

Stochastic Bregman Subgradient Methods for Nonsmooth Nonconvex Optimization Problems

Kuangyu Ding*, Kim-Chuan Toh†

April 29, 2024

Abstract

This paper focuses on the problem of minimizing a locally Lipschitz continuous function. Motivated by the effectiveness of Bregman gradient methods in training nonsmooth deep neural networks and the recent progress in stochastic subgradient methods for nonsmooth nonconvex optimization problems [11, 12, 50], we investigate the long-term behavior of stochastic Bregman subgradient methods in such context, especially when the objective function lacks Clarke regularity. We begin by exploring a general framework for Bregman-type methods, establishing their convergence by a differential inclusion approach. For practical applications, we develop a stochastic Bregman subgradient method that allows the subproblems to be solved inexactly. Furthermore, we demonstrate how a single timescale momentum can be integrated into the Bregman subgradient method with slight modifications to the momentum update. Additionally, we introduce a Bregman proximal subgradient method for solving composite optimization problems possibly with constraints, whose convergence can be guaranteed based on the general framework. Numerical experiments on training nonsmooth neural networks are conducted to validate the effectiveness of our proposed methods.

Keywords: Nonsmooth nonconvex optimization, Clarke regularity, Conservative field, Bregman subgradient methods, Deep learning.

1 Introduction

In this paper, we focus on exploring the application of the Bregman distance based stochastic subgradient methods for solving nonsmooth nonconvex optimization problems. We consider the following unconstrained optimization problem,

$$\min_{x \in \mathbb{R}^n} f(x), \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally Lipschitz continuous function (possibly nonconvex and nonsmooth). This class of problems encompasses many important applications, particularly in training nonsmooth neural networks where nonsmooth activate functions, such as the rectified linear unit (ReLU), are employed.

*Department of Mathematics, National University of Singapore, Singapore 119076 (kuangyud@u.nus.edu).

†Department of Mathematics, and Institute of Operations Research and Analytics, National University of Singapore, Singapore 119076 (mattohkc@nus.edu.sg).

First order methods are commonly used to solve (1). In many scenarios, only noisy gradients or subgradients are available. During the last several decades, a great number of stochastic first order methods have been proposed. Among these, stochastic gradient descent (SGD) could be the most fundamental method. Based on SGD, a great number of its variants have also been developed to obtain some benefits on speed-up, stability, or memory efficiency. For example, the heavy-ball SGD [40], signSGD [9], and normalized SGD [21, 53, 54]. These methods have demonstrated great efficiency and competitive generalization performance in various tasks in deep learning. Despite the progress in developing SGD-type methods, the conventional convergence analysis largely pertains to scenarios where the objective function f exhibits certain regularity properties, namely, differentiability or weakly convexity. However, the prevalence of nonsmooth activation functions, such as ReLU or Leaky ReLU, in neural network architectures results in loss functions that often lack Clarke regularity (e.g. differentiability, weak convexity). Consequently, the conventional convergence analysis of SGD-type methods is not applicable in the context of training nonsmooth neural networks. Towards this issue, Bolte and Pauwels [12] introduced the concept of conservative field, which generalizes the concept of Clarke subdifferential and admits chain rule and sum rule even for functions without Clarke regularity. Leveraging on the concept of conservative field, subsequent studies [11, 12, 17, 24, 34, 50, 51] have utilized a differential inclusion approach [7, 15] to establish convergence for SGD-type methods in the training of nonsmooth deep neural networks.

Beyond SGD-type methods, stochastic Bregman gradient methods, also known as mirror descent methods, have gained increasing interest recently. Initially introduced by Nemirovski and Yudin [39] for solving constrained convex problems, the fundamental update scheme of Bregman gradient method is given as follows,

$$x_{k+1} = \operatorname{argmin}_{x \in \mathbb{R}^n} \left\{ \langle g_k, x - x_k \rangle + \frac{1}{\eta_k} \mathcal{D}_\phi(x, x_k) \right\}, \quad (2)$$

where \mathcal{D}_ϕ represents the Bregman distance, a generalized measure of distance provided by a kernel function ϕ , with its definition to be presented in Section 2. A distinctive feature of Bregman gradient methods is the utilization of a broader selection of kernel functions, extending beyond the classical kernel function of $\frac{1}{2} \|\cdot\|^2$ used in SGD. Therefore, SGD is a special instance of Bregman gradient methods. The application of Bregman gradient methods has expanded to encompass both convex and nonconvex problems, as well as deterministic and stochastic contexts, as demonstrated by subsequent works [5, 6, 13, 23, 25, 29, 37, 52, 56]. Bregman gradient methods have found many applications in optimization problems associated with probability constraints, such as optimal transport problem [8, 18] and reinforcement learning [33, 55]. Recent works have increasingly highlighted the potential of Bregman gradient methods in neural network training. Notably, [23] has shown that Bregman gradient methods can achieve better generalization performance and enhanced robustness in stepsize tuning compared to SGD for certain deep learning tasks. Additionally, many recent works, such as [4, 28, 35, 46, 49], have demonstrated that Bregman gradient methods exhibit implicit regularization, leading to improved generalization performance with the selection of an appropriate kernel function. Additionally, [3, 27, 36] provide an interpretation of the classical gradient descent method on reparameterized models from the perspective of Bregman gradient methods. This perspective has led to some applications in deep learning, including neural network quantization, as illustrated by [1].

However, existing convergence analysis for Bregman gradient methods has been limited to objective functions that are either differentiable or weakly convex. Given the increasing interest in

applying Bregman gradient methods to train nonsmooth neural networks, and the current limited understanding of the convergence properties of Bregman subgradient methods for solving (1), this paper aims to provide a theoretical convergence guarantee for Bregman subgradient methods applied to nonsmooth nonconvex problems, particularly for the training of nonsmooth neural networks. Moreover, we propose practical Bregman subgradient methods specifically for training nonsmooth neural networks while ensuring their convergence.

To summarize, the contributions of this paper are as follows:

- **General Bregman differential inclusion:** We first investigate a general Bregman differential inclusion, whose discrete update scheme is consistent with that of Bregman-type methods and allows for biased evaluations of the abstract set-valued mapping. We establish the convergence for the discrete update scheme associated with this Bregman differential inclusion. Specifically, we demonstrate that any cluster point of the sequence generated by the discrete update scheme lies in the stable set of the Bregman differential inclusion, and the Lyapunov function values converge. A key aspect of our approach is the utilization of linear interpolation of the dual sequence induced by the kernel function.
- **Applications of the general Bregman differential inclusion:** By exploiting the flexibility of choosing the kernel function and the set valued mapping in the general framework, we introduce three types of stochastic Bregman subgradient methods for different scenarios. First, we consider the vanilla stochastic Bregman subgradient method (SBG) for unconstrained optimization problem, and establish its convergence. We further show that under certain regularity conditions of the kernel function, a preconditioned subgradient method can be regarded as an inexact Bregman subgradient method, thus fitting it within our proposed method. Moreover, we propose a single timescale momentum based stochastic Bregman subgradient method (MSBG), and establish its convergence by the sophisticated choice of the kernel function and set-valued mapping. Lastly, we consider the stochastic Bregman proximal subgradient method (SBPG) for constrained composite optimization problem and establish its convergence.
- **Numerical experiments:** To evaluate the performance of our proposed stochastic Bregman subgradient methods, we employ a block-wise polynomial kernel function in our proposed Bregman subgradient methods. We conduct numerical experiments to compare SGD, the momentum based stochastic Bregman subgradient method (MSBG), and its inexact version (iMSBG), for training nonsmooth neural networks. The results illustrate that our Bregman subgradient methods can achieve generalization performance comparable to that of SGD as well as the enhanced robustness of stepsize tuning.

The remaining sections of the paper are structured as follows. In Section 2, we provide preliminary materials on notations, set-valued mapping, conservative field, and Bregman proximal mapping. Section 3 is dedicated to establishing the convergence of the general Bregman-type method based on the associated differential inclusion. The specific applications of the general Bregman-type method are presented in Section 4, where we first consider the vanilla stochastic Bregman subgradient method and then propose a single timescale momentum based Bregman subgradient method. The latter part of Section 4 explores the stochastic Bregman proximal subgradient method. Section 5 conducts numerical experiments to demonstrate the performance of our proposed Bregman subgradient methods. In the last section, we give a conclusion on this paper.

2 Preliminary

2.1 Notations

Given a proper and lower semicontinuous function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$, we denote its domain as $\text{dom } f = \{x : f(x) < \infty\}$. The Fenchel conjugate function of f is defined as $f^*(y) := \sup\{\langle x, y \rangle - f(x) : x \in \mathbb{R}^n\}$. For a set $\mathcal{S} \subset \mathbb{R}^n$, we use $\text{cl } \mathcal{S}$ to denote its closure, $\text{int } \mathcal{S}$ to denote the set of its interior points, and $\text{conv}(\mathcal{S})$ to denote its convex hull. A function $f : \mathcal{S} \rightarrow \mathbb{R}$ is said to be of class $\mathcal{C}^k(\mathcal{S})$ if it is k times differentiable and the k -th derivative is continuous on \mathcal{S} . When there is no ambiguity regarding the domain, we simply use the notation \mathcal{C}^k . We let $\mathcal{C}(A, B)$ be the set of continuous mappings from set A to set B . We use $\|\cdot\|$ to denote the Euclidean norm for vectors and the Frobenius matrix norm for matrices. The d -dimensional unit ball is denoted by \mathbb{B}^d . The distance between a point w and a set A is denoted by $\text{dist}(w, A) := \inf\{\|w - u\| : u \in A\}$. We use the convention $\text{dist}(w, \emptyset) := \infty$. The Minkowski sum of two sets A and B is denoted as $A + B := \{u + v : u \in A, v \in B\}$. We use $\alpha A := \{\alpha u : u \in A\}$ to denote the scaled set of A by a given scalar α . For a positive sequence $\{\eta_k\}$, we define $\lambda_\eta(0) := 0$, $\lambda_\eta(k) := \sum_{i=0}^{k-1} \eta_i$ for $k \geq 1$, and $\Lambda_\eta(t) := \sup\{k : \lambda_\eta(k) \leq t\}$. In other word, $\Lambda_\eta(t) = k$ if and only if $\lambda_\eta(k) \leq t < \lambda_\eta(k+1)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider a stochastic process $\{\xi_k\}_{k \geq 0}$ and a filtration $\{\mathcal{F}_k\}_{k \geq 0}$, where \mathcal{F}_k is defined by the σ -algebra $\mathcal{F}_k := \sigma(\xi_0, \dots, \xi_k)$ on Ω , the conditional expectation is denoted as $\mathbb{E}[\cdot | \mathcal{F}_k]$.

2.2 Set-valued mapping and Clarke subdifferential

In this subsection, we present definitions and concepts from set-valued analysis, mainly based on [41], and recall the concept of Clarke subdifferential [19].

Definition 2.1. A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a mapping from \mathbb{R}^n to a collection of subsets of \mathbb{R}^m . S is said to be closed if its graph, defined by

$$\text{graph}(S) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in S(x)\} \quad (3)$$

is a closed set in $\mathbb{R}^n \times \mathbb{R}^m$. Give a nonnegative δ , the δ -perturbed set of S is defined by

$$S^\delta(x) := \bigcup_{\{y \in \mathbb{R}^m : \|y - x\| \leq \delta\}} (S(y) + \delta \mathbb{B}^m).$$

Definition 2.2. Let $\mathcal{X} \subset \mathbb{R}^n$. A set-valued mapping $S : \mathcal{X} \rightrightarrows \mathbb{R}^m$ is called outer semicontinuous at $\bar{x} \in \mathcal{X}$ if for any sequence $x_i \xrightarrow{\mathcal{X}} \bar{x}$ and $v_i \in S(x_i)$ converging to some $\bar{v} \in \mathbb{R}^m$, we have $\bar{v} \in S(\bar{x})$. S is said to be outer semicontinuous if it is outer semicontinuous everywhere over \mathcal{X} .

Proposition 2.1. ([41, Theorem 5.7, Proposition 5.12]) Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. If $\text{graph}(S)$ is closed, then S is outer semicontinuous. In addition, if $S(\bar{x})$ is closed, then for any $\rho > 0$ and $\epsilon > 0$, there is a neighborhood V of \bar{x} , such that

$$S(x) \cap \rho \mathbb{B}^m \subset S(\bar{x}) + \epsilon \mathbb{B}^m, \text{ for all } x \in V.$$

Definition 2.3. Let $\mathcal{X} \subset \mathbb{R}^n$ be a closed set, the regular normal cone at $\bar{x} \in \mathcal{X}$ is defined as $\hat{N}_{\mathcal{X}}(\bar{x}) := \{v \in \mathbb{R}^n : \langle v, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|), \text{ for } x \in \mathcal{X}\}$. The limiting normal cone is defined by $N_{\mathcal{X}}^L(\bar{x}) := \{v \in \mathbb{R}^n : x_k \xrightarrow{\mathcal{X}} \bar{x}, v_k \in \hat{N}_{\mathcal{X}}(x_k), v_k \rightarrow v\}$. The normal cone is defined by $N_{\mathcal{X}}(\bar{x}) := \text{cl conv}(N_{\mathcal{X}}^L(\bar{x}))$.

According to [41, Proposition 6.6], the limiting normal cone $N_{\mathcal{X}}^L$ is outer semicontinuous, while this is not necessarily true for $N_{\mathcal{X}}$.

Definition 2.4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function. The generalized directional derivative of f at $x \in \mathbb{R}^n$ along the direction $d \in \mathbb{R}^n$ is defined by

$$f^\circ(x; d) := \lim_{y \rightarrow x, t \downarrow 0} \sup \frac{f(y + td) - f(y)}{t}.$$

The Clarke subdifferential of f at x is defined by

$$\partial f(x) := \{v \in \mathbb{R}^n : \langle v, d \rangle \leq f^\circ(x; d), \text{ for all } d \in \mathbb{R}^n\}.$$

Definition 2.5. We say that f is Clarke regular, if for any $d \in \mathbb{R}^n$, its one-side directional derivative, defined by

$$f^*(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t},$$

exists and $f^\circ(x; d) = f^*(x; d)$.

Clarke regularity excludes functions whose graph has upwards corners, such as $-|x|$. Notably, some basic calculus rules, including the sum rule and chain rule, may not be applicable to the Clarke subdifferential in the absence of Clarke regularity. For further details, readers may refer to [19, Chapter 2].

2.3 Conservative field, path differentiability

In this subsection, we briefly introduce some relevant materials on conservative field, which are mainly based on [12].

Definition 2.6. (Conservative field and path-differentiability) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function. A set-valued mapping $D_f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a conservative field for f if it is nonempty closed valued, and has closed graph. For any absolutely continuous curve $\gamma : [0, 1] \rightarrow \mathbb{R}^n$, f admits a chain rule with respect to D_f , i.e.

$$\frac{d}{dt}(f \circ \gamma)(t) = \langle v, \dot{\gamma}(t) \rangle, \text{ for all } v \in D_f(\gamma(t)) \text{ and almost all } t \in [0, 1]. \quad (4)$$

If a locally Lipschitz function f admits a conservative field D_f , then we say that f is path-differentiable, and f is the potential function of D_f .

More generally, we can define conservative mappings for vector-valued functions, which serve as a generalization of the Jacobian.

Definition 2.7. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a locally Lipschitz continuous function. $J_F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m \times n}$ is called a conservative mapping for F , if for any absolutely continuous curve $\gamma : [0, 1] \rightarrow \mathbb{R}^n$, the following chain rule holds for almost all $t \in [0, 1]$:

$$\frac{d}{dt}F(\gamma(t)) = V\dot{\gamma}(t), \text{ for any } V \in J_F(\gamma(t)).$$

Definition 2.8. Given a closed set $\mathcal{X} \subset \mathbb{R}^n$, we say that \mathcal{X} admits a chain rule, if for any absolutely continuous curve $\gamma : [0, 1] \rightarrow \mathcal{X}$, it holds that

$$\langle N_{\mathcal{X}}(\gamma(t)), \dot{\gamma}(t) \rangle = \{0\}, \text{ for almost all } t \in [0, 1].$$

As demonstrated in [22], when \mathcal{X} is Whitney stratifiable (e.g. \mathcal{X} is definable), then \mathcal{X} admits the chain rule. We also use $\frac{d}{dt}(f \circ \gamma)(t) = \langle D_f(\gamma(t)), \dot{\gamma}(t) \rangle$ for almost all $t \in [0, 1]$ to represent that the chain rule (4) is valid for almost all $t \in [0, 1]$. As stated in [12, Remark 3], if D_f is a conservative field for f , then $\text{conv}(D_f)$ is also a conservative field for f , and D_f is locally bounded. Consequently, from Proposition 2.1, it follows that for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, there exist a neighborhood V of x , such that $\cup_{y \in V} D_f(y) \subset D_f(x) + \varepsilon \mathbb{B}^n$. This property is a key ingredient for establishing the convergence of subgradient methods based on the concept of conservative field. It is worth noting that the conservative field for a function f is not unique. The following two lemmas provide insights into the relationship between the conservative field and the Clarke subdifferential. In particular, the latter lemma indicates that the Clarke subdifferential serves as the smallest conservative field among all convex valued conservative fields. We say that x is a D_f -critical point if $0 \in D_f(x)$, to differentiate it from the conventional critical point defined in terms of Clarke subdifferential.

Lemma 2.1. ([12, Theorem 1]) Consider a locally Lipschitz continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, with D_f as its conservative field. Then, $D_f(x) = \{\nabla f(x)\}$ for almost all $x \in \mathbb{R}^n$.

Lemma 2.2. ([12, Corollary 1]) Consider a locally Lipschitz continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and its Clarke subdifferential ∂f , as well as its any convex valued conservative field D_f . Then, ∂f is a conservative field of f . Moreover, for any $x \in \mathbb{R}^n$, it holds that $\partial f(x) \subset D_f(x)$.

The next two lemmas provide the key motivation for introducing the concept of conservative field, as it highlights that conservative fields preserve some basic calculus rules such as the chain rule that do not necessarily hold for Clarke subdifferential without Clarke regularity.

Lemma 2.3. Let $F_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $F_2 : \mathbb{R}^m \rightarrow \mathbb{R}^l$ be locally Lipschitz continuous vector-valued functions, and $J_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m \times n}$ and $J_2 : \mathbb{R}^m \rightrightarrows \mathbb{R}^{l \times m}$ be the conservative mappings of F_1 and F_2 respectively. Then the mapping $J_2 \circ J_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^{l \times n}$ is a conservative mapping for $F_2 \circ F_1$.

Lemma 2.4. Let D_{f_i} be a conservative field for f_i , $i = 1, \dots, N$. Then, $D_f = \sum_{i=1}^N D_{f_i}$ is a conservative field of $f = \sum_{i=1}^N f_i$.

As previously mentioned, the Clarke subdifferential may not adhere to certain calculus rules, whereas the conservative field does as demonstrated by Lemma 2.3 and Lemma 2.4. These favorable properties of the conservative field enable the well-definedness of many gradient methods based on automatic differentiation (AD) in deep learning. The range of path-differentiable functions covers a wide range of objective functions in real-world applications. According to [12, Proposition 2], if f or $-f$ is Clarke regular, then f is path-differentiable. Therefore, functions with upwards corners, such as $f(x) = -|x|$, are path-differentiable despite not being Clarke regular at 0. Another important class of path-differentiable are semi-algebraic functions, or more broadly, definable functions in an o-minimal structure [20, 48]. Definable functions encompass a vast majority of objective functions encountered in real-world applications. For instance, commonly used nonlinear activation functions and loss functions in deep learning, such as the sigmoid, softmax, ReLU activate functions, l_1 loss, cross-entropy loss, hinge loss, and logistic loss, are all definable. Furthermore, definability can

be preserved under composition, finite summation, and set-valued integration under certain mild conditions, as demonstrated in [11, 12, 22]. This preservation implies that neural networks built by definable blocks are themselves definable functions, thereby rendering gradient methods employed in deep learning, such as those based on automatic differentiation (AD), to be well-defined in terms of the conservative field rather than the Clarke subdifferential. For a detailed discussion on the definition of definable functions within o-minimal structures, readers can refer to [11, Section 4.1], with [20, 48] providing comprehensive insights into this theory. While the class of definable functions covers a wide range of function classes, it exhibits a relatively simple geometric structure known as Whitney stratification. This inherent simplicity in the geometric structure of definable functions enables the analysis of nonsmooth optimization algorithms [10, 12, 22].

We end this subsection by introducing the concept of functional convergence over any compact set, as discussed in literature such as [7, 22, 26]. Given a sequence of mappings $f_n \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$, we say that f_n converges to f in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$ if, for any $T > 0$, it holds that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|f_n(t) - f(t)\| = 0. \quad (5)$$

This notion of convergence is equivalent to the convergence in the metric

$$d(f, g) = \sum_{t=1}^{\infty} 2^{-t} \sup_{s \in [0, t]} \min\{\|f(s) - g(s)\|, 1\},$$

which makes the space of continuous functions $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$ a complete metric space.

2.4 Bregman proximal mapping

In this subsection, we review some concepts related to Bregman proximal mappings. Given that this paper mainly focuses on unconstrained problems, we restrict our discussion on the entire space \mathbb{R}^n . For more general concepts about Bregman gradient mapping, readers can refer to works such as [5, 13, 23].

Definition 2.9. (*Kernel function and Bregman distance over \mathbb{R}^n*). A function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a kernel function over \mathbb{R}^n , if ϕ is convex and $\phi \in \mathcal{C}^1(\mathbb{R}^n)$. The Bregman distance [16] generated by ϕ is denoted as $\mathcal{D}_\phi(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, +\infty)$, where

$$\mathcal{D}_\phi(x, y) = \phi(x) - \phi(y) - \langle \nabla \phi(y), x - y \rangle.$$

Definition 2.10. (*Legendre kernel over \mathbb{R}^n*). Let ϕ be a kernel function over \mathbb{R}^n , such that $\lim_{k \rightarrow \infty} \|\nabla \phi(x_k)\| = \infty$ whenever $\{x_k\}_{k \in \mathbb{N}}$ satisfies $\lim_{k \rightarrow \infty} \|x_k\| = \infty$. The function ϕ is called a Legendre function over \mathbb{R}^n if it is also strictly convex on \mathbb{R}^n .

Definition 2.11. Given a locally Lipschitz continuous function R and a Legendre kernel function $\phi \in \mathcal{C}^1(\mathbb{R}^n)$, we denote the Bregman proximal mapping by $\text{Prox}_R^\phi := (\nabla \phi + \partial R)^{-1} \nabla \phi$, which is a set-valued mapping defined as follows,

$$\text{Prox}_R^\phi(x) := \operatorname{argmin}_{u \in \mathbb{R}^n} \{R(u) + \mathcal{D}_\phi(u, x)\}. \quad (6)$$

Under mild conditions, $\text{Prox}_R^\phi(x)$ is a nonempty compact set for any $x \in \mathbb{R}^n$, which will ensure the well-posedness of our methods. We have the following lemma, which directly follows from Weierstrass's theorem, we omit its proof for simplicity.

Lemma 2.5. Let R be a continuous function and ϕ be a Legendre function over \mathbb{R}^n . Suppose $\alpha R + \phi$ is supercoercive, i.e. $\lim_{\|x\| \rightarrow \infty} \frac{\alpha R(x) + \phi(x)}{\|x\|} = \infty$, for any $\alpha > 0$. Then, for any $x \in \mathbb{R}^n$, the set $\text{Prox}_R^\phi(x)$ is a nonempty compact subset of \mathbb{R}^n .

In the context of Bregman proximal gradient method, we focus on the Bregman forward-backward splitting operator $T_{\alpha,R}^\phi : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by:

$$T_{\alpha,R}^\phi(x, v) := \operatorname{argmin}_{z \in \mathbb{R}^n} \left\{ \langle v, z - x \rangle + R(z) + \frac{1}{\alpha} \mathcal{D}_\phi(z, x) \right\}.$$

For any given $v \in \mathbb{R}^n$ and $\alpha > 0$, under the same assumption in Lemma 2.5, $T_{\alpha,R}^\phi(x, v)$ is also well defined.

3 Bregman differential inclusion

In this section, we investigate the Bregman-type differential inclusion and its discrete approximation, which coincides with the iterative Bregman subgradient methods. The analysis tools employed in this section draw inspiration from various works on stochastic approximation, such as [7, 15, 22, 26, 45]. To establish the convergence of the discrete sequence $\{x_k\}$, a key idea is to show that the linear interpolation of the sequence $\{x_k\}$ defined by

$$x(t) := x_k + \frac{t - \lambda_\eta(k)}{\eta_k} (x_{k+1} - x_k), \quad t \in [\lambda_\eta(k), \lambda_\eta(k+1)) \quad (7)$$

is a perturbed solution [7] to the associated differential inclusion, where $\{\eta_k\}$ serves as the stepsize in the subgradient methods. However, due to the non-Euclidean nature of Bregman subgradient methods, this methodology requires modification. Given a kernel function ϕ and a general set-valued mapping \mathcal{H} , we consider the following general differential inclusion:

$$\frac{d}{dt} \nabla \phi(x(t)) \in -\mathcal{H}(x(t)), \quad \text{for almost all } t \geq 0. \quad (8)$$

Any absolutely continuous solution to (8) is termed a trajectory of (8). The stable set of (8) is defined as

$$\mathcal{H}^{-1}(0) := \{x \in \mathbb{R}^n : 0 \in \mathcal{H}(x)\}. \quad (9)$$

For any \mathcal{C}^2 convex function ϕ , the differential inclusion (8) can be interpreted as a gradient flow equipped with the Riemannian metric induced by $\langle \cdot, \cdot \rangle_{\nabla^2 \phi(x)}$, as demonstrated in works such as [2, 14]. The corresponding discrete scheme of (8) is given by

$$\nabla \phi(x_{k+1}) = \nabla \phi(x_k) - \eta_k(d_k + \xi_k), \quad (10)$$

where d_k is an evaluation of $\mathcal{H}(x_k)$ with possible inexactness, and ξ_k is the stochastic noise. This formulation, referred to as the general Bregman-type method, is notable for the versatile choices of ϕ and \mathcal{H} . For the Bregman counterpart of the interpolated process (7) in the Euclidean setting, we introduce the linear interpolation for the dual sequence $\{\nabla \phi(x_k)\}$:

$$x(t) := \nabla \phi^* \left(\nabla \phi(x_k) + \frac{t - \lambda_\eta(k)}{\eta_k} (\nabla \phi(x_{k+1}) - \nabla \phi(x_k)) \right), \quad t \in [\lambda_\eta(k), \lambda_\eta(k+1)). \quad (11)$$

If $\phi^* \in C^1(\mathbb{R}^n)$, then (11) is well defined. Let $x^t(\cdot)$ denote the time-shifted curve of the interpolated process, i.e. $x^t(\cdot) = x(t + \cdot)$. We make the following assumptions on (8) and (10) to ensure that the iterative sequence generated by (10) tracks a trajectory of (8) asymptotically.

Assumption 3.1. 1. ϕ is a supercoercive Legendre kernel function over \mathbb{R}^n , and $\nabla\phi$ is differentiable almost everywhere.

2. The sequences $\{x_k\}$, $\{\nabla\phi(x_k)\}$ and $\{d_k\}$ are uniformly bounded.
3. The stepsize $\{\eta_k\}$ satisfies $\sum_{k=0}^{\infty} \eta_k = \infty$ and $\lim_{k \rightarrow \infty} \eta_k = 0$.
4. For any $T > 0$, the noise sequence $\{\xi_k\}$ satisfies

$$\lim_{s \rightarrow \infty} \sup_{s \leq i \leq \Lambda_{\eta}(\lambda_{\eta}(s) + T)} \left\| \sum_{k=s}^i \eta_k \xi_k \right\| = 0. \quad (12)$$

5. The set-valued mapping \mathcal{H} has a closed graph. Additionally, for any unbounded increasing sequence $\{k_j\}$ such that $\{x_{k_j}\}$ converges to \bar{x} , it holds that

$$\lim_{N \rightarrow \infty} \text{dist} \left(\frac{1}{N} \sum_{j=1}^N d_{k_j}, \mathcal{H}(\bar{x}) \right) = 0. \quad (13)$$

Here are some remarks on Assumption 3.1.

Remark 3.1. 1. If the Legendre function ϕ is supercoercive, i.e. $\lim_{\|u\| \rightarrow \infty} \frac{\phi(u)}{\|u\|} = \infty$, then by [42, Theorem 26.5, Corollary 13.3.1], $\phi^* \in C^1(\mathbb{R}^n)$ is strictly convex, and $(\nabla\phi)^{-1} = \nabla\phi^*$.

2. Uniform boundedness of either $\{x_k\}$ or $\{\nabla\phi(x_k)\}$ may suffice under mild conditions. For example, if ϕ is locally strongly convex, then the uniform boundedness of $\{\nabla\phi(x_k)\}$ implies the uniform boundedness of $\{x_k\}$. Conversely, if $\nabla\phi$ is locally Lipschitz continuous, then the uniform boundedness of $\{x_k\}$ leads to the uniform boundedness of $\{\nabla\phi(x_k)\}$.
3. For a martingale difference noise sequence $\{\xi_k\}$, i.e. $\mathbb{E}[\xi_{k+1} | \mathcal{F}_k] = 0$ holds almost surely, as shown in [7], (12) can be ensured almost surely under one of the following conditions:

- (a) Uniform boundedness of $\{\xi_k\}$ with $\eta_k = o\left(\frac{1}{\log k}\right)$ as utilized in [17, 50, 51].
- (b) Variance-bounded $\{\xi_k\}$, i.e. $\mathbb{E}[\|\xi_k\|^2 | \mathcal{F}_k] \leq \sigma^2$ for some $\sigma > 0$, and square-summable $\{\eta_k\}$, i.e. $\sum_{k=0}^{\infty} \eta_k^2 < \infty$, as employed in [22].

To establish under Assumption 3.1 that, the discrete sequence generated by the general Bregman-type method (10) tracks a trajectory of the differential inclusion (8), we introduce a piecewise constant mapping $y(\cdot)$ defined by $y(s) = d_k$ for any $s \in [\lambda_{\eta}(k), \lambda_{\eta}(k+1))$. We also define a time-shifted solution $x_t(\cdot)$ to the following ordinary differential equation:

$$\frac{d}{ds} \nabla\phi(x_t(s)) = -y(s) \text{ for all } s \geq t, \text{ with initial condition } \nabla\phi(x_t(t)) = \nabla\phi(x(t)).$$

The following lemma suggests that the interpolated process $x(t)$ defined by (11) asymptotically approximates this time-shifted solution. We remind the reader to note the difference between $x_t(\cdot)$ and $x^t(\cdot)$.

Lemma 3.1. Suppose Assumption 3.1 holds, then for any $T > 0$, it holds that

$$\lim_{t \rightarrow \infty} \sup_{s \in [t, t+T]} \|\nabla \phi(x(s)) - \nabla \phi(x_t(s))\| = 0. \quad (14)$$

Proof. Fix an arbitrary $s \in [t, t+T]$, let $\tau_t = \Lambda_\eta(t)$, $\tau_s = \Lambda_\eta(s)$. By the definition of $x_t(\cdot)$, we have

$$\begin{aligned} \nabla \phi(x_t(s)) &= \nabla \phi(x(t)) - \int_t^{\lambda_\eta(\tau_t)} y(u) du - \int_{\lambda_\eta(\tau_t)}^{\lambda_\eta(\tau_s)} y(u) du - \int_{\lambda_\eta(\tau_s)}^s y(u) du \\ &= \nabla \phi(x(\lambda_\eta(\tau_t))) - \sum_{i=\tau_t}^{\tau_s-1} \eta_i d_i + \left(\nabla \phi(x(t)) - \int_t^{\lambda_\eta(\tau_t)} y(u) du - \nabla \phi(x(\lambda_\eta(\tau_t))) \right) - \int_{\lambda_\eta(\tau_s)}^s y(u) du \\ &= \nabla \phi(x(\lambda_\eta(\tau_s))) + \sum_{i=\tau_t}^{\tau_s-1} \eta_i \xi_i + \left(\nabla \phi(x(t)) - \int_t^{\lambda_\eta(\tau_t)} y(u) du - \nabla \phi(x(\lambda_\eta(\tau_t))) \right) - \int_{\lambda_\eta(\tau_s)}^s y(u) du \\ &= \nabla \phi(x(s)) + \sum_{i=\tau_t}^{\tau_s-1} \eta_i \xi_i + \left(\nabla \phi(x(t)) - \int_t^{\lambda_\eta(\tau_t)} y(u) du - \nabla \phi(x(\lambda_\eta(\tau_t))) \right) \\ &\quad + \left(\nabla \phi(x(\lambda_\eta(\tau_s))) - \int_{\lambda_\eta(\tau_s)}^s y(u) du - \nabla \phi(x(s)) \right). \end{aligned}$$

Note that

$$\begin{aligned} &\left\| \nabla \phi(x(t)) - \int_t^{\lambda_\eta(\tau_t)} y(u) du - \nabla \phi(x(\lambda_\eta(\tau_t))) \right\| \\ &\leq \|\nabla \phi(x(t)) - \nabla \phi(x(\lambda_\eta(\tau_t)))\| + \int_t^{\lambda_\eta(\tau_t)} \|y(u)\| du \\ &\leq \|\nabla \phi(x(\lambda_\eta(\tau_t+1))) - \nabla \phi(x(\lambda_\eta(\tau_t)))\| + \int_t^{\lambda_\eta(\tau_t)} \|y(u)\| du \\ &\leq \eta_{\tau_t} (\|\xi_{\tau_t}\| + 2\|d_{\tau_t}\|), \end{aligned}$$

and similarly $\|\nabla \phi(x(\lambda_\eta(\tau_s))) - \int_{\lambda_\eta(\tau_s)}^s y(u) du - \nabla \phi(x(s))\| \leq \eta_{\tau_s} (\|\xi_{\tau_s}\| + 2\|d_{\tau_s}\|)$. By Assumption 3.1, we have that $\limsup_{t \rightarrow \infty} \eta_{\tau_t} (\|\xi_{\tau_t}\| + 2\|d_{\tau_t}\|) = 0$, $\limsup_{s \rightarrow \infty} \eta_{\tau_s} (\|\xi_{\tau_s}\| + 2\|d_{\tau_s}\|) = 0$, and $\lim_{t \rightarrow \infty} \sup_{s \in [t, t+T]} \sum_{i=\tau_t}^{\tau_s-1} \eta_i \xi_i = 0$. Therefore, it holds that $\lim_{t \rightarrow \infty} \sup_{s \in [t, t+T]} \|\nabla \phi(x(s)) - \nabla \phi(x_t(s))\| = 0$, which completes the proof. \square

Theorem 3.1. Suppose Assumptions 3.1 holds. Then, for any sequence $\{\tau_k\}_{k=1}^\infty \subset \mathbb{R}_+$, the set of shifted sequence $\{\nabla \phi(x^{\tau_k}(\cdot))\}_{k=1}^\infty$ is relatively compact in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. If $\lim_{k \rightarrow \infty} \tau_k = \infty$, then any cluster point $\bar{x}^*(\cdot)$ of $\{\nabla \phi(x^{\tau_k}(\cdot))\}_{k=1}^\infty$ belongs to $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Define $\bar{x}(\cdot) := \nabla \phi^*(\bar{x}^*(\cdot))$. Then there exists a measurable $\bar{y}(t) \in \mathcal{H}(\bar{x}(t))$ satisfying

$$\nabla \phi(\bar{x}(t)) = \nabla \phi(\bar{x}(0)) - \int_0^t \bar{y}(\tau) d\tau \quad \text{for all } t \geq 0. \quad (15)$$

Equivalently, we have

$$\frac{d}{dt} \nabla \phi(\bar{x}(t)) \in -\mathcal{H}(\bar{x}(t)), \quad \text{for almost all } t \geq 0.$$

Proof. By the definition of $x_t(\cdot)$, it follows that $\nabla\phi(x_t(s)) = \nabla\phi(x(t)) - \int_t^s y(u)du$, for all $s \geq t$. By the boundedness of $y(s)$, Arzelà-Ascoli's theorem [43] ensures that $\{\nabla\phi(x_t(t+\cdot))\}_{t \in \mathbb{R}_+}$ is relatively compact in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. For any subset $\{\tau_k\} \subset \mathbb{R}_+$, we consider the sequence $\{\nabla\phi(x^{\tau_k}(\cdot))\}_{k \in \mathbb{N}_+}$. There are two cases to consider. Case (i): the sequence $\{\tau_k\}$ has a cluster point t . Without loss of generality, assume that $\lim_{k \rightarrow \infty} \tau_k = t$. By the definition of $x(\cdot)$ in (11) and uniform boundedness assumption, it holds that $\nabla\phi(x(\cdot))$ is Lipschitz continuous. Thus, $\nabla\phi(x^{\tau_k}(\cdot))$ converges to $\nabla\phi(x^t(\cdot))$ in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Case (ii): $\lim_{k \rightarrow \infty} \tau_k = \infty$. Suppose that $\nabla\phi(x^{\tau_k}(\cdot))$ does not have any cluster point in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Since $\{\nabla\phi(x_{\tau_k}(\tau_k + \cdot))\}$ is relatively compact in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$, without loss of generality, we assume that $\lim_{k \rightarrow \infty} \nabla\phi(x_{\tau_k}(\tau_k + \cdot)) = \bar{x}^*(\cdot)$. Then, for any compact set $C \subset \mathbb{R}_+$, it follows from Lemma 3.1 that

$$\begin{aligned} & \lim_{k \rightarrow \infty, s \in C} \|\nabla\phi(x^{\tau_k}(s)) - \bar{x}^*(s)\| \\ & \leq \lim_{k \rightarrow \infty, s \in C} \|\nabla\phi(x^{\tau_k}(s)) - \nabla\phi(x_{\tau_k}(\tau_k + s))\| + \lim_{k \rightarrow \infty, s \in C} \|\nabla\phi(x_{\tau_k}(\tau_k + s)) - \bar{x}^*(s)\| = 0, \end{aligned}$$

which contradicts that $\nabla\phi(x^{\tau_k}(\cdot))$ does not have any cluster point in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Thus, in both cases, $\{\nabla\phi(x^{\tau_k}(\cdot))\}$ is relatively compact in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Because $\{\tau_k\}$ is an arbitrary subset in \mathbb{R}_+ , we have that $\{\nabla\phi(x^t(\cdot))\}$ is relatively compact in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$.

Next, we aim to construct a trajectory of the differential inclusion. Define the shifts $y^t(\cdot) = y(t + \cdot)$. Consider $\{\tau_k\}$ satisfying $\tau_k \rightarrow \infty$, and fix $T > 0$. Without loss of generality, we assume that $\nabla\phi(x^{\tau_k}(\cdot))$ converges to $\bar{x}^*(\cdot)$ in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$, otherwise, we choose its convergent subsequence. The set $\mathcal{Y}_T := \{y^{\tau_k}(s), s \in [0, T]\}_{k \in \mathbb{N}} \subset L^2([0, T])$ is bounded. Therefore, it follows from the Banach-Alaoglu theorem [44] that \mathcal{Y}_T is weakly sequentially compact, i.e. there exists a subsequence $\{\tau_{k_j}\}$ and $\bar{y}(\cdot) \in L^2([0, T])$ such that $y^{\tau_{k_j}}(\cdot) \rightarrow \bar{y}(\cdot)$ weakly in $L^2([0, T])$. On the other hand, by Lemma 3.1, we have $\nabla\phi(x_{\tau_{k_j}}(\tau_{k_j} + \cdot))$ converges to $\bar{x}^*(\cdot)$ in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. For any $\tau \in [0, T]$, by the definition of $x_t(\cdot)$, we have

$$\nabla\phi(x_t(t + \tau)) = \nabla\phi(x_t(t)) - \int_0^\tau y^t(s)ds.$$

Setting $t = \tau_{k_j}$ and taking the limit as $k_j \rightarrow \infty$, we deduce that

$$\bar{x}^*(\tau) = \bar{x}^*(0) - \int_0^\tau \bar{y}(s)ds.$$

Let $\bar{x}(\cdot) = \nabla\phi^*(\bar{x}^*(\cdot))$, since $T > 0$ is arbitrary, we get (15).

The remaining step is to verify that $\bar{y}(s) \in \mathcal{H}(\bar{x}(s))$ for almost all $s \geq 0$. We again fix an arbitrary $T > 0$. Given that $\mathcal{Y}_T \subset L^2([0, T])$ is bounded, the Banach-Saks theorem [44] implies that for τ_k (choosing a subsequence if necessary), $\frac{1}{N} \sum_{k=1}^N y^{\tau_k}(s)$ strongly converges to $\bar{y}(s)$ in $L^2([0, T])$. By the definition of $y(\cdot)$, we have $y^{\tau_k}(s) = d_{\Lambda_\eta(\tau_k+s)}$. Now for any $s \in [0, T]$, we have

$$\begin{aligned} & \|\nabla\phi(x(\lambda_\eta(\Lambda_\eta(\tau_k + s)))) - \nabla\phi(\bar{x}(s))\| \\ & \leq \|\nabla\phi(x(\lambda_\eta(\Lambda_\eta(\tau_k + s)))) - \nabla\phi(x(\tau_k + s))\| + \|\nabla\phi(x^{\tau_k}(s)) - \nabla\phi(\bar{x}(s))\| \\ & \leq \|\nabla\phi(x(\lambda_\eta(\Lambda_\eta(\tau_k + s) + 1))) - \nabla\phi(x(\lambda_\eta(\Lambda_\eta(\tau_k + s))))\| + \|\nabla\phi(x^{\tau_k}(s)) - \nabla\phi(\bar{x}(s))\| \\ & \leq \eta_{\Lambda_\eta(\tau_k+s)} (\|\xi_{\Lambda_\eta(\tau_k+s)}\| + \|d_{\Lambda_\eta(\tau_k+s)}\|) + \|\nabla\phi(x^{\tau_k}(s)) - \nabla\phi(\bar{x}(s))\|, \end{aligned}$$

which converges to zero as $k \rightarrow \infty$. By the continuity of $\nabla\phi^*$, we have that $x(\lambda_\eta(\Lambda_\eta(\tau_k + s)))$

converges to $\bar{x}(s)$ in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. By Assumption 3.1, for almost any $s \in [0, T]$, we have

$$\begin{aligned} \text{dist}(\bar{y}(s), \mathcal{H}(\bar{x}(s))) &\leq \left\| \frac{1}{N} \sum_{k=1}^N y^{\tau_k}(s) - \bar{y}(s) \right\| + \text{dist} \left(\frac{1}{N} \sum_{k=1}^N y^{\tau_k}(s), \mathcal{H}(\bar{x}(s)) \right) \\ &= \left\| \frac{1}{N} \sum_{k=1}^N y^{\tau_k}(s) - \bar{y}(s) \right\| + \text{dist} \left(\frac{1}{N} \sum_{k=1}^N d_{\Lambda_\eta(\lambda_\eta(\tau_k+s))}, \mathcal{H}(\bar{x}(s)) \right) \rightarrow 0. \end{aligned}$$

Since T is arbitrary and $\mathcal{H}(\bar{x}(s))$ is a closed set, we conclude that $\bar{y}(s) \in \mathcal{H}(\bar{x}(s))$ for almost all $s \geq 0$. This completes the proof. \square

The following assumption ensures that the trajectory subsequently converges to the stable set of (8), and the Lyapunov function values converge.

Assumption 3.2. *There exists a continuous function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$, such that the following conditions hold:*

1. (Weak Morse-Sard). *The set $\{\Psi(x) : \text{for } x \text{ such that } 0 \in \mathcal{H}(x)\}$ has empty interior in \mathbb{R} .*
2. (Lyapunov function). *Ψ is lower bounded, i.e. $\liminf_{x \in \mathbb{R}^n} \Psi(x) > -\infty$. For any trajectory $z(t)$ of the differential inclusion (8) with $z(0) \notin \mathcal{H}^{-1}(0)$, there exists $T > 0$ such that*

$$\Psi(z(T)) < \sup_{t \in [0, T]} \Psi(z(t)) \leq \Psi(z(0)).$$

We say that a continuous function Ψ , satisfying Assumption 3.2.2, is a Lyapunov function for the differential inclusion (8) with a stable set $\mathcal{H}^{-1}(0)$.

Lemma 3.2. *Suppose Assumption 3.1 is satisfied. Then, it holds that*

$$\liminf_{t \rightarrow \infty} \Psi(x(t)) = \liminf_{k \rightarrow \infty} \Psi(x_k), \quad \limsup_{t \rightarrow \infty} \Psi(x(t)) = \limsup_{k \rightarrow \infty} \Psi(x_k). \quad (16)$$

Proof. For simplicity, we only prove the case for \liminf , the argument for \limsup follows similarly. By Assumption 3.1, we have that

$$\lim_{k \rightarrow \infty} \|\nabla \phi(x_{k+1}) - \nabla \phi(x_k)\| = 0. \quad (17)$$

Let $\tau_i \rightarrow \infty$ be an arbitrary sequence with $x(\tau_i) \rightarrow x^*$. By the definition of λ_η and Λ_η , it follows that

$$\begin{aligned} \|\nabla \phi(x_{\Lambda_\eta(\tau_i)}) - \nabla \phi(x^*)\| &\leq \|\nabla \phi(x_{\Lambda_\eta(\tau_i)}) - \nabla \phi(x(\tau_i))\| + \|\nabla \phi(x(\tau_i)) - \nabla \phi(x^*)\| \\ &\leq \|\nabla \phi(x_{\Lambda_\eta(\tau_i)}) - \nabla \phi(x_{\Lambda_\eta(\tau_i)+1})\| + \|\nabla \phi(x(\tau_i)) - \nabla \phi(x^*)\|. \end{aligned}$$

The right-hand side converges to zero. By Remark 3.1.1, we have that $\Psi \circ \nabla \phi^*$ is continuous, so $\lim_{i \rightarrow \infty} \Psi(x_{\Lambda_\eta(\tau_i)}) = \lim_{i \rightarrow \infty} \Psi \circ \nabla \phi^* \circ \nabla \phi(x_{\Lambda_\eta(\tau_i)}) = \Psi \circ \nabla \phi^* \circ \nabla \phi(x^*) = \Psi(x^*)$. By choosing $\tau_i \rightarrow \infty$ as the sequence realizing $\liminf_{t \rightarrow \infty} \Psi(x(t))$, and assuming without loss of generality that $x(\tau_i) \rightarrow x^*$, we get

$$\liminf_{k \rightarrow \infty} \Psi(x_k) \leq \lim_{i \rightarrow \infty} \Psi(x_{\Lambda_\eta(\tau_i)}) = \Psi(x^*) = \liminf_{t \rightarrow \infty} \Psi(x(t)).$$

This completes the proof. \square

The following proposition demonstrates that the function value converges along the interpolated process defined in (11). The non-escape argument in the proof is adapted from those of [22, Proposition 3.5] and [26, Theorem 3.20], with particular attention paid to the dual map $\nabla\phi$ and its inverse $\nabla\phi^*$.

Proposition 3.1. *Suppose Assumption 3.1 and 3.2 hold, then function value $\Psi(x(t))$ converges as $t \rightarrow \infty$.*

Proof. Assuming $\liminf_{t \rightarrow \infty} \Psi(x(t)) = 0$, we define the level set $\mathcal{L}_r := \{x \in \mathbb{R}^n : \Psi(x) \leq r\}$. Choose any $\epsilon > 0$ such that $\epsilon \notin \Psi(\mathcal{H}^{-1}(0))$. The weak Morse-Sard condition in Assumption 3.1 implies that ϵ can be chosen arbitrarily small, and Lemma 3.2 implies that there are infinitely many k such that $x_k \in \mathcal{L}_\epsilon$. For any $x_k \in \mathcal{L}_\epsilon$, by the continuity of Ψ , we have that there exists $\alpha > 0$ such that $\text{dist}(x_k, \mathbb{R}^n \setminus \mathcal{L}_{2\epsilon}) > \alpha$. By (17), for sufficiently large k , we have that $\|x_{k+1} - x_k\| < \alpha$. Therefore, for all large k , $x_k \in \mathcal{L}_\epsilon$ implies that $x_{k+1} \in \mathcal{L}_{2\epsilon}$. Now, we define the last entrance and the first subsequent exit times,

$$k_i = \max\{m \geq j_{i-1} : x_m \in \mathcal{L}_\epsilon\}, \quad j_i = \min\{m \geq k_i : x_m \in \mathbb{R}^n \setminus \mathcal{L}_{2\epsilon}\}. \quad (18)$$

We prove that such upcrossing occurs for finite times. Otherwise, if there exists $\{k_i\}$ such that $\lim_{i \rightarrow \infty} k_i = \infty$, then Theorem 3.1 indicates that, up to a subsequence, $\nabla\phi(x^{\lambda_\eta(k_i)}(\cdot))$ converges to $\nabla\phi(\bar{x}(\cdot))$, where $\bar{x}(\cdot)$ is a trajectory of (8). By the definition of k_i , we have $\Psi(x_{k_i}) \leq \epsilon$, $\Psi(x_{k_i+1}) > \epsilon$. By (17) and the continuity of $\Psi \circ \nabla\phi^*$, we have that $\lim_{i \rightarrow \infty} \Psi(x_{k_i}) = \lim_{i \rightarrow \infty} \Psi(x_{k_i+1}) = \epsilon$. Recall that $x^{\lambda_\eta(k_i)}(0) = x_{k_i}$, therefore, $\Psi(\bar{x}(0)) = \lim_{i \rightarrow \infty} \Psi(x_{k_i}) = \epsilon$. Since $\bar{x}(0)$ is not in the stable set, there exists $T > 0$, such that

$$\Psi(\bar{x}(T)) < \sup_{s \in [0, T]} \Psi(\bar{x}(s)) \leq \Psi(\bar{x}(0)) = \epsilon.$$

Then, there exists $\delta > 0$, such that $\Psi(\bar{x}(T)) \leq \epsilon - 2\delta$. Moreover, for sufficiently large i , we have

$$\sup_{s \in [0, T]} \Psi(x^{\lambda_\eta(k_i)}(s)) \leq \sup_{s \in [0, T]} \Psi(\bar{x}(s)) + \sup_{s \in [0, T]} |\Psi(x^{\lambda_\eta(k_i)}(s)) - \Psi(\bar{x}(s))| \leq 2\epsilon.$$

The last inequality comes from the uniform convergence of $\{\nabla\phi(x^{\lambda_\eta(k_i)}(\cdot))\}$ in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. This implies that for all large i , $\{x(\lambda_\eta(k_i) + s) : s \in [0, T]\} \subset \mathcal{L}_{2\epsilon}$. Thus it holds that $\lambda_\eta(j_i) > \lambda_\eta(k_i) + T$. Let $l_i = \max\{m : \lambda_\eta(k_i) \leq \lambda_\eta(m) \leq \lambda_\eta(k_i) + T\}$. Then $\|\nabla\phi(x_{l_i}) - \nabla\phi(x^{\lambda_\eta(k_i)}(T))\| \leq \|\nabla\phi(x_{l_i}) - \nabla\phi(x_{l_i+1})\| \rightarrow 0$, and hence $\|\nabla\phi(x_{l_i}) - \nabla\phi(\bar{x}(T))\| \rightarrow 0$ as $i \rightarrow \infty$. By the continuity of $\Psi \circ \nabla\phi^*$, we have that $\Psi(x_{l_i}) \leq \epsilon - \delta$ for all large i . By the definition of k_i and j_i , we have that $\lambda_\eta(j_i) < \lambda_\eta(l_i) \leq \lambda_\eta(k_i) + T$, which leads to a contradiction. Therefore, for all large k , $\Psi(x_k) \leq 2\epsilon$. Since ϵ can be chosen arbitrarily small, it holds that $\lim_{k \rightarrow \infty} \Psi(x_k) = 0$. This completes the proof. \square

Now, we are ready to present the main theorem in this section.

Theorem 3.2. *Suppose Assumptions 3.1 and 3.2 hold. Then any cluster point of $\{x_k\}$ lies in $\mathcal{H}^{-1}(0)$ and the function values $\{\Psi(x_k)\}_{k \geq 1}$ converge.*

Proof. Since $\{x_k\}$ is bounded, let x^* be any cluster point of $\{x_k\}$, and $\lim_{i \rightarrow \infty} x_{k_i} = x^*$. By Theorem 3.1, up to a subsequence, $\nabla\phi(x^{\lambda_\eta(k_i)}(\cdot)) \rightarrow \nabla\phi(\bar{x}(\cdot))$ for some $\bar{x}(\cdot) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Note

that $x_{k_i} = x^{\lambda_\eta(k_i)}(0)$, so we have

$$\begin{aligned}\|\bar{x}(0) - x^*\| &\leq \limsup_{i \rightarrow \infty} \left(\|\bar{x}(0) - x^{\lambda_\eta(k_i)}(0)\| + \|x^{\lambda_\eta(k_i)}(0) - x_{k_i}\| + \|x_{k_i} - x^*\| \right) \\ &= \limsup_{i \rightarrow \infty} \left(\|\nabla\phi^*(\nabla\phi(\bar{x}(0))) - \nabla\phi^*(\nabla\phi(x^{\lambda_\eta(k_i)}(0)))\| + \|x^{\lambda_\eta(k_i)}(0) - x_{k_i}\| + \|x_{k_i} - x^*\| \right) \\ &= 0,\end{aligned}$$

where the last equality comes from the continuity of $\nabla\phi^*$. Hence $\bar{x}(0) = x^*$. Suppose $x^* \notin \mathcal{H}^{-1}(0)$, then by Assumption 3.2, there exists $T > 0$, such that

$$\Psi(\bar{x}(T)) < \sup_{t \in [0, T]} \Psi(\bar{x}(t)) \leq \Psi(x^*).$$

On the other hand, by Proposition 3.1, $\Psi(x(t))$ converges as $t \rightarrow \infty$. Therefore, we obtain

$$\Psi(\bar{x}(T)) = \lim_{i \rightarrow \infty} \Psi(x^{\lambda_\eta(k_i)}(T)) = \lim_{i \rightarrow \infty} \Psi(x(\lambda_\eta(k_i) + T)) = \lim_{t \rightarrow \infty} \Psi(x(t)) = \Psi(x^*),$$

which leads to a contradiction. Therefore, $0 \in \mathcal{H}(x^*)$. This completes the proof. \square

4 Applications

Based on the framework of the general Bregman-type method as outlined in (8), in this section, we consider three specific types of stochastic Bregman subgradient methods by choosing different types of kernel function ϕ and set-valued mapping \mathcal{H} . In the first two parts, we consider vanilla and single timescale momentum based stochastic Bregman subgradient methods for unconstrained optimization problems. Subsequently, we extend our methods to the stochastic Bregman proximal subgradient method for solving constrained composite optimization problems.

4.1 Stochastic Bregman subgradient method

In this subsection, we consider the following stochastic Bregman subgradient update scheme:

$$\begin{aligned}x_{k+1} &\approx \arg \min_{x \in \mathbb{R}^n} \left\{ \langle g_k, x - x_k \rangle + \frac{1}{\eta_k} \mathcal{D}_\phi(x, x_k) \right\}, \\ \text{s.t. } &\left\| g_k + \frac{\nabla\phi(x_{k+1}) - \nabla\phi(x_k)}{\eta_k} \right\| \leq \nu_k.\end{aligned}\tag{SBG}$$

where $g_k = d_k + \xi_k$, $d_k \in D_f^{\delta_k}(x_k)$, and ξ_k is the stochastic noise. The associated differential inclusion is

$$\frac{d}{dt} \nabla\phi(x(t)) \in -D_f(x(t)).\tag{19}$$

Given the allowance for inexact solutions in the SBG framework, we illustrate that a kernel Hessian preconditioned subgradient method fits within our framework, akin to the approach in the recently proposed ABPG in [47]. The concept of Hessian preconditioning has also been examined in continuous settings as seen in [2, 14]. Assuming the absence of the nonsmooth term in ABPG,

the existence and nonsingularity of $\nabla^2\phi$ everywhere, and that $\sup_k \|(\nabla^2\phi(x_k))^{-1}\| \leq c$ for some $c > 0$, the ABPG updates scheme in [47] is given by

$$x_k^+ = x_k - \eta_k (\nabla^2\phi(x_k))^{-1} g_k. \quad (20)$$

This yields

$$\begin{aligned} & \nabla\phi(x_k^+) - (\nabla\phi(x_k) - \eta_k g_k) \\ &= \nabla\phi(x_k - \eta_k (\nabla^2\phi(x_k))^{-1} g_k) - (\nabla\phi(x_k) - \eta_k g_k) \\ &= \nabla\phi(x_k) - \eta_k \nabla^2\phi(x_k) (\nabla^2\phi(x_k))^{-1} g_k + o(\eta_k) - (\nabla\phi(x_k) - \eta_k g_k) \\ &= o(\eta_k), \end{aligned}$$

indicating that $\lim_{k \rightarrow \infty} \left\| g_k + \frac{\nabla\phi(x_k^+) - \nabla\phi(x_k)}{\eta_k} \right\| = 0$, which implies that x_k^+ is an approximate solution to the SBG subproblem.

We make the following assumptions on (SBG).

Assumption 4.1. 1. ϕ is a supercoercive Legendre kernel function over \mathbb{R}^n , and $\nabla\phi$ is differentiable almost everywhere. Moreover, for any absolutely continuous mapping $z(\cdot) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$, $\nabla^2\phi(z(s))$ is positive definite for almost all $s \geq 0$.

2. The sequences $\{x_k\}$, $\{\nabla\phi(x_k)\}$ and $\{d_k\}$ are uniformly bounded almost surely.
3. $\{\xi_k\}$ is a martingale difference noise, i.e. $\mathbb{E}[\xi_{k+1} | \mathcal{F}_k] = 0$ holds almost surely. $\sum_{k=0}^{\infty} \eta_k = \infty$, and the stepsize and noise satisfy one of the following two conditions:
 - (a) $\{\xi_k\}$ is uniformly bounded and $\eta_k = o\left(\frac{1}{\log k}\right)$.
 - (b) $\{\xi_k\}$ has bounded variance, i.e. $\mathbb{E}[\|\xi_{k+1}\|^2 | \mathcal{F}_k] \leq \sigma^2 < \infty$, and $\sum_{k=0}^{\infty} \eta_k^2 < \infty$.
4. $\lim_{k \rightarrow \infty} \delta_k = 0$, and $\lim_{k \rightarrow \infty} \nu_k = 0$.

To ensure the convergence of (SBG), we make the following assumptions on f and kernel ϕ .

Assumption 4.2. 1. f is lower bounded, i.e. $\liminf_{x \in \mathbb{R}^n} f(x) > -\infty$. Moreover, f is a potential function that admits D_f as its convex valued conservative field.

2. The critical value set $\{f(x) : 0 \in D_f(x)\}$ has empty interior in \mathbb{R} .

We have the following two propositions.

Lemma 4.1. Suppose Assumption 4.1 and 4.2 hold. For any d_k^e such that $\|d_k^e\| \leq \nu_k$, and any increasing sequence $\{k_j\}$ such that $\{x_{k_j}\}$ converges to \bar{x} , it holds that

$$\lim_{N \rightarrow \infty} \text{dist} \left(\frac{1}{N} \sum_{j=1}^N (d_{k_j} + d_{k_j}^e), D_f(\bar{x}) \right) = 0.$$

Proof. By the inexact condition in (SBG), it follows that there exists d_k^e , such that $\|d_k^e\| \leq \nu_k$ and

$$\nabla\phi(x_{k+1}) = \nabla\phi(x_k) - \eta_k (d_k + d_k^e + \xi_k).$$

Define $\tilde{d}_k := d_k + d_k^e \in D_f^{\tilde{\delta}_k}(x_k)$, where $\tilde{\delta}_k = \delta_k + \nu_k$. Note that $\lim_{k \rightarrow \infty} \tilde{\delta}_k = 0$. Since D_f has a closed graph, then for any $\{x_{k_j}\}$ converging to \bar{x} , it holds that $\lim_{j \rightarrow \infty} \text{dist}(\tilde{d}_{k_j}, D_f(\bar{x})) = 0$. Note that $D_f(\bar{x})$ is a convex set, by Jensen's inequality, we have

$$\lim_{N \rightarrow \infty} \text{dist}\left(\frac{1}{N} \sum_{j=1}^N \tilde{d}_{k_j}, D_f(\bar{x})\right) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \text{dist}(\tilde{d}_{k_j}, D_f(\bar{x})) = 0.$$

This completes the proof. \square

Proposition 4.1. *Suppose Assumption 4.1 and 4.2 hold. Then f is a Lyapunov function for the differential inclusion (19) with the stable set $\{x \in \mathbb{R}^n : 0 \in D_f(x)\}$.*

Proof. Consider any trajectory $z(\cdot)$ for the differential inclusion (19) with $0 \notin D_f(z(0))$. We have that for almost all $s \geq 0$,

$$\frac{d}{ds} f(z(s)) = \langle D_f(z(s)), \dot{z}(s) \rangle \ni -\langle \nabla^2 \phi(z(s)) \dot{z}(s), \dot{z}(s) \rangle$$

Therefore, for any $t \geq 0$, it holds that

$$f(z(t)) - f(z(0)) = - \int_0^t \langle \nabla^2 \phi(z(s)) \dot{z}(s), \dot{z}(s) \rangle ds \leq - \int_0^t \lambda_{\min}(\nabla^2 \phi(z(s))) \|\dot{z}(s)\|^2 ds \leq 0.$$

We now prove the required result by contradiction. Suppose for any $t \geq 0$, $f(z(t)) = f(z(0))$, then, we have $\lambda_{\min}(\nabla^2 \phi(z(s))) \|\dot{z}(s)\|^2 = 0$ for almost all $s \geq 0$. Since $\lambda_{\min}(\nabla^2 \phi(z(\cdot))) > 0$ almost everywhere in \mathbb{R}_+ , then $\dot{z}(s) = 0$ for almost all $s \geq 0$. Since $z(\cdot)$ is absolutely continuous, therefore, $z(t) \equiv z(0)$ for any $t \geq 0$. Then, $0 = \frac{d}{dt} \nabla \phi(z(t)) \in -D_f(z(t)) = -D_f(z(0))$. This is contradictory to the fact that $z(0)$ is not a D_f -critical point of f . Therefore, there exists $T > 0$, such that $f(z(T)) < \sup_{t \in [0, T]} f(z(t)) \leq f(z(0))$. This completes the proof. \square

By Lemma 4.1, Proposition 4.1 and Theorem 3.2, we can directly derive the following convergence results for (SBG).

Theorem 4.1. *Suppose Assumption 4.1 and 4.2 hold. Then almost surely, any cluster point of $\{x_k\}$ generated by (SBG) is a D_f -critical point and the function values $\{f(x_k)\}$ converge.*

4.2 Momentum based stochastic Bregman subgradient method

In this section, we introduce a momentum based stochastic Bregman subgradient method. For a chosen kernel function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, the momentum based update scheme is given as follows:

$$\begin{cases} x_{k+1} = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \langle m_k, x - x_k \rangle + \frac{1}{\eta_k} \mathcal{D}_\varphi(x, x_k) \right\} \\ \text{s.t. } \left\| m_k + \frac{1}{\eta_k} (\nabla \varphi(x_{k+1}) - \nabla \varphi(x_k)) \right\| \leq \nu_k, \\ m_{k+1} = m_k - \theta_k P(x_k)(m_k - g_k), \end{cases} \quad (\text{MSBG})$$

where $g_k = d_k + \xi_k$, $d_k \in D_f^{\delta_k}(x_k)$, and $P(x_k) \in \mathbb{R}^{n \times n}$ denotes a preconditioning matrix. Similar to (20), the MSBG subproblem can also be solved in an inexact manner by adopting a preconditioned

subgradient strategy as shown in (20). For the ease of presentation, we omit the discussion. Our MSBG method is a single timescale method in the sense that the stepsize η_k for the primal variable and the stepsize θ_k for the momentum decay at the same rate.

We make the following assumptions on (MSBG).

Assumption 4.3. 1. $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$ is a supercoercive Legendre kernel function over \mathbb{R}^n , and $\nabla^2\varphi(\cdot)$ is positive definite everywhere. Moreover, $P(\cdot) = (\nabla^2\varphi(\cdot))^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$.

2. The sequences $\{x_k\}$, $\{\nabla\phi(x_k)\}$, $\{d_k\}$, and $\{m_k\}$ are uniformly bounded almost surely.
3. $\{\xi_k\}$ is a martingale difference noise, $\sum_{k=0}^{\infty} \eta_k = \infty$, and the stepsize and noise satisfy one of the following two conditions:
 - (a) $\{\xi_k\}$ is uniformly bounded and $\eta_k = o\left(\frac{1}{\log k}\right)$.
 - (b) $\{\xi_k\}$ has bounded variance, i.e. $\mathbb{E}[\|\xi_{k+1}\|^2 | \mathcal{F}_k] \leq \sigma^2 < \infty$, and $\sum_{k=0}^{\infty} \eta_k^2 < \infty$.
4. $\lim_{k \rightarrow \infty} \delta_k = 0$ and $\lim_{k \rightarrow \infty} \nu_k = 0$.
5. There exists a positive τ such that $\lim_{k \rightarrow \infty} \frac{\theta_k}{\eta_k} = \tau$.

Assumption 4.4. 1. f is lower bounded, path-differential, and admits a convex valued conservative field D_f .

2. The critical value set $\{f(x) : 0 \in D_f(x)\}$ has empty interior in \mathbb{R} .

Consider the following differential inclusion,

$$\frac{d}{dt} \begin{bmatrix} \nabla\varphi(x(t)) \\ m(t) \end{bmatrix} \in - \begin{bmatrix} m(t) \\ \tau(\nabla^2\varphi(x(t)))^{-1}(m(t) - D_f(x(t))) \end{bmatrix}, \text{ for almost all } t \geq 0. \quad (21)$$

Define

$$\phi(x, m) := \varphi(x) + \frac{1}{2}\|m\|^2, \quad \mathcal{H}(x, m) := \begin{bmatrix} m \\ \tau(\nabla^2\varphi(x))^{-1}(m - D_f(x)) \end{bmatrix}.$$

Then, (21) can be reformulated in the form of (8) as:

$$\frac{d}{dt} \nabla\phi(x(t), m(t)) \in -\mathcal{H}(x(t), m(t)), \text{ for almost all } t \geq 0. \quad (22)$$

The stable set of (22) is given by $\mathcal{H}^{-1}(0) = \{(x, m) : m = 0, 0 \in D_f(x)\}$. Based on the differential inclusion (22), (MSBG) can be reformulated in the form of (10) as:

$$\begin{cases} \nabla\varphi(x_{k+1}) = \nabla\varphi(x_k) - \eta_k(m_k + d_k^e) \\ m_{k+1} = m_k - \eta_k \cdot \frac{\theta_k}{\eta_k} (\nabla^2\varphi(x_k))^{-1}(m_k - d_k), \end{cases} \quad (23)$$

where $\|d_k^e\| \leq \nu_k$.

Lemma 4.2. Suppose Assumption 4.3 and 4.4 hold. Let $\{(x_k, m_k)\}$ be the sequence generated by (MSBG), $d_{x,k} := m_k + d_k^e$, and $d_{m,k} := \frac{\theta_k}{\eta_k} (\nabla^2\varphi(x_k))^{-1}(m_k - d_k)$, where $\|d_k^e\| \leq \nu_k$. For any increasing sequence $\{k_j\}$ such that (x_{k_j}, m_{k_j}) converges to (\bar{x}, \bar{m}) , it holds that

$$\lim_{N \rightarrow \infty} \text{dist} \left(\frac{1}{N} \sum_{j=1}^N (d_{x,k_j}, d_{m,k_j}), \mathcal{H}(\bar{x}, \bar{m}) \right) = 0.$$

Proof. By Assumption 4.3, it holds that

$$\lim_{j \rightarrow \infty} \text{dist} \left(m_{k_j} + d_{k_j}^e, \bar{m} \right) \leq \lim_{j \rightarrow \infty} \text{dist}(m_{k_j}, \bar{m}) + \nu_{k_j} = 0.$$

By Assumption 4.3.1, we have $(\nabla^2 \varphi(x))^{-1}$ is continuous, and hence

$$\lim_{j \rightarrow \infty} \text{dist} \left((\nabla^2 \varphi(x_{k_j}))^{-1}(m_{k_j} - d_{k_j}), (\nabla^2 \varphi(\bar{x}))^{-1}(\bar{m} - D_f(\bar{x})) \right) = 0.$$

Since $D_f(\bar{x})$ is a compact set, and $\lim_{k \rightarrow \infty} \frac{\theta_k}{\eta_k} = \tau$, it holds that

$$\lim_{j \rightarrow \infty} \text{dist} \left(\frac{\theta_{k_j}}{\eta_{k_j}} (\nabla^2 \varphi(x_{k_j}))^{-1}(m_{k_j} - d_{k_j}), \tau (\nabla^2 \varphi(\bar{x}))^{-1}(\bar{m} - D_f(\bar{x})) \right) = 0.$$

By the fact that $D_f(\bar{x})$ is a convex set and Jensen's inequality, we have that

$$\lim_{N \rightarrow \infty} \text{dist} \left(\frac{1}{N} \sum_{j=1}^N (d_{x,k_j}, d_{m,k_j}), \mathcal{H}(\bar{m}, \bar{x}) \right) \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \text{dist} \left((d_{x,k_j}, d_{m,k_j}), \mathcal{H}(\bar{m}, \bar{x}) \right) = 0.$$

This completes the proof. \square

Proposition 4.2. Suppose Assumption 4.3 and 4.4 hold. Then $h(x, m) = f(x) + \frac{1}{2\tau} \|m\|^2$ is a Lyapunov function for (21) with the stable set $\mathcal{B} := \{(x, m) \in \mathbb{R}^n \times \mathbb{R}^n : m = 0, 0 \in D_f(x)\}$.

Proof. Consider any trajectory $(x(t), m(t))$ for the differential inclusion (21) with $(x(0), m(0)) \notin \mathcal{B}$. There exists measurable $d_f(s) \in D_f(x(s))$, such that for almost all $s \geq 0$,

$$\begin{aligned} & \frac{d}{ds} h(x(s), m(s)) \\ &= \langle D_f(x(s)), \dot{x}(s) \rangle + \left\langle \frac{m(s)}{\tau}, \dot{m}(s) \right\rangle \\ &\ni -\langle d_f(s), (\nabla^2 \varphi(x(s)))^{-1} m(s) \rangle - \langle m(s), (\nabla^2 \varphi(x(s)))^{-1} (m(s) - d_f(s)) \rangle \\ &= -\langle m(s), (\nabla^2 \varphi(x(s)))^{-1} m(s) \rangle. \end{aligned}$$

Thus, for any $t \geq 0$, $h(x(t), m(t)) \leq h(x(0), m(0))$. For any $(x(0), m(0)) \notin \mathcal{B}$, either $m(0) \neq 0$ or $m(0) = 0$ and $0 \notin D_f(x(0))$. If $m(0) \neq 0$, then the continuity of $m(\cdot)$ ensures the existence of $T > 0$ and $\alpha > 0$ where $\|m(s)\| \geq \alpha$ for $s \in [0, T]$. Thus we have

$$h(x(T), m(T)) - h(x(0), m(0)) \leq - \int_0^T \langle m(s), (\nabla^2 \varphi(x(s)))^{-1} m(s) \rangle ds < 0.$$

Now consider the case $m(0) = 0$ and $0 \notin D_f(x(0))$. By the outer semicontinuity of D_f and Assumption 4.3.1, there exists $\tilde{T} > 0$, such that for almost all $t \in [0, \tilde{T}]$, it holds that $0 \notin (\nabla^2 \varphi(x(t)))^{-1} D_f(x(t))$. Now suppose for all $t \geq 0$, $h(x(t), m(t)) = h(x(0), m(0))$, then we have $m(s) = 0$ for almost all $s \geq 0$. Since m is continuous, it holds that $m \equiv 0$. Note that for almost any $t \geq 0$, $\dot{m}(t) \in -\tau (\nabla^2 \varphi(x(t)))^{-1} (m(t) - D_f(x(t)))$, thus $0 \in (\nabla^2 \varphi(x(t)))^{-1} D_f(x(t))$ holds for almost all $t \geq 0$, which leads to a contradiction. Therefore, for both cases, there exists $T > 0$ such that $h(x(T), m(T)) < h(x(0), m(0))$. This completes the proof. \square

By Lemma 4.2, Proposition 4.2, and Theorem 3.2, we have the following convergence results for (MSBG).

Theorem 4.2. *Suppose Assumption 4.3 and 4.4 hold. Then almost surely, any cluster point of $\{x_k\}$ generated by (MSBG) is a D_f -critical point of f , $\lim_{k \rightarrow \infty} m_k = 0$, and the function values $\{f(x_k)\}$ converge.*

Proof. Theorem 3.2 implies that any cluster point of $\{(x_k, m_k)\}$ lies in $\{(x, m) : 0 \in D_f(x), m = 0\}$, and $\{f(x_k) + \frac{1}{2\tau} \|m_k\|^2\}$ converges. For any convergent subsequence $x_{k_j} \rightarrow \bar{x}$, since $\{m_k\}$ is bounded, then there exist subsequence $\{m_{k_{j_i}}\}$ such that $m_{k_{j_i}} \rightarrow \bar{m}$. Therefore, $(x_{k_{j_i}}, m_{k_{j_i}}) \rightarrow (\bar{x}, \bar{m})$. Then, it holds that $0 \in D_f(\bar{x})$. Similarly, we can prove that for any convergent subsequence $\{m_{k_j}\}$ such that $m_{k_j} \rightarrow \bar{m}$, we have that $\bar{m} = 0$. Therefore, $\lim_{k \rightarrow \infty} m_k = 0$. Then, $\lim_{k \rightarrow \infty} f(x_k) = \lim_{k \rightarrow \infty} f(x_k) + \frac{1}{2\tau} \|m_k\|^2$. This completes the proof. \square

4.3 Stochastic Bregman proximal subgradient method

In this section, we consider solving the following constrained composite optimization problem:

$$\min_{x \in \mathcal{X}} h(x) := f(x) + R(x), \quad (24)$$

where \mathcal{X} is a closed subset of \mathbb{R}^n , and R is a locally Lipschitz function with an efficiently computable conservative field. In many applications, R serves as the regularization function, which is usually Clarke regular, and ∂R is efficient to compute. We consider applying the follow Bregman proximal subgradient method to solve (24),

$$\begin{cases} x_{k+1} \approx \operatorname{argmin}_{x \in \mathcal{X}} \left\{ \langle g_k, x - x_k \rangle + \frac{1}{\eta_k} \mathcal{D}_\phi(x, x_k) + R(x) \right\}, \\ \text{s.t. } \langle g_k, x_{k+1} - x_k \rangle + \frac{1}{\eta_k} \mathcal{D}_\phi(x_{k+1}, x_k) + R(x_{k+1}) \leq R(x_k), \text{ and} \\ \text{dist} \left(0, g_k + \frac{1}{\eta_k} (\nabla \phi(x_{k+1}) - \nabla \phi(x_k)) + D_R(x_{k+1}) + N_{\mathcal{X}}^L(x_{k+1}) \right) \leq \nu_k. \end{cases} \quad (\text{SBPG})$$

where $g_k = d_{f,k} + \xi_k$, $d_{f,k} \in D_f^{\delta_k}(x_k)$. We can reformulate (SBPG) in the form of (10) as follows:

$$\nabla \phi(x_{k+1}) = \nabla \phi(x_k) - \eta_k (d_{f,k} + d_{R,k} + d_{\mathcal{X},k} + d_{e,k} + \xi_k),$$

where $d_{f,k} \in D_f^{\delta_k}(x_k)$, $d_{R,k} \in D_R(x_{k+1})$, $d_{\mathcal{X},k} \in N_{\mathcal{X}}^L(x_{k+1})$, and $\|d_{e,k}\| \leq \nu_k$. When $\mathcal{X} = \mathbb{R}^n$, with $\delta_k = 0$ and $\nu_k = 0$, it follows that $x_{k+1} = T_{\eta_k, R}^\phi(x_k, g_k)$. Let $d_k := d_{f,k} + d_{R,k} + d_{\mathcal{X},k} + d_{e,k}$. We can easily verify that there exists $\{\tilde{\delta}_k\}$ such that $\lim_{k \rightarrow \infty} \tilde{\delta}_k = 0$, and $d_k \in \mathcal{H}^{\tilde{\delta}_k}(x_k)$, where $\mathcal{H} := D_f + D_R + N_{\mathcal{X}}$. This leads to a differential inclusion for the proximal updates given by

$$\frac{d}{dt} \nabla \phi(x(t)) \in -\mathcal{H}(x(t)), \text{ where } \mathcal{H} = D_f + D_R + N_{\mathcal{X}}. \quad (25)$$

The momentum technique can also be integrated into (SBPG), as illustrated in Section 4.2. For the sake of readability, we omit this extension. Note that neither $N_{\mathcal{X}}$ nor $N_{\mathcal{X}}^L$ is locally bounded, thus the results that rely on local boundedness assumption such as those presented in [24, 51] cannot be directly applied. We make the following assumptions on (SBPG).

Assumption 4.5. 1. ϕ is a supercoercive Legendre kernel function over \mathbb{R}^n . Moreover, ϕ is locally strongly convex and $\nabla\phi$ is locally Lipschitz continuous. Additionally, for any absolutely continuous mapping $z(\cdot) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$, $\nabla^2\phi(z(s))$ is positive definite for almost all $s \geq 0$.

2. The sequences $\{x_k\}$, $\{\nabla\phi(x_k)\}$, $\{d_{f,k}\}$ and $\{d_{R,k}\}$ are uniformly bounded almost surely.
3. $\{\xi_k\}$ is a uniformly bounded martingale difference noise and $\sup_{k \geq 0} \|\xi_k\| < \infty$. The stepsize sequence $\{\eta_k\}$ satisfies $\sum_{k=0}^{\infty} \eta_k = \infty$ and $\eta_k = o\left(\frac{1}{\log k}\right)$.
4. $\lim_{k \rightarrow \infty} \delta_k = 0$, and $\lim_{k \rightarrow \infty} \nu_k = 0$.
5. For any $\eta > 0$, $\eta R + \phi$ is supercoercive.

Assumption 4.6. 1. h is lower bounded. Moreover, the locally Lipschitz continuous functions f and R are potential functions that admit convex valued D_f and D_R as their conservative fields, respectively. \mathcal{X} admits the chain rule as described in Definition 2.8.

2. The critical value set $\{h(x) : 0 \in D_f(x) + D_R(x) + N_{\mathcal{X}}(x)\}$ has empty interior in \mathbb{R} .

Lemma 4.3. Suppose Assumption 4.5 and 4.6 hold. Let $d_k := d_{f,k} + d_{R,k} + d_{\mathcal{X},k} + d_{e,k}$. For any increasing sequence $\{k_j\}$ such that $\{x_{k_j}\}$ converges to \bar{x} , it holds that

$$\lim_{N \rightarrow \infty} \text{dist}\left(\frac{1}{N} \sum_{j=1}^N d_{k_j}, \mathcal{H}(\bar{x})\right) = 0,$$

where $\mathcal{H} := D_f + D_R + N_{\mathcal{X}}$.

Proof. Given the condition

$$R(x_{k+1}) + \langle d_{f,k} + \xi_k, x_{k+1} - x_k \rangle + \frac{1}{\eta_k} \mathcal{D}_{\phi}(x_{k+1}, x_k) \leq R(x_k),$$

we derive that

$$\frac{\mathcal{D}_{\phi}(x_{k+1}, x_k)}{\|x_{k+1} - x_k\|} \leq \eta_k \frac{|R(x_{k+1}) - R(x_k)|}{\|x_{k+1} - x_k\|} + \eta_k (\|d_{f,k} + \xi_k\|).$$

Assumption 4.5 ensures that $\sup_{k \geq 0} \frac{1}{\eta_k} \|\nabla\phi(x_{k+1}) - \nabla\phi(x_k)\| < \infty$. Moreover, we have that for some $\|d_{e,k}\| \leq \nu_k$,

$$\frac{\nabla\phi(x_{k+1}) - \nabla\phi(x_k)}{\eta_k} = -(d_{f,k} + d_{R,k} + d_{\mathcal{X},k} + d_{e,k} + \xi_k).$$

Note that the left hand side is uniformly bounded, $\{d_{f,k}\}$, $\{d_{R,k}\}$, $\{d_{e,k}\}$ and $\{\xi_k\}$ are all uniformly bounded, therefore, it holds that $\{d_{\mathcal{X},k}\}$ is also uniformly bounded. For any $\{x_{k_j}\}$ converging to \bar{x} , by the outer semicontinuity of D_f , D_R and $N_{\mathcal{X}}^L$, along with Proposition 2.1, it holds that

$$\lim_{j \rightarrow \infty} \text{dist}(d_{x,k_j}, D_f(\bar{x})) = 0, \quad \lim_{j \rightarrow \infty} \text{dist}(d_{R,k_j}, D_R(\bar{x})) = 0, \quad \text{and} \quad \lim_{j \rightarrow \infty} \text{dist}(d_{\mathcal{X},k_j}, N_{\mathcal{X}}^L(\bar{x})) = 0.$$

Note that $N_{\mathcal{X}}^L \subset N_{\mathcal{X}}$, therefore, $\lim_{j \rightarrow \infty} \text{dist}(d_{k_j}, \mathcal{H}(\bar{x})) = 0$. By Jensen's inequality, we prove this lemma. \square

Proposition 4.3. Suppose Assumption 4.5 and 4.6 hold. Then the function h in (24) is a Lyapunov function for the differential inclusion (25) with stable set $\{x \in \mathbb{R}^n : 0 \in D_f(x) + D_R(x) + N_{\mathcal{X}}(x)\}$.

Proof. Consider any trajectory $z(t)$ of (25) with $0 \notin \mathcal{H}(z(0))$. By the chain rule, it holds that for almost all $t \geq 0$,

$$f(z(t))' = \langle D_f(z(t)), \dot{z}(t) \rangle, \quad R(z(t))' = \langle D_R(z(t)), \dot{z}(t) \rangle, \quad 0 = \langle N_{\mathcal{X}}(z(t)), \dot{z}(t) \rangle.$$

Then, we have that $h(z(t))' = \langle \mathcal{H}(z(t)), \dot{z}(t) \rangle$ for almost all $t \geq 0$. Note that for almost all $t \geq 0$, it holds that $\nabla^2 \phi(z(t)) \dot{z}(t) \in -\mathcal{H}(z(t))$. Then, we have

$$\begin{aligned} h(z(t)) - h(z(0)) &= \int_0^t \langle \mathcal{H}(z(s)), \dot{z}(s) \rangle ds = - \int_0^t \langle \nabla^2 \phi(z(s)) \dot{z}(s), \dot{z}(s) \rangle ds \\ &\leq - \int_0^t \lambda_{\min}(\nabla^2 \phi(z(s))) \|\dot{z}(s)\|^2 ds. \end{aligned}$$

If there exists no $t > 0$, such that $h(z(t)) < h(z(0))$. Then, we have that $\dot{z}(t) = 0$ for almost all $t \geq 0$. Thus, $z(t) \equiv z(0)$. Therefore, we have that $0 = \frac{d}{dt} \nabla \phi(z(s)) \in -\mathcal{H}(z(0))$, which is contradictory to the fact that $0 \notin \mathcal{H}(z(0))$. This completes the proof. \square

By Lemma 4.3, Proposition 4.3 and Theorem 3.2, we can directly derive the following convergence results.

Theorem 4.3. Suppose Assumption 4.5 and 4.6 hold. Then almost surely, any cluster point of $\{x_k\}$ generated by (SBPG) is a \mathcal{H} -critical point and the function values $\{f(x_k) + R(x_k)\}$ converge.

5 Numerical experiments

In this section, we conduct preliminary numerical experiments to illustrate the performance of our proposed methods, focusing on training nonsmooth neural networks for image classification and language modeling tasks. These experiments are performed using an NVIDIA RTX 3090 GPU, and implemented in Python 3.9 with PyTorch version 1.12.0.

Our experiments employ a polynomial kernel-based stochastic Bregman subgradient method to train nonsmooth neural networks. Specifically, we use a blockwise polynomial kernel function $\phi(x) = \sum_{i=1}^L p_i(\|x_i\|)$, where $x = (x_1, \dots, x_L)$ represents the concatenation of all layers' parameters in a neural network with L layers, and each p_i is a univariate polynomial of degree at least 2. When $p_i(\lambda) = \frac{1}{2}\lambda^2$, this approach becomes equivalent to SGD. The polynomial $p_i(\lambda) = \frac{1}{2}\lambda^2 + \frac{\sigma}{r}\lambda^r$ with $r \geq 4$, as discussed in the prior work [23], is applied in our numerical experiments. In this case, the update scheme is defined as follows

$$\begin{aligned} x_{k+1} &= \nabla \phi^*(\nabla \phi(x_k) - \eta_k m_k) \\ m_{k+1} &= m_k - \theta_k (\nabla^2 \phi(x_k))^{-1} (m_k - g_k), \end{aligned} \tag{MSBG}$$

where the calculation of x_{k+1} involves solving a nonlinear equation. Given the allowance of inexact solutions for the subproblems of MSBG, as mentioned earlier, we consider an alternative preconditioned update scheme,

$$\begin{aligned} x_{k+1} &= x_k - \eta_k (\nabla^2 \phi(x_k))^{-1} m_k \\ m_{k+1} &= m_k - \theta_k (\nabla^2 \phi(x_k))^{-1} (m_k - g_k), \end{aligned} \tag{iMSBG}$$

which avoids solving a nonlinear equation in the first step of (MSBG). By the Sherman-Morrison formula, we have

$$(\nabla|_{x_i}^2 p_i(\|x_i\|))^{-1} = \frac{1}{\sigma \|x_i\|^{r-2}} I - \frac{\sigma(r-2) \|x_i\|^{r-4}}{(1+\sigma \|x_i\|^{r-2})^2 + \sigma(r-2)(1+\sigma \|x_i\|^{r-2}) \|x_i\|^2} x_i x_i^T,$$

and $\nabla^2 \phi(x) = \text{diag}((\nabla|_{x_1}^2 p_1(\|x_1\|))^{-1}, \dots, (\nabla|_{x_L}^2 p_L(\|x_L\|))^{-1})$ is block diagonal. We employ this kernel function and use the notation MSBGK/iMSBGK to denote MSBG/iMSBG with the polynomial degree parameter r set to K . Our experiments focus on two main applications: training Convolutional Neural Networks (CNNs) for image classification and Long Short-Term Memory (LSTM) [31] networks for language modeling. Specifically, our image classification experiments include training Resnet14 and ResNet34 [30] on CIFAR-10 and CIFAR-100 datasets [32]. Our language modeling experiments focus on 1-layer, 2-layer, and 3-layer LSTM networks applied to the Penn Treebank dataset [38].

CNNs on image classification For the CNN experiments, we set the stepsize η_s for each epoch s as $\eta_s = \frac{\eta_0}{1+(\log(s+1))^{1.1}}$, where η_0 is the initial stepsize. The momentum parameters are all set to $\theta_s = \frac{0.1}{1+\log(s+1)^{1.1}}$. For MSBG4 and iMSBG4, we choose $\sigma = 0.01$, and for iMSBG6, $\sigma = 0.0001$. We search the initial stepsize η_0 among the grid $\{0.001, 0.01, 0.1, 1.0\}$ and select the value that achieves the highest test accuracy. The results are shown in Figure 1 and Figure 2. We can observe that by selecting a proper kernel function, our Bregman subgradient methods can outperform SGD in terms of test accuracy. Moreover, we can see that MSBG4 and iMSBG4 have similar performance, although iMSBG4 solves the subproblem inexactly.

Additionally, we evaluate the robustness of the selection of initial stepsize η_0 , as demonstrated in Figure 3. We can see that the peak test accuracies of all methods are similar, yet our Bregman subgradient methods demonstrate a wider effective initial step size range, indicating a reduced sensitivity to the choice of initial step size – a benefit attributable to the kernel function.

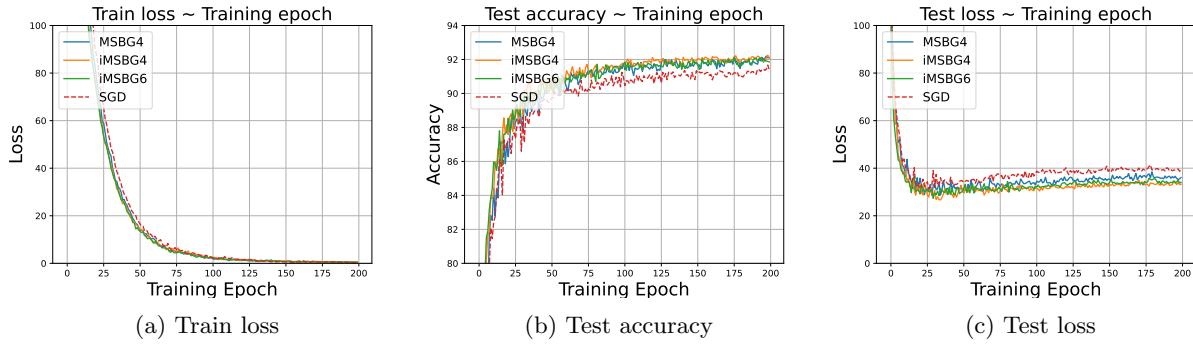


Figure 1: Resnet 14 on CIFAR10.

LSTMs on language modeling For the LSTM experiments, we initially set the stepsize as a constant. The stepsize is then decreased to 0.1 times its previous value at both the 150th and 300th epochs. After 300 epochs, we set $\eta_s = \frac{0.01\eta_0}{1+\log(s-300)^{1.1}}$, with s representing the epoch number.

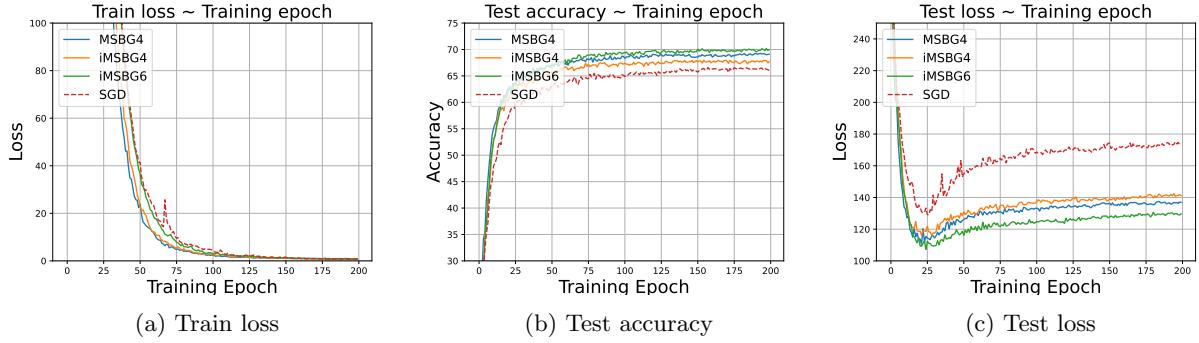


Figure 2: Resnet 34 on CIFAR100.

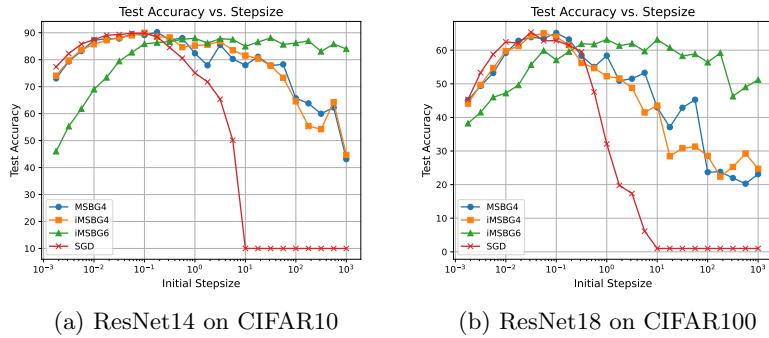


Figure 3: Robustness for initial stepsize. Figure (a) reports the test accuracy for 40 epochs. Figure (b) reports the test accuracy for 30 epochs.

Here η_0 is the initial stepsize. Within the s -th epoch, η_k takes the constant value η_s . Similarly, the momentum parameters are all set to $\theta_s = \frac{0.1}{1+\log(s+1)^{1.1}}$. For MSBG4 and iMSBG4, we set $\sigma = 10^{-6}$. We search η_0 among the grid $\{1, 10, 20, 40, 80, 100\}$ and report the results based on achieving the highest test accuracy. The results are shown in Figures 4, 5, and 6. We can observe that selecting an appropriate kernel function enables our Bregman subgradient methods to achieve superior test accuracy compared to SGD.

We also compare the one-epoch runtime for all considered methods over all tasks. We can observe in Table 1 that the proposed inexact Bregman subgradient methods are nearly as efficient as SGD, largely because iMSBG circumvents the need to solve nonlinear equations in computing the Bregman proximal mapping.

6 Conclusion

This paper explores Bregman subgradient methods for solving nonsmooth nonconvex optimization problems, particularly focusing on path-differentiable functions. We introduce a comprehensive stochastic Bregman framework that accommodates inexact evaluations of the abstract set-valued mapping. Employing a differential inclusion strategy and linear interpolation of dual sequences,

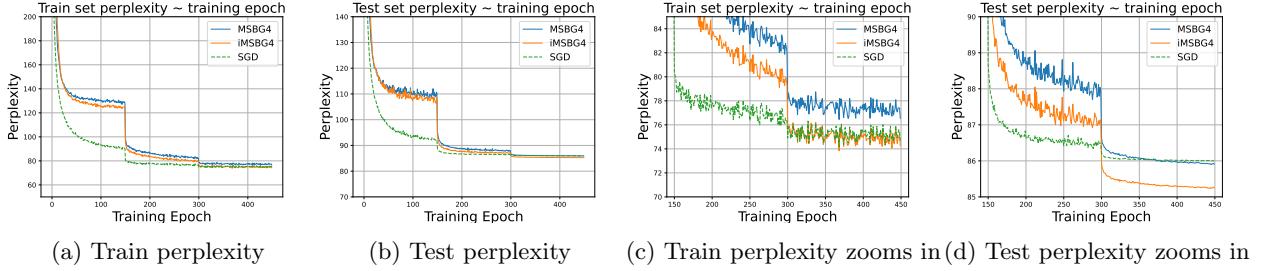


Figure 4: 1-layer LSTM on Penn Treebank dataset.

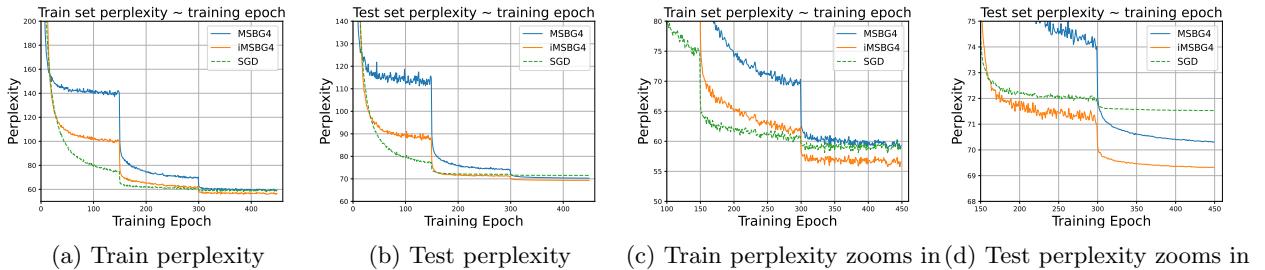


Figure 5: 2-layer LSTM on Penn Treebank dataset.

we establish convergence results for our stochastic Bregman-type methods. This ensures that the discrete sequence subsequently converges to the stable set of the differential inclusion, as well as the convergence of the Lyapunov function values. For applications, we demonstrate that stochastic Bregman subgradient methods, even with subproblems being solved inexactly, fit within our general framework, and we establish their convergence properties. Moreover, we integrate a momentum technique into the stochastic Bregman subgradient methods. Additionally, we extend our methodology to a proximal variant of the stochastic Bregman subgradient methods for solving constrained composite optimization problems and establish its convergence results. Finally, we conduct numerical experiments on training nonsmooth neural networks to evaluate the performance of our proposed stochastic Bregman subgradient methods. Our experimental results validate the practical benefits and effectiveness of our approaches in deep learning.

References

- [1] T. Ajanthan, K. Gupta, P. Torr, R. Hartley, and P. Dokania. Mirror descent view for neural network quantization. In *International Conference on Artificial Intelligence and Statistics*, pages 2809–2817. PMLR, 2021.
- [2] F. Alvarez, J. Bolte, and O. Brahic. Hessian Riemannian gradient flows in convex programming. *SIAM J. Control and Optimization*, 43(2):477–501, 2004.
- [3] E. Amid and M. K. Warmuth. Reparameterizing mirror descent as gradient descent. *Advances in Neural Information Processing Systems*, 33:8430–8439, 2020.

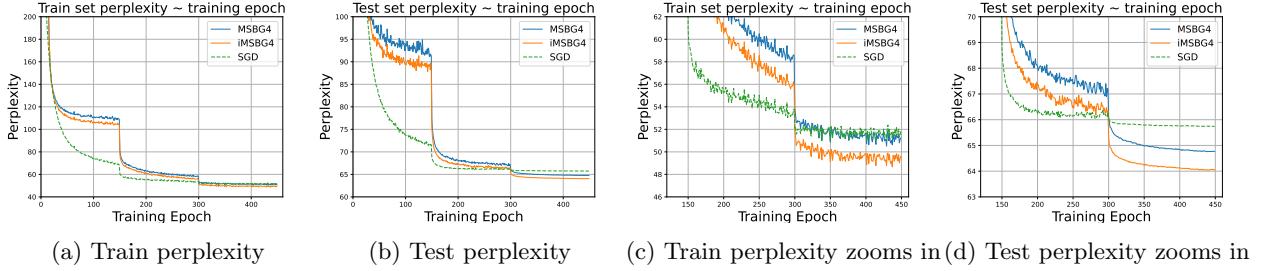


Figure 6: 3-layer LSTM on Penn Treebank dataset.

Method	Task 1	Task 2	Task 3	Task 4	Task 5
SGD	18.11 ± 0.32	36.28 ± 0.07	5.56 ± 0.03	17.24 ± 0.03	30.11 ± 0.06
MSBG4	20.57 ± 0.12	40.43 ± 0.16	6.48 ± 0.11	18.97 ± 0.12	32.81 ± 0.23
iMSBG4	18.30 ± 0.04	37.25 ± 0.07	5.62 ± 0.01	18.45 ± 0.04	31.68 ± 0.27
iMSBG6	18.38 ± 0.06	37.24 ± 0.12	5.66 ± 0.04	18.52 ± 0.04	31.45 ± 0.31

Table 1: Computation Time of Each Epoch (in seconds). Task 1 is training ResNet14 on CIFAR10. Task 2 is training ResNet18 on CIFAR100. Task 3,4,5 are training one-layer, two-layer, three-layer LSTM on n Penn Treebank dataset, respectively.

- [4] N. Azizan, S. Lale, and B. Hassibi. Stochastic mirror descent on overparameterized nonlinear models. *IEEE Transactions on Neural Networks and Learning Systems*, 33(12):7717–7727, 2021.
- [5] H. H. Bauschke, J. Bolte, and M. Teboulle. A descent lemma beyond Lipschitz gradient continuity: first-order methods revisited and applications. *Mathematics of Operations Research*, 42(2):330–348, 2017.
- [6] A. Beck and M. Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. *Operations Research Letters*, 31(3):167–175, 2003.
- [7] M. Benaïm, J. Hofbauer, and S. Sorin. Stochastic approximations and differential inclusions. *SIAM J. Control and Optimization*, 44(1):328–348, 2005.
- [8] J.-D. Benamou, G. Carlier, M. Cuturi, L. Nenna, and G. Peyré. Iterative Bregman projections for regularized transportation problems. *SIAM J. Scientific Computing*, 37(2):A1111–A1138, 2015.
- [9] J. Bernstein, Y.-X. Wang, K. Azizzadenesheli, and A. Anandkumar. signs gd: Compressed optimisation for non-convex problems. In *International Conference on Machine Learning*, pages 560–569. PMLR, 2018.
- [10] J. Bolte, A. Daniilidis, A. Lewis, and M. Shiota. Clarke subgradients of stratifiable functions. *SIAM J. Optimization*, 18(2):556–572, 2007.
- [11] J. Bolte, T. Le, and E. Pauwels. Subgradient sampling for nonsmooth nonconvex minimization. *arXiv preprint arXiv:2202.13744*, 2022.

- [12] J. Bolte and E. Pauwels. Conservative set valued fields, automatic differentiation, stochastic gradient methods and deep learning. *Mathematical Programming*, 188:19–51, 2021.
- [13] J. Bolte, S. Sabach, M. Teboulle, and Y. Vaisbourd. First order methods beyond convexity and lipschitz gradient continuity with applications to quadratic inverse problems. *SIAM J. Optimization*, 28(3):2131–2151, 2018.
- [14] J. Bolte and M. Teboulle. Barrier operators and associated gradient-like dynamical systems for constrained minimization problems. *SIAM J. Control and Optimization*, 42(4):1266–1292, 2003.
- [15] V. S. Borkar. *Stochastic approximation: a dynamical systems viewpoint*, volume 48. Springer, 2009.
- [16] L. M. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR Computational Mathematics and Mathematical Physics*, 7(3):200–217, 1967.
- [17] C. Castera, J. Bolte, C. Févotte, and E. Pauwels. An inertial newton algorithm for deep learning. *J. of Machine Learning Research*, 22(1):5977–6007, 2021.
- [18] H. T. Chu, L. Liang, K.-C. Toh, and L. Yang. An efficient implementable inexact entropic proximal point algorithm for a class of linear programming problems. *Computational Optimization and Applications*, 85(1):107–146, 2023.
- [19] F. H. Clarke. *Optimization and nonsmooth analysis*. SIAM, 1990.
- [20] M. Coste. *An introduction to o-minimal geometry*. Istituti editoriali e poligrafici internazionali Pisa, 2000.
- [21] A. Cutkosky and H. Mehta. Momentum improves normalized sgd. In *International Conference on Machine Learning*, pages 2260–2268. PMLR, 2020.
- [22] D. Davis, D. Drusvyatskiy, S. Kakade, and J. D. Lee. Stochastic subgradient method converges on tame functions. *Foundations of Computational Mathematics*, 20(1):119–154, 2020.
- [23] K. Ding, J. Li, and K.-C. Toh. Nonconvex Stochastic Bregman Proximal Gradient Method with Application to Deep Learning. *arXiv preprint arXiv:2306.14522*, 2023.
- [24] K. Ding, N. Xiao, and K.-C. Toh. Adam-family methods with decoupled weight decay in deep learning. *arXiv preprint arXiv:2310.08858*, 2023.
- [25] R.-A. Dragomir, A. B. Taylor, A. dAspremont, and J. Bolte. Optimal complexity and certification of Bregman first-order methods. *Mathematical Programming*, pages 1–43, 2021.
- [26] J. C. Duchi and F. Ruan. Stochastic methods for composite and weakly convex optimization problems. *SIAM J. Optimization*, 28(4):3229–3259, 2018.
- [27] U. Ghai, Z. Lu, and E. Hazan. Non-convex online learning via algorithmic equivalence. *Advances in Neural Information Processing Systems*, 35:22161–22172, 2022.

- [28] S. Gunasekar, J. Lee, D. Soudry, and N. Srebro. Characterizing implicit bias in terms of optimization geometry. In *International Conference on Machine Learning*, pages 1832–1841. PMLR, 2018.
- [29] F. Hanzely and P. Richtárik. Fastest rates for stochastic mirror descent methods. *Computational Optimization and Applications*, 79:717–766, 2021.
- [30] K. He, X. Zhang, S. Ren, and J. Sun. Deep residual learning for image recognition. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 770–778, 2016.
- [31] S. Hochreiter and J. Schmidhuber. Long short-term memory. *Neural Computation*, 9(8):1735–1780, 1997.
- [32] A. Krizhevsky, G. Hinton, et al. Learning multiple layers of features from tiny images. 2009.
- [33] G. Lan. Policy mirror descent for reinforcement learning: Linear convergence, new sampling complexity, and generalized problem classes. *Mathematical Programming*, 198(1):1059–1106, 2023.
- [34] T. Le. Nonsmooth nonconvex stochastic heavy ball. *arXiv preprint arXiv:2304.13328*, 2023.
- [35] Y. Li, C. Ju, E. X. Fang, and T. Zhao. Implicit regularization of Bregman proximal point algorithm and mirror descent on separable data. *arXiv preprint arXiv:2108.06808*, 2021.
- [36] Z. Li, T. Wang, J. D. Lee, and S. Arora. Implicit bias of gradient descent on reparametrized models: On equivalence to mirror descent. *Advances in Neural Information Processing Systems*, 35:34626–34640, 2022.
- [37] H. Lu, R. M. Freund, and Y. Nesterov. Relatively smooth convex optimization by first-order methods, and applications. *SIAM J. Optimization*, 28(1):333–354, 2018.
- [38] M. Marcus, B. Santorini, and M. A. Marcinkiewicz. Building a large annotated corpus of english: The penn treebank. 1993.
- [39] A. S. Nemirovskij and D. B. Yudin. Problem complexity and method efficiency in optimization. *Wiley-Interscience*, 1983.
- [40] B. T. Polyak. Some methods of speeding up the convergence of iteration methods. *Ussr Computational Mathematics and Mathematical Physics*, 4(5):1–17, 1964.
- [41] R. Rockafellar and R. J.-B. Wets. *Variational Analysis*. Springer Verlag, Heidelberg, Berlin, New York, 1998.
- [42] R. T. Rockafellar. *Convex analysis*, volume 11. Princeton University Press, 1997.
- [43] W. Rudin. *Principles of mathematical analysis*. McGraw-Hill, Inc., New York, 1964.
- [44] W. Rudin. *Functional analysis 2nd ed.* McGraw-Hill, Inc., New York, 1991.
- [45] A. Ruszczyński. Convergence of a stochastic subgradient method with averaging for nonsmooth nonconvex constrained optimization. *Optimization Letters*, 14(7):1615–1625, 2020.

- [46] H. Sun, K. Gatmiry, K. Ahn, and N. Azizan. A unified approach to controlling implicit regularization via mirror descent. *J. of Machine Learning Research*, 24(393):1–58, 2023.
- [47] S. Takahashi and A. Takeda. Approximate Bregman proximal gradient algorithm for relatively smooth nonconvex optimization. *arXiv preprint arXiv:2311.07847*, 2023.
- [48] L. Van den Dries and C. Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 1996.
- [49] F. Wu and P. Rebescini. A continuous-time mirror descent approach to sparse phase retrieval. *Advances in Neural Information Processing Systems*, 33:20192–20203, 2020.
- [50] N. Xiao, X. Hu, X. Liu, and K.-C. Toh. Adam-family methods for nonsmooth optimization with convergence guarantees. *arXiv preprint arXiv:2305.03938*, 2023.
- [51] N. Xiao, X. Hu, and K.-C. Toh. Convergence guarantees for stochastic subgradient methods in nonsmooth nonconvex optimization. *arXiv preprint arXiv:2307.10053*, 2023.
- [52] L. Yang and K.-C. Toh. Bregman proximal point algorithm revisited: A new inexact version and its inertial variant. *SIAM J. Optimization*, 32(3):1523–1554, 2022.
- [53] Y. You, I. Gitman, and B. Ginsburg. Large batch training of convolutional networks. *arXiv preprint arXiv:1708.03888*, 2017.
- [54] Y. You, J. Li, S. Reddi, J. Hseu, S. Kumar, S. Bhojanapalli, X. Song, J. Demmel, K. Keutzer, and C.-J. Hsieh. Large batch optimization for deep learning: Training bert in 76 minutes. *arXiv preprint arXiv:1904.00962*, 2019.
- [55] W. Zhan, S. Cen, B. Huang, Y. Chen, J. D. Lee, and Y. Chi. Policy mirror descent for regularized reinforcement learning: A generalized framework with linear convergence. *SIAM J. Optimization*, 33(2):1061–1091, 2023.
- [56] S. Zhang and N. He. On the convergence rate of stochastic mirror descent for nonsmooth nonconvex optimization. *arXiv preprint arXiv:1806.04781*, 2018.