# EXTREMAL PROBLEMS FOR INTERSECTING FAMILIES OF SUBSPACES WITH A MEASURE 

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#### Abstract

We introduce a measure for subspaces of a vector space over a $q$-element field, and propose some extremal problems for intersecting families. These are $q$-analogue of Erdős-Ko-Rado type problems, and we answer some of the basic questions.


## 1. Introduction

The purpose of the paper is to introduce a measure on the set of subspaces of a vector space over a finite field and propose some extremal problems for intersecting families of subspaces. We then show some Erdős-Ko-Rado type results for vector spaces with this measure.

To motivate our problems, we begin with the Erdős-Ko-Rado theorem and its measure version. Let $n$ and $k$ be positive integers with $n \geqslant k$. Let $X_{n}:=\{1,2, \ldots, n\}$, and let $2^{X_{n}}$ and $\binom{X_{n}}{k}$ denote the power set of $X_{n}$ and the set of $k$-element subsets of $X_{n}$, respectively. A family $U \subset 2^{X_{n}}$ of subsets is called intersecting if $x \cap y \neq \emptyset$ for all $x, y \in U$.
Theorem A ([3]). Let $\frac{k}{n} \leqslant \frac{1}{2}$. If a family $U \subset\binom{X_{n}}{k}$ is intersecting, then

$$
|U| /\binom{n}{k} \leqslant \frac{k}{n} .
$$

Moreover, if $|U| /\binom{n}{k}=\frac{k}{n}$ and if $\frac{k}{n}<\frac{1}{2}$, then there exists $i \in X_{n}$ such that

$$
U=\left\{x \in\binom{X_{n}}{k}: i \in x\right\} .
$$

This result has a measure counterpart. Let $p$ be a real number with $0<p<1$. We define a $p$-biased measure $\tilde{\mu}_{p}: 2^{2^{X_{n}}} \rightarrow[0,1]$ by

$$
\tilde{\mu}_{p}(U):=\sum_{x \in U} p^{|x|}(1-p)^{n-|x|}
$$

for $U \subset 2^{X_{n}}$. This is a probability measure and

$$
\tilde{\mu}_{p}\left(2^{X_{n}}\right)=\sum_{k=0}^{n}\binom{n}{k} p^{k}(1-p)^{n-k}=(p+(1-p))^{n}=1
$$

by the binomial theorem.
Theorem B ([1). Let $p \leqslant \frac{1}{2}$. If a family $U \subset 2^{X_{n}}$ is intersecting, then

$$
\tilde{\mu}_{p}(U) \leqslant p
$$

|  | uniform version | measure version |
| :---: | :---: | :---: |
| subsets | Theorem $\overline{\mathrm{A}}$ | Theorem $\overline{\mathrm{B}}$ |
| subspaces | Theorem $\overline{\mathrm{C}}$ | $?$ |

Table 1.

Moreover, if $\tilde{\mu}_{p}(U)=p$ and if $p<\frac{1}{2}$, then there exists $i \in X_{n}$ such that

$$
U=\left\{x \in 2^{X_{n}}: i \in x\right\}
$$

Now, we switch to work on subspaces from subsets. Throughout the paper, we fix a prime power $q$. Let $\mathbb{F}_{q}$ be the $q$-element field, and let $\mathbb{F}_{q}^{n}$ denote the $n$-dimensional vector space over $\mathbb{F}_{q}$. Let $\Omega_{n}$ and $\Omega_{n}^{(k)}$ denote the set of all subspaces of $\mathbb{F}_{q}^{n}$ and the set of $k$-dimensional subspaces of $\mathbb{F}_{q}^{n}$, respectively. Define $[n]:=\frac{q^{n}-1}{q-1},[n]!:=\prod_{j=1}^{n}[j]$, and $\left[\begin{array}{l}n \\ k\end{array}\right]:=\frac{[n]!}{[k]![n-k]!}$. Then, $\left|\Omega_{n}^{(k)}\right|=\left[\begin{array}{l}n \\ k\end{array}\right]$. A family $U \subset \Omega_{n}$ of subspaces is called intersecting if $x \cap y \neq 0$ for all $x, y \in U$.
Theorem C ([6]). Let $\frac{k}{n} \leqslant \frac{1}{2}$. If a family $U \subset \Omega_{n}^{(k)}$ is intersecting, then

$$
|U| /\left[\begin{array}{l}
n \\
k
\end{array}\right] \leqslant \frac{[k]}{[n]}
$$

Moreover, if $|U| /\left[\begin{array}{l}n \\ k\end{array}\right]=\frac{[k]}{[n]}$ and if $\frac{k}{n}<\frac{1}{2}$, then there exists $y \in \Omega_{n}^{(1)}$ such that

$$
U=\left\{x \in \Omega_{n}^{(k)}: y \subset x\right\}
$$

So far, we have mentioned three Erdős-Ko-Rado type results. It seems natural to expect a result that is a $q$-analogue of Theorem B and at the same time, a measure version corresponding to Theorem C (see Table 1). To find such a result, we first need to introduce a measure on $\Omega_{n}$. Let $\sigma$ be a positive real number, and let

$$
\phi_{\sigma, n}(k):=\frac{\sigma^{k} q^{\binom{k}{2}}}{(-\sigma ; q)_{n}},
$$

where

$$
(-\sigma ; q)_{n}:=\prod_{j=0}^{n-1}\left(1+\sigma q^{j}\right)=\sum_{k=0}^{n}\left[\begin{array}{l}
n  \tag{1}\\
k
\end{array}\right] \sigma^{k} q^{\binom{k}{2}} .
$$

The second identity in (11) is known as the $q$-binomial theorem. (This is an identity as polynomials in $\sigma$ and hence is valid when, e.g., $\sigma<0$.) Then, define a $\sigma$-biased measure $\mu_{\sigma}: 2^{\Omega_{n}} \rightarrow[0,1]$ by

$$
\mu_{\sigma}(U):=\sum_{x \in U} \phi_{\sigma, n}(\operatorname{dim} x)
$$

for $U \subset \Omega_{n}$. This is a probability measure and

$$
\mu_{\sigma}\left(\Omega_{n}\right)=\sum_{k=0}^{n}\left[\begin{array}{l}
n \\
k
\end{array}\right] \phi_{\sigma, n}(k)=1
$$

We note that

$$
\lim _{q \rightarrow 1}\left[\begin{array}{l}
n \\
k
\end{array}\right] \phi_{\sigma, n}(k)=\binom{n}{k}\left(\frac{\sigma}{1+\sigma}\right)^{k}\left(1-\frac{\sigma}{1+\sigma}\right)^{n-k} .
$$

This suggests that the measure $\mu_{\sigma}$ is a $q$-analogue of the measure $\tilde{\mu}_{p}$, and the maximum $\mu_{\sigma^{-}}$ biased measure of an intersecting family of subspaces (in a possible result corresponding to Theorem (B) should be $\frac{\sigma}{1+\sigma}$, which plays the role of $p$ when $q \rightarrow 1$. Indeed, we have the following.

Theorem 1. Let

$$
\sigma \leqslant q^{-2\left\lfloor\frac{n-1}{2}\right\rfloor-1}= \begin{cases}q^{-n} & \text { if } n \text { is odd } \\ q^{-n+1} & \text { if } n \text { is even } .\end{cases}
$$

If a family $U \subset \Omega_{n}$ is intersecting, then

$$
\mu_{\sigma}(U) \leqslant \frac{\sigma}{1+\sigma}
$$

Moreover, if $\mu_{\sigma}(U)=\frac{\sigma}{1+\sigma}$ and if $\sigma<q^{-2\left\lfloor\frac{n-1}{2}\right\rfloor-1}$, then there exists $y \in \Omega_{n}^{(1)}$ such that

$$
U=\left\{x \in \Omega_{n}: y \subset x\right\}
$$

Write $a=q^{-2\left\lfloor\frac{n-1}{2}\right\rfloor-1}$ for brevity. Then, that $\sigma \leqslant q^{a}$ is equivalent to $\frac{\sigma}{1+\sigma} \leqslant \frac{q^{a}}{1+q^{a}}$, and we have $\frac{q^{a}}{1+q^{a}} \rightarrow \frac{1}{2}$ when $q \rightarrow 1$ (irrespective of the actual value of $a$ ). Our proof of Theorem 1 allows us to take the limit $q \rightarrow 1$, and we thus restore Theorem B. However, unlike the bound $p \leqslant \frac{1}{2}$ in Theorem B , the bound $\sigma \leqslant q^{a}$ in Theorem 1 is not best possible in general. Let $\frac{\sigma}{1+\sigma}=\frac{[p n]}{[n]}$, or equivalently, $\sigma=\frac{[p n]}{[n]-[p n]}$ for a fixed $p$ with $0<p<1$, where we extend the notation $[\lambda]=\frac{q^{\lambda}-1}{q-1}$ to any $\lambda \in \mathbb{R}$. Then, we have $\frac{\sigma}{1+\sigma}=\frac{q^{p n}-1}{q^{n}-1} \rightarrow p$ as $q \rightarrow 1$.
Conjecture 1. For every $0<p<\frac{1}{2}$, there exists $n_{0}$ such that the following holds for all $n>n_{0}$ and $\sigma=\frac{[p n]}{[n]-[p n]}:$ if a family $U \subset \Omega_{n}$ is intersecting, then

$$
\mu_{\sigma}(U) \leqslant \frac{\sigma}{1+\sigma}
$$

with equality if and only if there exists $y \in \Omega_{n}^{(1)}$ such that

$$
U=\left\{x \in \Omega_{n}: y \subset x\right\}
$$

Note that Conjecture 1 does not cover Theorem 1. Indeed, if $p$ is fixed and $n$ is sufficiently large, then

$$
\begin{equation*}
\sigma=\frac{[p n]}{[n]-[p n]} \sim q^{-(1-p) n} \tag{2}
\end{equation*}
$$

Thus, Conjecture 1 applies to the case roughly $q^{-n}<\sigma<q^{-\frac{n}{2}}$. We were unable to prove (or disprove) this conjecture. Instead, we present some weaker results supporting it under more general settings.

Let $t$ be a fixed positive integer. A family $U \subset \Omega_{n}$ of subspaces is called $t$-intersecting if $\operatorname{dim}(x \cap y) \geqslant t$ for all $x, y \in U$. Fix $y \in \Omega_{n}^{(t)}$, and define a $t$-intersecting family $A_{n}^{(t)}$ by

$$
\begin{equation*}
A_{n}^{(t)}:=\left\{x \in \Omega_{n}: y \subset x\right\} \tag{3}
\end{equation*}
$$

Note that $A_{n}^{(1)}$ is an optimal (1-)intersecting family in Theorem 1 and Conjecture 1, Using (11) (with $\sigma$ replaced by $\sigma q^{t}$ ) and $\sigma^{k} q^{\binom{k}{2}}=\sigma^{t} q^{\binom{t}{2}} \cdot\left(\sigma q^{t}\right)^{k-t} q^{\binom{k-t}{2}}$, we have

$$
\sum_{k=t}^{n}\left[\begin{array}{l}
n-t \\
k-t
\end{array}\right] \sigma^{k} q^{\binom{k}{2}}=\sigma^{t} q^{\binom{t}{2}}\left(-\sigma q^{t} ; q\right)_{n-t}
$$

and so

$$
\mu_{\sigma}\left(A_{n}^{(t)}\right)=\sum_{k=t}^{n}\left[\begin{array}{l}
n-t  \tag{4}\\
k-t
\end{array}\right] \phi_{\sigma, n}(k)=\frac{\sigma^{t} q^{\binom{t}{2}}}{(-\sigma ; q)_{t}}=\left(-\frac{1}{\sigma} ; \frac{1}{q}\right)_{t}^{-1}
$$

In particular, $\mu_{\sigma}\left(A_{n}^{(1)}\right)=\frac{\sigma}{1+\sigma}$. We are interested in the maximum $\sigma$-biased measure of $t$-intersecting families, and let

$$
f(n, t, \sigma):=\max \left\{\mu_{\sigma}(U)^{\frac{1}{n}}: U \subset \Omega_{n} \text { is } t \text {-intersecting }\right\}
$$

Problem 1. Find a condition for $\sigma$ that guarantees $f(n, t, \sigma)=\mu_{\sigma}\left(A_{n}^{(t)}\right)^{\frac{1}{n}}$.
Based on (2), we define

$$
\sigma_{\theta, n}:=q^{-(1-\theta) n}
$$

and write

$$
\mu_{\theta, n}:=\mu_{\sigma_{\theta, n}}
$$

Then, for $0<\theta<\frac{1}{2}$, we have

$$
\lim _{n \rightarrow \infty} \mu_{\theta, n}\left(A_{n}^{(t)}\right)^{\frac{1}{n}}=q^{-(1-\theta) t}
$$

(see Lemma 2 in Section 3), and this is the best we can do approximately as shown below.
Theorem 2. We have

$$
\lim _{n \rightarrow \infty} f\left(n, t, \sigma_{\theta, n}\right)= \begin{cases}q^{-(1-\theta) t} & \text { if } 0<\theta<\frac{1}{2} \\ 1 & \text { if } \frac{1}{2}<\theta<1\end{cases}
$$

Conjecture 2. We have

$$
\lim _{n \rightarrow \infty} f\left(n, t, \sigma_{\frac{1}{2}, n}\right)=q^{-\frac{1}{2} t}
$$

Two families $U, W \subset \Omega_{n}$ are called cross $t$-intersecting if $\operatorname{dim}(x \cap y) \geqslant t$ for all $x \in U$ and $y \in W$. Let

$$
g\left(n, t, \sigma_{1}, \sigma_{2}\right):=\max \left\{\left(\mu_{\sigma_{1}}(U) \mu_{\sigma_{2}}(W)\right)^{\frac{1}{n}}: U, W \subset \Omega_{n} \text { are cross } t \text {-intersecting }\right\}
$$

Theorem 3. If $0<\theta_{1}, \theta_{2}<\frac{1}{2}$, then $\lim _{n \rightarrow \infty} g\left(n, t, \sigma_{\theta_{1}, n}, \sigma_{\theta_{2}, n}\right)=q^{-\left(2-\theta_{1}-\theta_{2}\right) t}$.
If $U \subset \Omega_{n}$ is $t$-intersecting, then $U, U$ are cross $t$-intersecting. This gives us that

$$
f(n, t, \sigma)^{2} \leqslant g(n, t, \sigma, \sigma)
$$

Thus, Theorem 3 implies Theorem 2 for the case $0<\theta<\frac{1}{2}$.
In Section 2, we prove Theorem [1, To this end, we first translate the problem into a semidefinite programming problem and then solve it by computing eigenvalues of related matrices. In Section 3, we prove Theorem 2 by a probabilistic approach. For this, we use that the distribution $\left[\begin{array}{l}n \\ k\end{array}\right] \phi_{\sigma, n}(k)$ on the points $k=0,1, \ldots, n$ is concentrated around $\theta n$.

We also use the result (Theorem (D) about the maximum size of $t$-intersecting families of subspaces of dimension $k$ due to Frankl and Wilson [4]. In a similar way, we prove Theorem 3 in Section 4, where we need the result (Theorem (G) about cross $t$-intersecting uniform families due to Cao, Lu, Lv, and Wang [2]. We mention that Theorem G] partly generalizes the result (Theorem (F) about the case $t=1$ due to Suda and Tanaka [11], which was also proved by solving the corresponding semidefinite programming problem.

## 2. Proof of Theorem 1

For a non-empty finite set $\Lambda$, let $\mathbb{R}^{\Lambda}$ be the set of real column vectors with coordinates indexed by $\Lambda$. For two non-empty finite sets $\Lambda$ and $\Xi$, we also identify $\mathbb{R}^{\Lambda \times \Xi}$ with the set of real matrices with rows indexed by $\Lambda$ and columns indexed by $\Xi$. When $\Lambda \subset \Lambda^{\prime}$ and $\Xi \subset \Xi^{\prime}$, we often view $\mathbb{R}^{\Lambda}$ (resp. $\mathbb{R}^{\Lambda \times \Xi}$ ) as a subspace of $\mathbb{R}^{\Lambda^{\prime}}$ (resp. $\mathbb{R}^{\Lambda^{\prime} \times \Xi^{\prime}}$ ) in the obvious manner. Define $W_{k, \ell}, \bar{W}_{k, \ell} \in \mathbb{R}^{\Omega_{n}^{(k)} \times \Omega_{n}^{(\ell)}}$ by

$$
\left(W_{k, \ell}\right)_{x, y}=\left\{\begin{array}{ll}
1 & \text { if } x \subset y \text { or } x \supset y, \\
0 & \text { otherwise },
\end{array} \quad\left(\bar{W}_{k, \ell}\right)_{x, y}= \begin{cases}1 & \text { if } x \cap y=0 \\
0 & \text { otherwise }\end{cases}\right.
$$

for $x \in \Omega_{n}^{(k)}, y \in \Omega_{n}^{(\ell)}$.
We define the subspaces $\boldsymbol{U}_{i}\left(0 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor\right)$ of $\mathbb{R}^{\Omega_{n}}$ by

$$
\boldsymbol{U}_{i}=\left\{\boldsymbol{u} \in \mathbb{R}^{\Omega_{n}^{(i)}}: W_{i-1, i} \boldsymbol{u}=0\right\} \quad\left(0 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor\right)
$$

where $W_{-1,0}:=0$. Since

$$
W_{i-1, i} W_{i, i-1}=W_{i-1, i-2} W_{i-2, i-1}+q^{i-1}\left[\begin{array}{c}
n-2 i+2 \\
1
\end{array}\right] W_{i-1, i-1} \quad\left(1 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor\right)
$$

and the RHS is positive definite, it follows that the matrices $W_{i-1, i}\left(1 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor\right)$ have full rank $\left[\begin{array}{c}n \\ i-1\end{array}\right]$ and hence

$$
\operatorname{dim} \boldsymbol{U}_{i}=d_{i}:=\left[\begin{array}{c}
n \\
i
\end{array}\right]-\left[\begin{array}{c}
n \\
i-1
\end{array}\right] \quad\left(0 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor\right) .
$$

For $0 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor$, we fix an orthonormal basis $\boldsymbol{u}_{i, 1}, \boldsymbol{u}_{i, 2}, \ldots, \boldsymbol{u}_{i, d_{i}}$ of $\boldsymbol{U}_{i}$, and define

$$
\boldsymbol{u}_{i, r}^{k}=q^{-\frac{i(k-i)}{2}}\left[\begin{array}{c}
n-2 i  \tag{5}\\
k-i
\end{array}\right]^{-\frac{1}{2}} W_{k, i} \boldsymbol{u}_{i, r} \quad\left(1 \leqslant r \leqslant d_{i}, i \leqslant k \leqslant n-i\right) .
$$

In [11, it is shown that the $\boldsymbol{u}_{i, r}^{k}$ form an orthonormal basis of $\mathbb{R}^{\Omega_{n}}$, and that

$$
\begin{equation*}
\bar{W}_{k, \ell} \mathbf{u}_{i, r}^{\ell}=\theta_{i}^{k, \ell} \boldsymbol{u}_{i, r}^{k}, \tag{6}
\end{equation*}
$$

where $\boldsymbol{u}_{i, r}^{k}:=0$ if $k<i$ or $k>n-i$, and

$$
\begin{aligned}
\theta_{i}^{k, \ell} & =(-1)^{i} q^{\binom{i}{2}+k \ell-\frac{i(k+\ell)}{2}}\left[\begin{array}{c}
n-k-i \\
\ell-i
\end{array}\right]\left[\begin{array}{c}
n-2 i \\
k-i
\end{array}\right]^{\frac{1}{2}}\left[\begin{array}{c}
n-2 i \\
\ell-i
\end{array}\right]^{-\frac{1}{2}} \\
& =(-1)^{i} \frac{q^{i}\binom{i}{2}+k \ell-\frac{i(k+\ell)}{2}}{(q ; q)_{n-k-\ell}}\left(\frac{(q ; q)_{n-k-i}(q ; q)_{n-\ell-i}}{(q ; q)_{k-i}(q ; q)_{\ell-i}}\right)^{\frac{1}{2}}
\end{aligned}
$$

We note that

$$
\theta_{i}^{k, \ell}=\theta_{i}^{\ell, k} \quad(i \leqslant k, \ell \leqslant n-i)
$$

See also [4]. Thus, for $0 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor$ and $1 \leqslant r \leqslant d_{i}$, the subspace

$$
\boldsymbol{V}_{i, r}=\operatorname{span}\left\{\boldsymbol{u}_{i, r}^{i}, \boldsymbol{u}_{i, r}^{i+1}, \ldots, \boldsymbol{u}_{i, r}^{n-i}\right\}
$$

of $\mathbb{R}^{\Omega_{n}}$ is invariant under all the $\bar{W}_{k, \ell}$, and we have

$$
\mathbb{R}^{\Omega_{n}}=\bigoplus_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \bigoplus_{r=1}^{d_{i}} \boldsymbol{V}_{i, r} \quad \text { (orthogonal direct sum) }
$$

We have $d_{0}=1$, and without loss of generality, we set $\boldsymbol{u}_{0,1}$ to be the vector with 1 in coordinate $0 \in \Omega_{n}^{(0)}$, and 0 in all other coordinates.

Let $\Delta \in \mathbb{R}^{\Omega_{n} \times \Omega_{n}}$ be the diagonal matrix with diagonal entries

$$
\Delta_{x, x}=\mu_{\sigma}(x)=\phi_{\sigma, n}(\operatorname{dim} x) \quad\left(x \in \Omega_{n}\right),
$$

and let $J \in \mathbb{R}^{\Omega_{n} \times \Omega_{n}}$ be the matrix all of whose entries are 1 . Let $S \mathbb{R}^{\Omega_{n} \times \Omega_{n}}$ denote the set of symmetric matrices in $\mathbb{R}^{\Omega_{n} \times \Omega_{n}}$. Following [8, 10], we formulate the problem of maximizing the $\mu_{\sigma}$-biased measure of an intersecting family into a semidefinite programming problem as follows:

$$
\begin{array}{rll}
(\mathrm{P}): & \text { maximize } & \operatorname{tr}(\Delta J \Delta X) \\
\text { subject to } & \operatorname{tr}(\Delta X)=1, X \succcurlyeq 0, X \geqslant 0 \\
& X_{x, y}=0 \text { for } x, y \in \Omega_{n}, x \cap y=0,
\end{array}
$$

where $X \in S \mathbb{R}^{\Omega_{n} \times \Omega_{n}}$ is the variable, tr means trace, and $X \succcurlyeq 0$ (resp. $X \geqslant 0$ ) means that $X$ is positive semidefinite (resp. nonnegative). Indeed, if $\boldsymbol{x} \in \mathbb{R}^{\Omega_{n}}$ is the characteristic vector of a non-empty intersecting family $U \subset \Omega_{n}$, then the matrix

$$
\begin{equation*}
X:=\mu_{\sigma}(U)^{-1} \boldsymbol{x} \boldsymbol{x}^{\top} \in S \mathbb{R}^{\Omega_{n} \times \Omega_{n}} \tag{7}
\end{equation*}
$$

satisfies all the constraints and we have $\operatorname{tr}(\Delta J \Delta X)=\mu_{\sigma}(U)$. We recommend the introductory paper [14] on semidefinite programming. We note that the above semidefinite programming problem can be generalized to handle cross-intersecting families. See [11, 12], and also [7]. The dual problem for $(\mathrm{P})$ is then given by

$$
\begin{array}{lll}
\text { (D): } & \text { minimize } & \alpha \\
& \text { subject to } & S:=\alpha \Delta-\Delta J \Delta+A-Z \succcurlyeq 0, Z \geqslant 0, \\
& A_{x, y}=0 \text { for } x, y \in \Omega_{n}, x \cap y \neq 0,
\end{array}
$$

where $\alpha \in \mathbb{R}$ and $A, Z \in S \mathbb{R}^{\Omega_{n} \times \Omega_{n}}$ are the variables. For any feasible solutions to (P) and (D), we have

$$
\begin{equation*}
\alpha-\operatorname{tr}(\Delta J \Delta X)=\operatorname{tr}((\alpha \Delta-\Delta J \Delta) X)=\operatorname{tr}((S-A+Z) X) \geqslant 0 \tag{8}
\end{equation*}
$$

since $\operatorname{tr}(S X) \geqslant 0, \operatorname{tr}(Z X) \geqslant 0$, and $\operatorname{tr}(A X)=0$, and hence $\alpha$ gives an upper bound on $\mu_{\sigma}(U)$.

Our goal now is to find a feasible solution $(\alpha, A, Z)$ to (D) with

$$
\alpha:=\frac{\sigma}{1+\sigma} .
$$

Instead of working directly with the matrix $S$ above, we consider the positive semidefiniteness of

$$
S^{\prime}:=\Delta^{-\frac{1}{2}} S \Delta^{-\frac{1}{2}}=\alpha I-\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}}+A^{\prime}-Z^{\prime}
$$

where $Z^{\prime} \geqslant 0$ and $\left(A^{\prime}\right)_{x, y}=0$ whenever $x \cap y \neq 0$. We set

$$
Z^{\prime}:=0
$$

and choose $A^{\prime}$ of the form

$$
\begin{equation*}
A^{\prime}=\sum_{k+\ell \leqslant n} \frac{a_{k, \ell}^{\prime}}{\theta_{0}^{k, \ell}} \bar{W}_{k, \ell}, \tag{9}
\end{equation*}
$$

where $a_{k, \ell}^{\prime} \in \mathbb{R}$. Let $\mathbf{1} \in \mathbb{R}^{\Omega_{n}}$ be the all-ones vector and recall our choice of the vector $\boldsymbol{u}_{0,1} \in \boldsymbol{U}_{0}=\mathbb{R}^{\Omega_{n}^{(0)}}$. Then, we have (cf. (5) )

$$
\mathbf{1}=\sum_{k=0}^{n} W_{k, 0} \boldsymbol{u}_{0,1}=\sum_{k=0}^{n}\left[\begin{array}{l}
n \\
k
\end{array}\right]^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{k},
$$

so that

$$
\Delta^{\frac{1}{2}} \mathbf{1}=\sum_{k=0}^{n} \phi_{k}^{\frac{1}{2}}\left[\begin{array}{l}
n  \tag{10}\\
k
\end{array}\right]^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{k}
$$

where we abbreviate

$$
\phi_{k}:=\phi_{\sigma, n}(k) \quad(0 \leqslant k \leqslant n) .
$$

Let $\mathbb{R}^{(n+1) \times(n+1)}$ be the set of real matrices with rows and columns indexed by $0,1, \ldots, n$. Let $C \in \mathbb{R}^{(n+1) \times(n+1)}$ be the lower triangular matrix defined by

$$
C_{k, \ell}=\left[\begin{array}{l}
k \\
\ell
\end{array}\right] \quad(0 \leqslant \ell \leqslant k \leqslant n) .
$$

Then, it follows that

$$
\left(C^{-1}\right)_{k, \ell}=\left[\begin{array}{l}
k \\
\ell
\end{array}\right](-1)^{k-\ell} q^{\binom{k-\ell}{2}} \quad(0 \leqslant \ell \leqslant k \leqslant n) .
$$

Indeed, for $0 \leqslant \ell \leqslant k \leqslant n$, we have

$$
\sum_{j=\ell}^{k} C_{k, j}\left[\begin{array}{l}
j \\
\ell
\end{array}\right](-1)^{j-\ell} q^{\binom{2-\ell}{2}}=\left[\begin{array}{l}
k \\
\ell
\end{array}\right] \sum_{j=\ell}^{k}\left[\begin{array}{c}
k-\ell \\
j-\ell
\end{array}\right](-1)^{j-\ell} q^{\left(\frac{j-\ell}{2}\right)}=\left[\begin{array}{c}
k \\
\ell
\end{array}\right](1 ; q)_{k-\ell}=\delta_{k, \ell}
$$

by (1) (with $\sigma=-1$ ). Now, let $G \in \mathbb{R}^{(n+1) \times(n+1)}$ be the upper triangular matrix given by

$$
G_{k, \ell}=\left[\begin{array}{l}
n-k \\
n-\ell
\end{array}\right](-1)^{k} \sigma^{\ell} q^{\binom{k}{2}+\binom{\ell}{2}} \quad(0 \leqslant k \leqslant \ell \leqslant n)
$$

and let

$$
\begin{equation*}
F=C G C^{-1} \tag{11}
\end{equation*}
$$

Note that the diagonal entries of $G$ are

$$
(-1)^{k} \sigma^{k} q^{k(k-1)} \quad(0 \leqslant k \leqslant n)
$$

and these are the eigenvalues of $G$, and hence of $F$. We will set

$$
a_{k, \ell}^{\prime}:=\frac{F_{k, \ell}}{1+\sigma} \cdot \frac{\phi_{k}^{\frac{1}{2}}\left[\begin{array}{l}
n  \tag{12}\\
k
\end{array}\right]^{\frac{1}{2}}}{\phi_{\ell}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
\ell
\end{array}\right]^{\frac{1}{2}}} \quad(0 \leqslant k \leqslant n, 0 \leqslant \ell \leqslant n-k)
$$

in (9), and show that the corresponding $S$ gives an optimal feasible solution to (D).
To describe the matrix $F$, we will use the following lemma.
Lemma 1. For integers $a, b$, and $c$ such that $a \geqslant b \geqslant 0$ and $c \geqslant 0$, we have

$$
\sum_{j=0}^{b}\left[\begin{array}{l}
b \\
j
\end{array}\right]\left[\begin{array}{c}
a-j \\
c
\end{array}\right](-1)^{j} q^{\left(\frac{j}{2}\right)}=q^{b(a-c)}\left[\begin{array}{l}
a-b \\
c-b
\end{array}\right]
$$

In particular, the LHS above vanishes if $b>c$.
Proof. First, we have

$$
\left[\begin{array}{c}
a-j \\
c
\end{array}\right]=\sum_{d=0}^{c} q^{d(a-b-c+d)}\left[\begin{array}{l}
a-b \\
c-d
\end{array}\right]\left[\begin{array}{c}
b-j \\
d
\end{array}\right] \quad(0 \leqslant j \leqslant b) .
$$

To see this, fix $z \in \Omega_{a-j}^{(a-b)}$ and count $x \in \Omega_{a-j}^{(c)}$ such that $\operatorname{dim}(x \cap z)=c-d$ for each $d$ $(0 \leqslant d \leqslant c)$. Then, using (1) (with $\sigma=-1$ ), we have

$$
\begin{aligned}
\sum_{j=0}^{b}\left[\begin{array}{l}
b \\
j
\end{array}\right]\left[\begin{array}{c}
a-j \\
c
\end{array}\right](-1)^{j} q^{\binom{2}{2}} & =\sum_{d=0}^{c} q^{d(a-b-c+d)}\left[\begin{array}{l}
a-b \\
c-d
\end{array}\right] \sum_{j=0}^{b}\left[\begin{array}{c}
b \\
j
\end{array}\right]\left[\begin{array}{c}
b-j \\
d
\end{array}\right](-1)^{j} q^{\binom{j}{2}} \\
& =\sum_{d=0}^{c} q^{d(a-b-c+d)}\left[\begin{array}{l}
a-b \\
c-d
\end{array}\right]\left[\begin{array}{l}
b \\
d
\end{array}\right] \sum_{j=0}^{b}\left[\begin{array}{c}
b-d \\
j
\end{array}\right](-1)^{j} q^{\binom{j}{2}} \\
& =\sum_{d=0}^{c} q^{d(a-b-c+d)}\left[\begin{array}{l}
a-b \\
c-d
\end{array}\right]\left[\begin{array}{l}
b \\
d
\end{array}\right](1 ; q)_{b-d} \\
& =q^{b(a-c)}\left[\begin{array}{l}
a-b \\
c-b
\end{array}\right]
\end{aligned}
$$

as desired.
For $0 \leqslant k, \ell \leqslant n$, we have

$$
\begin{align*}
F_{k, \ell} & =\sum_{j, m=0}^{n} C_{k, j} G_{j, m}\left(C^{-1}\right)_{m, \ell}  \tag{13}\\
& =\sum_{m=0}^{n}\left[\begin{array}{c}
m \\
\ell
\end{array}\right](-1)^{m-\ell} \sigma^{m} q^{\binom{m}{2}+\binom{m-\ell}{2}} \sum_{j=0}^{n}\left[\begin{array}{c}
k \\
j
\end{array}\right]\left[\begin{array}{c}
n-j \\
n-m
\end{array}\right](-1)^{j} q^{\binom{j}{2}} \\
& =\sum_{m=0}^{n}\left[\begin{array}{c}
n-k \\
m
\end{array}\right]\left[\begin{array}{c}
m \\
\ell
\end{array}\right](-1)^{m-\ell} \sigma^{m} q^{k m+\binom{m}{2}+\binom{m-\ell}{2}} \\
& =\left[\begin{array}{c}
n-k \\
\ell
\end{array}\right] \sigma^{\ell} q^{k \ell+\binom{\ell}{2}} \sum_{h=0}^{n-k-\ell}\left[\begin{array}{c}
n-k-\ell \\
h
\end{array}\right](-1)^{h} \sigma^{h} q^{h(h+k+\ell-1)}
\end{align*}
$$

by Lemma (with $(a, b, c)=(n, k, n-m)$ ), where we set $h=m-\ell$ in the last line above. In particular, it follows that $F$ is upper anti-triangular, i.e., $F_{k, \ell}=0$ whenever $k+\ell>n$. Moreover, it is immediate to see that

$$
F_{k, \ell} \cdot \phi_{k}\left[\begin{array}{l}
n \\
k
\end{array}\right]=F_{\ell, k} \cdot \phi_{\ell}\left[\begin{array}{l}
n \\
\ell
\end{array}\right] \quad(0 \leqslant k, \ell \leqslant n)
$$

Thus, if we define the matrix $A^{\prime}$ in (91) by (12), then $A^{\prime}$ is symmetric since

$$
a_{k, \ell}^{\prime}=\frac{F_{\ell, k}}{1+\sigma} \cdot \frac{\phi_{\ell}\left[\begin{array}{l}
n \\
\ell
\end{array}\right]}{\phi_{k}\left[\begin{array}{l}
n \\
k
\end{array}\right]} \cdot \frac{\phi_{k}^{\frac{1}{2}}\left[\begin{array}{c}
n \\
k
\end{array}\right]^{\frac{1}{2}}}{\phi_{\ell}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
\ell
\end{array}\right]^{\frac{1}{2}}}=a_{\ell, k}^{\prime} \quad(0 \leqslant k \leqslant n, 0 \leqslant \ell \leqslant n-k) .
$$

It seems that the entries of $F$ have no simpler expression in general. However, if we let $q \rightarrow 1$, then

$$
\lim _{q \rightarrow 1} G_{k, \ell}=\binom{n-k}{n-\ell}(-1)^{k} \sigma^{\ell} \quad(0 \leqslant k \leqslant \ell \leqslant n)
$$

and we also have

$$
\lim _{q \rightarrow 1} F_{k, \ell}=\binom{n-k}{\ell} \sigma^{\ell}(1-\sigma)^{n-k-\ell} \quad(0 \leqslant k \leqslant n, 0 \leqslant \ell \leqslant n-k)
$$

The relation (11) after taking the limit $q \rightarrow 1$ was shown earlier in [9, Lemmas 2.9, 2.21]. We also note that the above limit of $F$ is closely related to the matrix $A^{(n)}$ (with $p=\frac{\sigma}{1+\sigma}$ ) considered in [5].

A 1-eigenvector of $G$ is given by $(1,0, \ldots, 0)^{\top} \in \mathbb{R}^{n+1}$, since $G$ is upper triangular and $G_{0,0}=1$. Then, a 1-eigenvector of $F=C G C^{-1}$ is

$$
C(1,0, \ldots, 0)^{\top}=(1,1, \ldots, 1)^{\top} .
$$

It follows from (6) that the matrix $A^{\prime}$ in (19) satisfies

$$
A^{\prime} \boldsymbol{u}_{0,1}^{\ell}=\sum_{k=0}^{n-\ell} a_{k, \ell}^{\prime} \boldsymbol{u}_{0,1}^{k} \quad(0 \leqslant \ell \leqslant n) .
$$

In other words, the matrix $A_{0}^{\prime} \in \mathbb{R}^{(n+1) \times(n+1)}$ representing the action of $A^{\prime}$ on the subspace $\boldsymbol{V}_{0,1}$ with respect to the basis $\boldsymbol{u}_{0,1}^{0}, \boldsymbol{u}_{0,1}^{1}, \ldots, \boldsymbol{u}_{0,1}^{n}$ is upper anti-triangular and is given by

$$
\left(A_{0}^{\prime}\right)_{k, \ell}=\left\{\begin{array}{ll}
a_{k, \ell}^{\prime} & \text { if } k+\ell \leqslant n, \\
0 & \text { if } k+\ell>n,
\end{array} \quad(0 \leqslant k, \ell \leqslant n)\right.
$$

From now on, we define $A^{\prime}$ by (12). Then, since $F$ is also upper anti-triangular, we have

$$
\begin{equation*}
A_{0}^{\prime}=\frac{1}{1+\sigma} D_{0} F D_{0}^{-1} \tag{14}
\end{equation*}
$$

where $D_{0} \in \mathbb{R}^{(n+1) \times(n+1)}$ is the diagonal matrix with diagonal entries $\left(D_{0}\right)_{k, k}=\phi_{k}^{\frac{1}{2}}\left[\begin{array}{l}n \\ k\end{array}\right]^{\frac{1}{2}}$ $(0 \leqslant k \leqslant n)$. In particular, $A_{0}^{\prime}$ has eigenvalues

$$
\frac{(-1)^{k} \sigma^{k} q^{k(k-1)}}{1+\sigma} \quad(0 \leqslant k \leqslant n)
$$

Moreover, it follows from the above comment that a $\frac{1}{1+\sigma}$-eigenvector of $A_{0}^{\prime}$ is given by

$$
\boldsymbol{w}_{0}:=D_{0}(1,1, \ldots, 1)^{\top}=\left(\phi_{0}^{\frac{1}{2}}\left[\begin{array}{l}
n  \tag{15}\\
0
\end{array}\right]^{\frac{1}{2}}, \phi_{1}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
1
\end{array}\right]^{\frac{1}{2}}, \ldots, \phi_{n}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
n
\end{array}\right]^{\frac{1}{2}}\right)^{\top}
$$

By (10), the matrix representing the action of $\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}}=\left(\Delta^{\frac{1}{2}} 1\right)\left(\Delta^{\frac{1}{2}} 1\right)^{\top}$ on the subspace $\boldsymbol{V}_{0,1}$ with respect to the same basis $\boldsymbol{u}_{0,1}^{0}, \boldsymbol{u}_{0,1}^{1}, \ldots, \boldsymbol{u}_{0,1}^{n}$ is $\boldsymbol{w}_{0}\left(\boldsymbol{w}_{0}\right)^{\top}$. Indeed, we have

$$
\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{\ell}=\phi_{\ell}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
\ell
\end{array}\right]^{\frac{1}{2}} \Delta^{\frac{1}{2}} \mathbf{1}=\sum_{k=0}^{n} \phi_{k}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]^{\frac{1}{2}} \phi_{\ell}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
\ell
\end{array}\right]^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{k} \quad(0 \leqslant \ell \leqslant n)
$$

Since

$$
\left(\boldsymbol{w}_{0}\right)^{\top} \boldsymbol{w}_{0}=\sum_{k=0}^{n} \phi_{k}\left[\begin{array}{l}
n \\
k
\end{array}\right]=\mu_{\sigma}\left(\Omega_{n}\right)=1
$$

the matrix $\boldsymbol{w}_{0}\left(\boldsymbol{w}_{0}\right)^{\top}$ has $\boldsymbol{w}_{0}$ as a 1-eigenvector. Since $\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}}$ is a rank-one matrix, this is the only nontrivial action of $\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}}$, i.e., all the other eigenvalues are zero. Recall our choice of $\alpha$ and $Z^{\prime}$. The vector $\boldsymbol{w}_{0}$ is an eigenvector of the action of

$$
S^{\prime}=\frac{\sigma}{1+\sigma} I-\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}}+A^{\prime}
$$

on $\boldsymbol{V}_{0,1}$ with eigenvalue

$$
\frac{\sigma}{1+\sigma}-1+\frac{1}{1+\sigma}=0
$$

The other $n$ eigenvalues of $S^{\prime}$ on $\boldsymbol{V}_{0,1}$ are given by

$$
\begin{equation*}
\frac{\sigma}{1+\sigma}+\frac{(-1)^{k} \sigma^{k} q^{k(k-1)}}{1+\sigma} \quad(1 \leqslant k \leqslant n) \tag{16}
\end{equation*}
$$

Next, we consider the actions of $S^{\prime}$ on the other subspaces $\boldsymbol{V}_{i, r}$, where $1 \leqslant i \leqslant\left\lfloor\frac{n}{2}\right\rfloor$ and $1 \leqslant r \leqslant d_{i}$. By (6), the matrix $A_{i}^{\prime}$, indexed by $i, i+1, \ldots, n-i$, representing the action of $A^{\prime}$ on $\boldsymbol{V}_{i, r}$ with respect to the basis $\boldsymbol{u}_{i, r}^{i}, \boldsymbol{u}_{i, r}^{i+1}, \ldots, \boldsymbol{u}_{i, r}^{n-i}$ is given by

$$
\left(A_{i}^{\prime}\right)_{k, \ell}=a_{k, \ell}^{\prime} \cdot \frac{\theta_{i}^{k, \ell}}{\theta_{0}^{k, \ell}}=a_{k, \ell}^{\prime} \cdot(-1)^{i} q^{\binom{i}{2}-\frac{i(k+\ell)}{2}}\left(\frac{(q ; q)_{n-k-i}(q ; q)_{n-\ell-i}(q ; q)_{k}(q ; q)_{\ell}}{(q ; q)_{n-k}(q ; q)_{n-\ell}(q ; q)_{k-i}(q ; q)_{\ell-i}}\right)^{\frac{1}{2}}
$$

for $i \leqslant k, \ell \leqslant n-i$. We then have (cf. (12))

$$
\begin{equation*}
A_{i}^{\prime}=\frac{1}{1+\sigma} D_{i} F_{i} D_{i}^{-1} \tag{17}
\end{equation*}
$$

where

$$
\left(F_{i}\right)_{k, \ell}=F_{k, \ell} \cdot(-1)^{i} q^{\binom{i}{2}-i k} \frac{(q ; q)_{n-k-i}(q ; q)_{\ell}}{(q ; q)_{n-k}(q ; q)_{\ell-i}} \quad(i \leqslant k, \ell \leqslant n-i)
$$

and $D_{i}$ is diagonal with diagonal entries

$$
\left(D_{i}\right)_{k, k}=\phi_{k}^{\frac{1}{2}}\left[\begin{array}{c}
n \\
k
\end{array}\right]^{\frac{1}{2}} \cdot q^{\frac{i k}{2}}\left(\frac{(q ; q)_{n-k}(q ; q)_{k}}{(q ; q)_{n-k-i}(q ; q)_{k-i}}\right)^{\frac{1}{2}} \quad(i \leqslant k \leqslant n-i)
$$

If we write $F=F_{n ; \sigma}$ to specify the parameters, then using (13),

$$
\left[\begin{array}{c}
n-k \\
\ell
\end{array}\right]=\left[\begin{array}{c}
n-k-i \\
\ell-i
\end{array}\right] \frac{(q ; q)_{n-k}(q ; q)_{\ell-i}}{(q ; q)_{n-k-i}(q ; q)_{\ell}},
$$

and

$$
k \ell=(k-i)(\ell-i)+i(k+\ell)-i^{2}, \quad\binom{\ell}{2}=\binom{\ell-i}{2}+i \ell-\binom{i+1}{2}
$$

it is routinely verified that

$$
\begin{equation*}
\left(F_{i}\right)_{k, \ell}=\left(F_{n-2 i ; \sigma q^{2 i}}\right)_{k-i, \ell-i} \cdot(-1)^{i} \sigma^{i} q^{i(i-1)} \quad(i \leqslant k, \ell \leqslant n-i) \tag{18}
\end{equation*}
$$

where we note that the rows and columns of $F_{n-2 i ; \sigma q^{2 i}}$ are indexed by $0,1, \ldots, n-2 i$. It follows that the eigenvalues of $A^{\prime}$ on $\boldsymbol{V}_{i, r}$ are given by

$$
(-1)^{h}\left(\sigma q^{2 i}\right)^{h} q^{h(h-1)} \cdot \frac{(-1)^{i} \sigma^{i} q^{i(i-1)}}{1+\sigma}=\frac{(-1)^{k} \sigma^{k} q^{k(k-1)}}{1+\sigma} \quad(i \leqslant k \leqslant n-i)
$$

where $h=k-i$, and therefore those of $S^{\prime}$ are

$$
\begin{equation*}
\frac{\sigma}{1+\sigma}+\frac{(-1)^{k} \sigma^{k} q^{k(k-1)}}{1+\sigma} \quad(i \leqslant k \leqslant n-i) . \tag{19}
\end{equation*}
$$

For the matrix $S^{\prime}$ to be positive semidefinite, all the eigenvalues in (16) and (19) must be nonnegative. This is equivalent to

$$
\sigma^{k} q^{k(k-1)} \leqslant \sigma \quad(1 \leqslant k \leqslant n, k: \text { odd }),
$$

which then simplifies to the condition given in Theorem 1 (when $n \geqslant 3$ ). If this condition is satisfied, then the matrix $S=\Delta^{\frac{1}{2}} S^{\prime} \Delta^{\frac{1}{2}}$ gives a feasible solution to (D) with objective value $\frac{\sigma}{1+\sigma}$, which is attained by $A_{n}^{(1)}$ defined by (3).

For the rest of the proof, assume that $\sigma<q^{-2\left\lfloor\frac{n-1}{2}\right\rfloor-1}$. Let $U \subset \Omega_{n}$ be an intersecting family such that $\mu_{\sigma}(U)=\frac{\sigma}{1+\sigma}$. Let $\boldsymbol{x} \in \mathbb{R}^{\Omega_{n}}$ be the characteristic vector of $U$, and let the matrix $X \in S \mathbb{R}^{\Omega_{n} \times \Omega_{n}}$ be as in (7). Then, equality is attained in (8), and hence it follows that $\operatorname{tr}(S X)=0$, or equivalently, $S^{\prime} \Delta^{\frac{1}{2}} \boldsymbol{x}=0$.

Recall that $\boldsymbol{w}_{0}$ is a 0 -eigenvector of the action of $S^{\prime}$ on $\boldsymbol{V}_{0,1}$. The corresponding 0 eigenvector of $S^{\prime}\left(\right.$ in $\left.\mathbb{R}^{\Omega_{n}}\right)$ is (cf. (10))

$$
\boldsymbol{v}_{0}:=\Delta^{\frac{1}{2}} \mathbf{1}=\sum_{k=0}^{n} \phi_{k}^{\frac{1}{2}}\left[\begin{array}{l}
n  \tag{20}\\
k
\end{array}\right]^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{k}
$$

The eigenvalues of $S^{\prime}$ in (19) are zero if and only if $(i, k)=(1,1)$, in which case, we can similarly see that the corresponding 0 -eigenvectors of $S^{\prime}$ are of the form

$$
\boldsymbol{v}_{r}=\sum_{k=1}^{n-1} \eta_{k} \boldsymbol{u}_{1, r}^{k} \quad\left(1 \leqslant r \leqslant d_{1}\right)
$$

where $\eta_{k} \neq 0$ for all $k$. More specifically,

$$
\left(\eta_{1}, \eta_{2}, \ldots, \eta_{n-1}\right)^{\top}=D_{1}(1,1, \ldots, 1)^{\top}=\left(\left(D_{1}\right)_{1,1},\left(D_{1}\right)_{2,2}, \ldots,\left(D_{1}\right)_{n-1, n-1}\right)^{\top}
$$

See (17) and (18). On the other hand, the eigenvalues of $S^{\prime}$ in (16) are zero if and only if $k=1$. Since $G$ is upper triangular, and $G_{0,0}=1$ and $G_{1,1}=-\sigma$, a $(-\sigma)$-eigenvector
of $G$ is given by $(\nu, 1,0, \ldots, 0)^{\top} \in \mathbb{R}^{n+1}$, where $\nu=-\frac{G_{0,1}}{1+\sigma}$. Then, a $(-\sigma)$-eigenvector of $F=C G C^{-1}$ is

$$
C(\nu, 1,0, \ldots, 0)^{\top}=\left(\nu+\left[\begin{array}{l}
0 \\
1
\end{array}\right], \nu+\left[\begin{array}{l}
1 \\
1
\end{array}\right], \nu+\left[\begin{array}{l}
2 \\
1
\end{array}\right], \ldots, \nu+\left[\begin{array}{l}
n \\
1
\end{array}\right]\right)^{\top}
$$

and the corresponding 0 -eigenvector of $S^{\prime}$ becomes (cf. (14))

$$
\boldsymbol{v}_{0}^{\prime}=\sum_{k=0}^{n} \phi_{k}^{\frac{1}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]^{\frac{1}{2}}\left(\nu+\left[\begin{array}{l}
k \\
1
\end{array}\right]\right) \boldsymbol{u}_{0,1}^{k} .
$$

Since $S^{\prime} \Delta^{\frac{1}{2}} \boldsymbol{x}=0$, the vector $\Delta^{\frac{1}{2}} \boldsymbol{x}$ must be a linear combination of the above vectors:

$$
\Delta^{\frac{1}{2}} \boldsymbol{x}=c_{0} \boldsymbol{v}_{0}+c_{1} \boldsymbol{v}_{1}+\cdots+c_{d_{1}} \boldsymbol{v}_{d_{1}}+c_{0}^{\prime} \boldsymbol{v}_{0}^{\prime}
$$

Expand the RHS above in terms of the $\boldsymbol{u}_{i, r}^{k}$. Note that $U$ does not contain $0 \in \Omega_{n}^{(0)}$, so that the coefficient of $\boldsymbol{u}_{0,1}^{0}$ is zero, i.e., $c_{0}+c_{0}^{\prime} \nu=0$. Suppose now that $U \cap \Omega_{n}^{(1)}=\emptyset$. Then, the coefficients of $\boldsymbol{u}_{0,1}^{1}$ and $\boldsymbol{u}_{1, r}^{1}\left(1 \leqslant r \leqslant d_{1}\right)$ are all zero because these vectors form a basis of $\mathbb{R}^{\Omega_{n}^{(1)}}$. That the coefficient of $\boldsymbol{u}_{0,1}^{1}$ equals zero is equivalent to $c_{0}+c_{0}^{\prime}(\nu+1)=0$. Combining this with $c_{0}+c_{0}^{\prime} \nu=0$, we have $c_{0}=c_{0}^{\prime}=0$. Moreover, the coefficient of $\boldsymbol{u}_{1, r}^{1}$ equals $c_{r} \eta_{1}$ for $1 \leqslant r \leqslant d_{1}$, and hence $c_{1}=\cdots=c_{d_{1}}=0$ since $\eta_{1} \neq 0$. It follows that $\Delta^{\frac{1}{2}} \boldsymbol{x}=0$, which is absurd. We have now shown that $U \cap \Omega_{n}^{(1)} \neq \emptyset$. Since $U$ is a maximal intersecting family, we must have $U=A_{n}^{(1)}$ for some $y \in \Omega_{n}^{(1)}$. This completes the proof of Theorem 1 .

## 3. Proof of Theorem 2

In this section, we abbreviate

$$
\mu:=\mu_{\theta, n}, \quad \sigma:=\sigma_{\theta, n}
$$

except in the statement of lemmas. We will also write $\phi=\phi_{\theta, n}:=\phi_{\sigma_{\theta, n}, n}$.
Lemma 2. If $0<\theta<\frac{1}{2}$, then

$$
\lim _{n \rightarrow \infty} \mu_{\theta, n}\left(A_{n}^{(t)}\right)^{\frac{1}{n}}=q^{-(1-\theta) t}
$$

Proof. By (4), we have

$$
\begin{equation*}
\mu\left(A_{n}^{(t)}\right)=\prod_{j=0}^{t-1}\left(1+q^{(1-\theta) n-j}\right)^{-1} \tag{21}
\end{equation*}
$$

Since $\left(1+q^{(1-\theta) n-j}\right)^{-\frac{1}{n}} \rightarrow q^{-(1-\theta)}$ for $0 \leqslant j \leqslant t-1$, we have $\mu\left(A_{n}^{(t)}\right)^{\frac{1}{n}} \rightarrow q^{-(1-\theta) t}$.
The next result shows that the distribution

$$
\Phi(k)=\Phi_{\theta, n}(k):=\left[\begin{array}{l}
n \\
k
\end{array}\right] \phi_{\theta, n}(k) \quad(k=0,1, \ldots, n)
$$

concentrates around $k \sim n \theta$.

EXTREMAL PROBLEMS FOR INTERSECTING FAMILIES OF SUBSPACES WITH A MEASURE 13
Lemma 3. Let $0<\theta<1$. For every $\epsilon>0$, there exists $L>1$ such that

$$
\sum_{|k-\theta n|>L} \Phi_{\theta, n}(k)<\epsilon
$$

for sufficiently large $n$, where the sum is over all $k=0,1, \ldots, n$ such that $|k-\theta n|>L$. Proof. Define a probability measure $\Psi=\Psi_{\theta, n}$ on the points $q^{k-\theta n}(k=0,1, \ldots, n)$ by

$$
\Psi\left(q^{k-\theta n}\right)=\Phi(k) \quad(k=0,1, \ldots, n) .
$$

By (1), the mean is computed as

$$
\mathbb{E}[X]=\sum_{k=0}^{n} q^{k-\theta n} \Phi(k)=\frac{(-\sigma q ; q)_{n}}{q^{\theta n}(-\sigma ; q)_{n}}=\frac{1+\sigma q^{n}}{q^{\theta n}(1+\sigma)},
$$

which converges to 1 when $n \rightarrow \infty$. Also,

$$
\mathbb{E}\left[X^{2}\right]=\sum_{k=0}^{n} q^{2 k-2 \theta n} \Phi(k)=\frac{\left(-\sigma q^{2} ; q\right)_{n}}{q^{2 \theta n}(-\sigma ; q)_{n}}=\frac{\left(1+\sigma q^{n}\right)\left(1+\sigma q^{n+1}\right)}{q^{2 \theta n}(1+\sigma)(1+\sigma q)}
$$

which converges to $q$ when $n \rightarrow \infty$. Hence, the variance satisfies

$$
\mathbb{V}[X]=\mathbb{E}\left[X^{2}\right]-\mathbb{E}[X]^{2} \rightarrow q-1
$$

In what follows, let $n$ be sufficiently large so that $\mathbb{E}[X]<2$ and $\mathbb{V}[X]<q$. There exists $L>1$ such that $\left(q^{L}-2\right)^{2} \epsilon>2 q$. Then, we have

$$
q>\mathbb{V}[X] \geqslant \sum_{k>\theta n+L}\left(q^{k-\theta n}-\mathbb{E}[X]\right)^{2} \Phi(k) \geqslant\left(q^{L}-2\right)^{2} \sum_{k>\theta n+L} \Phi(k),
$$

from which it follows that

$$
\sum_{k>\theta n+L} \Phi(k)<\frac{q}{\left(q^{L}-2\right)^{2}}<\frac{\epsilon}{2}
$$

Next, we have

$$
\mathbb{E}\left[X^{-1}\right]=\frac{q^{\theta n}\left(1+\sigma q^{-1}\right)}{1+\sigma q^{n-1}} \rightarrow q, \quad \mathbb{E}\left[X^{-2}\right]=\frac{q^{2 \theta n}\left(1+\sigma q^{-2}\right)\left(1+\sigma q^{-1}\right)}{\left(1+\sigma q^{n-2}\right)\left(1+\sigma q^{n-1}\right)} \rightarrow q^{3}
$$

so

$$
\mathbb{V}\left[X^{-1}\right]=\mathbb{E}\left[X^{-2}\right]-\mathbb{E}\left[X^{-1}\right]^{2} \rightarrow q^{3}-q^{2}
$$

By a similar argument, we can show that there exists $L^{\prime}>1$ such that

$$
\sum_{k<\theta n-L^{\prime}} \Phi(k)<\frac{\epsilon}{2}
$$

for sufficiently large $n$. The result now follows by replacing $L$ by $L^{\prime}$ if $L^{\prime}>L$.
For our purpose, we need a stronger tail bound.
Claim 4. Let $0<\theta<\frac{1}{2}$. We have

$$
\begin{equation*}
\sum_{k>\frac{n}{2}} \Phi_{\theta, n}(k)=o\left(q^{-(1-\theta) t n}\right), \tag{22}
\end{equation*}
$$

where the sum is over all integers $k$ with $\frac{n}{2}<k \leqslant n$.

Proof. We write $m:=\left\lceil\frac{n}{2}\right\rceil$ and $s:=\lceil\theta n\rceil$ for typographical reasons. First, we claim that

$$
\begin{equation*}
\Phi(m)<q^{-\frac{1}{2}(m-s)^{2}+O(n)}=q^{-\frac{1}{2}\left(\frac{1}{2}-\theta\right)^{2} n^{2}+O(n)} \tag{23}
\end{equation*}
$$

In view of Lemma 3, we estimate (cf. (1))

$$
\Phi(m)=\frac{\left[\begin{array}{c}
n  \tag{24}\\
m
\end{array}\right] \sigma^{m} q^{\binom{m}{2}}}{\sum_{k=0}^{n}\left[\begin{array}{l}
n \\
k
\end{array}\right] \sigma^{k} q^{\binom{k}{2}}}<\frac{\left[\begin{array}{l}
n \\
m
\end{array}\right] \sigma^{m} q^{\binom{m}{2}}}{\left[\begin{array}{l}
n \\
s
\end{array}\right] \sigma^{s} q^{\binom{s}{2}}}=\frac{\left[\begin{array}{c}
n \\
m
\end{array}\right]}{\left[\begin{array}{c}
n \\
s
\end{array}\right]} \sigma^{m-s} q^{\binom{m}{2}-\binom{s}{2}} .
$$

For further estimation of the RHS above, we note that

$$
\frac{[n-m]}{[s]}=\frac{q^{n-m}-1}{q^{s}-1}<q^{n-m-s+1}=q^{m-s+O(1)}
$$

and so

$$
\frac{\left[\begin{array}{l}
n  \tag{25}\\
m
\end{array}\right]}{\left[\begin{array}{c}
n \\
s
\end{array}\right]}=\frac{[s]![n-s]!}{[m]![n-m]!}=\prod_{j=1}^{m-s} \frac{[n-m+j]}{[s+j]}<\left(\frac{[n-m]}{[s]}\right)^{m-s}<q^{(m-s)^{2}+O(n)}
$$

for sufficiently large $n$. We also have

$$
\begin{equation*}
\sigma^{m-s} q^{\binom{m}{2}-\binom{s}{2}}=\left(q^{-(1-\theta) n}\right)^{m-s} q^{\frac{m^{2}}{2}-\frac{s^{2}}{2}+O(n)}=q^{-\frac{3}{2}(m-s)^{2}+O(n)} . \tag{26}
\end{equation*}
$$

Substituting (25) and (26) into the RHS of (24), we get (23).
Next, we verify that $\Phi(k)$ is decreasing in $k$ for $k \geqslant m$. Indeed, since $\frac{[n-k]}{[k+1]}<q^{n-2 k}$ as above, it follows that

$$
\frac{\Phi(k+1)}{\Phi(k)}=\frac{\left[\begin{array}{c}
n \\
k+1
\end{array}\right] \sigma^{k+1} q^{\binom{k+1}{2}}}{\left[\begin{array}{c}
n \\
k
\end{array}\right] \sigma^{k} q^{\left(\begin{array}{c}
k
\end{array}\right)}}=\frac{[n-k]}{[k+1]} \sigma q^{k}<q^{\theta n-k} \leqslant q^{\theta n-m}<1
$$

Hence, it follows from (23) that

$$
\sum_{k>\frac{n}{2}} \Phi(k)<m \Phi(m)<q^{-\frac{1}{2}\left(\frac{1}{2}-\theta\right)^{2} n^{2}+O(n)}=o\left(q^{-(1-\theta) t n}\right)
$$

for sufficiently large $n$.
Remark 1. The RHS of (22) can be replaced by $o\left(q^{-R n}\right)$ for any fixed $R>0$. For our purpose, we need $R \geqslant(1-\theta) t$.

We now invoke the following result.
Theorem D (Frankl-Wilson [4]). Let $n \geqslant 2 k$. If a family $U \subset \Omega_{n}^{(k)}$ is t-intersecting, then $|U| \leqslant\left[\begin{array}{c}n-t \\ k-t\end{array}\right]$.

For the characterization of the optimal families in Theorem (1) see [6, 13].
Lemma 5. If $0<\theta<\frac{1}{2}$, then $\lim _{n \rightarrow \infty} f\left(n, t, \sigma_{\theta, n}\right)=q^{-(1-\theta) t}$.
Proof. By Lemma 2, we have

$$
f(n, t, \sigma) \geqslant \mu\left(A_{n}^{(t)}\right)^{\frac{1}{n}} \rightarrow q^{-(1-\theta) t}
$$

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On the other hand, let $U_{n} \subset \Omega_{n}$ be a $t$-intersecting family satisfying $\mu\left(U_{n}\right)=f(n, t, \sigma)^{n}$, and let $U_{n}^{(k)}=U_{n} \cap \Omega_{n}^{(k)}$. We have

$$
\mu\left(U_{n}\right)=\sum_{k=0}^{n}\left|U_{n}^{(k)}\right| \phi(k) \leqslant \sum_{k \leqslant \frac{n}{2}}\left|U_{n}^{(k)}\right| \phi(k)+\sum_{k>\frac{n}{2}} \Phi(k) .
$$

By Theorem D and (21),

$$
\sum_{k \leqslant \frac{n}{2}}\left|U_{n}^{(k)}\right| \phi(k) \leqslant \sum_{k \leqslant \frac{n}{2}}\left[\begin{array}{l}
n-t \\
k-t
\end{array}\right] \phi(k) \leqslant \mu\left(A_{n}^{(t)}\right)=\prod_{j=0}^{t-1}\left(1+q^{(1-\theta) n-j}\right)^{-1}
$$

and by Claim 4,

$$
\sum_{k>\frac{n}{2}} \Phi(k)=o\left(q^{-(1-\theta) t n}\right) .
$$

Thus, we have

$$
\begin{aligned}
\mu\left(U_{n}\right) & \leqslant \mu\left(A_{n}^{(t)}\right)+o\left(q^{-(1-\theta) t n}\right) \\
& =\mu\left(A_{n}^{(t)}\right)\left(1+o\left(\prod_{j=0}^{t-1}\left(q^{-(1-\theta) n}+q^{-j}\right)\right)\right) \\
& =\mu\left(A_{n}^{(t)}\right)(1+o(1)) .
\end{aligned}
$$

Finally, it follows from Lemma 2 that

$$
f(n, t, \sigma)=\mu\left(U_{n}\right)^{\frac{1}{n}} \leqslant \mu\left(A_{n}^{(t)}\right)^{\frac{1}{n}}(1+o(1))^{\frac{1}{n}} \rightarrow q^{-(1-\theta) t} .
$$

Claim 6. Let $\frac{1}{2}<\theta<1$. There exists $\delta>0$ such that

$$
\sum_{k<\frac{n+t}{2}} \Phi_{\theta, n}(k)<q^{-\delta n^{2}}
$$

for sufficiently large $n$, where the sum is over all integers $k$ with $0 \leqslant k<\frac{n+t}{2}$.
Proof. The proof is similar to that of Claim 4. We write $m^{\prime}:=\left\lceil\frac{n+t}{2}\right\rceil$ and $s:=\lceil\theta n\rceil$. We first claim that

$$
\begin{equation*}
\Phi\left(m^{\prime}\right)<q^{-\frac{1}{2}\left(s-m^{\prime}\right)^{2}+O(n)}=q^{-\frac{1}{2}\left(\theta-\frac{1}{2}\right)^{2} n^{2}+O(n)} . \tag{27}
\end{equation*}
$$

We have (cf. (1))

$$
\left.\Phi\left(m^{\prime}\right)=\frac{\left[\begin{array}{c}
n \\
m^{\prime}
\end{array}\right] \sigma^{m^{\prime}} q^{\binom{m^{\prime}}{2}}}{\sum_{k=0}^{n}\left[\begin{array}{c}
n \\
k
\end{array}\right] \sigma^{k} q^{\binom{k}{2}}}<\frac{\left[\begin{array}{c}
n \\
m^{\prime}
\end{array}\right] \sigma^{m^{\prime}} q^{\binom{m^{\prime}}{2}}}{\left[\begin{array}{c}
n \\
s
\end{array}\right] \sigma^{s} q^{(s)} 2} \begin{array}{l}
\text { a }
\end{array}\right] \frac{\left[\begin{array}{c}
n \\
m^{\prime}
\end{array}\right]}{\left[\begin{array}{c}
n \\
s
\end{array}\right]} \sigma^{m^{\prime}-s} q^{\binom{m^{\prime}}{2}-\binom{s}{2}} .
$$

Since $\frac{\left[m^{\prime}\right]}{[n-s]}<q^{m^{\prime}-n+s+1}=q^{s-m^{\prime}+O(1)}$, it follows that

$$
\frac{\left[\begin{array}{c}
n \\
m^{\prime}
\end{array}\right]}{\left[\begin{array}{c}
n \\
s
\end{array}\right]}=\frac{[s]![n-s]!}{\left[m^{\prime}\right]!\left[n-m^{\prime}\right]!}=\prod_{j=1}^{s-m^{\prime}} \frac{\left[m^{\prime}+j\right]}{[n-s+j]}<\left(\frac{\left[m^{\prime}\right]}{[n-s]}\right)^{s-m^{\prime}}=q^{\left(s-m^{\prime}\right)^{2}+O(n)}
$$

for sufficiently large $n$. We also have

$$
\sigma^{m^{\prime}-s} q^{\binom{m^{\prime}}{2}-\binom{s}{2}}=\left(q^{-(1-\theta) n}\right)^{m^{\prime}-s} q^{\frac{\left(m^{\prime}\right)^{2}}{2}-\frac{s^{2}}{2}+O(n)}=q^{-\frac{3}{2}\left(s-m^{\prime}\right)^{2}+O(n)} .
$$

Hence, we get (27).
Next, we verify that $\Phi(k)$ is increasing in $k$ for $k \leqslant m^{\prime}$, provided that $n$ is sufficiently large. This follows from $\frac{[k]}{[n-k+1]}<q^{2 k-n}$ and

$$
\frac{\Phi(k-1)}{\Phi(k)}=\frac{\left[\begin{array}{c}
n \\
k-1
\end{array}\right] \sigma^{k-1} q^{\binom{k-1}{2}}}{\left[\begin{array}{c}
n \\
k
\end{array}\right] \sigma^{k} q^{\binom{k}{2}}}=\frac{[k]}{[n-k+1] \sigma q^{k-1}}<q^{k-\theta n+1} \leqslant q^{m^{\prime}-\theta n+1}<1 .
$$

Hence, if we choose $\delta$ such that $0<\delta<\frac{1}{2}\left(\theta-\frac{1}{2}\right)^{2}$, then we have

$$
\sum_{k<\frac{n+t}{2}} \Phi(k)<m^{\prime} \Phi\left(m^{\prime}\right)<q^{-\delta n^{2}}
$$

for sufficiently large $n$.
Lemma 7. If $\frac{1}{2}<\theta<1$, then $\lim _{n \rightarrow \infty} f\left(n, t, \sigma_{\theta, n}\right)=1$.
Proof. Clearly, we have $f(n, t, \sigma) \leqslant 1$. So we need to show that $\underline{\lim }_{n \rightarrow \infty} f(n, t, \sigma) \geqslant 1$. Define a $t$-intersecting family $B_{n}$ by

$$
B_{n}:=\left\{x \in \Omega_{n}: \operatorname{dim} x \geqslant \frac{n+t}{2}\right\} .
$$

By Claim 6, we have

$$
\mu\left(B_{n}\right)=1-\sum_{k<\frac{n+t}{2}} \Phi(k)>1-q^{-\delta n^{2}}
$$

for sufficiently large $n$. Then, the desired inequality follows from

$$
f(n, t, \sigma) \geqslant \mu\left(B_{n}\right)^{\frac{1}{n}} \geqslant\left(1-q^{-\delta n^{2}}\right)^{\frac{1}{n}} \rightarrow 1 .
$$

Proof of Theorem 园 Immediate from Lemma 5 and Lemma 7

## 4. Proof of Theorem 3

Here, we list some results concerning cross $t$-intersecting families of uniform subspaces. We omit the descriptions of the optimal families.
Theorem E (Tokushige [15]). Let $n \geqslant 2 k$. If $U \subset \Omega_{n}^{(k)}$ and $W \subset \Omega_{n}^{(k)}$ are cross $t$ intersecting, then $|U||W| \leqslant\left[\begin{array}{c}n-t \\ k-t\end{array}\right]^{2}$.

Theorem F (Suda-Tanaka [11). Let $n \geqslant 2 k$ and $n \geqslant 2 \ell$. If $U \subset \Omega_{n}^{(k)}$ and $W \subset \Omega_{n}^{(\ell)}$ are cross 1-intersecting, then $|U||W| \leqslant\left[\begin{array}{c}n-1 \\ k-1\end{array}\right]\left[\begin{array}{c}n-1 \\ \ell-1\end{array}\right]$.

Theorem G (Cao-Lu-Lv-Wang [2]). Let $n \geqslant k+\ell+t+1$. If $U \subset \Omega_{n}^{(k)}$ and $W \subset \Omega_{n}^{(\ell)}$ are cross t-intersecting, then $|U||W| \leqslant\left[\begin{array}{c}n-t \\ k-t\end{array}\right]\left[\begin{array}{c}n-t \\ \ell-t\end{array}\right]$.

For the proof of Theorem 3, we use Theorem G, Note that while Theorem $G$ is the most general result so far, it does not fully contain Theorem E and Theorem E

The next claim can be shown exactly in the same way (with slightly more cumbersome computation) as Claim 4, and we omit the proof. See also Remark 1 ,

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Claim 8. Let $0<\theta<\frac{1}{2}$, and let $t$ be a fixed positive integer. Then, we have

$$
\sum_{k>\frac{n-t-1}{2}} \Phi_{\theta, n}(k)=o\left(q^{-2 t n}\right),
$$

where the sum is over all integers $k$ with $\frac{n-t-1}{2}<k \leqslant n$.
Proof of Theorem [3. For $i=1,2$, we write $\sigma_{i}:=\sigma_{\theta_{i}, n}, \phi_{i}:=\phi_{\theta_{i}, n}$, and $\mu_{i}:=\mu_{\theta_{i}, n}$ for brevity. Suppose that cross $t$-intersecting families $U_{n}, W_{n} \subset \Omega_{n}$ satisfy

$$
g\left(n, t, \sigma_{1}, \sigma_{2}\right)^{n}=\mu_{1}\left(U_{n}\right) \mu_{2}\left(W_{n}\right)
$$

and let $U_{n}^{(k)}=U_{n} \cap \Omega_{n}^{(k)}$ and $W_{n}^{(\ell)}=W_{n} \cap \Omega_{n}^{(\ell)}$. If $k, \ell \leqslant \frac{n-t-1}{2}$, then $n \geqslant k+\ell+t+1$, and we can apply Theorem G to $U_{n}^{(k)}$ and $W_{n}^{(\ell)}$. By Claim 8, we may write

$$
\begin{aligned}
\mu_{1}\left(U_{n}\right) & =\sum_{k \leqslant \frac{n-t-1}{2}}\left|U_{n}^{(k)}\right| \phi_{1}(k)+o\left(q^{-2 t n}\right), \\
\mu_{2}\left(W_{n}\right) & =\sum_{\ell \leqslant \frac{n-t-1}{2}}\left|W_{n}^{(\ell)}\right| \phi_{2}(\ell)+o\left(q^{-2 t n}\right) .
\end{aligned}
$$

Then, by Theorem G, we have

$$
\begin{aligned}
&\left(\sum_{k \leqslant \frac{n-t-1}{2}}\left|U_{n}^{(k)}\right| \phi_{1}(k)\right)\left(\sum_{\ell \leqslant \frac{n-t-1}{2}}\left|W_{n}^{(\ell)}\right| \phi_{2}(\ell)\right) \\
&=\sum_{k \leqslant \frac{n-t-1}{2}} \sum_{\ell \leqslant \frac{n-t-1}{2}}\left|U_{n}^{(k)}\right|\left|W_{n}^{(\ell)}\right| \phi_{1}(k) \phi_{2}(\ell) \\
& \leqslant \sum_{k \leqslant \frac{n-t-1}{2}} \sum_{\ell \leqslant \frac{n-t-1}{2}}\left[\begin{array}{c}
n-t \\
k-t
\end{array}\right]\left[\begin{array}{c}
n-t \\
\ell-t
\end{array}\right] \phi_{1}(k) \phi_{2}(\ell) \\
&=\left(\sum_{k \leqslant \frac{n-t-1}{2}}\left[\begin{array}{c}
n-t \\
k-t
\end{array}\right] \phi_{1}(k)\right)\left(\sum_{\ell \leqslant \frac{n-t-1}{2}}\left[\begin{array}{c}
n-t \\
\ell-t
\end{array}\right] \phi_{2}(\ell)\right) \\
& \leqslant \mu_{1}\left(A_{n}^{(t)}\right) \mu_{2}\left(A_{n}^{(t)}\right),
\end{aligned}
$$

where $A_{n}^{(t)}$ is defined by (3). It follows that

$$
\begin{aligned}
\mu_{1}\left(U_{n}\right) \mu_{2}\left(W_{n}\right) & =\left(\sum_{k \leqslant \frac{n-t-1}{2}}\left|U_{n}^{(k)}\right| \phi_{1}(k)\right)\left(\sum_{\ell \leqslant \frac{n-t-1}{2}}\left|W_{n}^{(\ell)}\right| \phi_{2}(\ell)\right)+o\left(q^{-2 t n}\right) \\
& \leqslant \mu_{1}\left(A_{n}^{(t)}\right) \mu_{2}\left(A_{n}^{(t)}\right)+o\left(q^{-2 t n}\right)
\end{aligned}
$$

Thus, by using Lemma 2, we have

$$
g\left(n, t, \sigma_{1}, \sigma_{2}\right)=\left(\mu_{1}\left(U_{n}\right) \mu_{2}\left(W_{n}\right)\right)^{\frac{1}{n}} \leqslant\left(\mu_{1}\left(A_{n}^{(t)}\right) \mu_{2}\left(A_{n}^{(t)}\right)+o\left(q^{-2 t n}\right)\right)^{\frac{1}{n}} \rightarrow q^{-\left(2-\theta_{1}-\theta_{2}\right) t}
$$

The opposite inequality follows from

$$
g\left(n, t, \sigma_{1}, \sigma_{2}\right) \geqslant\left(\mu_{1}\left(A_{n}^{(t)}\right) \mu_{2}\left(A_{n}^{(t)}\right)\right)^{\frac{1}{n}} \rightarrow q^{-\left(2-\theta_{1}-\theta_{2}\right) t}
$$

This completes the proof of Theorem 3.

## 5. Concluding Remarks

5.1. Theorem 1 from Hoffman's bound. Here, we show that the inequality in Theorem 1 can also be interpreted as an application of the so-called Hoffman's bound. Let $\mathcal{G}=(\Omega, E)$ be a finite simple graph with vertex set $\Omega$ and edge set $E$, and let $\phi: \Omega \rightarrow[0,1]$ be a weight function such that $\sum_{x \in \Omega} \phi(x)=1$. Let $\mu: 2^{\Omega} \rightarrow[0,1]$ be the probability measure on $\Omega$ defined by $\mu(U):=\sum_{x \in U} \phi(x)$. We say that $U \subset \Omega$ is independent if no two vertices in $U$ are adjacent in $\mathcal{G}$. The independence measure of $\mathcal{G}$, denoted by $\alpha(\mathcal{G})$, is the maximum of $\mu(U)$ over independent sets $U \subset \Omega$. We say that a symmetric matrix $B \in \mathbb{R}^{\Omega \times \Omega}$ reflects adjacency in $\mathcal{G}$ if

- $B_{x, y}=0$ whenever $\{x, y\} \notin E$, and
- $B$ has an eigenvector $\boldsymbol{w} \in \mathbb{R}^{\Omega}$ such that $\boldsymbol{w}_{x}=\sqrt{\phi(x)}=\sqrt{\mu(x)}(x \in \Omega)$.

Let $\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \ldots, \boldsymbol{u}_{|\Omega|}$ be eigenvectors of $B$. We may assume that they form an orthonormal basis with respect to the standard inner product, and that $\boldsymbol{u}_{1}=\boldsymbol{w}$. Let $\lambda_{i}$ denote the eigenvalue for $\boldsymbol{u}_{i}$, and let $\lambda_{\min }:=\min \left\{\lambda_{i}: 2 \leqslant i \leqslant|\Omega|\right\}$. Under these assumptions, we have the following upper bound for $\alpha(\mathcal{G})$ in terms of $\lambda_{1}$ and $\lambda_{\text {min }}$.
Lemma 9 (Hoffman's bound). We have $\alpha(\mathcal{G}) \leqslant \frac{-\lambda_{\min }}{\lambda_{1}-\lambda_{\min }}$.
The original version of Hoffman's bound assumes that $\mathcal{G}$ is a regular graph and that $\mu$ is the uniform measure on $\Omega$, but its proof works for the above general version: evaluate $\boldsymbol{v}^{\top} B \boldsymbol{v}$ in two ways for an independent set $U \subset \Omega$, where $\boldsymbol{v} \in \mathbb{R}^{\Omega}$ is defined by

$$
\boldsymbol{v}_{x}=\left\{\begin{array}{ll}
\sqrt{\mu(x)} & \text { if } x \in U, \\
0 & \text { if } x \notin U,
\end{array} \quad(x \in \Omega)\right.
$$

Let the vertex set of $\mathcal{G}$ be $\Omega:=\Omega_{n}$, where two distinct vertices $x, y \in \Omega_{n}$ are adjacent whenever $x \cap y=0$. Observe that $U \subset \Omega_{n}$ is independent in $\mathcal{G}$ if and only if $U$ is an intersecting family. Let $\phi:=\phi_{\sigma, n}, \mu:=\mu_{\sigma}$, and $B:=A^{\prime}$ from (9) with (12). Then it follows that $A^{\prime}$ reflects the adjacency, and that $\boldsymbol{w}=\Delta^{\frac{1}{2}} \mathbf{1}=\boldsymbol{v}_{0}$ is a $\frac{1}{1+\sigma}$-eigenvector of $A^{\prime}$ (see (15) and (20)). Since $\lambda_{1}=\frac{1}{1+\sigma}$ and $\lambda_{\min }=-\frac{\sigma}{1+\sigma}$ by (16) and (19), Hoffman's bound yields $\alpha(\mathcal{G}) \leqslant \frac{\sigma}{1+\sigma}$. Moreover, in this case, $\alpha(\mathcal{G})$ is the maximum $\mu_{\sigma}$-biased measure of intersecting families in $\Omega_{n}$, and so we obtain the inequality in Theorem 1. In fact, we can apply our semidefinite programming formulation conversely to prove Hoffman's bound in general. This was already done in [8] for the original version. We adopted this formulation in this paper for possible extendability; see, e.g., [12].
5.2. Comparison of Theorem $G$ with the corresponding subset version. For the case $t=1$, Theorem $G$ reads as follows.

Corollary 1. Let $n \geqslant k+\ell+2$. If $U \subset \Omega_{n}^{(k)}$ and $W \subset \Omega_{n}^{(\ell)}$ are cross 1 -intersecting, then $|U||W| \leqslant\left[\begin{array}{c}n-1 \\ k-1\end{array}\right]\left[\begin{array}{c}n-1 \\ \ell-1\end{array}\right]$.

If we replace $\Omega_{n}^{(k)}$ and $\Omega_{n}^{(\ell)}$ in the above result with $\binom{X_{n}}{k}$ and $\binom{X_{n}}{\ell}$, respectively, then the situation becomes more complicated; see [16]. Indeed, for any large $M>0$, we can
find $n, k, \ell$ with $n=k+\ell+M$ and cross 1-intersecting families $U \subset\binom{X_{n}}{k}$ and $W \subset\binom{X_{n}}{\ell}$ such that

$$
\begin{equation*}
|U||W|>\binom{n-1}{k-1}\binom{n-1}{\ell-1} . \tag{28}
\end{equation*}
$$

To see this, let

$$
U:=\left\{x \in\binom{X_{n}}{k}: x \cap X_{2} \neq \emptyset\right\}, \quad W:=\left\{x \in\binom{X_{n}}{\ell}: X_{2} \subset x\right\} .
$$

Then, $U$ and $W$ are cross 1-intersecting, and $|U|=\binom{n}{k}-\binom{n-2}{k},|W|=\binom{n-2}{\ell-2}$. In this case, (28) is equivalent to $(2 n-k-1) k(\ell-1)>(n-1)^{2}$, and this condition is satisfied, for example, if $k \geqslant 2, \ell=9 k, n=17 k$. This means that for any given $M=7 k$, we can construct families satisfying (28), and so the condition $n \geqslant k+\ell+M$ is not sufficient to get the upper bound $\binom{n-1}{k-1}\binom{n-1}{\ell-1}$ for the product of the sizes of cross 1-intersecting families.

It is interesting to determine whether the condition $n \geqslant k+\ell+2$ in the corollary is sharp or not. We cannot replace the condition with $n \geqslant k+\ell$. To see this, let $n=k+\ell$, $k=1, \ell \geqslant 3$, and fix $z \in \Omega_{n}^{(2)}$. Then

$$
U:=\left\{x \in \Omega_{n}^{(1)}: x \subset z\right\}, \quad W:=\left\{x \in \Omega_{n}^{(\ell)}: z \subset x\right\}
$$

are cross 1-intersecting, and

$$
|U||W|=\left[\begin{array}{l}
2 \\
1
\end{array}\right]\left[\begin{array}{l}
n-2 \\
\ell-2
\end{array}\right]=\left[\begin{array}{l}
2 \\
1
\end{array}\right]\left[\begin{array}{c}
\ell-1 \\
1
\end{array}\right]>\left[\begin{array}{l}
\ell \\
1
\end{array}\right]=\left[\begin{array}{l}
n-1 \\
k-1
\end{array}\right]\left[\begin{array}{l}
n-1 \\
\ell-1
\end{array}\right] .
$$

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