end{aligned} \tag{45}$$

$$\begin{aligned}\langle \delta\lambda^*(r_1,t_1)\delta\lambda(r_2,t_2)\rangle &= (\nu'(r_1) + \lambda'(r_1))(\nu'(r_2) + \lambda'(r_2))[\bar{\mathcal{K}}^*(\gamma_{r_1}) - X^*_1(r_1))(\bar{\mathcal{K}}(\gamma_{r_2}) - X_1(r_2)) \\ &\quad e^{(\gamma^*_{r_1} - X^*(r_1))t_1 + (\gamma_{r_2} - X_1(r_2))t_2} \int e^{X^*_1(r_1) - \gamma^*_{r_1})t''_1 + (X_1(r_2) - \gamma_{r_2})t''_2} x^*_1(r_1) x_1(r_2) \\ &\quad \langle \delta_s v^*(r_1,t'_1)\delta_s v(r_2,t'_2)\rangle dt'_1 dt'_2 dt''_1 dt''_2 - (\nu'(r_1) + \lambda'(r_1))(\nu'(r_2) + \lambda'(r_2))(\tilde{\mathcal{K}}^*(\gamma_{r_1}) x^*_1(r_1) \\ &\quad + e^{\lambda(r_1) - \nu(r_1)})(\tilde{\mathcal{K}}(\gamma_{r_2}) x_1(r_2) + e^{\lambda(r_2) - \nu(r_2)}) \int \langle \delta_s v^*(r_1,t'_1)\delta_s v(r_2,t'_2)\rangle dt'_1 dt'_2 \end{aligned} \tag{46}$$

$$\begin{aligned}\langle \delta\nu^*(r_1,t_1)\delta\nu(r_2,t_2)\rangle &= \int \mathcal{W}^*(r'_1,t_1,t''_1)\mathcal{W}(r'_2,t'_2,t''_2)\langle \delta_s v^*(r'_1,t'_1)\delta_s v(r'_2,t'_2)\rangle dt'_1 dt''_2 dr'_1 + \\ &\quad 16\pi^2 \int \langle \delta_s p^*(r'_1,t_1)\delta_s p(r'_2,t_2)\rangle dr'_1 dr'_2 \end{aligned} \tag{47}$$

Similar relation for $\langle \delta\nu^*(r_1,t_1)\delta\lambda(r_1,t_1)\rangle$ etc can also be written. The point separated correlations in the co-incident limit can be used to find out root mean square of the perturbations that give an estimate of their magnitudes. However our main interest here lies in the the two point or point separated form of the expressions which will act as the building block to study extended properties of the compact configuration. The two points can be separated for a large distance (keeping causality condition in mind) such that correlations probe extended structures inside the dense fluid matter giving access to non- local properties and phenomena. Further discussion of results is given in the concluding section.

In the next section we give results for polar perturbations in the non-rotating configuration,

C. Polar Perturbations

In this section we solve the sourced Einstein's equations for polar perturbations. The induced perturbations may be deterministic or stochastic depending on the source . Similar to the case of radial perturbations we can have these two possibilities here. We work out the deterministic induced perturbations in this article for the polar perturbations, stochastic polar perturbations can be worked out in an analogous way. For a non-rotating geometry we proceed as follows.

1. *Modelling source term for induced polar perturbations*

We will present the polar perturbatons in Regge-Wheeler gauge. The source term τ^{ab} can be shown to have polar decomposition of τ_{ab} as a pull back on the 2-sphere which takes the form

$$\tau_{ij} = e_{ij}\tau^{scalar}_{lm} Y_{lm} + \tau^{polar}_{lm} \nabla_i \nabla_j Y_{lm} \tag{48}$$

where e_{ij} is the metric restricted to the 2-sphere. The indices i,j are reserved for θ,ϕ, while a,b run for $\{t,r,\theta,\phi\}$. In the Regge-Wheeler gauge $\tau^{polar}_{lm} = 0$ corresponding to the decomposition of the metric (see appendix). The projection for $t-t$ and $r-r$ and $t-r$ components on the 2-sphere can be given by,

$$\begin{aligned}&\tau^t{}_{t\,lm} Y_{lm} = \delta_s \epsilon_{lm}(r,t) Y_{lm}(\theta,\phi); \tau^r{}_{r\,lm} Y_{lm} = \delta_s p_{lm}(r,t) Y_{lm}(\theta,\phi); \\ &\tau^t{}_{r\,lm} = e^{\nu-\lambda}(\epsilon + p)\delta_s v_{lm}(r,t) Y_{lm}\end{aligned} \tag{49}$$

The θ, ϕ components have the form,

$$\tau^{\theta}{}_{\theta\, lm}(r,t)Y_{lm}(\theta,\phi) = \delta_s p_{lm}(r,t)Y_{lm}(\theta,\phi)$$
$$\tau^{\phi}{}_{\phi\, lm}(r,t)Y_{lm}(\theta,\phi) = \delta_s p_{lm}(r,t)Y_{lm}(\theta,\phi) \quad (50)$$

As we see further, these form the sources for induced perturbations . Evaluating in this article for $m = 0$ case, the non-zero components of the source are given as,

$$\tau^{t}{}_{t} = \delta_s \epsilon_l(r,t)P_l(\cos\theta); \tau^{r}{}_{r} = \delta_s p_l(r,t)P_l(\cos\theta); \tau^{t}{}_{r} = e^{\nu-\lambda}(\epsilon+p)\delta_s v_l(r,t)P_l(\cos\theta) \quad (51)$$
$$\tau^{\theta}{}_{\theta} = \delta_s p_l(r,t)P_l(\cos\theta)$$
$$\tau^{\phi}{}_{\phi} = \delta_s p_l(r,t)P_l(\cos\theta) \quad (52)$$

We consider in this subsection $\delta_s p, \delta_s \epsilon$ and $\delta_s v$ of the form $\delta_s s(r)e^{i\omega t}$, which are deterministic sources. Similar results can be obtained for stochastic noise. We have worked out similar results for the Einstein-Langevin equation in [17] as a stochastic case, and with a different response kernel. Our results in the article are based on the Einstein's sourced equation with the specific and more resonable response kernel, and are presented as deterministic oscillations. These results will have significance in exploring mesoscopic scale effects in the interiors of dense matter for various new aspects in rheological modelling of the exotic fluids.

2. The sourced Einstein's equations for polar perturbations

The sourced Einstein's equation for the polar perturbations, borrows the standard notation for the perturbed quantities in the field equation, with the new response kernel and source term incorporated according to the proposed equation (2).The response kernel is of the same form as in subsection (III A 3), and we also assume the perturbations below, H_0, H_1, K, W and V are of the form $\mathcal{S}(r)e^{\gamma_r t}$.

We have used the standard form (as given in appendix to denote the perturbations) of metric and fluid perturbations. Writing, the non-zero components of the sourced field equations as follows,

The $t-t$ component is given by,

$$\delta G^{t}{}_{t}[h,x) - 8\pi\delta T^{t}{}_{t}[h,x) - 8\pi\int \mathcal{K}_0(x-x')\delta T^{t}{}_{t}[\xi,x')d^4x' = \tau^{t}{}_{t}[g,x) \quad (53)$$

which with the polar decomposition reads,

$$[\{-e^{-2\lambda}\partial_r^2 - e^{-2\lambda}(\frac{3}{r}-\lambda')\partial_r + [\frac{1}{2r^2}(l-1)(l+2)+8\pi(\epsilon+p)]\}K + \{\frac{e^{-2\lambda}}{r}\partial_r - e^{-2\lambda}$$
$$(\frac{1}{r^2}-\frac{2}{r}\lambda'+\frac{e^{2\lambda}}{2r^2}l(l+1)) + 4\pi(\epsilon+p)\}H_0 + \tilde{\mathcal{K}}_0(\gamma)8\pi(\frac{e^{-\lambda}}{r^2}((\epsilon+p)\partial_r+\epsilon')W + (\epsilon+p)\frac{l(l+1)}{r^2}V)]P_l(\cos\theta) =$$
$$-\delta_s\epsilon_l(r)e^{i\omega t}P_{\tilde{l}}(\cos\theta) \quad (54)$$

The $t-r$ component reads

$$\delta G^{t}{}_{r}[h,x) - 8\pi\delta T^{t}{}_{r}[h;\xi,x) = \tau^{t}{}_{r}(x) \quad (55)$$

given by

$$[e^{-\lambda+\nu}(\partial_r+2\nu')H_0 - e^{-\lambda+\nu}\partial_r K + \mathcal{K}(\gamma)(\epsilon+p)\frac{e^{-2\nu+\lambda}}{r^2}\partial_t W]P_l(\cos\theta) = 8\pi e^{(\nu-\lambda)}(\epsilon+p)\delta_s v_l(r,t)P_{\tilde{l}}(\cos\theta) \quad (56)$$

The $r-r$ component is given by,

$$\delta G^{r}{}_{r}[h,x) - 8\pi\delta T^{r}{}_{r}[h,x) - 8\pi\int \mathcal{K}_1(x-x')\delta T^{r}{}_{r}[\xi,x')d^4x' = \tau^{r}{}_{r}[g,x) \quad (57)$$

with its projection on the two sphere reads,

$$[\{e^{-2\nu}\partial_t^2 + e^{-2\lambda}(\frac{1}{r}-\nu')\partial_r - 8\pi\Gamma_1 p + \frac{1}{r}(l-1)(l-2)\}K - \{\frac{e^{-2\lambda}}{r} + \frac{1}{r^2}(e^{-2\lambda}-1) + \frac{1}{r}(l-1)(l+2) +$$
$$4\pi\Gamma_1 p\}H_0 - 8\pi\tilde{\mathcal{K}}_1(\gamma)[\Gamma_1 p\frac{e^{-\lambda}}{r^2}\partial_r + \frac{p'}{r^2}e^{-\lambda}]W - 8\pi\tilde{\mathcal{K}}(\gamma)\Gamma_1 p\frac{l(l+1)}{r^2}V]P_l(\cos\theta) = \delta_s p_{\tilde{l}}(r)e^{i\omega t}P_{\tilde{l}}(\cos\theta). \quad (58)$$

The $\theta-\theta$ component is given by,

$$\delta G^{\theta}{}_{\theta}[h,x) - 8\pi\delta T^{\theta}{}_{\theta}[h,x) - 8\pi\int \mathcal{K}_1(x-x')\delta T^{\theta}{}_{\theta}[\xi,x')d^4x' = \tau^{\theta}{}_{\theta}[g,x) \tag{59}$$

which reads,

$$\begin{aligned}
&[[e^{-2\nu}\partial_t^2 - \frac{e^{-2\lambda}}{2}\partial_r^2 - \frac{1}{2}e^{-2\lambda}(\frac{2}{r}\nu' - 2\lambda' + \frac{1}{r} - 2e^{2\lambda}\partial_r)\partial_r - 8\pi\Gamma_1 p]K \\
&[-e^{-2\lambda}(\partial_r - \frac{1}{r} - \lambda')(\partial_r + 2\nu') + \frac{1}{2}e^{-2\nu}\partial_t^2 + \frac{1}{2}e^{-2\lambda}\partial_r^2 + \frac{e^{-2\lambda}}{2}(\frac{2}{r} + \\
&3\nu' - \lambda')\partial_r - 8\pi p - 4\pi\Gamma_1 p]H_0 + 8\pi\mathcal{K}_1(\gamma)[\Gamma_1 p\frac{e^{-\lambda}}{r^2}\partial_r + \frac{p'}{r^2}e^{-\lambda}]W + \\
&8\pi\mathcal{K}_1(\gamma)\Gamma_1 p\frac{l(l+1)}{r^2}V]P_l(\cos\theta) = \delta_s p_{\tilde{l}}(r)e^{i\omega t}P_{\tilde{l}}(\cos\theta)
\end{aligned} \tag{60}$$

From (56)

$$\begin{aligned}
W(r,t)P_l(\cos\theta) &= \frac{r^2\gamma e^{2\nu-\lambda}}{\tilde{\mathcal{K}}(\gamma)(\epsilon+p)}[\{-e^{-\lambda+\nu}(\partial_r + 2\nu')H_0 P_l(\cos\theta) + e^{-\lambda+\nu}\partial_r K P_l(\cos\theta)\} \\
&\quad + 8\pi e^{\nu-\lambda}(\epsilon+p)\delta_s v_{\tilde{l}}(r)e^{i\omega t}P_{\tilde{l}}\cos\tilde{\theta}]
\end{aligned} \tag{61}$$

Substituting this in equation (54),

$$\begin{aligned}
V(r,t)P_l(\cos\theta) &= \frac{r^2}{l(l+1)(\epsilon+p)}\{-[-e^{-2\lambda}\partial_r^2 - e^{-2\lambda}(\frac{3}{r} - \lambda')\partial_r + [\frac{1}{2r^2}(l-1)(l+2) + 8\pi(\epsilon+p)] + \\
&\quad \tilde{\mathcal{K}}_0(\gamma)8\pi(\frac{e^{-\lambda}}{r^2}((\epsilon+p)\partial_r + \epsilon'))\frac{r^2\gamma e^{2\nu-\lambda}}{\tilde{\mathcal{K}}(\gamma)(\epsilon+p)}e^{-\lambda+\nu}\partial_r]\}KP_l(\cos\theta) \\
&\quad + [\frac{e^{-2\lambda}}{r}\partial_r - e^{-2\lambda}(\frac{1}{r^2} - \frac{2}{r}\lambda' + \frac{e^{2\lambda}}{2r^2}l(l+1)) + 4\pi(\epsilon+p) + \\
&\quad \tilde{\mathcal{K}}_0(\gamma)8\pi(\frac{e^{-\lambda}}{r^2}((\epsilon+p)\partial_r + \epsilon'))\frac{r^2\gamma e^{2\nu-\lambda}}{\tilde{\mathcal{K}}(\gamma)(\epsilon+p)}(\partial_r + 2\nu')]H_0 P_l(\cos\theta) \\
&\quad + (8\pi)^2\tilde{\mathcal{K}}_0(\gamma)(\frac{e^{-\lambda}}{r^2}(\epsilon+p)\partial_r + \epsilon')r^2\gamma\frac{e^{3\nu-2\lambda}}{\tilde{\mathcal{K}}(\gamma)}(\delta_s v(r) - \delta_s\epsilon_{\tilde{l}})(r)e^{i\omega t}P_{\tilde{l}}(\cos\tilde{\theta})
\end{aligned} \tag{62}$$

Using the above in equation (58),

$$\begin{aligned}
&[\{e^{-2\nu}\partial_t^2 + e^{-2\lambda}(\frac{1}{r} - \nu')\partial_r - 8\pi\Gamma_1 p + \frac{1}{r}(l-1)(l-2)\} - 8\pi\tilde{\mathcal{K}}_1[\Gamma_1 p e^{-\lambda}r^2\partial_r + \\
&\frac{p'}{r^2}e^{-\lambda}]\frac{r^2\gamma e^{3\nu-2\lambda}}{\tilde{\mathcal{K}}(\gamma)(\epsilon+p)}\partial_r + 8\pi\tilde{\mathcal{K}}_0(\gamma)(\frac{e^{-\lambda}}{r^2}((\epsilon+p)\partial_r + \epsilon')\frac{r^2\gamma e^{3\nu-2\lambda}}{\tilde{\mathcal{K}}(\gamma)(\epsilon+p)}\partial_r]KP_l\cos\theta \\
&+[\{\frac{e^{-2\lambda}}{r} + \frac{1}{r^2}(e^{-2\lambda}-1) - \frac{1}{r}(l-1)(l+2) + 4\pi\Gamma_1 p - e^{-\lambda+\nu}(\partial_r + 2\nu') + \frac{e^{-2\lambda}}{r}\partial_r - e^{-2\lambda}(\frac{1}{r^2} - \frac{2}{r}\lambda' \\
&+\frac{e^{2\lambda}}{2r^2}l(l+1)) + 4\pi(\epsilon+p) + \tilde{\mathcal{K}}_0(\gamma)8\pi(\frac{e^{-\lambda}}{r^2}((\epsilon+p)\partial_r + \epsilon')\frac{r^2\gamma e^{2\nu-\lambda}(\partial_r + 2\nu')}{\tilde{\mathcal{K}}(\gamma)(\epsilon+p)}(\partial_r + 2\nu')]H_0 P_l\cos\theta \\
&= -(8\pi)^2\{\tilde{\mathcal{K}}_1(\gamma)e^{-\lambda}r^2(\Gamma_1 p\partial_r + p') + \tilde{\mathcal{K}}_0(\gamma)(\frac{e^{-\lambda}}{r^2}(\epsilon+p)\partial_r + \epsilon')\}\frac{r^2\gamma e^{3\nu-2\lambda}}{\tilde{\mathcal{K}}(\gamma)}\delta_s v_{\tilde{l}}(r)e^{i\omega t}P_{\tilde{l}}(\cos\tilde{\theta}) - \\
&(\delta_s\epsilon_{\tilde{l}}(r) + \delta_s p_{\tilde{l}}(r))e^{-i\omega t}P_{\tilde{l}}(\cos\tilde{\theta})
\end{aligned} \tag{63}$$

which can be written in compact form as,

$$\{X_l^1 K(r,t) + Y_l^1 H_0(r,t)\}P_l\cos\theta = \{(-F(r)\delta_s v_{\tilde{l}}(r) - \delta_s\epsilon_{\tilde{l}}(r) + \delta_s p_{\tilde{l}}(r))e^{-i\omega t}\}P_{\tilde{l}}(\cos\tilde{\theta}) \tag{64}$$

The coefficients $X_l^1, Y_l^1, F(r)$ can be matched with the terms in the above equations.

We also have from equation (59),

$$
\begin{aligned}
&\{e^{-2\nu}\partial_t^2 - e^{-2\lambda}2\partial_r^2 - \frac{1}{2}e^{-2\lambda}(\frac{2}{r}\nu' - 2\lambda' + \frac{1}{r} - 2e^{2\lambda}\partial_r)\partial_r - 8\pi\Gamma_1 p + 8\pi\tilde{\mathcal{K}_1}(\gamma)[\Gamma_1 p\frac{e^{-\lambda}}{r^2}\partial_r + \frac{p'}{r^2}e^{-2\lambda}]\frac{r^2\gamma e^{3\nu-2\lambda}}{\tilde{K}(\gamma)(\epsilon+p)}e^{-\lambda+\nu}\partial_r \\
&-8\pi\frac{\mathcal{K}_1(\gamma)}{(\epsilon+p)}[-e^{-2\lambda}\partial_r^2 - e^{-2\lambda}(\frac{3}{r} - \lambda')\partial_r + [\frac{1}{2r^2}(l-1)(l+2) + 8\pi(\epsilon+p)] + \tilde{\mathcal{K}}_0(\gamma)8\pi(\frac{e^{-\lambda}}{r^2}((\epsilon+p))\partial_r + \epsilon')\frac{r^2\gamma e^{3\nu-2\lambda}}{\tilde{K}(\gamma)(\epsilon+p)}\partial_r]\}K \\
&+\{[-e^{-2\lambda}(\partial_r - \frac{1}{r} - \lambda')(\partial_r + 2\nu') + \frac{1}{2}e^{-2\nu}\partial_t^2 + \frac{1}{2}e^{-2\lambda}\partial_r^2 + \frac{e^{-2\lambda}}{2}(\partial 2r + 3\nu' - \lambda')\partial_r - 8\pi p - 4\pi\Gamma_1 p] - \\
&8\pi\mathcal{K}_1(\gamma)[\Gamma_1 p\frac{e^{-\lambda}}{r^2}\partial_r + \frac{p'}{r^2}e^{-2\lambda}]r^2\gamma\frac{e^{3\nu-2\lambda}}{\tilde{K}(\gamma)(\epsilon+p)}\{e^{-\lambda+\nu}(\partial_r + 2\nu')\} + [e^{-2\lambda}r\partial_r - e^{-2\lambda}(\frac{1}{r^2} - \\
&\frac{2\lambda'}{r} + \frac{e^{2\lambda}}{2r^2}l(l+1)) + 4\pi(\epsilon+p) + \tilde{\mathcal{K}}_0(\gamma)8\pi(\frac{e^{-\lambda}}{r^2}((\epsilon+p)\partial_r + \epsilon')\frac{r^2\gamma e^{2\nu-\lambda}}{\tilde{K}(\gamma)(\epsilon+p)}(\partial_r + 2\nu')]\}H_0 = \\
&\{(\delta_s p_{\tilde{l}}(r)) + \delta_s\epsilon_{\tilde{l}}(r)\}e^{i\omega t}P_{\tilde{l}}(\cos\tilde{\theta}) - (8\pi)^2\{\tilde{\mathcal{K}}_1(\gamma)(\Gamma_1 p\frac{e^{-\lambda}}{r^2}\partial_r + \frac{p'}{r^2}e^{-2\lambda} + \tilde{\mathcal{K}}_0(\gamma)(\frac{e^{-\lambda}}{r^2}(\epsilon+p)\partial_r + \epsilon')\} \\
&\frac{r^2\gamma e^{3\nu-2\lambda}}{\tilde{\mathcal{K}}(\gamma)}\delta_s v_{\tilde{l}}(r)e^{i\omega t}P_{\tilde{l}}(\cos\tilde{\theta})
\end{aligned}
\tag{65}
$$

which is written in compact form as,

$$\{X_l^2 K(r,t) + Y_l^2 H_0(r,t)\}P_l\cos\theta = (\delta_s p_{\tilde{l}}(r) + \delta_s\epsilon_{\tilde{l}}(r) - F(r)\delta_s v_{\tilde{l}})e^{i\omega t}P_{\tilde{l}}\cos(\tilde{\theta}) \tag{66}$$

The coefficients $X_l^2, Y_l^2, F(r)$ can be matched with the terms in the above equation. From equations (63), (64) and (65), (66) one can evaluate H_0 and K while W and V are given by (61) and (62) respectively. This will need numerical methods to solve them for H_0 and K, we have presented the analytical form of the basic equations here. However from the expressions it is clear that these will be expressed in terms of some combination of $\delta_s\epsilon_{\tilde{l}}(r,t), \delta_s p_{\tilde{l}}(r,t), \delta_s v_{\tilde{l}}(r,t)$. Thus it is the same term which induces the polar perturbations with characteristic modes, that can be analysed in future work to understand the new scales in compact matter which are yet untouched. We expect new structural properties as well as new phenomena at these scales to emerge as a result of our speculations.

IV. CONCLUSION

In this article we have solved the inhomogeneous sourced Einstein's equations for a spherically symmetric non-rotating relativistic star with cold dense matter fluid in a near equilibrium configuration. We have given a first principles account of perturbations induced at mesoscopic scales in a relativistic star, due to internal sources in the cold dense compact matter. The radial perturbations are of significance to study stability properties, while the polar perturbations connect to interests in gravitational waves. The significance of the induced perturbations lies in realizing that they are different from the implicit perturbations that arise from homogenous solutions of the Einstein's equation. Here the inhomogeneous source term is of central importance to characterize the internal structure and mechanism at mesoscopic scales for the dense compact matter. Hence with our approach and formulations, we are trying to build up a new theory for sub-hydro mesoscopic scale physics for the exotic fluids that has not yet been touched upon. The mesoscopic scales show their typical behaviour characterized by the modes of oscillations both for deterministic and stochastic cases. Through the basic new mathematical developments that we do in this article, new features of the astrophysical system can be touched upon for further investigations. In future work we will move towards characterizing turbulence in the superfluid state of the exotic matter, with emphasis on the stochastically induced perturbations for qualitative as well as quantitative analysis. This will need further formulations to relate the realistic configurations that provide us with a new theoretical base for possible observations. It is expected that this will give access to dynamical effects and interior structure of the rotating and gravitationally radiating compact star configuration with more refined features than are currently under focus. The work done in this article is a refinement of the basic new principles in order to establish a sub-hydro meso-scale theory for exotic dense matter present . We expect that the extreme physical conditions existing inside compact stars show up distinct results in the sub-hydro meso scales which connect the microscales to macroscopic scales while retaining their own specific dynamical and structural properties. This also gives way for the intermediate scale physics in extended non-local strong gravity regions.

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[1] J.M.Lattimer and M.Prakash, ApJ 550, 426 (2001).
[2] J.L.Zdunik, M.Fortin and P.Haensel, A&A , 599 (2017).
[3] Jianing Li et. al ., Phys Rev D , 106,083021 (2022)
[4] Golam Mortuza Hossain, Susobhan Mandal, Phys. Rev D 104, 123005 (2021)
[5] T.Celora, et.al, Phys.Rev D , 103016 (2022).
[6] Tuhin Malik et.at., The Astrophys. J, 930:17 (15pp) (2022)
[7] Andreas Schmitt., Lect.Notes. Phys. 811, Springer, Berlin Heidelberg (2010).
[8] Schmitt.A, Shternin.P, Astro. Sp.ScoLib, 457 (2018)
[9] T.Celora et.al, Phys.Rev.D 104,084090 (2021).
[10] T.Celora et.al, MNRAS, 527(2), (2024).
[11] S.Seveso et.al, MNRAS, 455(4) (2016).
[12] P.M.Pizzochero et.al, Nature Astronomy 1 (0134) (2017)
[13] Nicki Mullins et al, Phys Rev D 108,076013 (2023).
[14] Seema Satin.,Gen.Rel.Grav. 55(37) (2023)
[15] Seema Satin., CQG, 40 (5), 055010(2023).
[16] Bei Lok Hu and Enric Verdaguer,Liv.Rev.Rel. 11 (3) (2008).
[17] Seema Satin, IJMPD 33(02), 2450007 (2024).

APPENDIX

Review of polar perturbations in spherically symmetric star

The polar perturbations are even-parity perturbations on the 2-sphere of the metric. The perturbed metric in Regge Wheeler gauge then is given by

$$h_{\alpha\beta} = \begin{bmatrix} e^{2\nu}H_0Y_{lm} & H_1Y_{lm} & 0 & 0 \\ Sym & e^{2\lambda}H_2Y_{lm} & 0 & 0 \\ Sym & Sym & r^2KY_{lm} & 0 \\ Sym & Sym & Sym & r^2K\sin^2\theta Y_{lm} \end{bmatrix}$$

Eventually the independent metric variables are commonly taken to be H_0, H_1, and K as it turns out that $H_2 = H_0$ using the Einstein's tensors $\delta G^\theta_\theta - \delta G^\phi_\phi = 0$.

$$0 \;=\; \delta G^\theta_\theta - \delta G^\phi_\phi = (H_2 - H_0)\frac{1}{2r^2}(\partial^2_\theta - \cot\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial^2_\phi)Y_{lm}(\theta,\phi) \tag{67}$$

or $H_0 = H_2$. Also from

$$\delta G_{r\theta} = 0 \tag{68}$$

$$e^{-2\nu}\partial_t H_1 = H_0' + 2\nu' H_0 - K' \tag{69}$$

The perturbed fluid four-velocity with Y_{lm} replaced by the condition $m = 0$.

$$\delta u^t = -e^{\nu}[1 - \frac{1}{2}H_0 P_l(\cos\theta)] \quad (70)$$

$$\delta u^r = -\frac{e^{-(\nu+\lambda}}{r^2} W_{,t} P_l(\cos\theta) \quad (71)$$

$$\delta u^\theta = e^{\nu}\frac{1}{r^2} V_{,t}\partial_\theta P_l(\cos\theta) \quad (72)$$

$$\delta u^\phi = 0 \quad (73)$$

The Lagrangian change in number density of baryons is Δn, then

$$\Delta n/n = \{-\frac{e^{-\lambda}}{r^2}W' - \frac{l(l+1)}{r^2}V + \frac{1}{2}H_2 + K\}P_l(\cos\theta). \quad (74)$$

The corresponding Eulerian changes in energy density and pressure are

$$\delta\epsilon = (\epsilon + p)(\Delta n/n) - \epsilon' e^{-\lambda} W P_l(\cos\theta) \quad (75)$$

$$\delta p = \Gamma p(\Delta n/n) - p'\frac{e^{-\lambda}}{r^2} W P_l(\cos\theta). \quad (76)$$

The Eulerian changes in the stress-energy tensor are

$$\delta T^t_t = -\delta\epsilon, \delta T^r_r = \delta T^\theta_\theta = \delta T^\phi_\phi = \delta p, \quad (77)$$

$$\delta T^r_t = (\epsilon + p)u_t\delta u^r, \delta T^t_r = (\epsilon + p)u_r\delta u^t, \quad (78)$$

$$\delta T^\theta_t = (\epsilon + p)u_t\delta u^\theta, \delta T^t_\theta = (\epsilon + p)u_\theta\delta u^t \quad (79)$$

Rest of the components of δT^α_β vanish.