EXTREMAL PROBLEMS FOR INTERSECTING FAMILIES OF SUBSPACES WITH A MEASURE

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ABSTRACT. We introduce a measure for subspaces of a vector space over a q-element field, and propose some extremal problems for intersecting families. These are q-analogue of Erdős–Ko–Rado type problems, and we answer some of the basic questions.

1. INTRODUCTION

The purpose of the paper is to introduce a measure on the set of subspaces of a vector space over a finite field and propose some extremal problems for intersecting families of subspaces. We then show some Erdős–Ko–Rado type results for vector spaces with this measure.

To motivate our problems, we begin with the Erdős–Ko–Rado theorem and its measure version. Let n and k be positive integers with $n \ge k$. Let $X_n := \{1, 2, \ldots, n\}$, and let 2^{X_n} and $\binom{X_n}{k}$ denote the power set of X_n and the set of k-element subsets of X_n , respectively. A family $U \subset 2^{X_n}$ of subsets is called intersecting if $x \cap y \ne \emptyset$ for all $x, y \in U$.

Theorem A ([3]). Let $\frac{k}{n} \leq \frac{1}{2}$. If a family $U \subset \binom{X_n}{k}$ is intersecting, then

$$|U| / \binom{n}{k} \leqslant \frac{k}{n}$$

Moreover, if $|U|/\binom{n}{k} = \frac{k}{n}$ and if $\frac{k}{n} < \frac{1}{2}$, then there exists $i \in X_n$ such that

$$U = \left\{ x \in \binom{X_n}{k} : i \in x \right\}.$$

This result has a measure counterpart. Let p be a real number with 0 . We define a <math>p-biased measure $\tilde{\mu}_p : 2^{2^{X_n}} \to [0, 1]$ by

$$\tilde{\mu}_p(U) := \sum_{x \in U} p^{|x|} (1-p)^{n-|x|}$$

for $U \subset 2^{X_n}$. This is a probability measure and

$$\tilde{\mu}_p(2^{X_n}) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} = (p+(1-p))^n = 1$$

by the binomial theorem.

Theorem B ([1]). Let $p \leq \frac{1}{2}$. If a family $U \subset 2^{X_n}$ is intersecting, then

$$\tilde{\mu}_p(U) \leqslant p$$

	uniform version	measure version
subsets	Theorem A	Theorem B
subspaces	Theorem C	?

TABLE 1.

Moreover, if $\tilde{\mu}_p(U) = p$ and if $p < \frac{1}{2}$, then there exists $i \in X_n$ such that

$$U = \{ x \in 2^{X_n} : i \in x \} .$$

Now, we switch to work on subspaces from subsets. Throughout the paper, we fix a prime power q. Let \mathbb{F}_q be the q-element field, and let \mathbb{F}_q^n denote the n-dimensional vector space over \mathbb{F}_q . Let Ω_n and $\Omega_n^{(k)}$ denote the set of all subspaces of \mathbb{F}_q^n and the set of k-dimensional subspaces of \mathbb{F}_q^n , respectively. Define $[n] := \frac{q^n-1}{q-1}$, $[n]! := \prod_{j=1}^n [j]$, and $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$. Then, $|\Omega_n^{(k)}| = \begin{bmatrix} n \\ k \end{bmatrix}$. A family $U \subset \Omega_n$ of subspaces is called intersecting if $x \cap y \neq 0$ for all $x, y \in U$.

Theorem C ([6]). Let $\frac{k}{n} \leq \frac{1}{2}$. If a family $U \subset \Omega_n^{(k)}$ is intersecting, then

$$|U| / \binom{n}{k} \leqslant \frac{[k]}{[n]}$$

Moreover, if $|U|/{n \brack k} = \frac{[k]}{[n]}$ and if $\frac{k}{n} < \frac{1}{2}$, then there exists $y \in \Omega_n^{(1)}$ such that $U = \left\{ x \in \Omega_n^{(k)} : y \subset x \right\}.$

So far, we have mentioned three Erdős–Ko–Rado type results. It seems natural to expect a result that is a q-analogue of Theorem B, and at the same time, a measure version corresponding to Theorem C (see Table 1). To find such a result, we first need to introduce a measure on Ω_n . Let σ be a positive real number, and let

$$\phi_{\sigma,n}(k) := \frac{\sigma^k q^{\binom{k}{2}}}{(-\sigma;q)_n},$$

where

(1)
$$(-\sigma;q)_n := \prod_{j=0}^{n-1} (1+\sigma q^j) = \sum_{k=0}^n {n \brack k} \sigma^k q^{\binom{k}{2}}$$

The second identity in (1) is known as the q-binomial theorem. (This is an identity as polynomials in σ and hence is valid when, e.g., $\sigma < 0$.) Then, define a σ -biased measure $\mu_{\sigma}: 2^{\Omega_n} \to [0, 1]$ by

$$\mu_{\sigma}(U) := \sum_{x \in U} \phi_{\sigma,n}(\dim x)$$

for $U \subset \Omega_n$. This is a probability measure and

$$\mu_{\sigma}(\Omega_n) = \sum_{k=0}^n {n \brack k} \phi_{\sigma,n}(k) = 1.$$

We note that

$$\lim_{k \to 1} {n \brack k} \phi_{\sigma,n}(k) = {n \choose k} \left(\frac{\sigma}{1+\sigma}\right)^k \left(1-\frac{\sigma}{1+\sigma}\right)^{n-k}$$

This suggests that the measure μ_{σ} is a *q*-analogue of the measure $\tilde{\mu}_p$, and the maximum μ_{σ} biased measure of an intersecting family of subspaces (in a possible result corresponding to Theorem B) should be $\frac{\sigma}{1+\sigma}$, which plays the role of *p* when $q \to 1$. Indeed, we have the following.

Theorem 1. Let

$$\sigma \leqslant q^{-2\lfloor \frac{n-1}{2} \rfloor - 1} = \begin{cases} q^{-n} & \text{if } n \text{ is odd,} \\ q^{-n+1} & \text{if } n \text{ is even.} \end{cases}$$

If a family $U \subset \Omega_n$ is intersecting, then

$$\mu_{\sigma}(U) \leqslant \frac{\sigma}{1+\sigma}$$

Moreover, if $\mu_{\sigma}(U) = \frac{\sigma}{1+\sigma}$ and if $\sigma < q^{-2\lfloor \frac{n-1}{2} \rfloor - 1}$, then there exists $y \in \Omega_n^{(1)}$ such that $U = \{x \in \Omega_n : y \subset x\}.$

Write $a = q^{-2\lfloor \frac{n-1}{2} \rfloor - 1}$ for brevity. Then, that $\sigma \leq q^a$ is equivalent to $\frac{\sigma}{1+\sigma} \leq \frac{q^a}{1+q^a}$, and we have $\frac{q^a}{1+q^a} \to \frac{1}{2}$ when $q \to 1$ (irrespective of the actual value of a). Our proof of Theorem 1 allows us to take the limit $q \to 1$, and we thus restore Theorem B. However, unlike the bound $p \leq \frac{1}{2}$ in Theorem B, the bound $\sigma \leq q^a$ in Theorem 1 is not best possible in general. Let $\frac{\sigma}{1+\sigma} = \frac{[pn]}{[n]}$, or equivalently, $\sigma = \frac{[pn]}{[n]-[pn]}$ for a fixed p with $0 , where we extend the notation <math>[\lambda] = \frac{q^{\lambda}-1}{q-1}$ to any $\lambda \in \mathbb{R}$. Then, we have $\frac{\sigma}{1+\sigma} = \frac{q^{pn}-1}{q^{n}-1} \to p$ as $q \to 1$.

Conjecture 1. For every $0 , there exists <math>n_0$ such that the following holds for all $n > n_0$ and $\sigma = \frac{[pn]}{[n]-[pn]}$: if a family $U \subset \Omega_n$ is intersecting, then

$$\mu_{\sigma}(U) \leqslant \frac{\sigma}{1+\sigma},$$

with equality if and only if there exists $y \in \Omega_n^{(1)}$ such that

$$U = \{ x \in \Omega_n : y \subset x \}.$$

Note that Conjecture 1 does not cover Theorem 1. Indeed, if p is fixed and n is sufficiently large, then

(2)
$$\sigma = \frac{[pn]}{[n] - [pn]} \sim q^{-(1-p)n}$$

Thus, Conjecture 1 applies to the case roughly $q^{-n} < \sigma < q^{-\frac{n}{2}}$. We were unable to prove (or disprove) this conjecture. Instead, we present some weaker results supporting it under more general settings.

Let t be a fixed positive integer. A family $U \subset \Omega_n$ of subspaces is called t-intersecting if $\dim(x \cap y) \ge t$ for all $x, y \in U$. Fix $y \in \Omega_n^{(t)}$, and define a t-intersecting family $A_n^{(t)}$ by

(3)
$$A_n^{(t)} := \{ x \in \Omega_n : y \subset x \}.$$

Note that $A_n^{(1)}$ is an optimal (1-)intersecting family in Theorem 1 and Conjecture 1. Using (1) (with σ replaced by σq^t) and $\sigma^k q^{\binom{k}{2}} = \sigma^t q^{\binom{t}{2}} \cdot (\sigma q^t)^{k-t} q^{\binom{k-t}{2}}$, we have

$$\sum_{k=t}^{n} {n-t \brack k-t} \sigma^{k} q^{\binom{k}{2}} = \sigma^{t} q^{\binom{t}{2}} (-\sigma q^{t}; q)_{n-t},$$

and so

(4)
$$\mu_{\sigma}(A_{n}^{(t)}) = \sum_{k=t}^{n} {n-t \brack k-t} \phi_{\sigma,n}(k) = \frac{\sigma^{t} q^{\binom{t}{2}}}{(-\sigma;q)_{t}} = \left(-\frac{1}{\sigma}; \frac{1}{q}\right)_{t}^{-1}.$$

In particular, $\mu_{\sigma}(A_n^{(1)}) = \frac{\sigma}{1+\sigma}$. We are interested in the maximum σ -biased measure of *t*-intersecting families, and let

$$f(n,t,\sigma) := \max\{\mu_{\sigma}(U)^{\frac{1}{n}} : U \subset \Omega_n \text{ is } t\text{-intersecting}\}$$

Problem 1. Find a condition for σ that guarantees $f(n,t,\sigma) = \mu_{\sigma}(A_n^{(t)})^{\frac{1}{n}}$.

Based on (2), we define

$$\sigma_{\theta,n} := q^{-(1-\theta)n},$$

and write

$$\mu_{\theta,n} := \mu_{\sigma_{\theta,n}}.$$

Then, for $0 < \theta < \frac{1}{2}$, we have

$$\lim_{n \to \infty} \mu_{\theta,n} (A_n^{(t)})^{\frac{1}{n}} = q^{-(1-\theta)t}$$

(see Lemma 2 in Section 3), and this is the best we can do approximately as shown below. **Theorem 2.** We have

$$\lim_{n \to \infty} f(n, t, \sigma_{\theta, n}) = \begin{cases} q^{-(1-\theta)t} & \text{if } 0 < \theta < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < \theta < 1. \end{cases}$$

Conjecture 2. We have

$$\lim_{n \to \infty} f(n, t, \sigma_{\frac{1}{2}, n}) = q^{-\frac{1}{2}t}.$$

Two families $U, W \subset \Omega_n$ are called cross t-intersecting if $\dim(x \cap y) \ge t$ for all $x \in U$ and $y \in W$. Let

$$g(n, t, \sigma_1, \sigma_2) := \max\{(\mu_{\sigma_1}(U)\mu_{\sigma_2}(W))^{\frac{1}{n}} : U, W \subset \Omega_n \text{ are cross } t \text{-intersecting}\}.$$

Theorem 3. If $0 < \theta_1, \theta_2 < \frac{1}{2}$, then $\lim_{n \to \infty} g(n, t, \sigma_{\theta_1, n}, \sigma_{\theta_2, n}) = q^{-(2-\theta_1 - \theta_2)t}$.

If $U \subset \Omega_n$ is t-intersecting, then U, U are cross t-intersecting. This gives us that

$$f(n,t,\sigma)^2 \leqslant g(n,t,\sigma,\sigma).$$

Thus, Theorem 3 implies Theorem 2 for the case $0 < \theta < \frac{1}{2}$.

In Section 2, we prove Theorem 1. To this end, we first translate the problem into a semidefinite programming problem and then solve it by computing eigenvalues of related matrices. In Section 3, we prove Theorem 2 by a probabilistic approach. For this, we use that the distribution $\binom{n}{k}\phi_{\sigma,n}(k)$ on the points $k = 0, 1, \ldots, n$ is concentrated around θn .

We also use the result (Theorem D) about the maximum size of t-intersecting families of subspaces of dimension k due to Frankl and Wilson [4]. In a similar way, we prove Theorem 3 in Section 4, where we need the result (Theorem G) about cross t-intersecting uniform families due to Cao, Lu, Lv, and Wang [2]. We mention that Theorem G partly generalizes the result (Theorem F) about the case t = 1 due to Suda and Tanaka [11], which was also proved by solving the corresponding semidefinite programming problem.

2. Proof of Theorem 1

For a non-empty finite set Λ , let \mathbb{R}^{Λ} be the set of real column vectors with coordinates indexed by Λ . For two non-empty finite sets Λ and Ξ , we also identify $\mathbb{R}^{\Lambda \times \Xi}$ with the set of real matrices with rows indexed by Λ and columns indexed by Ξ . When $\Lambda \subset \Lambda'$ and $\Xi \subset \Xi'$, we often view \mathbb{R}^{Λ} (resp. $\mathbb{R}^{\Lambda \times \Xi}$) as a subspace of $\mathbb{R}^{\Lambda'}$ (resp. $\mathbb{R}^{\Lambda' \times \Xi'}$) in the obvious manner. Define $W_{k,\ell}, \overline{W}_{k,\ell} \in \mathbb{R}^{\Omega_n^{(k)} \times \Omega_n^{(\ell)}}$ by

$$(W_{k,\ell})_{x,y} = \begin{cases} 1 & \text{if } x \subset y \text{ or } x \supset y, \\ 0 & \text{otherwise,} \end{cases} \quad (\overline{W}_{k,\ell})_{x,y} = \begin{cases} 1 & \text{if } x \cap y = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $x \in \Omega_n^{(k)}$, $y \in \Omega_n^{(\ell)}$. We define the subspaces U_i $(0 \leq i \leq \lfloor \frac{n}{2} \rfloor)$ of \mathbb{R}^{Ω_n} by

$$\boldsymbol{U}_i = \{ \boldsymbol{u} \in \mathbb{R}^{\Omega_n^{(i)}} : W_{i-1,i} \boldsymbol{u} = 0 \} \qquad (0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor)$$

where $W_{-1,0} := 0$. Since

$$W_{i-1,i}W_{i,i-1} = W_{i-1,i-2}W_{i-2,i-1} + q^{i-1} \begin{bmatrix} n-2i+2\\1 \end{bmatrix} W_{i-1,i-1} \qquad (1 \le i \le \lfloor \frac{n}{2} \rfloor)$$

and the RHS is positive definite, it follows that the matrices $W_{i-1,i}$ $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$ have full rank $\begin{bmatrix} n \\ i-1 \end{bmatrix}$ and hence

$$\dim \boldsymbol{U}_i = d_i := \begin{bmatrix} n\\ i \end{bmatrix} - \begin{bmatrix} n\\ i-1 \end{bmatrix} \qquad (0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor)$$

For $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we fix an orthonormal basis $u_{i,1}, u_{i,2}, \ldots, u_{i,d_i}$ of U_i , and define

(5)
$$\boldsymbol{u}_{i,r}^{k} = q^{-\frac{i(k-i)}{2}} \begin{bmatrix} n-2i\\ k-i \end{bmatrix}^{-\frac{1}{2}} W_{k,i} \boldsymbol{u}_{i,r} \qquad (1 \leqslant r \leqslant d_i, \ i \leqslant k \leqslant n-i).$$

In [11], it is shown that the $\boldsymbol{u}_{i,r}^k$ form an orthonormal basis of \mathbb{R}^{Ω_n} , and that

(6)
$$\overline{W}_{k,\ell}\boldsymbol{u}_{i,r}^{\ell} = \theta_i^{k,\ell}\boldsymbol{u}_{i,r}^k,$$

where $\boldsymbol{u}_{i,r}^k := 0$ if k < i or k > n - i, and

$$\theta_i^{k,\ell} = (-1)^i q^{\binom{i}{2}+k\ell-\frac{i(k+\ell)}{2}} {n-k-i \brack \ell-i} {n-2i \brack k-i}^{\frac{1}{2}} {n-2i \brack \ell-i}^{-\frac{1}{2}}$$
$$= (-1)^i \frac{q^{\binom{i}{2}+k\ell-\frac{i(k+\ell)}{2}}}{(q;q)_{n-k-\ell}} \left(\frac{(q;q)_{n-k-i}(q;q)_{n-\ell-i}}{(q;q)_{k-i}(q;q)_{\ell-i}}\right)^{\frac{1}{2}}.$$

We note that

$$\theta_i^{k,\ell} = \theta_i^{\ell,k} \qquad (i \leqslant k, \ell \leqslant n-i).$$

See also [4]. Thus, for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq r \leq d_i$, the subspace

$$oldsymbol{V}_{i,r} = ext{span} \{oldsymbol{u}_{i,r}^i, oldsymbol{u}_{i,r}^{i+1}, \dots, oldsymbol{u}_{i,r}^{n-i}\}$$

of \mathbb{R}^{Ω_n} is invariant under all the $\overline{W}_{k,\ell}$, and we have

$$\mathbb{R}^{\Omega_n} = \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{r=1}^{d_i} V_{i,r} \qquad \text{(orthogonal direct sum)}.$$

We have $d_0 = 1$, and without loss of generality, we set $\boldsymbol{u}_{0,1}$ to be the vector with 1 in coordinate $0 \in \Omega_n^{(0)}$, and 0 in all other coordinates. Let $\Delta \in \mathbb{R}^{\Omega_n \times \Omega_n}$ be the diagonal matrix with diagonal entries

$$\Delta_{x,x} = \mu_{\sigma}(x) = \phi_{\sigma,n}(\dim x) \qquad (x \in \Omega_n),$$

and let $J \in \mathbb{R}^{\Omega_n \times \Omega_n}$ be the matrix all of whose entries are 1. Let $S\mathbb{R}^{\Omega_n \times \Omega_n}$ denote the set of symmetric matrices in $\mathbb{R}^{\Omega_n \times \Omega_n}$. Following [8, 10], we formulate the problem of maximizing the μ_{σ} -biased measure of an intersecting family into a semidefinite programming problem as follows: imize $tr(\Lambda I \Lambda Y)$

(P): maximize
$$\operatorname{tr}(\Delta J \Delta X)$$

subject to $\operatorname{tr}(\Delta X) = 1, \ X \succeq 0, \ X \ge 0,$
 $X_{x,y} = 0 \text{ for } x, y \in \Omega_n, \ x \cap y = 0,$

where $X \in S\mathbb{R}^{\Omega_n \times \Omega_n}$ is the variable, tr means trace, and $X \succeq 0$ (resp. $X \ge 0$) means that X is positive semidefinite (resp. nonnegative). Indeed, if $x \in \mathbb{R}^{\Omega_n}$ is the characteristic vector of a non-empty intersecting family $U \subset \Omega_n$, then the matrix

(7)
$$X := \mu_{\sigma}(U)^{-1} \boldsymbol{x} \boldsymbol{x}^{\mathsf{T}} \in S \mathbb{R}^{\Omega_n \times \Omega_n}$$

satisfies all the constraints and we have $tr(\Delta J \Delta X) = \mu_{\sigma}(U)$. We recommend the introductory paper [14] on semidefinite programming. We note that the above semidefinite programming problem can be generalized to handle cross-intersecting families. See [11, 12], and also [7]. The dual problem for (P) is then given by

(D): minimize
$$\alpha$$

subject to $S := \alpha \Delta - \Delta J \Delta + A - Z \succcurlyeq 0, \ Z \ge 0,$
 $A_{x,y} = 0$ for $x, y \in \Omega_n, \ x \cap y \ne 0,$

where $\alpha \in \mathbb{R}$ and $A, Z \in S \mathbb{R}^{\Omega_n \times \Omega_n}$ are the variables. For any feasible solutions to (P) and (D), we have

(8)
$$\alpha - \operatorname{tr}(\Delta J \Delta X) = \operatorname{tr}((\alpha \Delta - \Delta J \Delta)X) = \operatorname{tr}((S - A + Z)X) \ge 0,$$

since $tr(SX) \ge 0$, $tr(ZX) \ge 0$, and tr(AX) = 0, and hence α gives an upper bound on $\mu_{\sigma}(U).$

Our goal now is to find a feasible solution (α, A, Z) to (D) with

$$\alpha := \frac{\sigma}{1+\sigma}.$$

Instead of working directly with the matrix ${\cal S}$ above, we consider the positive semidefiniteness of

$$S' := \Delta^{-\frac{1}{2}} S \Delta^{-\frac{1}{2}} = \alpha I - \Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}} + A' - Z',$$

where $Z' \ge 0$ and $(A')_{x,y} = 0$ whenever $x \cap y \neq 0$. We set

$$Z' := 0,$$

and choose A' of the form

(9)
$$A' = \sum_{k+\ell \leqslant n} \frac{a'_{k,\ell}}{\theta_0^{k,\ell}} \overline{W}_{k,\ell},$$

where $a'_{k,\ell} \in \mathbb{R}$. Let $\mathbf{1} \in \mathbb{R}^{\Omega_n}$ be the all-ones vector and recall our choice of the vector $\mathbf{u}_{0,1} \in \mathbf{U}_0 = \mathbb{R}^{\Omega_n^{(0)}}$. Then, we have (cf. (5))

$$\mathbf{1} = \sum_{k=0}^{n} W_{k,0} \boldsymbol{u}_{0,1} = \sum_{k=0}^{n} {n \brack k}^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{k},$$

so that

(10)
$$\Delta^{\frac{1}{2}} \mathbf{1} = \sum_{k=0}^{n} \phi_{k}^{\frac{1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{k},$$

where we abbreviate

$$\phi_k := \phi_{\sigma,n}(k) \qquad (0 \leqslant k \leqslant n).$$

Let $\mathbb{R}^{(n+1)\times(n+1)}$ be the set of real matrices with rows and columns indexed by $0, 1, \ldots, n$. Let $C \in \mathbb{R}^{(n+1)\times(n+1)}$ be the lower triangular matrix defined by

$$C_{k,\ell} = \begin{bmatrix} k \\ \ell \end{bmatrix} \qquad (0 \leqslant \ell \leqslant k \leqslant n).$$

Then, it follows that

$$(C^{-1})_{k,\ell} = \begin{bmatrix} k\\ \ell \end{bmatrix} (-1)^{k-\ell} q^{\binom{k-\ell}{2}} \qquad (0 \le \ell \le k \le n).$$

Indeed, for $0 \leq \ell \leq k \leq n$, we have

$$\sum_{j=\ell}^{k} C_{k,j} {j \brack \ell} (-1)^{j-\ell} q^{\binom{j-\ell}{2}} = {k \brack \ell} \sum_{j=\ell}^{k} {k-\ell \brack j-\ell} (-1)^{j-\ell} q^{\binom{j-\ell}{2}} = {k \brack \ell} (1;q)_{k-\ell} = \delta_{k,\ell}$$

by (1) (with $\sigma = -1$). Now, let $G \in \mathbb{R}^{(n+1) \times (n+1)}$ be the upper triangular matrix given by

$$G_{k,\ell} = \begin{bmatrix} n-k\\ n-\ell \end{bmatrix} (-1)^k \sigma^\ell q^{\binom{k}{2} + \binom{\ell}{2}} \qquad (0 \leqslant k \leqslant \ell \leqslant n),$$

and let

(11)
$$F = CGC^{-1}$$

Note that the diagonal entries of G are

$$(-1)^k \sigma^k q^{k(k-1)} \qquad (0 \leqslant k \leqslant n),$$

and these are the eigenvalues of G, and hence of F. We will set

(12)
$$a'_{k,\ell} := \frac{F_{k,\ell}}{1+\sigma} \cdot \frac{\phi_k^{\frac{1}{2}} {n \choose k}^{\frac{1}{2}}}{\phi_\ell^{\frac{1}{2}} {n \choose \ell}^{\frac{1}{2}}} \qquad (0 \le k \le n, \ 0 \le \ell \le n-k)$$

in (9), and show that the corresponding S gives an optimal feasible solution to (D). To describe the matrix F, we will use the following lemma.

Lemma 1. For integers a, b, and c such that $a \ge b \ge 0$ and $c \ge 0$, we have

$$\sum_{j=0}^{b} \begin{bmatrix} b\\ j \end{bmatrix} \begin{bmatrix} a-j\\ c \end{bmatrix} (-1)^{j} q^{\binom{j}{2}} = q^{b(a-c)} \begin{bmatrix} a-b\\ c-b \end{bmatrix}.$$

In particular, the LHS above vanishes if b > c.

Proof. First, we have

$$\begin{bmatrix} a-j\\c \end{bmatrix} = \sum_{d=0}^{c} q^{d(a-b-c+d)} \begin{bmatrix} a-b\\c-d \end{bmatrix} \begin{bmatrix} b-j\\d \end{bmatrix} \qquad (0 \le j \le b).$$

To see this, fix $z \in \Omega_{a-j}^{(a-b)}$ and count $x \in \Omega_{a-j}^{(c)}$ such that $\dim(x \cap z) = c - d$ for each d $(0 \leq d \leq c)$. Then, using (1) (with $\sigma = -1$), we have

$$\sum_{j=0}^{b} \begin{bmatrix} b \\ j \end{bmatrix} \begin{bmatrix} a-j \\ c \end{bmatrix} (-1)^{j} q^{\binom{j}{2}} = \sum_{d=0}^{c} q^{d(a-b-c+d)} \begin{bmatrix} a-b \\ c-d \end{bmatrix} \sum_{j=0}^{b} \begin{bmatrix} b \\ j \end{bmatrix} \begin{bmatrix} b-j \\ d \end{bmatrix} (-1)^{j} q^{\binom{j}{2}}$$
$$= \sum_{d=0}^{c} q^{d(a-b-c+d)} \begin{bmatrix} a-b \\ c-d \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} \sum_{j=0}^{b} \begin{bmatrix} b-d \\ j \end{bmatrix} (-1)^{j} q^{\binom{j}{2}}$$
$$= \sum_{d=0}^{c} q^{d(a-b-c+d)} \begin{bmatrix} a-b \\ c-d \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} (1;q)_{b-d}$$
$$= q^{b(a-c)} \begin{bmatrix} a-b \\ c-b \end{bmatrix},$$

as desired.

For $0 \leq k, \ell \leq n$, we have

(13)
$$F_{k,\ell} = \sum_{j,m=0}^{n} C_{k,j} G_{j,m} (C^{-1})_{m,\ell}$$
$$= \sum_{m=0}^{n} {m \brack \ell} (-1)^{m-\ell} \sigma^{m} q^{\binom{m}{2} + \binom{m-\ell}{2}} \sum_{j=0}^{n} {k \brack j} {n-j \brack n-m} (-1)^{j} q^{\binom{j}{2}}$$
$$= \sum_{m=0}^{n} {n-k \brack m} {m \brack \ell} (-1)^{m-\ell} \sigma^{m} q^{km + \binom{m}{2} + \binom{m-\ell}{2}}$$
$$= {n-k \brack \ell} \sigma^{\ell} q^{k\ell + \binom{\ell}{2}} \sum_{h=0}^{n-k-\ell} {n-k-\ell \brack h} (-1)^{h} \sigma^{h} q^{h(h+k+\ell-1)}$$

by Lemma 1 (with (a, b, c) = (n, k, n - m)), where we set $h = m - \ell$ in the last line above. In particular, it follows that F is upper anti-triangular, i.e., $F_{k,\ell} = 0$ whenever $k + \ell > n$. Moreover, it is immediate to see that

$$F_{k,\ell} \cdot \phi_k \begin{bmatrix} n \\ k \end{bmatrix} = F_{\ell,k} \cdot \phi_\ell \begin{bmatrix} n \\ \ell \end{bmatrix} \qquad (0 \leqslant k, \ell \leqslant n).$$

Thus, if we define the matrix A' in (9) by (12), then A' is symmetric since

$$a'_{k,\ell} = \frac{F_{\ell,k}}{1+\sigma} \cdot \frac{\phi_{\ell} {n \brack \ell}}{\phi_{k} {n \atop k}} \cdot \frac{\phi_{k}^{\frac{1}{2}} {n \atop \ell}}{\phi_{\ell}^{\frac{1}{2}} {n \atop \ell}}^{\frac{1}{2}} = a'_{\ell,k} \qquad (0 \leqslant k \leqslant n, \ 0 \leqslant \ell \leqslant n-k).$$

It seems that the entries of F have no simpler expression in general. However, if we let $q \rightarrow 1$, then

$$\lim_{q \to 1} G_{k,\ell} = \binom{n-k}{n-\ell} (-1)^k \sigma^\ell \qquad (0 \le k \le \ell \le n),$$

and we also have

$$\lim_{q \to 1} F_{k,\ell} = \binom{n-k}{\ell} \sigma^{\ell} (1-\sigma)^{n-k-\ell} \qquad (0 \le k \le n, \, 0 \le \ell \le n-k).$$

The relation (11) after taking the limit $q \to 1$ was shown earlier in [9, Lemmas 2.9, 2.21]. We also note that the above limit of F is closely related to the matrix $A^{(n)}$ (with $p = \frac{\sigma}{1+\sigma}$) considered in [5].

A 1-eigenvector of G is given by $(1, 0, ..., 0)^{\mathsf{T}} \in \mathbb{R}^{n+1}$, since G is upper triangular and $G_{0,0} = 1$. Then, a 1-eigenvector of $F = CGC^{-1}$ is

$$C(1, 0, \dots, 0)^{\mathsf{T}} = (1, 1, \dots, 1)^{\mathsf{T}}.$$

It follows from (6) that the matrix A' in (9) satisfies

$$A'\boldsymbol{u}_{0,1}^{\ell} = \sum_{k=0}^{n-\ell} a'_{k,\ell} \boldsymbol{u}_{0,1}^{k} \qquad (0 \leqslant \ell \leqslant n).$$

In other words, the matrix $A'_0 \in \mathbb{R}^{(n+1)\times(n+1)}$ representing the action of A' on the subspace $V_{0,1}$ with respect to the basis $\boldsymbol{u}_{0,1}^0, \boldsymbol{u}_{0,1}^1, \dots, \boldsymbol{u}_{0,1}^n$ is upper anti-triangular and is given by

$$(A'_0)_{k,\ell} = \begin{cases} a'_{k,\ell} & \text{if } k + \ell \leqslant n, \\ 0 & \text{if } k + \ell > n, \end{cases} \qquad (0 \leqslant k, \ell \leqslant n)$$

From now on, we define A' by (12). Then, since F is also upper anti-triangular, we have

(14)
$$A_0' = \frac{1}{1+\sigma} D_0 F D_0^{-1},$$

where $D_0 \in \mathbb{R}^{(n+1)\times(n+1)}$ is the diagonal matrix with diagonal entries $(D_0)_{k,k} = \phi_k^{\frac{1}{2}} {n \choose k}^{\frac{1}{2}} (0 \leq k \leq n)$. In particular, A'_0 has eigenvalues

$$\frac{(-1)^k \sigma^k q^{k(k-1)}}{1+\sigma} \qquad (0 \leqslant k \leqslant n).$$

Moreover, it follows from the above comment that a $\frac{1}{1+\sigma}$ -eigenvector of A'_0 is given by

(15)
$$\boldsymbol{w}_0 := D_0(1, 1, \dots, 1)^{\mathsf{T}} = \left(\phi_0^{\frac{1}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix}^{\frac{1}{2}}, \phi_1^{\frac{1}{2}} \begin{bmatrix} n \\ 1 \end{bmatrix}^{\frac{1}{2}}, \dots, \phi_n^{\frac{1}{2}} \begin{bmatrix} n \\ n \end{bmatrix}^{\frac{1}{2}} \right)^{\mathsf{T}}.$$

By (10), the matrix representing the action of $\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}} = (\Delta^{\frac{1}{2}} \mathbf{1}) (\Delta^{\frac{1}{2}} \mathbf{1})^{\mathsf{T}}$ on the subspace $V_{0,1}$ with respect to the same basis $\boldsymbol{u}_{0,1}^0, \boldsymbol{u}_{0,1}^1, \ldots, \boldsymbol{u}_{0,1}^n$ is $\boldsymbol{w}_0(\boldsymbol{w}_0)^{\mathsf{T}}$. Indeed, we have

$$\Delta^{\frac{1}{2}} J \Delta^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{\ell} = \phi_{\ell}^{\frac{1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}^{\frac{1}{2}} \Delta^{\frac{1}{2}} \mathbf{1} = \sum_{k=0}^{n} \phi_{k}^{\frac{1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}} \phi_{\ell}^{\frac{1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}^{\frac{1}{2}} \boldsymbol{u}_{0,1}^{k} \qquad (0 \leq \ell \leq n).$$

Since

$$(\boldsymbol{w}_0)^{\mathsf{T}} \boldsymbol{w}_0 = \sum_{k=0}^n \phi_k \begin{bmatrix} n \\ k \end{bmatrix} = \mu_{\sigma}(\Omega_n) = 1,$$

the matrix $\boldsymbol{w}_0(\boldsymbol{w}_0)^{\mathsf{T}}$ has \boldsymbol{w}_0 as a 1-eigenvector. Since $\Delta^{\frac{1}{2}}J\Delta^{\frac{1}{2}}$ is a rank-one matrix, this is the only nontrivial action of $\Delta^{\frac{1}{2}}J\Delta^{\frac{1}{2}}$, i.e., all the other eigenvalues are zero. Recall our choice of α and Z'. The vector \boldsymbol{w}_0 is an eigenvector of the action of

$$S' = \frac{\sigma}{1+\sigma}I - \Delta^{\frac{1}{2}}J\Delta^{\frac{1}{2}} + A$$

on $V_{0,1}$ with eigenvalue

$$\frac{\sigma}{1+\sigma} - 1 + \frac{1}{1+\sigma} = 0$$

The other n eigenvalues of S' on $V_{0,1}$ are given by

(16)
$$\frac{\sigma}{1+\sigma} + \frac{(-1)^k \sigma^k q^{k(k-1)}}{1+\sigma} \qquad (1 \leqslant k \leqslant n).$$

Next, we consider the actions of S' on the other subspaces $V_{i,r}$, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq r \leq d_i$. By (6), the matrix A'_i , indexed by $i, i + 1, \ldots, n - i$, representing the action of A' on $V_{i,r}$ with respect to the basis $\boldsymbol{u}_{i,r}^i, \boldsymbol{u}_{i,r}^{i+1}, \ldots, \boldsymbol{u}_{i,r}^{n-i}$ is given by

$$(A'_{i})_{k,\ell} = a'_{k,\ell} \cdot \frac{\theta_{i}^{k,\ell}}{\theta_{0}^{k,\ell}} = a'_{k,\ell} \cdot (-1)^{i} q^{\binom{i}{2} - \frac{i(k+\ell)}{2}} \left(\frac{(q;q)_{n-k-i}(q;q)_{n-\ell-i}(q;q)_{k}(q;q)_{\ell}}{(q;q)_{n-k}(q;q)_{n-\ell}(q;q)_{k-i}(q;q)_{\ell-i}}\right)^{\frac{1}{2}}$$

for $i \leq k, \ell \leq n-i$. We then have (cf. (12))

(17)
$$A'_{i} = \frac{1}{1+\sigma} D_{i} F_{i} D_{i}^{-1},$$

where

$$(F_i)_{k,\ell} = F_{k,\ell} \cdot (-1)^i q^{\binom{i}{2} - ik} \frac{(q;q)_{n-k-i}(q;q)_\ell}{(q;q)_{n-k}(q;q)_{\ell-i}} \qquad (i \le k, \ell \le n-i),$$

and D_i is diagonal with diagonal entries

$$(D_i)_{k,k} = \phi_k^{\frac{1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}} \cdot q^{\frac{ik}{2}} \left(\frac{(q;q)_{n-k}(q;q)_k}{(q;q)_{n-k-i}(q;q)_{k-i}} \right)^{\frac{1}{2}} \qquad (i \le k \le n-i).$$

If we write $F = F_{n;\sigma}$ to specify the parameters, then using (13),

$$\begin{bmatrix} n-k\\ \ell \end{bmatrix} = \begin{bmatrix} n-k-i\\ \ell-i \end{bmatrix} \frac{(q;q)_{n-k}(q;q)_{\ell-i}}{(q;q)_{n-k-i}(q;q)_{\ell}},$$

and

$$k\ell = (k-i)(\ell-i) + i(k+\ell) - i^2, \qquad \binom{\ell}{2} = \binom{\ell-i}{2} + i\ell - \binom{i+1}{2},$$

it is routinely verified that

(18)
$$(F_i)_{k,\ell} = (F_{n-2i;\sigma q^{2i}})_{k-i,\ell-i} \cdot (-1)^i \sigma^i q^{i(i-1)} \qquad (i \le k, \ell \le n-i),$$

where we note that the rows and columns of $F_{n-2i;\sigma q^{2i}}$ are indexed by $0, 1, \ldots, n-2i$. It follows that the eigenvalues of A' on $V_{i,r}$ are given by

$$(-1)^{h} (\sigma q^{2i})^{h} q^{h(h-1)} \cdot \frac{(-1)^{i} \sigma^{i} q^{i(i-1)}}{1+\sigma} = \frac{(-1)^{k} \sigma^{k} q^{k(k-1)}}{1+\sigma} \qquad (i \le k \le n-i)$$

where h = k - i, and therefore those of S' are

(19)
$$\frac{\sigma}{1+\sigma} + \frac{(-1)^k \sigma^k q^{k(k-1)}}{1+\sigma} \qquad (i \le k \le n-i)$$

For the matrix S' to be positive semidefinite, all the eigenvalues in (16) and (19) must be nonnegative. This is equivalent to

$$\sigma^k q^{k(k-1)} \leqslant \sigma \qquad (1 \leqslant k \leqslant n, \ k : \text{odd}),$$

which then simplifies to the condition given in Theorem 1 (when $n \ge 3$). If this condition is satisfied, then the matrix $S = \Delta^{\frac{1}{2}} S' \Delta^{\frac{1}{2}}$ gives a feasible solution to (D) with objective value $\frac{\sigma}{1+\sigma}$, which is attained by $A_n^{(1)}$ defined by (3).

For the rest of the proof, assume that $\sigma < q^{-2\lfloor \frac{n-1}{2} \rfloor - 1}$. Let $U \subset \Omega_n$ be an intersecting family such that $\mu_{\sigma}(U) = \frac{\sigma}{1+\sigma}$. Let $\boldsymbol{x} \in \mathbb{R}^{\Omega_n}$ be the characteristic vector of U, and let the matrix $X \in S\mathbb{R}^{\Omega_n \times \Omega_n}$ be as in (7). Then, equality is attained in (8), and hence it follows that $\operatorname{tr}(SX) = 0$, or equivalently, $S'\Delta^{\frac{1}{2}}\boldsymbol{x} = 0$.

Recall that \boldsymbol{w}_0 is a 0-eigenvector of the action of S' on $\boldsymbol{V}_{0,1}$. The corresponding 0eigenvector of S' (in \mathbb{R}^{Ω_n}) is (cf. (10))

(20)
$$\boldsymbol{v}_0 := \Delta^{\frac{1}{2}} \mathbf{1} = \sum_{k=0}^n \phi_k^{\frac{1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}} \boldsymbol{u}_{0,1}^k.$$

The eigenvalues of S' in (19) are zero if and only if (i, k) = (1, 1), in which case, we can similarly see that the corresponding 0-eigenvectors of S' are of the form

$$\boldsymbol{v}_r = \sum_{k=1}^{n-1} \eta_k \boldsymbol{u}_{1,r}^k \qquad (1 \leqslant r \leqslant d_1),$$

where $\eta_k \neq 0$ for all k. More specifically,

$$(\eta_1, \eta_2, \dots, \eta_{n-1})^{\mathsf{T}} = D_1(1, 1, \dots, 1)^{\mathsf{T}} = ((D_1)_{1,1}, (D_1)_{2,2}, \dots, (D_1)_{n-1,n-1})^{\mathsf{T}}.$$

See (17) and (18). On the other hand, the eigenvalues of S' in (16) are zero if and only if k = 1. Since G is upper triangular, and $G_{0,0} = 1$ and $G_{1,1} = -\sigma$, a $(-\sigma)$ -eigenvector

of G is given by $(\nu, 1, 0, \dots, 0)^{\mathsf{T}} \in \mathbb{R}^{n+1}$, where $\nu = -\frac{G_{0,1}}{1+\sigma}$. Then, a $(-\sigma)$ -eigenvector of $F = CGC^{-1}$ is

$$C(\nu, 1, 0, \dots, 0)^{\mathsf{T}} = \left(\nu + \begin{bmatrix} 0\\1 \end{bmatrix}, \nu + \begin{bmatrix} 1\\1 \end{bmatrix}, \nu + \begin{bmatrix} 2\\1 \end{bmatrix}, \dots, \nu + \begin{bmatrix} n\\1 \end{bmatrix}\right)^{\mathsf{T}},$$

and the corresponding 0-eigenvector of S' becomes (cf. (14))

$$oldsymbol{v}_0' = \sum_{k=0}^n \phi_k^{rac{1}{2}} igg[n \\ k igg]^{rac{1}{2}} igg(
u + igg[k \\ 1 igg] igg) oldsymbol{u}_{0,1}^k.$$

Since $S'\Delta^{\frac{1}{2}}\boldsymbol{x} = 0$, the vector $\Delta^{\frac{1}{2}}\boldsymbol{x}$ must be a linear combination of the above vectors:

$$\Delta^{\frac{1}{2}}\boldsymbol{x} = c_0\boldsymbol{v}_0 + c_1\boldsymbol{v}_1 + \dots + c_{d_1}\boldsymbol{v}_{d_1} + c'_0\boldsymbol{v}'_0.$$

Expand the RHS above in terms of the $\boldsymbol{u}_{i,r}^k$. Note that U does not contain $0 \in \Omega_n^{(0)}$, so that the coefficient of $\boldsymbol{u}_{0,1}^0$ is zero, i.e., $c_0 + c'_0 \nu = 0$. Suppose now that $U \cap \Omega_n^{(1)} = \emptyset$. Then, the coefficients of $\boldsymbol{u}_{0,1}^1$ and $\boldsymbol{u}_{1,r}^1$ $(1 \leq r \leq d_1)$ are all zero because these vectors form a basis of $\mathbb{R}^{\Omega_n^{(1)}}$. That the coefficient of $\boldsymbol{u}_{0,1}^1$ equals zero is equivalent to $c_0 + c'_0(\nu + 1) = 0$. Combining this with $c_0 + c'_0\nu = 0$, we have $c_0 = c'_0 = 0$. Moreover, the coefficient of $\boldsymbol{u}_{1,r}^1$ equals $c_r\eta_1$ for $1 \leq r \leq d_1$, and hence $c_1 = \cdots = c_{d_1} = 0$ since $\eta_1 \neq 0$. It follows that $\Delta^{\frac{1}{2}}\boldsymbol{x} = 0$, which is absurd. We have now shown that $U \cap \Omega_n^{(1)} \neq \emptyset$. Since U is a maximal intersecting family, we must have $U = A_n^{(1)}$ for some $y \in \Omega_n^{(1)}$. This completes the proof of Theorem 1.

3. Proof of Theorem 2

In this section, we abbreviate

 $\mu := \mu_{\theta,n}, \qquad \sigma := \sigma_{\theta,n},$

except in the statement of lemmas. We will also write $\phi = \phi_{\theta,n} := \phi_{\sigma_{\theta,n},n}$.

Lemma 2. If $0 < \theta < \frac{1}{2}$, then

$$\lim_{n \to \infty} \mu_{\theta, n} (A_n^{(t)})^{\frac{1}{n}} = q^{-(1-\theta)t}.$$

Proof. By (4), we have

(21)
$$\mu(A_n^{(t)}) = \prod_{j=0}^{t-1} \left(1 + q^{(1-\theta)n-j}\right)^{-1}.$$

Since $(1+q^{(1-\theta)n-j})^{-\frac{1}{n}} \to q^{-(1-\theta)}$ for $0 \leq j \leq t-1$, we have $\mu(A_n^{(t)})^{\frac{1}{n}} \to q^{-(1-\theta)t}$.

The next result shows that the distribution

$$\Phi(k) = \Phi_{\theta,n}(k) := \begin{bmatrix} n \\ k \end{bmatrix} \phi_{\theta,n}(k) \qquad (k = 0, 1, \dots, n)$$

concentrates around $k \sim n\theta$.

Lemma 3. Let $0 < \theta < 1$. For every $\epsilon > 0$, there exists L > 1 such that

$$\sum_{|k-\theta n|>L} \Phi_{\theta,n}(k) < \epsilon$$

for sufficiently large n, where the sum is over all k = 0, 1, ..., n such that $|k - \theta n| > L$. Proof. Define a probability measure $\Psi = \Psi_{\theta,n}$ on the points $q^{k-\theta n}$ (k = 0, 1, ..., n) by

$$\Psi(q^{k-\theta n}) = \Phi(k) \qquad (k = 0, 1, \dots, n).$$

By (1), the mean is computed as

$$\mathbb{E}[X] = \sum_{k=0}^{n} q^{k-\theta n} \Phi(k) = \frac{(-\sigma q; q)_n}{q^{\theta n} (-\sigma; q)_n} = \frac{1+\sigma q^n}{q^{\theta n} (1+\sigma)},$$

which converges to 1 when $n \to \infty$. Also,

$$\mathbb{E}[X^2] = \sum_{k=0}^n q^{2k-2\theta n} \Phi(k) = \frac{(-\sigma q^2; q)_n}{q^{2\theta n} (-\sigma; q)_n} = \frac{(1+\sigma q^n)(1+\sigma q^{n+1})}{q^{2\theta n} (1+\sigma)(1+\sigma q)},$$

which converges to q when $n \to \infty$. Hence, the variance satisfies

$$\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \to q - 1.$$

In what follows, let n be sufficiently large so that $\mathbb{E}[X] < 2$ and $\mathbb{V}[X] < q$. There exists L > 1 such that $(q^L - 2)^2 \epsilon > 2q$. Then, we have

$$q > \mathbb{V}[X] \ge \sum_{k > \theta n + L} (q^{k - \theta n} - \mathbb{E}[X])^2 \Phi(k) \ge (q^L - 2)^2 \sum_{k > \theta n + L} \Phi(k).$$

from which it follows that

$$\sum_{k>\theta n+L} \Phi(k) < \frac{q}{(q^L-2)^2} < \frac{\epsilon}{2}.$$

Next, we have

$$\mathbb{E}[X^{-1}] = \frac{q^{\theta n}(1 + \sigma q^{-1})}{1 + \sigma q^{n-1}} \to q, \qquad \mathbb{E}[X^{-2}] = \frac{q^{2\theta n}(1 + \sigma q^{-2})(1 + \sigma q^{-1})}{(1 + \sigma q^{n-2})(1 + \sigma q^{n-1})} \to q^3,$$

 \mathbf{SO}

$$\mathbb{V}[X^{-1}] = \mathbb{E}[X^{-2}] - \mathbb{E}[X^{-1}]^2 \to q^3 - q^2.$$

By a similar argument, we can show that there exists L' > 1 such that

$$\sum_{k < \theta n - L'} \Phi(k) < \frac{\epsilon}{2},$$

for sufficiently large n. The result now follows by replacing L by L' if L' > L.

For our purpose, we need a stronger tail bound.

Claim 4. Let $0 < \theta < \frac{1}{2}$. We have

(22)
$$\sum_{k>\frac{n}{2}} \Phi_{\theta,n}(k) = o(q^{-(1-\theta)tn}),$$

where the sum is over all integers k with $\frac{n}{2} < k \leq n$.

Proof. We write $m := \lceil \frac{n}{2} \rceil$ and $s := \lceil \theta n \rceil$ for typographical reasons. First, we claim that

(23)
$$\Phi(m) < q^{-\frac{1}{2}(m-s)^2 + O(n)} = q^{-\frac{1}{2}(\frac{1}{2} - \theta)^2 n^2 + O(n)}$$

In view of Lemma 3, we estimate (cf. (1))

(24)
$$\Phi(m) = \frac{{\binom{n}{m}}\sigma^m q^{\binom{m}{2}}}{\sum_{k=0}^n {\binom{n}{k}}\sigma^k q^{\binom{k}{2}}} < \frac{{\binom{n}{m}}\sigma^m q^{\binom{m}{2}}}{{\binom{n}{s}}\sigma^s q^{\binom{s}{2}}} = \frac{{\binom{n}{m}}}{{\binom{n}{s}}}\sigma^{m-s} q^{\binom{m}{2}-\binom{s}{2}}$$

For further estimation of the RHS above, we note that

$$\frac{[n-m]}{[s]} = \frac{q^{n-m}-1}{q^s-1} < q^{n-m-s+1} = q^{m-s+O(1)},$$

and so

(25)
$$\frac{\binom{n}{m}}{\binom{n}{s}} = \frac{[s]![n-s]!}{[m]![n-m]!} = \prod_{j=1}^{m-s} \frac{[n-m+j]}{[s+j]} < \left(\frac{[n-m]}{[s]}\right)^{m-s} < q^{(m-s)^2 + O(n)}$$

for sufficiently large n. We also have

(26)
$$\sigma^{m-s}q^{\binom{m}{2}-\binom{s}{2}} = \left(q^{-(1-\theta)n}\right)^{m-s}q^{\frac{m^2}{2}-\frac{s^2}{2}+O(n)} = q^{-\frac{3}{2}(m-s)^2+O(n)}.$$

Substituting (25) and (26) into the RHS of (24), we get (23).

Next, we verify that $\Phi(k)$ is decreasing in k for $k \ge m$. Indeed, since $\frac{[n-k]}{[k+1]} < q^{n-2k}$ as above, it follows that

$$\frac{\Phi(k+1)}{\Phi(k)} = \frac{{\binom{n}{k+1}}\sigma^{k+1}q^{\binom{k+1}{2}}}{{\binom{n}{k}}\sigma^{k}q^{\binom{k}{2}}} = \frac{[n-k]}{[k+1]}\sigma q^{k} < q^{\theta n-k} \leqslant q^{\theta n-m} < 1.$$

Hence, it follows from (23) that

$$\sum_{k > \frac{n}{2}} \Phi(k) < m \Phi(m) < q^{-\frac{1}{2}(\frac{1}{2} - \theta)^2 n^2 + O(n)} = o(q^{-(1-\theta)tn})$$

for sufficiently large n.

Remark 1. The RHS of (22) can be replaced by $o(q^{-Rn})$ for any fixed R > 0. For our purpose, we need $R \ge (1 - \theta)t$.

We now invoke the following result.

Theorem D (Frankl–Wilson [4]). Let $n \ge 2k$. If a family $U \subset \Omega_n^{(k)}$ is t-intersecting, then $|U| \le {n-t \brack k-t}$.

For the characterization of the optimal families in Theorem D, see [6, 13].

Lemma 5. If $0 < \theta < \frac{1}{2}$, then $\lim_{n \to \infty} f(n, t, \sigma_{\theta, n}) = q^{-(1-\theta)t}$.

Proof. By Lemma 2, we have

$$f(n,t,\sigma) \ge \mu(A_n^{(t)})^{\frac{1}{n}} \to q^{-(1-\theta)t}.$$

On the other hand, let $U_n \subset \Omega_n$ be a *t*-intersecting family satisfying $\mu(U_n) = f(n, t, \sigma)^n$, and let $U_n^{(k)} = U_n \cap \Omega_n^{(k)}$. We have

$$\mu(U_n) = \sum_{k=0}^n |U_n^{(k)}| \phi(k) \leqslant \sum_{k \leqslant \frac{n}{2}} |U_n^{(k)}| \phi(k) + \sum_{k > \frac{n}{2}} \Phi(k).$$

By Theorem D and (21),

$$\sum_{k \leq \frac{n}{2}} |U_n^{(k)}| \phi(k) \leq \sum_{k \leq \frac{n}{2}} {n-t \brack k-t} \phi(k) \leq \mu(A_n^{(t)}) = \prod_{j=0}^{t-1} \left(1 + q^{(1-\theta)n-j}\right)^{-1},$$

and by Claim 4,

$$\sum_{k>\frac{n}{2}} \Phi(k) = o(q^{-(1-\theta)tn}).$$

Thus, we have

$$\mu(U_n) \leqslant \mu(A_n^{(t)}) + o(q^{-(1-\theta)tn})$$

= $\mu(A_n^{(t)}) \left(1 + o\left(\prod_{j=0}^{t-1} (q^{-(1-\theta)n} + q^{-j})\right) \right)$
= $\mu(A_n^{(t)})(1 + o(1)).$

Finally, it follows from Lemma 2 that

$$f(n,t,\sigma) = \mu(U_n)^{\frac{1}{n}} \leqslant \mu(A_n^{(t)})^{\frac{1}{n}} (1+o(1))^{\frac{1}{n}} \to q^{-(1-\theta)t}.$$

Claim 6. Let $\frac{1}{2} < \theta < 1$. There exists $\delta > 0$ such that

$$\sum_{k < \frac{n+t}{2}} \Phi_{\theta,n}(k) < q^{-\delta n^2}$$

for sufficiently large n, where the sum is over all integers k with $0 \leq k < \frac{n+t}{2}$.

Proof. The proof is similar to that of Claim 4. We write $m' := \lceil \frac{n+t}{2} \rceil$ and $s := \lceil \theta n \rceil$. We first claim that

(27)
$$\Phi(m') < q^{-\frac{1}{2}(s-m')^2 + O(n)} = q^{-\frac{1}{2}(\theta - \frac{1}{2})^2 n^2 + O(n)}$$

We have (cf. (1))

$$\Phi(m') = \frac{\binom{n}{m'}\sigma^{m'}q^{\binom{m'}{2}}}{\sum_{k=0}^{n}\binom{n}{k}\sigma^{k}q^{\binom{k}{2}}} < \frac{\binom{n}{m'}\sigma^{m'}q^{\binom{m'}{2}}}{\binom{n}{s}\sigma^{s}q^{\binom{s}{2}}} = \frac{\binom{n}{m'}}{\binom{n}{s}}\sigma^{m'-s}q^{\binom{m'}{2}-\binom{s}{2}}$$

Since $\frac{[m']}{[n-s]} < q^{m'-n+s+1} = q^{s-m'+O(1)}$, it follows that

$$\frac{\binom{n}{m'}}{\binom{n}{s}} = \frac{[s]![n-s]!}{[m']![n-m']!} = \prod_{j=1}^{s-m'} \frac{[m'+j]}{[n-s+j]} < \left(\frac{[m']}{[n-s]}\right)^{s-m'} = q^{(s-m')^2 + O(n)}$$

for sufficiently large n. We also have

$$\sigma^{m'-s}q^{\binom{m'}{2}-\binom{s}{2}} = \left(q^{-(1-\theta)n}\right)^{m'-s}q^{\frac{(m')^2}{2}-\frac{s^2}{2}+O(n)} = q^{-\frac{3}{2}(s-m')^2+O(n)}.$$

Hence, we get (27).

Next, we verify that $\Phi(k)$ is increasing in k for $k \leq m'$, provided that n is sufficiently large. This follows from $\frac{[k]}{[n-k+1]} < q^{2k-n}$ and

$$\frac{\Phi(k-1)}{\Phi(k)} = \frac{\binom{n}{k-1}\sigma^{k-1}q^{\binom{k-1}{2}}}{\binom{n}{k}\sigma^{k}q^{\binom{k}{2}}} = \frac{[k]}{[n-k+1]\sigma q^{k-1}} < q^{k-\theta n+1} \le q^{m'-\theta n+1} < 1.$$

Hence, if we choose δ such that $0 < \delta < \frac{1}{2}(\theta - \frac{1}{2})^2$, then we have

$$\sum_{k < \frac{n+t}{2}} \Phi(k) < m' \Phi(m') < q^{-\delta n^2}$$

for sufficiently large n.

Lemma 7. If $\frac{1}{2} < \theta < 1$, then $\lim_{n \to \infty} f(n, t, \sigma_{\theta, n}) = 1$.

Proof. Clearly, we have $f(n, t, \sigma) \leq 1$. So we need to show that $\underline{\lim}_{n\to\infty} f(n, t, \sigma) \geq 1$. Define a *t*-intersecting family B_n by

$$B_n := \left\{ x \in \Omega_n : \dim x \ge \frac{n+t}{2} \right\}.$$

By Claim 6, we have

$$\mu(B_n) = 1 - \sum_{k < \frac{n+t}{2}} \Phi(k) > 1 - q^{-\delta n^2}$$

for sufficiently large n. Then, the desired inequality follows from

$$f(n,t,\sigma) \ge \mu(B_n)^{\frac{1}{n}} \ge \left(1 - q^{-\delta n^2}\right)^{\frac{1}{n}} \to 1.$$

Proof of Theorem 2. Immediate from Lemma 5 and Lemma 7.

4. Proof of Theorem 3

Here, we list some results concerning cross t-intersecting families of uniform subspaces. We omit the descriptions of the optimal families.

Theorem E (Tokushige [15]). Let $n \ge 2k$. If $U \subset \Omega_n^{(k)}$ and $W \subset \Omega_n^{(k)}$ are cross tintersecting, then $|U||W| \le {n-t \choose k-t}^2$.

Theorem F (Suda–Tanaka [11]). Let $n \ge 2k$ and $n \ge 2\ell$. If $U \subset \Omega_n^{(k)}$ and $W \subset \Omega_n^{(\ell)}$ are cross 1-intersecting, then $|U||W| \le {n-1 \brack \ell-1} {n-1 \brack \ell-1}$.

Theorem G (Cao–Lu–Lv–Wang [2]). Let $n \ge k + \ell + t + 1$. If $U \subset \Omega_n^{(k)}$ and $W \subset \Omega_n^{(\ell)}$ are cross t-intersecting, then $|U||W| \le {n-t \brack k-t} {n-t \brack \ell-t}$.

For the proof of Theorem 3, we use Theorem G. Note that while Theorem G is the most general result so far, it does not fully contain Theorem E and Theorem F.

The next claim can be shown exactly in the same way (with slightly more cumbersome computation) as Claim 4, and we omit the proof. See also Remark 1.

Claim 8. Let $0 < \theta < \frac{1}{2}$, and let t be a fixed positive integer. Then, we have

$$\sum_{k>\frac{n-t-1}{2}} \Phi_{\theta,n}(k) = o(q^{-2tn}),$$

where the sum is over all integers k with $\frac{n-t-1}{2} < k \leq n$.

Proof of Theorem 3. For i = 1, 2, we write $\sigma_i := \sigma_{\theta_i,n}$, $\phi_i := \phi_{\theta_i,n}$, and $\mu_i := \mu_{\theta_i,n}$ for brevity. Suppose that cross t-intersecting families $U_n, W_n \subset \Omega_n$ satisfy

$$g(n, t, \sigma_1, \sigma_2)^n = \mu_1(U_n)\mu_2(W_n),$$

and let $U_n^{(k)} = U_n \cap \Omega_n^{(k)}$ and $W_n^{(\ell)} = W_n \cap \Omega_n^{(\ell)}$. If $k, \ell \leq \frac{n-t-1}{2}$, then $n \geq k + \ell + t + 1$, and we can apply Theorem G to $U_n^{(k)}$ and $W_n^{(\ell)}$. By Claim 8, we may write

$$\mu_1(U_n) = \sum_{k \leq \frac{n-t-1}{2}} |U_n^{(k)}| \phi_1(k) + o(q^{-2tn}),$$

$$\mu_2(W_n) = \sum_{\ell \leq \frac{n-t-1}{2}} |W_n^{(\ell)}| \phi_2(\ell) + o(q^{-2tn}).$$

Then, by Theorem G, we have

$$\begin{split} \left(\sum_{k\leqslant\frac{n-t-1}{2}} |U_n^{(k)}|\phi_1(k)\right) &\left(\sum_{\ell\leqslant\frac{n-t-1}{2}} |W_n^{(\ell)}|\phi_2(\ell)\right) \\ &= \sum_{k\leqslant\frac{n-t-1}{2}} \sum_{\ell\leqslant\frac{n-t-1}{2}} |U_n^{(k)}| |W_n^{(\ell)}|\phi_1(k)\phi_2(\ell) \\ &\leqslant \sum_{k\leqslant\frac{n-t-1}{2}} \sum_{\ell\leqslant\frac{n-t-1}{2}} {n-t \brack k-t} {n-t \brack \ell-t} \phi_1(k)\phi_2(\ell) \\ &= \left(\sum_{k\leqslant\frac{n-t-1}{2}} {n-t \brack k-t} \phi_1(k)\right) \left(\sum_{\ell\leqslant\frac{n-t-1}{2}} {n-t \brack \ell-t} \phi_2(\ell)\right) \\ &\leqslant \mu_1(A_n^{(t)})\mu_2(A_n^{(t)}), \end{split}$$

where $A_n^{(t)}$ is defined by (3). It follows that

$$\mu_1(U_n)\mu_2(W_n) = \left(\sum_{k \leq \frac{n-t-1}{2}} |U_n^{(k)}|\phi_1(k)\right) \left(\sum_{\ell \leq \frac{n-t-1}{2}} |W_n^{(\ell)}|\phi_2(\ell)\right) + o(q^{-2tn})$$
$$\leq \mu_1(A_n^{(t)})\mu_2(A_n^{(t)}) + o(q^{-2tn}).$$

Thus, by using Lemma 2, we have

 $g(n, t, \sigma_1, \sigma_2) = (\mu_1(U_n)\mu_2(W_n))^{\frac{1}{n}} \leqslant \left(\mu_1(A_n^{(t)})\mu_2(A_n^{(t)}) + o(q^{-2tn})\right)^{\frac{1}{n}} \to q^{-(2-\theta_1-\theta_2)t}.$ The opposite inequality follows from

$$g(n,t,\sigma_1,\sigma_2) \ge \left(\mu_1(A_n^{(t)})\mu_2(A_n^{(t)})\right)^{\frac{1}{n}} \to q^{-(2-\theta_1-\theta_2)t}.$$

This completes the proof of Theorem 3.

5. Concluding Remarks

5.1. Theorem 1 from Hoffman's bound. Here, we show that the inequality in Theorem 1 can also be interpreted as an application of the so-called Hoffman's bound. Let $\mathcal{G} = (\Omega, E)$ be a finite simple graph with vertex set Ω and edge set E, and let $\phi : \Omega \to [0, 1]$ be a weight function such that $\sum_{x \in \Omega} \phi(x) = 1$. Let $\mu : 2^{\Omega} \to [0, 1]$ be the probability measure on Ω defined by $\mu(U) := \sum_{x \in U} \phi(x)$. We say that $U \subset \Omega$ is independent if no two vertices in U are adjacent in \mathcal{G} . The independence measure of \mathcal{G} , denoted by $\alpha(\mathcal{G})$, is the maximum of $\mu(U)$ over independent sets $U \subset \Omega$. We say that a symmetric matrix $B \in \mathbb{R}^{\Omega \times \Omega}$ reflects adjacency in \mathcal{G} if

- $B_{x,y} = 0$ whenever $\{x, y\} \notin E$, and
- B has an eigenvector $\boldsymbol{w} \in \mathbb{R}^{\Omega}$ such that $\boldsymbol{w}_x = \sqrt{\phi(x)} = \sqrt{\mu(x)}$ $(x \in \Omega)$.

Let $u_1, u_2, \ldots, u_{|\Omega|}$ be eigenvectors of B. We may assume that they form an orthonormal basis with respect to the standard inner product, and that $u_1 = w$. Let λ_i denote the eigenvalue for u_i , and let $\lambda_{\min} := \min\{\lambda_i : 2 \leq i \leq |\Omega|\}$. Under these assumptions, we have the following upper bound for $\alpha(\mathcal{G})$ in terms of λ_1 and λ_{\min} .

Lemma 9 (Hoffman's bound). We have $\alpha(\mathcal{G}) \leq \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}}$.

The original version of Hoffman's bound assumes that \mathcal{G} is a regular graph and that μ is the uniform measure on Ω , but its proof works for the above general version: evaluate $\boldsymbol{v}^{\mathsf{T}} B \boldsymbol{v}$ in two ways for an independent set $U \subset \Omega$, where $\boldsymbol{v} \in \mathbb{R}^{\Omega}$ is defined by

$$\boldsymbol{v}_x = \begin{cases} \sqrt{\mu(x)} & \text{if } x \in U, \\ 0 & \text{if } x \notin U, \end{cases} \quad (x \in \Omega).$$

Let the vertex set of \mathcal{G} be $\Omega := \Omega_n$, where two distinct vertices $x, y \in \Omega_n$ are adjacent whenever $x \cap y = 0$. Observe that $U \subset \Omega_n$ is independent in \mathcal{G} if and only if U is an intersecting family. Let $\phi := \phi_{\sigma,n}, \mu := \mu_{\sigma}$, and B := A' from (9) with (12). Then it follows that A' reflects the adjacency, and that $\boldsymbol{w} = \Delta^{\frac{1}{2}} \mathbf{1} = \boldsymbol{v}_0$ is a $\frac{1}{1+\sigma}$ -eigenvector of A'(see (15) and (20)). Since $\lambda_1 = \frac{1}{1+\sigma}$ and $\lambda_{\min} = -\frac{\sigma}{1+\sigma}$ by (16) and (19), Hoffman's bound yields $\alpha(\mathcal{G}) \leq \frac{\sigma}{1+\sigma}$. Moreover, in this case, $\alpha(\mathcal{G})$ is the maximum μ_{σ} -biased measure of intersecting families in Ω_n , and so we obtain the inequality in Theorem 1. In fact, we can apply our semidefinite programming formulation conversely to prove Hoffman's bound in general. This was already done in [8] for the original version. We adopted this formulation in this paper for possible extendability; see, e.g., [12].

5.2. Comparison of Theorem G with the corresponding subset version. For the case t = 1, Theorem G reads as follows.

Corollary 1. Let $n \ge k + \ell + 2$. If $U \subset \Omega_n^{(k)}$ and $W \subset \Omega_n^{(\ell)}$ are cross 1-intersecting, then $|U||W| \le {n-1 \brack k-1} {n-1 \brack \ell-1}$.

If we replace $\Omega_n^{(k)}$ and $\Omega_n^{(\ell)}$ in the above result with $\binom{X_n}{k}$ and $\binom{X_n}{\ell}$, respectively, then the situation becomes more complicated; see [16]. Indeed, for any large M > 0, we can find n, k, ℓ with $n = k + \ell + M$ and cross 1-intersecting families $U \subset \binom{X_n}{k}$ and $W \subset \binom{X_n}{\ell}$ such that

(28)
$$|U||W| > {\binom{n-1}{k-1}} {\binom{n-1}{\ell-1}}$$

To see this, let

$$U := \left\{ x \in \binom{X_n}{k} : x \cap X_2 \neq \emptyset \right\}, \qquad W := \left\{ x \in \binom{X_n}{\ell} : X_2 \subset x \right\}.$$

Then, U and W are cross 1-intersecting, and $|U| = \binom{n}{k} - \binom{n-2}{k}$, $|W| = \binom{n-2}{\ell-2}$. In this case, (28) is equivalent to $(2n - k - 1)k(\ell - 1) > (n - 1)^2$, and this condition is satisfied, for example, if $k \ge 2$, $\ell = 9k$, n = 17k. This means that for any given M = 7k, we can construct families satisfying (28), and so the condition $n \ge k + \ell + M$ is not sufficient to get the upper bound $\binom{n-1}{\ell-1}\binom{n-1}{\ell-1}$ for the product of the sizes of cross 1-intersecting families.

It is interesting to determine whether the condition $n \ge k + \ell + 2$ in the corollary is sharp or not. We cannot replace the condition with $n \ge k + \ell$. To see this, let $n = k + \ell$, $k = 1, \ell \ge 3$, and fix $z \in \Omega_n^{(2)}$. Then

$$U := \{ x \in \Omega_n^{(1)} : x \subset z \}, \qquad W := \{ x \in \Omega_n^{(\ell)} : z \subset x \}$$

are cross 1-intersecting, and

$$|U||W| = \begin{bmatrix} 2\\1 \end{bmatrix} \begin{bmatrix} n-2\\\ell-2 \end{bmatrix} = \begin{bmatrix} 2\\1 \end{bmatrix} \begin{bmatrix} \ell-1\\1 \end{bmatrix} > \begin{bmatrix} \ell\\1 \end{bmatrix} = \begin{bmatrix} n-1\\k-1 \end{bmatrix} \begin{bmatrix} n-1\\\ell-1 \end{bmatrix}.$$

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