A Breiman's theorem for conditional dependent random vector and its applications to risk theory

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Abstract

In this paper, we give a Breiman's theorem for conditional dependent random vector, where one component has a regularly-varying-tailed distribution with the index $\alpha \geq 0$ and its slowly varying function satisfies a relaxed condition, while the other component is non-negative and its tail distribution is lighter than the former. This result substantially extends and improves Theorem 2.1 of Yang and Wang (Extremes, 2013). We also provide some concrete examples and some interesting properties of conditional dependent random vector. Further, we apply the above Breiman's theorem to risk theory, and obtain two asymptotic estimates of the finite-time ruin probability and the infinite-time ruin probability of a discrete-time risk model, in which the corresponding net loss and random discount are conditionally dependent.

Keywords: Conditional dependence; Breiman's theorem; Regular variation; Discrete-time risk model; Ruin probabilities; Asymptotic estimate

2000 Mathematics Subject Classification: 62H20; 62E20; 62P05

1 Introduction

It is well known that Breiman's theorem is an important tool to deal with the tail distribution of product of two components of a random vector, and it plays a key role in many application fields, such as risk theory, see Section 3 below.

In this paper, unless otherwise stated, let (X, Y) be a random vector with the marginal distributions F on $(-\infty, \infty)$ and G on $(0, \infty)$ respectively. Then $Z = XY$ represents the product of these two components and its distribution denoted by H. In risk theory, X and Y can be interpreted as the net loss and random discount, respectively. Both of them are the basic research objects of risk theory. The Breiman's theorem studies the asymptotics relations of the tail distributions of Z and X. Therefore, we should first introduce some concepts and symbols related to distribution.

Let V be a distribution, then we say that $\overline{V} = 1 - V$ is the tail distribution of V. If V is supported on $(-\infty, \infty)$ $\big($including $[0, \infty)$ and $(0, \infty)\big)$, then

$$\overline{V}(x) > 0 \quad \text{for all} \quad x \in (-\infty, \infty).$$

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Recall that a distribution V on $(-\infty,\infty)$ is said to be regularly-varying-tailed with the index $\alpha\geq 0$, denoted by $V\in\mathscr{R}_{-\alpha}$, if $\overline{V}(xy)\sim\overline{V}(x)y^{-\alpha}$, that is

$$\lim\overline{V}(xy)/\overline{V}(x)=y^{-\alpha}\qquad\text{for each } y>0.$$

Hereafter, all limits refer to $x\to\infty$ unless otherwise stated. For more details on regularly-varying-tailed distribution, see Bingham et al. [5] etc. For example,

$$V\in\mathscr{R}_{-\alpha}\quad\text{if and only if}\quad\overline{V}(x)\sim x^{-\alpha}L(x),$$

where $L(\cdot)$ is a positive slowly varying function at infinity satisfying

$$L(xy)\sim L(x)\qquad\text{for each}\ \ y>0.$$

The distribution class $\cup_{\alpha\geq 0}\mathscr{R}_{-\alpha}$ is a proper subclass of the following distribution class introduced by Chistyakov [10]. Say that V on $[0,\infty)$ $\big($including $(0,\infty)\big)$ belongs to the subexponential distribution class $\mathscr{S}$, if

$$\overline{V^{*2}}(x)\sim 2\overline{V}(x),$$

where V^{*2} is the convolution of V with itself. Say that V on $(-\infty,\infty)$ belongs to the class $\mathscr{S}$, if $V^+\in\mathscr{S}$, where $V^+(x)=V(x)\mathbf{1}_{[0,\infty)}(x)$ for all $x\in(-\infty,\infty)$.

Another distribution class introduced by Feller [15] also properly includes class $\cup_{\alpha\geq 0}\mathscr{R}_{-\alpha}$. Say that distribution V on $(-\infty,\infty)$ belongs to the dominantly-varying-tailed distribution class $\mathscr{D}$, if for each $t\in(0,1)$, $\overline{V}(tx)=O\big(\overline{V}(x)\big)$, that is

$$\limsup\overline{V}(tx)/\overline{V}(x)<\infty.$$

If $V\in\mathscr{D}$, then it has a useful property as follows, which plays a key role in many cases, see, for example, Theorems 1.C, 2.A, 2.B, 2.1, 3.1 and 3.2 below.

Proposition 1.A. A distribution $V\in\mathscr{D}$ if and only if for any distribution U on $(-\infty,\infty)$ satisfying $\overline{U}(x)=o(\overline{V}(x))$, that is

$$\lim\overline{U}(x)/\overline{V}(x)=0,$$

there exists a positive function $g(\cdot)$ on $[0,\infty)$ such that

$$g(x)\downarrow 0,\quad xg(x)\uparrow\infty\ \text{ and }\ \overline{U}\big(xg(x)\big)=o\big(\overline{V}(x)\big).\tag{1.1}$$

In the proposition, the proof of the necessity is given by Lemma 3.3 of Tang [39], and the proof of sufficiency is attributed to Proposition 3.1 of Zhou et al. [49].

In addition, the class $\mathscr{S}$ is properly included in the following distribution class. Say that distribution V on $(-\infty,\infty)$ belongs to the long-tailed distribution class $\mathscr{L}$, denoted by $V\in\mathscr{L}$, if for each $t\in(0,\infty)$,

$$\overline{V}(x-t)\sim\overline{V}(x).$$

Goldie [16] pointed out that class $\mathscr{L}$ and class $\mathscr{D}$ cannot contain each other. Further, for a distribution V, we denote a positive function class by

$$\mathscr{H}_V=\{h(\cdot):\ h(x)\uparrow\infty,\ \ x/g(x)\downarrow 0\ \text{ and }\ \overline{V}(x-t)/\overline{V}(x)\to 1\ \text{ uniformly for all }\ |t|\leq h(x)\}.$$

Then the following research way for distributions is often used in many occasions.

Proposition 1.B. (i) A distribution $V\in\mathscr{L}$ if and only if set $\mathscr{H}_V$ is not empty.

(ii) A distribution $V \in \mathscr{S}$ if and only if $V \in \mathscr{L}$ and for any $h(\cdot) \in \mathscr{H}_V$,

$$\int_{h(x)}^{x-h(x)} \overline{V}(x-y)V(dy) = o\big(\overline{V}(x)\big).$$

For the proofs of (i) and (ii), please refer to Lemma 2.5 of Cline and Samorodnitsky [11] and Proposition 1 of Asmussen et al. [3], respectively.

Now we return to the main research objects and objectives of this paper. We continue to use the above notation in the following text.

Many papers have been devoted to studying the tail asymptotics of the product $Z = XY$, especially the asymptotic relationship between $\overline{H}(x)$ and $\overline{F}(x)$.

In the case that X and Y are independent of each other, the following result was first stated by Breiman [6] with $\alpha \in [0, 1]$. And the complete conclusion can be found in Corollary 3.6. (iii) of Cline and Samorodnitsky [11].

Theorem 1.A. For the random vector (X, Y), assume that X and Y are independent of each other. If $F \in \mathscr{R}_{-\alpha}$ for some $\alpha \in [0, \infty)$ and $\mathrm{E}Y^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$, then $H \in \mathscr{R}_{-\alpha}$ and

$$\overline{H}(x) \sim \mathrm{E}Y^{\alpha}\overline{F}(x). \tag{1.2}$$

Since then, some papers have refined this result, see, for example, Jessen and Mikosch [22], Denisov and Zwart [13] and Resnick [34]. Among them, Proposition 2.1 of Denisov and Zwart [13] weakens the moment condition of r.v. Y under some restrictions for distributions F and G.

Theorem 1.B. For the random vector (X, Y), assume that X and Y are independent of each other and $F \in \mathscr{R}_{-\alpha}$ for some $\alpha \in [0, \infty)$ with a positive slowly varying function at infinity $L(\cdot)$. If $\mathrm{E}Y^{\alpha} < \infty$, $\overline{G}(x) = o(\overline{F}(x))$ and

$$\lim \sup_{1 \le y \le x} L(x/y)/L(x) < \infty, \tag{1.3}$$

then $H \in \mathscr{R}_{-\alpha}$ and (1.2) holds.

When F is not necessarily a regularly-varying-tailed, some corresponding results can be found in Hashorva et al. [21] for F belonging to the max-domain of attraction of the Gumbel or Weibull distribution and for G belonging to the max-domain of attraction of the Weibull distribution, Arendarczyk and Dębicki [1] for Weibull distributions F and G, and Hashorva and Li [20] and Cui et al. [12] for semi-regular-varying-tailed F, among others.

In addition, based on Embrechts and Goldie [14], Cline and Samorodnitsky [11] and Tang [38], Xu et al. [45] gave an equivalent condition for $H \in \mathscr{S}$ under the premise that $F \in \mathscr{S}$. In particular, if G is supported by $[0, a]$ for some $a > 0$, then a related result can also refer to Cline and Samorodnitsky [11]. Other qualitative results can be found in Tang [37], Liu and Tang [28] and so on.

In the case that X and Y are dependent, which is a common phenomenon in practice, Maulik et al. [29] extended Breiman's theorem to the case that Y is asymptotically independent of X (in a sense stronger than the usual concept of asymptotic independence), Jiang and Tang [23] extended it to the case that (X, Y) follows the two-dimensional Farlie-Gumbel-Morgenstern (FGM) distribution, see case 1 after Definition 1.1 below. Furthermore, Yang and Wang [46] considered the two-dimensional Sarmanov distribution, see Sarmanov [35], which is more general than the FGM distribution.

Definition 1.1. *Say that the random vector (X, Y) follows a bivariate Sarmanov distribution, if*

$$\mathrm{P}(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \big(1 + \theta\phi_1(u)\phi_2(v)\big) F(du)G(dv) \qquad \textit{for all } x, y \in (-\infty, \infty), \quad (1.4)$$

where $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are two measurable functions, called kernels of this distribution, and the θ is a real constant satisfying

$$\mathrm{E}\phi_1(X) = \mathrm{E}\phi_2(Y) = 0, \qquad (1.5)$$

and

$$1 + \theta\phi_1(x)\phi_2(y) \ge 0 \quad \text{ for all } x \in D_X \ \textit{ and } \ y \in D_Y, \qquad (1.6)$$

where

$$D_X = \{x \in \mathbb{R} : \ \mathrm{P}(X \in (x - \delta, x + \delta)) > 0 \quad \text{for each } \delta > 0\}$$

and

$$D_Y = \{y \ge 0 : \ \mathrm{P}(Y \in (y - \delta, y + \delta)) > 0 \quad \text{for each } \delta > 0\}.$$

Clearly, if $\theta = 0$, or $\phi_1(x) = 0$ for all $x \in D_X$, or $\phi_2(y) = 0$ for all $y \in D_Y$, then X and Y are independent. So, we say that a random vector (X, Y) follows a proper bivariate Sarmanov distribution, if the parameter $\theta \ne 0$, and the kernels $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are not identical to 0 in D_X and D_Y, respectively. For some more details on multivariate Sarmanov distributions, one can refer to Lee [26], Kotz et al. [25], among others. Three common choices for the kernels $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are listed as follows.

Case 1. $\phi_1(x) = 1 - 2F(x)$ and $\phi_2(y) = 1 - 2G(y)$ for all $x \in D_X$ and $y \in D_Y$, leading to the well-known FGM distribution;

Case 2. $\phi_1(x) = (e^{-x} - c_1)\mathbf{1}_{[0,\infty)}(x)$ with $c_1 = \mathrm{E}e^{-X}\mathbf{1}_{\{X \ge 0\}}/\mathrm{P}(X \ge 0)$ and $\phi_2(y) = e^{-y} - \mathrm{E}e^{-Y}$ for all $x \in D_X$ and $y \in D_Y$;

Case 3. $\phi_1(x) = x^p - EX^p$ and $\phi_2(y) = y^p - EY^p$ for all $x \in D_X$ and $y \in D_Y$.

Proposition 1.1 of Yang and Wang [46] remarks that for any proper bivariate Sarmanov distribution, the kernels are bounded.

Proposition 1.C. Assume that random vector (X, Y) follows a proper bivariate Sarmanov distribution of the form (1.4). Then there exist two positive constants b_1 and b_2 such that

$$|\phi_1(x)| \le b_1 \ \text{ for all } x \in D_X \ \text{ and } \ |\phi_2(y)| \le b_2 \ \text{ for all } \ y \in D_Y.$$

Motivated by Theorem 1.B and Jiang and Tang [23], using Proposition 1.A, Theorem 2.1 of Yang and Wang [46] gives the following result.

Theorem 1.C. Let (X, Y) be a random vector with a bivariate Sarmanov distribution defined by (1.4) satisfying (1.5) and (1.6). Assume that $F \in \mathscr{R}_{-\alpha}$ for some $\alpha \ge 0$ with a positive slowly varying function at infinity $L(\cdot)$ and the limit $\lim \phi_1(x) = d_1$ exists. If either (i) $\mathrm{E}Y^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$; or (ii) $\mathrm{E}Y^{\alpha} < \infty$, $\overline{G}(x) = o\big(\overline{F}(x)\big)$ and there exist a positive function $g(\cdot)$ such that (1.1) holds and

$$\lim \sup_{1 \le y \le xg(x)} L(x/y)/L(x) < \infty, \qquad (1.7)$$

then it holds that

$$\overline{H}(x) \sim \mathrm{E}\big(s(Y)Y^{\alpha}\big)\overline{F}(x), \qquad (1.8)$$

where $s(y) = 1 + \theta d_1 \phi_2(y)$ for all $y \in D_Y$.

Further, Remark 2.1 of Yang and Wang [46] shows that both Cases 1 and Case 2 satisfy all the conditions of Theorem 1.C. And Remark 2.2 of Yang and Wang [46] notes that condition (1.7) is properly weaker than condition (1.3).

We might as well call Theorems 1.A, 1.B and 1.C the Breiman's theorem. Naturally, we prefer to get a Breiman's theorem in a general dependent structure between X and Y. In addition, we intuitively believe that there should be some relationship between moment of Y and function $s(\cdot)$.

In Section 2, we introduce a conditional dependent structure of random vector (X, Y), and we give some concrete examples of conditional dependent random vectors, which show that the conditional dependent structure is relatively large. We also get some interesting properties of conditional dependent random vectors. Then, using a simple way, we prove a new Breiman's theorem for conditional dependent random vector (X, Y). Finally, in Section 3, according to this Breiman's theorem, we obtain the asymptotic estimations of finite-time ruin probability and infinite-time ruin probability in a discrete-time risk model with conditional dependent net loss and random discount.

Here, we especially point out that moment condition of the above results is related to some function $s(\cdot)$, see Definition 2.1 below. Thus, it is possible to reduce the order of these moments in many occasions.

2 Breiman's theorem for conditional dependent random vector

2.1 Concepts, Examples and Properties

We firstly introduce the concept of conditional dependent structure of random vector, for research and application of which, please see, for example, Badescu et al. [4], Li et al. [27], Asimit and Badescu [2], Jiang et al. [24].

Definition 2.1. *Let (X, Y) be a random vector. For each $x \in (-\infty, \infty)$ and $y \in D_Y$, assume that the two limits*

$$\lim_{t \downarrow 0} \frac{\mathrm{P}\big(X > x, Y \in (y-t, y]\big)}{\mathrm{P}\big(Y \in (y-t, y]\big)} \quad \text{and} \quad \lim_{t \downarrow 0} \frac{\mathrm{P}\big(X > x, Y \in [y, y+t)\big)}{\mathrm{P}\big(Y \in [y, y+t)\big)}$$

are exist and equal, denoted by $\mathrm{P}(X > x | Y = y)$ and called the conditional distribution of X under condition $Y = y$.

Say that (X, Y) is conditionally dependent (CD), if there exists a positive measurable function $s(\cdot)$ on $[0, \infty)$ such that

$$\mathrm{P}(X > x | Y = y) \sim \overline{F}(x) s(y)$$

uniformly for all $y \in D_Y$, that is

$$\lim \sup_{y \in D_Y} \big|\mathrm{P}(X > x | Y = y) / \big(\overline{F}(x) s(y)\big) - 1\big| = 0. \tag{2.1}$$

Then we give some concrete examples of CD random vector. For simplicity, in the following, we might as well set

$$\mathrm{P}(X > x | Y = y) = \lim_{t \downarrow 0} \mathrm{P}\big(X > x, Y \in [y, y+t)\big) / \mathrm{P}\big(Y \in [y, y+t)\big).$$

Proposition 2.1. *Let (X,Y) be a random vector with a bivariate Sarmanov joint distribution defined by (1.4) satisfying (1.5) and (1.6). If $\lim \phi_1(x) = d_1$ holds for some $d_1 \in (-\infty,\infty)$, $\phi_2(\cdot)$ is continuous on $(0,\infty)$ and*

$$s(y) = 1 + \theta d_1 \phi_2(y) > 0 \qquad \text{for all } y \in D_Y, \tag{2.2}$$

then (X,Y) is CD with the above function $s(\cdot)$ on D_Y.

Proof. For each $y \in D_Y$, because $\phi_2(\cdot)$ is continuous on y, then for any $\varepsilon \in (0,1)$, there exists a $\delta = \delta\big(\phi_2(\cdot), y, \varepsilon\big) > 0$ such that

$$|\phi_2(v) - \phi_2(y)| < \varepsilon \quad \text{for any } |v-y| < \delta.$$

Let $t \in (0,\delta)$, then

$$\begin{aligned}
\mathrm{P}\big(X > x, Y \in [y, y+t)\big) &= \int_x^\infty \int_{[y,y+t)} \big(1 + \theta\phi_1(u)\phi_2(y)\big) F(du)G(dv) \\
&\quad + \int_x^\infty \int_{[y,y+t)} \theta\phi_1(u)(\phi_2(v) - \phi_2(y)) F(du)G(dv) \\
< \ (\text{or} >)\ \mathrm{P}\big(Y \in [y,y+t)\big)\Big(&\int_x^\infty \big(1 + \phi_1(u)\phi_2(y)\big) F(du) \\
&+\varepsilon \ (\text{or} -\varepsilon) \int_x^\infty |\theta\phi_1(u)| F(du)\Big).
\end{aligned}$$

Further, according to Proposition 1.C, we have

$$\begin{aligned}
\mathrm{P}(X > x | Y = y) &= \lim_{t\downarrow 0} \mathrm{P}\big(X > x, Y \in [y,y+t)\big) / \mathrm{P}\big(Y \in [y,y+t)\big) \\
= &\int_x^\infty \big(1 + \theta\phi_1(u)\phi_2(y)\big) F(du) \\
= &\int_x^\infty \big(1 + \theta d_1 \phi_2(y)\big) F(du) + \int_x^\infty \theta\phi_2(y)\big(\phi_1(u) - d_1\big) F(du) \qquad (2.3)
\end{aligned}$$

Therefore, by (2.2), (2.3) and $\phi_1(x) \to d_1$, we have

$$\begin{aligned}
\limsup_{y \in D_Y} \Big|\frac{\mathrm{P}(X > x|Y=y)}{\overline{F}(x)s(y)} - 1\Big| &= \limsup_{y\in D_Y} \Big| \frac{\int_x^\infty \big(1+\theta d_1\phi_2(y)\big)F(du)}{\overline{F}(x)s(y)} \\
&\quad + \frac{\int_x^\infty \theta\phi_2(y)\big(\phi_1(u) - d_1\big)F(du)}{\overline{F}(x)s(y)} - 1\Big| \\
= \ 0,
\end{aligned}$$

that is (X,Y) is CD. □

Now we pay attention to another dependent structure of two-dimensional random vector, see, for example, Nelsen [30].

Definition 2.2. *Say that a random vector (X,Y) follows a bivariate Frank distribution, if for some $\theta > 0$,*

$$\mathrm{P}(X \le x, Y \le y) = -\frac{1}{\theta} \ln\left(1 + \frac{(e^{-\theta F(x)} - 1)(e^{-\theta G(y)} - 1)}{(e^{-\theta} - 1)}\right), \quad x,\ y \in (-\infty,\infty). \tag{2.4}$$

Proposition 2.2. *Let (X, Y) be the random vector with a bivariate Frank joint distribution defined by (2.4) for some $\theta > 0$, then (X, Y) is CD with the a positive measurable function $s(\cdot)$ on $(0, \infty)$ defined in (2.5) below.*

Proof. We first show that the following positive measurable function $s(\cdot)$ is positive and bounded on $(0, \infty)$. In fact, because $\theta > 0$,

$$0 < \frac{\theta e^{-\theta}}{1-e^{-\theta}} \leq s(y) = \frac{\theta e^{-\theta\overline{G}(y)}}{1-e^{-\theta}} \leq \frac{\theta}{1-e^{-\theta}} < \infty \qquad \text{for all } y > 0. \tag{2.5}$$

Secondly, we give the expression of $\mathrm{P}(X > x|Y = y)$. From (2.4) and

$$\lim_{v\to 0}(e^{-v} - 1)/v = -1,$$

for each pair $x \in (-\infty, \infty)$ and $y \in D_Y$, we have

$$\begin{aligned}
&\mathrm{P}(X > x|Y = y) = 1 - \lim_{t\downarrow 0} \frac{\mathrm{P}\big(X \leq x, Y \in [y, y+t]\big)}{\mathrm{P}\big(Y \in [y, y+t]\big)} \\
= \;& 1 + \lim_{t\downarrow 0} \frac{1}{\theta \mathrm{P}\big(Y \in [y, y+t]\big)} \ln\Big(\frac{(e^{-\theta}-1) + (e^{-\theta F(x)}-1)(e^{-\theta G(y+t)}-1)}{(e^{-\theta}-1) + (e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)}\Big) \\
= \;& 1 + \lim_{t\downarrow 0} \frac{1}{\theta P(Y \in [y, y+t])} \ln\Big(1 + \frac{(e^{-\theta F(x)}-1)(e^{-\theta G(y+t)} - e^{-\theta G(y)})}{(e^{-\theta}-1) + (e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)}\Big) \\
= \;& 1 + \lim_{t\downarrow 0} \frac{\big(e^{-\theta F(x)}-1\big)\big(e^{-\theta G(y+t)} - e^{-\theta G(y)}\big)}{\theta P(Y \in [y, y+t])\big((e^{-\theta}-1) + (e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)\big)} \\
= \;& 1 + \frac{(e^{-\theta F(x)}-1)e^{-\theta G(y)}}{(e^{-\theta}-1) + (e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)} \lim_{t\downarrow 0} \frac{e^{-\theta\big(G(y+t)-G(y)\big)} - 1}{\theta\big(G(y+t) - G(y)\big)} \\
= \;& 1 - \frac{(e^{-\theta F(x)}-1)e^{-\theta G(y)}}{(e^{-\theta}-1) + (e^{-\theta F(x)}-1)\big(e^{-\theta G(y)}-1\big)} \\
= \;& \frac{e^{-\theta} - e^{-\theta F(x)}}{(e^{-\theta}-1) + (e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)}.
\end{aligned}$$

Finally, we prove that (X, Y) is CD. By

$$\lim \big(e^{-\theta} - e^{-\theta F(x)}\big)/\overline{F}(x) = -\theta e^{-\theta}$$

and

$$\lim e^{-\theta F(x)} = e^{-\theta},$$

we know that, for any $\varepsilon > 0$, there exists $x_0 = x_0(F, \theta, \varepsilon) > 0$ such that, when $x \geq x_0$.

$$|(e^{-\theta} - e^{-\theta F(x)})/\theta e^{-\theta}\overline{F}(x) + 1| < \varepsilon$$

and

$$e^{-\theta F(x)} - e^{-\theta} < \varepsilon(1 - e^{-\theta})/(e^{\theta} - 1).$$

Thus, for each $x \geq x_0$ and the above ε, we have

$$
\begin{aligned}
& \sup_{y\in D_Y}\Big|\frac{\mathrm{P}(X>x|Y=y)}{\overline{F}(x)s(y)}-1\Big| = \sup_{y\in D_Y}\Big|\frac{\mathrm{P}(X>x|Y=y)-\overline{F}(x)s(y)}{\overline{F}(x)s(y)}\Big| \\
\le\ & \sup_{y\in D_Y}\big|\mathrm{P}(X>x|Y=y)-\overline{F}(x)s(y)\big|(1-e^{-\theta})\big/\big(\theta e^{-\theta}\overline{F}(x)\big) \\
=\ & \frac{e^{\theta}-1}{\theta\overline{F}(x)}\sup_{y\in D_Y}\Big|\frac{e^{-\theta}-e^{-\theta F(x)}}{(e^{-\theta}-1)+(e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)}-\frac{\overline{F}(x)\theta e^{-\theta\overline{G}(y)}}{1-e^{-\theta}}\Big| \\
=\ & \frac{e^{\theta}-1}{\theta\overline{F}(x)}\sup_{y>0}\Big|\frac{e^{-\theta}-e^{-\theta F(x)}+\theta e^{-\theta}\overline{F}(x)-\theta e^{-\theta}\overline{F}(x)}{(e^{-\theta}-1)+(e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)}-\frac{\overline{F}(x)\theta e^{-\theta\overline{G}(y)}}{1-e^{-\theta}}\Big| \\
\le\ & (1-e^{-\theta})\Big|\frac{e^{-\theta}-e^{-\theta F(x)}}{\theta e^{-\theta}\overline{F}(x)}+1\Big|\sup_{y\in D_Y}\Big|\frac{1}{(e^{-\theta}-1)+(e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)}\Big| \\
& \qquad +(1-e^{-\theta})\sup_{y\in D_Y}\Big|\frac{1}{(e^{-\theta}-1)+(e^{-\theta F(x)}-1)(e^{-\theta G(y)}-1)}+\frac{e^{\theta G(y)}}{1-e^{-\theta}}\Big| \\
<\ & \varepsilon+\sup_{y\in D_Y}\big|(e^{-\theta F(x)}-e^{-\theta})(e^{\theta G(y)}-1)\big|/(1-e^{-\theta}) \\
\le\ & \varepsilon+\sup_{y\in D_Y}\big|(e^{-\theta F(x)}-e^{-\theta})\big|(e^{\theta}-1)/(1-e^{-\theta}) \\
<\ & 2\varepsilon,
\end{aligned}
$$

which implies (X,Y) is CD with the above $s(\cdot)$ on $(0,\infty)$ by the arbitrariness of ε. $\square$

The third two-dimensional joint distribution is as follows, see, for example,Li et.al[27].

Definition 2.3. *Say that a random vector (X,Y) follows a bivariate Ali-Mikhail-Haq (AMH) distribution, if for some $\theta\in[-1,1)$,*

$$
\mathrm{P}(X\le x,Y\le y)=F(x)G(y)\big/\big(1-\theta\overline{F}(x)\overline{G}(x)\big),\qquad x,\ y\in(-\infty,\infty). \tag{2.6}
$$

Proposition 2.3. *Let (X,Y) be the random vector with a bivariate AMH joint distribution defined by (2.6) with some $\theta\in[-1,1)$. When $\theta\in(-1,1)$, (X,Y) is CD with the a positive measurable function*

$$
s(y)=1+\theta\big(1-2\overline{G}(y)\big)\qquad \text{for all}\ \ y>0. \tag{2.7}
$$

When $\theta=-1$,

$$
\mathrm{P}(X>x|Y=y)\sim\overline{F}(x)s(y)\quad \text{for each}\ \ y>0 \tag{2.8}
$$

and

$$
\lim\sup_{y\in D_Y\cap(0,y_0]}\big|\mathrm{P}(X>x|Y=y)/\big(\overline{F}(x)s(y)\big)-1\big|=0, \tag{2.9}
$$

where $s(y)=2\overline{G}(y)$ for all $y>0$ and

$$
y_0=\sup\{y>0:3\overline{G}^2(y)\ge 1\}.
$$

However, (X,Y) is not a CD.

Proof. Along the proof line of Proposition 2.2, for each $\theta \in [-1,1)$, $x \in (-\infty,\infty)$ and $y \in D_Y$, we first get that

$$\begin{aligned}
&\mathrm{P}(X>x|Y=y)=1-\lim_{t\downarrow 0}\mathrm{P}\big(X\le x, Y\in[y,y+t]\big)\big/\mathrm{P}\big(Y\in[y,y+t]\big)\\
=\ &\lim_{t\downarrow 0}\frac{1-\theta\overline{F}(x)\big(\overline{G}(y+t)+\overline{G}(y)\big)+\theta^2\overline{F}^2(x)\overline{G}(y+t)\overline{G}(y)-F(x)+\theta F(x)\overline{F}(x)}{\big(1-\theta\overline{F}(x)\overline{G}(y+t)\big)\big(1-\theta\overline{F}(x)\overline{G}(y)\big)}\\
=\ &\frac{\big(1+\theta-2\theta\overline{G}(y)-\theta\overline{F}(x)+\theta^2\overline{F}(x)\overline{G}^2(y)\big)\overline{F}(x)}{\big(1-\theta\overline{F}(x)\overline{G}(y)\big)^2}. \qquad (2.10)
\end{aligned}$$

Further, for any $\varepsilon>0$, we take $x_0=x_0(F,\varepsilon)>0$ such that $\overline{F}(x)\le\varepsilon$ for all $x\ge x_0$.

Next, we prove that (X,Y) is CD for $\theta\in(-1,1)$. In fact,

$$\begin{aligned}
&\sup_{y\in D_Y}\Big|\frac{\mathrm{P}(X>x|Y=y)}{\overline{F}(x)s(y)}-1\Big|\\
=\ &\sup_{y\in D_Y}\Big|\frac{\overline{G}(y)\big(2-\theta\overline{F}(x)\overline{G}(y)\big)\big(1+\theta(1-2\overline{G}(y))\big)-\big(1-\theta\overline{G}^2(y)\big)}{\big(1+\theta\big(1-2\overline{G}(y)\big)\big)\big(1-\theta\overline{F}(x)\overline{G}(y)\big)^2}\Big||\theta|\overline{F}(x)\\
\le\ &\sup_{y\in D_Y}\Big(\Big|\frac{\overline{G}(y)\big(2-\theta\overline{F}(x)\overline{G}(y)\big)}{\big(1-\theta\overline{F}(x)\overline{G}(y)\big)^2}\Big|+\Big|\frac{1-\theta\overline{G}^2(y)}{\big(1+\theta(1-2\overline{G}(y))\big)\big(1-\theta\overline{F}(x)\overline{G}(y)\big)^2}\Big|\Big)\overline{F}(x)\\
\le\ &\Big(\frac{2+|\theta|\varepsilon}{(1-|\theta|\varepsilon)^2}+\frac{1+|\theta|}{(1-|\theta|)(1-|\theta|\varepsilon)^2}\Big)\varepsilon.
\end{aligned}$$

Then we complete the proof by the arbitrariness of ε.

For $\theta=-1$, by (2.10), (2.8) holds clearly. (2.9) comes from the following fact. For the above $s(\cdot)=2\overline{G}(\cdot)$, ε and x_0, when $x\ge x_0$, we have

$$\begin{aligned}
&\sup_{y\in D_Y\cap(0,y_0]}\Big|\frac{\mathrm{P}(X>x|Y=y)}{\overline{F}(x)s(y)}-1\Big|=\sup_{y\in D_Y\cap(0,y_0]}\Big|\frac{3\overline{G}^2(y)+2\overline{F}(x)\overline{G}^3(y)-1}{2\overline{G}(y)\big(1+\overline{F}(x)\overline{G}(y)\big)^2}\Big|\overline{F}(x)\\
\le\ &\sup_{y\in D_Y\cap(0,y_0]}\Big(\big(3\overline{G}(y)+2\overline{F}(x)\overline{G}^2(y)\big)\Big/\big(1+\overline{F}(x)\overline{G}(y)\big)^2\Big)\overline{F}(x)/2\\
\le\ &(3+2\varepsilon)\varepsilon/2.
\end{aligned}$$

Finally, we prove (X,Y) is not a CD. For the above ε and x_0, we further set $\varepsilon<1/5$. For each $x\ge x_0$, there exists $y=y(F,G,\varepsilon)>y_0$ large enough such that

$$\overline{G}(y)<\overline{F}(x)(1-5\varepsilon)\big/\big(2\big(1+\varepsilon\big)^2\big),$$

which leads to that

$$\overline{G}(y)<\varepsilon \quad\text{and}\quad 1>3\overline{G}^2(y)+2\overline{F}(x)\overline{G}^3(y).$$

Therefore, for the above x and y, we have

$$\begin{aligned}
&\Big|\frac{\mathrm{P}(X>x|Y=y)}{\overline{F}(x)s(y)}-1\Big|=\frac{\big(1-3\overline{G}^2(y)-2\overline{F}(x)\overline{G}^3(y)\big)\overline{F}(x)}{2\overline{G}(y)\big(1+\overline{F}(x)\overline{G}(y)\big)^2}\\
\ge\ &(1-5\varepsilon)\overline{F}(x)\big/\big(2\big(1+\varepsilon\big)^2\overline{G}(y)\big)\\
>\ &1,
\end{aligned}$$

that is (X,Y) is not a CD. $\square$

In Propositions 2.1, 2.2 and 2.3, the functions $s(\cdot)$ are bounded. In fact, this is not an individual phenomenon.

Proposition 2.4. *Let (X,Y) be a CD random vector with a function $s(\cdot)$ in Definition 2.1. Then $s(\cdot)$ is bounded in $y \in D_Y$, that is there is a $C > 0$ such that*

$$s(y) \le C \qquad \text{for all } y \in D_Y.$$

Proof. First, according to Definition 2.1, $\mathrm{P}(X > x|Y = y) \le 1$ for all $y \in D_Y$. And because F on $(-\infty,\infty)$, $\overline{F}(x) > 0$ for each $x \in (-\infty,\infty)$. Thus, $s(\cdot)$ is finite by (2.1).

Secondly, we prove $s(\cdot)$ is bounded in D_Y. Otherwise, there exist $y_n \in D_Y,\ n \ge 1$ such that

$$y_n \uparrow \infty \ \text{ as } n \to \infty \ \text{ and } \ s(y_n) > n \ \text{ for all } \ n \ge 1.$$

And by (2.1), there exists $x_0 = x_0(X,Y) > 0$ such that, when $x \ge x_0$,

$$\overline{F}(x)s(y) < 2P(X > x|Y = y) \le 2 \qquad \text{for all } y \in D_Y.$$

Take an integer $n_0 = n_0(F,G,x_0)$ large enough such that $\overline{F}(x_0)n_0 > 2$. Then

$$\overline{F}(x_0)s(y_{n_0}) > \overline{F}(x_0)n_0 > 2,$$

which contradicts the previous inequality with $x = x_0$ and $y = y_{n_0}$. $\square$

In addition to the above three concrete examples, we also hope to find a more general way to construct some CD random vectors.

Proposition 2.5. *Let (ξ,η) be a random vector with two marginal distributions U and V on $(0,\infty)$. Assume that (ξ,η) is CD and the corresponding function $s_0(\cdot)$ on $(0,\infty)$ is continuous, that is there is a positive and continuous function $s_0(\cdot)$ on $(0,\infty)$ such that*

$$P(\xi > x|\eta = y) \sim \overline{U}(x)s_0(y) \qquad \text{for each } y \in D_\eta.$$

Further, let $X = \xi - \eta$ and $Y = \eta$. If $U \in \mathscr{L}$, then random vector (X,Y) still is CD with a positive and continuous function $s(\cdot)$ on $(0,\infty)$ satisfying

$$s(y) = s_0(y)\mathrm{E}s_0(Y) \qquad \text{for each } \ y \in D_Y = D_\eta.$$

Proof. Because (ξ,η) is CD, by $U \in \mathscr{L}$, we have

$$\overline{F}(x) = \int_{D_\eta} \mathrm{P}\big(\xi > x + v|\eta = v)V(dv) \sim \int_{D_\eta} \overline{U}(x+v)s_0(v)V(dv) \sim \overline{U}(x)\mathrm{E}s_0(\eta)$$

and

$$\mathrm{P}\big(X > x, Y \in [y,y+t)\big) = \mathrm{P}\big(\xi - \eta > x, \eta \in [y,y+t)\big) \sim \overline{U}(x)\int_{[y,y+t)} s_0(v)V(dv)$$

for any $t > 0$ and for each $y \in D_\eta$. Thus, according to Proposition 2.4 and by the continuity of $s(\cdot)$, for each $y \in D_\eta$, we know that $\mathrm{E}s_0(\eta) < \infty$ and

$$\begin{aligned}\mathrm{P}(X > x|Y = y) &= \lim_{t\downarrow 0} \mathrm{P}\big(\xi - \eta > x, \eta \in [y, y+t)\big)/\mathrm{P}\big(\eta \in [y, y+t)\big) \\ &\sim \overline{F}(x)s_0(y)\mathrm{E}s_0(\eta),\end{aligned}$$

that is (X, Y) is CD with $s(\cdot) = s_0(\cdot)\mathrm{E}s_0(\eta)$ on $D_Y = D_\eta$. □

We have reason to believe that there are many ways to construct CD random vectors. Therefore, we can say that there are many CD random vectors. More importantly, however, we find a somewhat surprising conclusion from the above proposition.

Proposition 2.6. *Let (X, Y) be a CD random vector with the corresponding function $s(\cdot)$ on $(0, \infty)$. If X is independent of Y, then $s(y) = 1$ for all $y \in D_Y$. On the contrary, $s(y) = 1$ for all $y \in D_Y$ cannot implies the independence of X and Y.*

Proof. We only need to prove the second conclusion. In fact, in Proposition 2.5, if ξ is independent of η, then $s_0(y) = 1$ for all $y \in D_\eta$. Thus, according the proposition, we know that $s(y) = s_0(y)\mathrm{E}s_0(\eta) = 1$ for all $y \in D_Y$. However, $X = \xi - \eta$ is not independent of $Y = \eta$ clearly. □

2.2 A new Breiman's theorem

In this subsection, we prove a new Breiman's theorem for CD random vector (X, Y).

Theorem 2.1. *Let (X, Y) be CD random vector. Assume that marginal distributions $F \in \mathscr{R}_{-\alpha}$ for some $\alpha \geq 0$ with a positive slowly varying function at infinity $L(\cdot)$ and G satisfies condition that $\overline{G}(x) = o\big(\overline{F}(x)\big)$. If either (i) for some $\varepsilon > 0$,*

$$\mathrm{E}Y^{\alpha+\varepsilon}s(Y) < \infty; \tag{2.11}$$

or (ii) $L(\cdot)$ satisfies (1.7) for some positive function $g(\cdot)$ in (1.1) and

$$\mathrm{E}Y^{\alpha}s(Y) < \infty, \tag{2.12}$$

then

$$\overline{H}(x) = \mathrm{P}(XY > x) \sim \mathrm{E}Y^{\alpha}s(Y)\overline{F}(x). \tag{2.13}$$

Proof. We firstly prove (ii).

Because $\mathscr{R}_{-\alpha} \subset \mathcal{D}$ and $\overline{G}(x) = o(\overline{F}(x))$, there is a positive function $g(\cdot)$ satisfying (1.1) according to Proposition 1.A. We take $g(\cdot)$ to sprite

$$\overline{H}(x) = \Big(\int_0^{xg(x)} + \int_{xg(x)}^{\infty}\Big)P(X > x/y \mid Y = y)G(dy) = \mathrm{P}_1(x) + \mathrm{P}_2(x). \tag{2.14}$$

For $\mathrm{P}_2(x)$, by (1.1) and $\overline{F}(x) = O\big(\overline{H}(x)\big)$, we have

$$\mathrm{P}_2(x) \leq \overline{G}\big(xg(x)\big) = o\big(\overline{F}(x)\big) = o\big(\overline{H}(x)\big). \tag{2.15}$$

For $\mathrm{P}_1(x)$, by (1.7) and (1.1), there exist four positive constants $x_0 = x_0(F)$ and

$$C_1 = C_1(x_0) \le C_2 = C_2(x_0) \le C_3 = C_3(x_0),$$

when $x \ge x_0$, it holds uniformly for all $y \in (0, xg(x)]$ that

$$\begin{aligned} &\mathrm{P}(X > x/y|Y = y)\mathbf{1}_{(0,xg(x)]}(y) \le P(X > x|Y = y)\mathbf{1}_{(0,1]}(y) \\ &\qquad\qquad\qquad\qquad + \mathrm{P}(X > x/y|Y = y)\mathbf{1}_{(1,xg(x)]}(y) \\ \le\ & C_1\big(\overline{F}(x)s(y)\mathbf{1}_{(0,1]}(y) + \overline{F}(x/y)s(y)\mathbf{1}_{(1,xg(x)]}(y)\big) \\ \le\ & C_2\big(s(y)\mathbf{1}_{(0,1]}(y) + y^{\alpha}s(y)L(x/y)/L(x)\mathbf{1}_{(1,xg(x)]}(y)\big)\overline{F}(x) \\ \le\ & C_3\big(s(y)\mathbf{1}_{(0,1]}(y) + y^{\alpha}s(y)\mathbf{1}_{(1,\infty]}(y)\big)\overline{F}(x). \end{aligned} \tag{2.16}$$

Thus, according to dominant convergence theorem, by (2.16) and (2.12), we know that

$$\begin{aligned} \lim \mathrm{P}_1(x)/\overline{F}(x) &= \int_0^{\infty} \lim \big(\mathrm{P}(X_1 > x/y|Y = y)/\overline{F}(x)\big)\mathbf{1}_{(0,xg(x)]}(y)G(dy) \\ = & \int_0^{\infty} \lim \big(\overline{F}(x/y)/\overline{F}(x)\big)\mathbf{1}_{(0,xg(x)]}(y)s(y)G(dy) \\ = & \ \mathrm{E}Y^{\alpha}s(Y). \end{aligned} \tag{2.17}$$

Combining (2.14), (2.15) and (2.17), we get (2.13) immediately.

Next, we prove (i). To this end, we only need to deal with $L(x/y)/L(x)$ in the third line of (2.16). According to Potter's theorem, see, for example, Theorem 1.5.6 of Bingham et al. [5], for the above x_0 large enough and the ε in (2.11), when $x \ge x_0$, there is a $C_3 = C_3(F, x_0, \varepsilon) \ge C_2$ such that

$$L(x/y)/L(x) \le C_3 y^{\alpha+\varepsilon}. \tag{2.18}$$

Thus, by (2.18) and (2.11), according to dominant convergence theorem, we know that (2.17) still holds.

Combining (2.14), (2.15) and (2.17), (2.13) still is holds. $\square$

According to Theorem 2.1 and Proposition 2.1, we can get the following results directly.

Corollary 2.1. *Let (X, Y) be a random vector with a bivariate Sarmanov joint distribution defined by (1.4) satisfying (1.5) and (1.6). Assume that all conditions of Proposition 2.1 are satisfied, then (2.13) holds with $s(\cdot)$ in (2.2).*

Remark 2.1. *(i) According to Proposition 1.C or Proposition 2.4, we know that, there are two positive constants C_1 and C_2 such that*

$$\mathrm{E}Y^{\alpha+\varepsilon}s(Y) \le C_1\mathrm{E}Y^{\alpha+\varepsilon} \quad \text{and} \quad \mathrm{E}Y^{\alpha}s(Y) \le C_2\mathrm{E}Y^{\alpha},$$

that is the order of $\mathrm{E}Y^{\alpha+\varepsilon}s(Y)$ (or $\mathrm{E}Y^{\alpha}s(Y)$) is not stronger than $\mathrm{E}Y^{\alpha+\varepsilon}$ (or $\mathrm{E}Y^{\alpha}$). In the following, however, we give a concrete random vector (X, Y), which satisfies all the conditions of Corollary 2.1 and the order of $\mathrm{E}Y^{\alpha}s(Y)$ is significantly lower than that of $\mathrm{E}Y^{\alpha}$. In other words, this result can substantially improve the moment condition of Theorem 1.C for many casses.

Example 2.1. *Let (X, Y) be a random vector with a bivariate Sarmanov joint distribution defined by (1.4).*

In (1.4), we take a positive measurable function $\phi_1(\cdot)$ on $(-\infty, \infty)$ such that $\mathrm{E}\phi_1(X) = 0$ and $\phi_1(x) \to d_1 > 0$. Further, if we take

$$\theta = 1/\big(d_1 \mathrm{E}(1 + 1/Y)\big) \quad \textit{and} \quad \phi_2(y) = 1/(1 + y) - 1/(\theta d_1) \qquad \textit{for all } y > 0,$$

then $\theta > 0$, $\phi_2(\cdot)$ is a continuous function on $(0, \infty)$,

$$s(y) = 1 + \theta d_1 \phi_2(y) = \theta d_1/(1 + y) > 0 \qquad \textit{for all } \ y > 0$$

and

$$\mathrm{E}\phi_2(Y) = \mathrm{E}1/(1 + Y) - 1/(\theta d_1) = 0.$$

Thus, (1.5) and (1.6) are satisfied. At this time,

$$\mathrm{E}Y^\alpha s(Y) = \mathrm{E}Y^\alpha/(1 + Y) \le G(1) + \mathrm{E}Y^{\alpha-1}\mathbf{1}_{\{Y>1\}}.$$

(ii) Theorem 1.C does not require conditions that $\phi_2(\cdot)$ to be continuous and $s(y) > 0$ for all $y > 0$. Thus, both Corollary 2.1 and Theorem 1.C have their own independent values. Of course, in our opinion, it is more substantial to reduce the order of moments of Y.

And according to Theorem 2.1, Proposition 2.2 and Proposition 2.5, by (2.5), we also have the two following results.

Corollary 2.2. *Let (X, Y) be a random vector with a bivariate Frank joint distribution defined by (2.4). Assume that all conditions of Proposition 2.2 are satisfied, then (2.13) holds with $s(\cdot)$ in (2.5).*

Corollary 2.3. *Let (X, Y) be a random vector with a bivariate AMH joint distribution defined by (2.6). Assume that all conditions of Proposition 2.3 are satisfied, then (2.13) holds with $s(\cdot)$ in the proposition.*

Corollary 2.4. *Let (X, Y) be a random vector defined by Proposition 2.5. Assume that all conditions of the proposition are satisfied, then (2.13) holds with $s(\cdot)$ in the proposition.*

3 Applications to risk theory

We first introduce a discrete-time risk model which was proposed by Nyrhinen [32, 33].

In the above model, within period i, the net insurance loss (i.e. the total claim amount minus the total premium income) is denoted by a r.v. X_i with a common distribution F on $(-\infty, \infty)$. And the random discount factor Y_i from time i to time $i-1$ with a common distribution G on $(0, \infty)$. As in Norberg [32], $X_n,\ n \ge 1$ and $Y_n,\ n \ge 1$ are called by insurance risks and financial risks, respectively. For each integer $n \ge 1$, the sum

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \tag{3.1}$$

represents the stochastic discount value of aggregate net losses up to time n. Let $x \geq 0$ be the initial wealth of the insurer, then the finite-time ruin probability at time n and the infinite-time ruin probability can be respectively defined by

$$\psi_n(x) = \mathrm{P}\big(\max_{1\leq k\leq n} S_k > x\big) \tag{3.2}$$

and

$$\psi(x) = \lim_{n\to\infty} \psi(x,n) = \mathrm{P}\big(\sup_{n\geq 1} S_n > x\big). \tag{3.3}$$

On the study of the asymptotics of $\psi_n(x)$ and $\psi(x)$, when $\{X_i : i \geq 1\}$ and $\{Y_i : i \geq 1\}$ are independent of each other, please refer to Tang and Tsitsiashvili [40, 41], Goovaerts et al. [19], Wang et al. [43], Wang and Tang [44], Zhang et al. [48], Chen and Yuen [8], Shen et al. [36], Gao and Wang [18], Tang et al. [42], Yi et al. [47], Zou et al. [49], Cheng et al. [9] and Xu et al. [45] among others.

Let vector (X, Y) be an independent copy of (X_i, Y_i) for all $i \geq 1$. When (X, Y) follows a certain dependent structure, there are also some corresponding results. Among them, Chen [7] has obtained the following result that, for each fixed n,

$$\psi(x,n) \sim \sum_{i=1}^{n} \mathrm{P}\Big(X_i \prod_{j=1}^{i} Y_j > x\Big) \tag{3.4}$$

under some conditions, where (X, Y) follows a common bivariate FGM distribution. And Theorem 4.1 and Theorem 4.2 of Yang and Wang [46] give the following two results for (X, Y) following a common bivariate Sarmanov distribution.

Hereafter, we denote by H_i the distribution function of $X_i \prod_{j=1}^{i} Y_j,\ i \geq 1$. Clearly, $H_1 = H$.

Theorem 3.A. In the described discrete-time risk model, assume that $\{(X, Y), (X_i, Y_i) : i \geq 1\}$ are i.i.d. random vectors. Under the conditions Theorem 1.C, for each fixed integer $n \geq 1$, it holds that

$$\psi(x,n) \sim \Big(\mathrm{E}\big(s(Y)Y^\alpha\big)\big(1 - \mathrm{E}^n Y^\alpha\big)/\big(1 - \mathrm{E}Y^\alpha\big)\Big)\overline{F}(x), \tag{3.5}$$

where $s(y) = 1 + \theta d_1 \phi_2(y),\ y \in D_Y$. Among them, we naturally agree that

$$(1 - \mathrm{E}^n Y^\alpha)/(1 - \mathrm{E}Y^\alpha) = n, \quad \text{ if } \ \mathrm{E}Y^\alpha = 1. \tag{3.6}$$

Theorem 3.B. In the described discrete-time risk model, assume that $\{(X, Y), (X_i, Y_i) : i \geq 1\}$ are i.i.d. random vectors. Under the conditions (i) of Theorem 1.C, for each fixed integer $n \geq 1$, it holds that

$$\psi(x) \sim \Big(\mathrm{E}\big(s(Y)Y^\alpha\big)/\big(1 - \mathrm{E}Y^\alpha\big)\Big)\overline{F}(x), \tag{3.7}$$

where $s(y) = 1 + \theta d_1 \phi_2(y),\ y \in D_Y$. Furthermore, (3.5) holds uniformly for all $n \geq 1$.

Clearly, these two results describe the concrete influence of Sarmanov dependent structure of (X, Y) on ruin probability.

In this section, using Theorem 2.1, we respectively give two asymptotic estimates of ruin probability of the discrete-time risk model, where the net loss and random discount are CD. Here, we can also find the influence of the CD structure on the ruin probability.

Theorem 3.1. *In the described discrete-time risk model, assume that $\{(X,Y),(X_i,Y_i), i \geq 1\}$ are i.i.d.random vectors. Under the conditions of Theorem 2.1, (3.5) holds with $s(\cdot)$ in definition 2.1 for each fixed integer $n \geq 1$.*

Proof. We deal with case (i) and case (ii) together.

To prove (3.5), we only need respectively to prove that

$$\overline{H_i}(x) = P(X_iY_i \cdots Y_1 > x) \sim \mathrm{E}^{i-1}Y^\alpha \mathrm{E}Y^\alpha s(Y)\overline{F}(x) \tag{3.8}$$

and

$$\psi(x,n) \sim \sum_{i=1}^{n} \overline{H_i}(x) \tag{3.9}$$

for all integers $1 \leq i \leq n$ and each $n \geq 1$ by induction.

We first prove (3.8). According to Theorem 2.1, (3.8) holds for $i = 1$ obviously, which implies $H_1 \in \mathscr{R}_{-\alpha}$ with a slowly varying function at infinity

$$L_1(x) \sim \mathrm{E}Y^\alpha s(Y)L(x),$$

where $L(\cdot)$ is a slowly varying function at infinity of distribution F.

Assume that (3.8) holds for each integer $i \geq 1$ and $H_i \in \mathscr{R}_{-\alpha}$ with a slowly varying function at infinity $L_i(\cdot)$ satisfying

$$L_i(x) \sim \mathrm{E}^{i-1}Y^\alpha \mathrm{E}Y^\alpha s(Y)L(x). \tag{3.10}$$

Clearly, replacing $L(\cdot)$ with $L_i(\cdot)$, condition (1.7) is still satisfied and

$$\overline{G}(x) = o\big(\overline{F}(x)\big) = o\big(\overline{H_i}(x)\big) \quad \text{for each } i \geq 1.$$

Recall that $(X_i, Y_1),\ i \geq 1$ are i.i.d. random vectors. Thus, according to Theorem 2.1, by induction assumption, it holds that

$$\overline{H_{i+1}}(x) = \mathrm{P}\big((X_{i+1}Y_{i+1}\cdots Y_2)Y_1 > x\big) \sim \mathrm{E}Y^\alpha \overline{H_i}(x) \sim \mathrm{E}^{(i+1)-1}Y^\alpha \mathrm{E}Y^\alpha s(Y)\overline{F}(x),$$

that is (3.8) and (3.10) hold for $i+1$ and $H_{i+1} \in \mathscr{R}_{-\alpha}$.

Secondly, we prove (3.9). To this end, we note that for each $n \geq 1$,

$$\mathrm{P}(S_n > x) \leq \psi(x,n) \leq \mathrm{P}\Big(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x\Big)$$

and (X^+, Y) also is CD. This indicates that if we can establish

$$\mathrm{P}(S_n > x) \sim \sum_{i=1}^{n} \overline{H_i}(x),$$

then the asymptotic formula

$$\mathrm{P}\left(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x\right) \sim \sum_{i=1}^{n} \overline{H_i}(x) \tag{3.11}$$

should hold as well. Furthermore, since $(X,Y),\ (X_i,Y_i),\ i \geq 1$ are i.i.d. random vectors, it holds that

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \overset{\mathrm{d}}{=} \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j = T_n, \quad n \geq 1,$$

where $\overset{\mathrm{d}}{=}$ stands for equality in distribution. Therefore, in order to prove (3.9), we only need to prove that

$$\mathrm{P}(T_n > x) = \overline{F_{T_n}}(x) \sim \sum_{i=1}^{n} \overline{H_i}(x) \sim \Big(\big(1 - \mathrm{E}^n Y^\alpha\big)\mathrm{E}Y^\alpha s(Y)/\big(1 - \mathrm{E}Y^\alpha\big)\Big)\overline{F}(x) \tag{3.12}$$

by induction on n.

Clearly, according to Theorem 2.1, by $F \in \mathscr{R}_{-\alpha}$, we know that (3.12) holds for $n = 1$ and $F_{T_1} = H_1 \in \mathscr{R}_{-\alpha}$.

For the sake of brevity, we only prove (3.12) for $n = 2$ in detail. By the method that has been used many times before, we have

$$\begin{aligned} \overline{F_{T_2}}(x) &\sim \int_0^{xg(x)} \mathrm{P}(X_1Y_1 + X_2 > x/y | Y_2 = y) G(dy) \\ &= \int_0^{xg(x)} \Big(\int_{-\infty}^{x/y} + \int_{x/y}^{\infty} \mathrm{P}(X_2 > x/y - z | Y_2 = y) H_1(dy)\Big) G(dy) \\ &= \mathrm{P}_1(x) + \mathrm{P}_2(x). \end{aligned} \tag{3.13}$$

Now we deal with $\mathrm{P}_1(x)$. To this end, by any $h(\cdot) \in \mathscr{H}_F$, we further split

$$\begin{aligned} \mathrm{P}_1(x) &= \int_0^{xg(x)} \Big(\int_{-\infty}^{-h(x/y)} + \int_{-h(x/y)}^{h(x/y)} + \int_{h(x/y)}^{x/y-h(x/y)} + \int_{x/y-h(x/y)}^{x/y}\Big) \\ &\qquad \cdot \mathrm{P}(X_2 > x/y - z | Y_2 = y) H_1(dz) G(dy) \\ &= \mathrm{P}_{11}(x) + \mathrm{P}_{12}(x) + \mathrm{P}_{13}(x) + \mathrm{P}_{14}(x). \end{aligned} \tag{3.14}$$

By $F \in \mathscr{R}_{-\alpha}$, we know that

$$\begin{aligned} \mathrm{P}_{11}(x) &\lesssim \int_0^{xg(x)} \overline{F}(x/y) s(y) H_1(-h(x/y) G(dy) \\ &\lesssim H_1\big(-h\big(1/g(x)\big)\overline{F}(x) \int_0^{xg(x)} \big(y^\alpha s(y) L(x/y)/L(x)\big) G(dy) \\ &= o(\overline{F}(x)). \end{aligned} \tag{3.15}$$

According to Proposition 1.B (i), by $F \in \mathscr{R}_{-\alpha}$, we have

$$\begin{aligned}
\mathrm{P}_{12}(x) &\sim \int_0^{xg(x)} s(y) \int_{-h(x/y)}^{h(x/y)} \mathrm{P}(X_2 > x/y - z) H_1(dy) G(dy) \\
&\sim \int_0^{xg(x)} \overline{F}(x/y) s(y) G(dy) \\
&= \mathrm{E}Y^\alpha s(Y) \overline{F}(x). \qquad (3.16)
\end{aligned}$$

And according to Proposition 1.B (ii), by (3.8) with $i = 1$ and $F \in \mathscr{R}_{-\alpha}$, using integration by parts, we have

$$\begin{aligned}
\mathrm{P}_{13}(x) &\sim \int_0^{xg(x)} s(y) \int_{h(x/y)}^{x/y-h(x/y)} \mathrm{P}(X_2 > x/y - z) H_1(dz) G(dy) \\
&\sim \int_0^{xg(x)} s(y) \int_{h(x/y)}^{x/y-h(x/y)} \overline{H_1}(x/y - u) F(du) G(dy) \\
&\sim \int_0^{xg(x)} s(y) o\big(\overline{F}(x/y)\big) G(dy) \\
&= o\big(\overline{F}(x)\big). \qquad (3.17)
\end{aligned}$$

Finally, by (3.8) with $i = 1$ and $F \in \mathscr{R}_{-\alpha}$ again, we get that

$$\begin{aligned}
\mathrm{P}_{14}(x) &\le \int_0^{xg(x)} \mathrm{P}\big(X_1 Y_1 \in (x/y - h(x/y), x/y]\big) G(dy) \\
&= \int_0^{xg(x)} o\big(\overline{H_1}(x/y)\big) G(dy) \\
&= o\big(\overline{F}(x)\big). \qquad (3.18)
\end{aligned}$$

Next, we deal with $\mathrm{P}_2(x)$. On the one hand, it is obvious to get that

$$\mathrm{P}_2(x) \le \mathrm{E}Y^\alpha \overline{H_1}(x) \sim \mathrm{E}Y^\alpha \mathrm{E}Y^\alpha s(Y) \overline{F}(x). \qquad (3.19)$$

On the other hand, according to Fatou Lemma, by (3.8), for any $a > 1$, we have

$$\begin{aligned}
\liminf \mathrm{P}_2(x)/\overline{H_1}(x) &\ge \int_{-\infty}^{\infty} \liminf \int_{-\infty}^{\infty} \mathrm{P}(X_2 > x/y - z | Y_2 = y) \\
&\qquad \cdot \mathbf{1}_{\{z > ax/y\}} H_1(dz) \mathbf{1}_{\{y \le xg(x)\}} / \overline{H_1}(x) G(dy) \\
&\ge \int_{-\infty}^{\infty} \liminf \int_{-\infty}^{\infty} \mathrm{P}\big(X_2 > -(a-1)x/y | Y_2 = y\big) \\
&\qquad \cdot \mathbf{1}_{\{z > ax/y\}} H_1(dz) \mathbf{1}_{\{y \le xg(x)\}} / \overline{H_1}(x) G(dy) \\
&= \int_{-\infty}^{\infty} \liminf \mathrm{P}\big(X_2 > -(a-1)x/y | y_2 = y\big) \overline{H_1}(ax/y) / \overline{H_1}(x) G(dy) \\
&= \int_{-\infty}^{\infty} \liminf \overline{H_1}(ax/y) / \overline{H_1}(x) G(dy) \\
&= a^{-\alpha} \mathrm{E}Y^\alpha s(Y). \qquad (3.20)
\end{aligned}$$

Let $a \downarrow 1$, combining with (3.19), we know that

$$\mathrm{P}_2(x) \sim \mathrm{E}Y^{\alpha}\mathrm{E}Y^{\alpha}s(Y)\overline{F}(x). \tag{3.21}$$

Hence by (3.13)-(3.18) and (3.21), (3.12) is holds for $n = 2$.

This completes the proof of Theorem 3.1. □

Theorem 3.2. *In the described discrete-time risk model, assume that* $\{(X, Y), (X_i, Y_i), i \geq 1\}$ *are i.i.d.random vectors. Under the conditions of Theorem 2.1 (i), (3.7) holds with* $s(\cdot)$ *in definition 2.1 for each fixed* $n \geq 1$.

The proof of this theorem is completely similar to the proof of Theorem 4.2 of Yang and Wang [46], and we have omitted its details. On the contrary, the proof of Theorem 3.1 is substantially different from the proof of Theorem 1.C.

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