

# Voting with Partial Orders: The Plurality and Anti-Plurality Classes

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## Abstract

The Plurality rule for linear orders selects the alternatives most frequently appearing in the first position of those orders, while the Anti-Plurality rule selects the alternatives least often occurring in the final position. We explore extensions of these rules to partial orders, offering axiomatic characterizations for these extensions.

## 1 Introduction

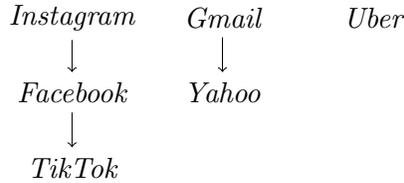
We consider the problem of a society choosing alternatives from a fixed set of finitely many alternatives. In a deviation from the model most commonly studied in social choice theory (Arrow et al., 2002; Zwicker, 2016), we focus on situations in which each individual’s preference is a *strict partial order*, allowing any given individual, for each pair of alternatives, to report either that she strictly prefers one over the other or that she does not wish to compare the two. The study of partial-order preferences deserves attention, as requiring voters to fully rank a (possibly) large number of alternatives may be overly demanding in many circumstances.

**Example 1.** Consider the following situation, inspired by an example first discussed by Terzopoulou and Endriss (2019). Suppose we ask an individual

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to rank different mobile apps, namely *Instagram*, *Facebook*, *TikTok*, *Uber*, *Gmail*, and *Yahoo*. She might be sure that she prefers *Instagram* to *Facebook*, and *Facebook* to *TikTok*, as she uses the three of them for posting videos; she also is sure that she prefers *Gmail* to *Yahoo*; but she might not be able to compare *Uber* to the rest of the apps, or *Gmail* to *Instagram*.



So it is clear that *Instagram* and *Gmail* are the most preferred apps, and that *TikTok* and *Yahoo* are the least preferred ones. But it is less clear whether *Uber* should be considered a good or a bad app, as it is not preferred to other alternatives, but there also are no alternatives that are preferred to it. Overall, it is not obvious what app should be considered *the best* for this individual, as the set of most preferred alternatives has a particular structure (e.g., *Instagram* beats two other apps, while *Uber* has not been compared to any other app). What is even less clear is how the preferences of multiple individuals should be aggregated in a situation like this.  $\triangle$

For the standard model of social choice, in which voters report strict linear orders, the most widely used rule in practice is the Plurality rule, which selects the alternative(s) most frequently ranked at the top of an individual preference order. While the Plurality rule has been rightfully criticised in the literature for its shortcomings (Laslier, 2011), it does have the significant advantage of being particularly simple: every voter simply awards a point to her most preferred alternative. Can we preserve some of this simplicity when moving from linear to partial orders? The answer is not obvious, as an individual who only reports a partial order might not have a uniquely identified most preferred alternative.

If we stick to the idea that an individual should not contribute to an alternative being selected when that alternative is not amongst her most preferred alternatives, and if we leave open the possibility of varying how points are assigned for the other alternatives, we arrive at what we shall call the *Plurality Class* of voting rules for partial orders. For instance, in the mobile apps example, *Instagram*, *Gmail*, and *Uber* are all top alternatives (in the sense that there are no alternatives more preferred to them), but they still differ in terms of the structure of the alternatives they dominate.

One natural rule might assign 1 point to each of them (and 0 to all others), while another might assign each top alternative as many points as there are alternatives it dominates (and still 0 to all others). Both of these rules belong to the Plurality Class, and both reduce to the standard Plurality rule for the special case of linear orders.

Our core contribution in this paper is to identify and axiomatically characterize the Plurality Class, as well as the closely related Anti-Plurality Class, which generalizes the well-known Anti-Plurality rule (also known as the Veto rule) for linear orders. In the standard setting, this is the rule that selects the alternative(s) least frequently ranked in the last position.

To characterize these two classes, we adapt many previously introduced axioms to this setting, while also proposing new ones. The standard Plurality and Anti-Plurality rules belong to the family of positional scoring rules, which are rules that assign scores to alternatives based on their positions in the linear orders, and then choose the alternative(s) that maximize the sum of the scores across all voters. Since Young (1975) characterized the scoring rules as the only ones satisfying Anonymity, Neutrality, Reinforcement, and Continuity, many different characterizations of the Plurality and Anti-Plurality rules for linear orders have been obtained. Let us briefly mention some of the characterizations of the Plurality rule that can be found in the literature:<sup>1</sup> Anonymity, Neutrality, Reinforcement, Continuity, and Reduction (Richelson, 1978);<sup>2</sup> Anonymity, Neutrality, Reinforcement, Monotonicity, and Bottom-invariance (Merlin and Naevé, 2000); and Anonymity, Neutrality, Reinforcement, Faithfulness, and Tops-only (Sekiguchi, 2012). For the case of the Anti-Plurality rule, we can mention the following characterizations for linear orders: Anonymity, Neutrality, Consistency, Monotonicity, and Top-invariance (Barberà and Dutta, 1982); Anonymity, Neutrality, Reinforcement, Continuity, and Minimal Veto (Baharad and Nitzan, 2005); and Anonymity, Neutrality, Reinforcement, Averseness, and Bottoms-only (Kurihara, 2018).

While we are not aware of any attempts to characterize Plurality-like rules for partial orders, there is a growing literature that recognizes

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<sup>1</sup>Definitions of these axioms can be found in Section 3, except for Anonymity, which requires that the names of the voters are not important for selecting a winning alternative; Reduction, which says that Pareto-dominated alternatives can be deleted; Monotonicity, which requires that if a voter improves the position of an already winning alternative, keeping everything else fixed, then that alternative becomes the only winner; and Minimal Veto, which says that every minority group should be assigned veto power.

<sup>2</sup>Ching (1996) later showed that Continuity, in fact, is not necessary for this characterization.

the practical significance of partial orders in the context of social choice (Terzopoulou, 2021). For instance, Pini et al. (2005) generalize both Arrow’s Theorem (1951) and the Muller-Satterthwaite Theorem (1977) to the case of partial orders. Terzopoulou and Endriss (2019) consider the problem of aggregating incomplete pairwise preferences where the voters have a weight that depends on the number of alternatives that they rank. Some work has focused on specific classes of partial orders, notably the top-truncated orders, where an individual strictly ranks her most preferred alternatives and we assume she is indifferent between the remaining alternatives. In this setting, Baumeister et al. (2012) provide complexity results on manipulation and bribery, while Terzopoulou and Endriss (2021) characterize different variants of the Borda rule. Further removed from our core interest here, Konczak and Lang (2005) introduced the notion of a *possible winner* for a profile of preferences, which is an alternative that would win under a given voting rule for at least one way in which to refine the profile into a profile of linear orders.

The remainder of this paper is organized as follows. Section 2 presents the model of voting we study, while Section 3 presents the axioms we use. Section 4 presents the characterization results for both classes of voting rules. Finally, Section 5 contains some concluding remarks.

## 2 The Model

In this section we present the model of voting with partial orders we study. It is a model of *variable electorates*, meaning that any finite subset of an infinite universe of voters might cast a ballot in any given election; and it is an *anonymous* model, meaning that all voters have the same degree of influence on the outcome.<sup>3</sup> Our model is one of the standard models of voting commonly studied in the literature, except that individual preferences are assumed to be strict partial orders rather than strict linear orders.

We also recall the definition of the family of positional scoring rules for this model and then introduce the Plurality Class and the Anti-Plurality Class of voting rules belonging to this family.

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<sup>3</sup>All results reported in this paper easily generalize to the setting where we drop this assumption of anonymity.

## 2.1 Voting with Partial Orders

Let  $A$  be a finite set of *alternatives*, with  $|A| = m \geq 3$ . A *preference* is a strict partial order over the set of alternatives, i.e., an irreflexive, antisymmetric, and transitive (but not necessarily complete) binary relation on  $A$ . We use  $\mathcal{D}$  to denote the *domain* of all such preferences. For  $a, b \in A$ , we say that  $a$  is (strictly) *preferred* to  $b$  in case  $a \succ b$ . If neither  $a \succ b$  nor  $b \succ a$  holds, we say that  $a$  is *incomparable* to  $b$ . We use  $\succ^{ab}$  to denote the preference  $\succ$  with alternatives  $a$  and  $b$  swapped, while everything else remains the same. The *top* of a preference  $\succ$  is the set of alternatives that are not dispreferred to any other alternative:

$$T(\succ) = \{a \in A \mid x \not\succeq a \text{ for all } x \in A\}$$

The *non-top* of a preference  $\succ$  is the set  $NT(\succ) = A \setminus T(\succ)$ . Analogously, the *bottom* of  $\succ$  is the set of alternatives that are not preferred to any other alternatives:

$$B(\succ) = \{b \in A \mid b \not\succeq x \text{ for all } x \in A\}$$

The *non-bottom* of  $\succ$  is the set  $NB(\succ) = A \setminus B(\succ)$ . Note that it is not necessarily the case that  $T(\succ) \cap B(\succ) = \emptyset$ . For instance, in Example 1, alternative *Uber* belongs to both the top and the bottom.

We consider elections in which a finite number  $n \in \mathbb{N}$  of *agents* (or *voters*) participate. Each agent  $i \leq n$  casts her vote by reporting a preference  $\succ_i \in \mathcal{D}$ , giving rise to a *profile*  $\succ = (\succ_1, \dots, \succ_n) \in \mathcal{D}^n$ . We use  $\succ_{-i}$  to denote the profile where the preference of the  $i$ th agent in profile  $\succ$  has been omitted; and we use  $(\succ'_i, \succ_{-i})$  to denote the profile we obtain when we replace the preference of the  $i$ th agent in  $\succ$  with  $\succ'_i$ . We further use the operator  $\oplus$  for the *concatenation* of profiles. That is, given two profiles  $\succ = (\succ_1, \dots, \succ_n) \in \mathcal{D}^n$  and  $\succ' = (\succ'_1, \dots, \succ'_{n'}) \in \mathcal{D}^{n'}$ , we define:

$$\succ \oplus \succ' = (\succ_1, \dots, \succ_n, \succ'_1, \dots, \succ'_{n'}) \in \mathcal{D}^{n+n'}$$

We stress that, as a consequence of adopting this way of modelling profiles, we are not able to specify agents “by name”. For example, if we concatenate a profile of length 20 and another of length 10, the 3rd agent of the second profile becomes the 23rd agent of the concatenated profile.

Finally, a *voting rule* is a function  $F : \bigcup_{n \in \mathbb{N}} \mathcal{D}^n \rightarrow 2^A \setminus \{\emptyset\}$ , mapping any given profile of preferences (of any finite length  $n \in \mathbb{N}$ ) to a nonempty set of alternatives, the *winners* of the election in question. We require voting rules to be *anonymous*, meaning that changing the order in which preferences

are listed in the input profile must never change the set of winners returned. Next, we introduce some concrete examples for natural definitions of voting rules in this setting.

## 2.2 Specific Classes of Voting Rules

For the standard model of voting with strict linear orders, the class of *positional scoring rules* includes some of the best-known and most widely used rules, such as the Borda rule and the Plurality rule. Following Kruger and Terzopoulou (2020), we now generalise the definition of this class to our model of voting with partial orders.

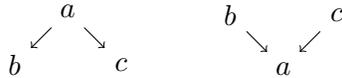
A *scoring function*  $s : A \times \mathcal{D} \rightarrow \mathbb{R}$  maps any given alternative  $a \in A$  in a given preference  $\succ \in \mathcal{D}$  to a real number  $s_\succ(a)$ , which we call *score* of  $a$ . We are specifically interested in scoring functions that are positional, meaning that the score of an alternative only depends on its position in the given preference. Formally, a scoring function  $s$  is *positional* if, for all permutations  $\sigma : A \rightarrow A$ , all preferences  $\succ \in \mathcal{D}$ , and all alternatives  $a \in A$ , it is the case that  $s_\succ(a) = s_{\sigma(\succ)}(a)$ , where  $\sigma(\succ) = \{(\sigma(a), \sigma(b)) \mid a \succ b\}$ .

Intuitively, a positional scoring function assigns scores to positions in a graph. Three well-known positional scoring functions for linear orders are the Borda scoring function, giving  $m - 1$  points to the first alternative,  $m - 2$  points to the second alternative, and so forth; the Plurality scoring function, assigning one point to the first alternative and zero to the rest; and the Anti-Plurality scoring function, assigning one point to all the alternatives except for the last-ranked one that gets zero points.

For every positional scoring function  $s$  there is an associated *positional scoring rule*  $F_s$ , which is the voting rule defined as follows:

$$F_s(\succ) = \operatorname{argmax}_{a \in A} \sum_{i \leq n} s_{\succ_i}(a) \quad \text{for any profile } \succ = (\succ_1, \dots, \succ_n).$$

**Example 2.** Consider the following profile, in which two agents report preferences regarding three alternatives:



A natural way in which one might generalize the Borda scoring function to the case of partial orders would be to define  $s_\succ(a)$  as the cardinality of the set  $\{x \in A \mid a \succ x\}$ , i.e., by equating the Borda score of an alternative  $a$  with the number of other alternatives dominated by  $a$ . Under this rule, for

the above profile, alternative  $a$  wins the election with a score of 2, while  $b$  and  $c$  each only receive a score of 1.

A natural way of generalizing the Plurality scoring function to our setting would be to set  $s_{\succ}(a)$  to 1 if  $\{x \in A \mid x \succ a\}$  is empty, and to set it to 0 otherwise. Under this rule, we obtain a three-way tie.  $\triangle$

But this particular way of generalizing the Plurality rule is just one of many options. For instance, we could also assign to each alternative in the top a score equal to the number of alternatives it dominates (and to each other alternative a score of 0). We now propose a class of positional scoring rules for strict partial orders that, we believe adequately captures the range of options one has available for generalizing the Plurality rule for linear orders.

**Definition 1.** *A positional scoring rule  $F_s$  belongs to the **Plurality Class** if the associated scoring function  $s$  satisfies the following property: Given a preference  $\succ \in \mathcal{D}$ , for all  $a \in T(\succ)$  and  $b \in NT(\succ)$ , it is the case that  $s_{\succ}(a) \geq s_{\succ}(b)$  and this inequality is strict in at least one case; and there exists a constant  $k \in \mathbb{R}$  such that  $s_{\succ}(c) = k$  for all  $c \in NT(\succ)$ .*

Thus, we require that all alternatives that are not in the top receive the same fixed score  $k$ , that no alternative in the top receives a score below  $k$ , and that at least one alternative in the top receives a score strictly above  $k$ . But we do not impose any further restrictions on how alternatives in the top are to be scored—except that the scoring function must be positional. Observe that on the domain of strict linear orders, every voting rule in the Plurality Class reduces to the familiar Plurality rule for linear orders.

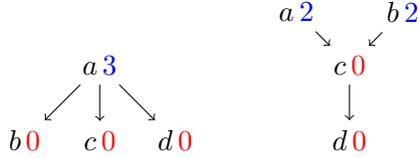
Note that applying an affine transformation to a scoring function does not change the voting rule that is being induced.<sup>4</sup> Indeed, while an affine transformation will change the scores each alternative receives, it will not change whether or not a given alternative receives the highest score and thus wins the election. So if  $s'$  is an affine transformation of  $s$ , then  $F_s = F_{s'}$ , meaning that for every given positional scoring rule  $F_s$  there are infinitely many ways of representing that one rule. In the specific context of Definition 1, this means that, w.l.o.g., we may assume that  $k = 0$ .

It is important to note, that for two partial orders with different graph structures, the scoring function can assign different scores to the alternatives. The following example illustrates this point.

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<sup>4</sup>Applying an affine transformation  $x \mapsto \alpha x + \beta$  with  $\alpha \in \mathbb{R}_{>0}$  and  $\beta \in \mathbb{R}$  to a scoring function  $s$  means replacing, for every preference  $\succ \in \mathcal{D}$  and every alternative  $a \in A$ , the original score  $s_{\succ}(a)$  with the new score  $\alpha \cdot s_{\succ}(a) + \beta$ .

**Example 3.** Let  $s$  be the scoring function that, for any alternative in the top assigns a score that is equal to the number of alternatives below it, and to every alternative not in the top assigns a score of 0. Here are the scores assigned to the alternatives under consideration for two possible preferences:



The positional scoring rule  $F_s$  induced by the scoring function  $s$  belongs to the Plurality Class.  $\triangle$

Next, we define two subclasses of the Plurality Class that are of particular interest due to their simplicity.

**Definition 2.** A positional scoring rule  $F_s$  in the Plurality Class belongs to the **Simple Plurality Class** if  $s_{\succ}(a) = s_{\succ}(b)$  for every preference  $\succ \in \mathcal{D}$  and every two alternatives  $a, b \in T(\succ)$ .

**Definition 3.** A positional scoring rule  $F_s$  in the Plurality Class belongs to the **Uniformly Simple Plurality Class** if there exists a  $k^* \in \mathbb{R}$  such that  $s_{\succ}(a) = k^*$  for every preference  $\succ \in \mathcal{D}$  and every alternative  $a \in T(\succ)$ .

Thus, a Plurality rule is *simple* if every agent (or, more precisely: every preference) assigns the same fixed score to all of its top alternatives (besides, of course, assigning the same score to all its non-top alternatives). Such a rule furthermore is *uniformly simple* if the score assigned to a top alternative does not depend on the structure of the preference but is uniform across all possible preferences. Note that, in view of what we said above about positional scoring rules being invariant under affine transformations, there in fact is only a single rule in the Uniformly Simple Plurality Class (which can be represented by infinitely many different scoring functions). We call this rule the *Uniformly Simple Plurality rule*. The canonical way of defining it would be as the rule that assigns 1 point to every alternative in the top and 0 points to all other alternatives.

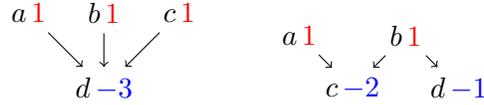
We also propose a class of positional scoring rules for strict partial orders that generalizes the Anti-Plurality rule for linear orders.

**Definition 4.** A positional scoring rule  $F_s$  belongs to the **Anti-Plurality Class** if the associated scoring function  $s$  satisfies the following property: Given a preference  $\succ \in \mathcal{D}$ , for all  $a \in NB(\succ)$  and  $b \in B(\succ)$ , it is the case

that  $s_{\succ}(a) \geq s_{\succ}(b)$  and this inequality is strict in at least one case; and there exists a constant  $k \in \mathbb{R}$  such that  $s_{\succ}(c) = k$  for all  $c \in NB(\succ)$ .

Also here it is easy to see that, indeed, when applied to a profile of strict linear orders, any rule in the Anti-Plurality Class reduces to the standard Anti-Plurality rule. But in the general case, the class is made up of many different rules.

**Example 4.** Consider the following profile of two preferences, with associated scores:



These scores might be generated by a positional scoring function  $s$ , associated with a voting rule  $F_s$  in the Anti-Plurality Class, under which the scores given to the alternatives at the bottom depend on the number of other alternatives that dominate them.  $\triangle$

Again, we define two subclasses of the Anti-Plurality Class that are of particular interest due to their simplicity.

**Definition 5.** A positional scoring rule  $F_s$  in the Anti-Plurality Class belongs to the **Simple Anti-Plurality Class** if  $s_{\succ}(a) = s_{\succ}(b)$  for every preference  $\succ \in \mathcal{D}$  and every two alternatives  $a, b \in B(\succ)$ .

**Definition 6.** A positional scoring rule  $F_s$  in the Anti-Plurality Class belongs to the **Uniformly Simple Anti-Plurality Class** if there exists a  $k^* \in \mathbb{R}$  such that  $s_{\succ}(a) = k^*$  for every preference  $\succ \in \mathcal{D}$  and every alternative  $a \in B(\succ)$ .

Also here, the Uniformly Simple Anti-Plurality Class only includes a single voting rule, the *Uniformly Simple Anti-Plurality* rule.

### 3 Axioms

In this section, we introduce a number of axioms, i.e., fundamental normative requirements for voting rules, which later will turn out to be useful for our characterization results. Every axiom postulates a property that, one might argue, any reasonable voting rules  $F$  should satisfy.

The first three axioms, Neutrality, Reinforcement, and Continuity, are very common in the literature on scoring rules and are used by Young (1975)

to characterize them. The first of these is a symmetry requirement that essentially states that the names of the alternatives should not be considered important for choosing a winner. We formulate it in terms of permutations  $\sigma : A \rightarrow A$  on the set of alternatives, which extend to sets  $S \subseteq A$  of alternatives as well as preferences  $\succ \in \mathcal{D}$  in the natural manner:  $\sigma(S) = \{\sigma(x) \mid x \in S\}$  and  $\sigma(\succ) = \{(\sigma(x), \sigma(y)) \mid x \succ y\}$ .

**Axiom 1** (Neutrality). *For any profile  $\succ = (\succ_1, \dots, \succ_n)$  and permutation  $\sigma : A \rightarrow A$ , it should be the case that  $\sigma(F(\succ_1, \dots, \succ_n)) = F(\sigma(\succ_1), \dots, \sigma(\succ_n))$ .*

The next axiom, which is also known as Consistency, states that, if two disjoint electorates agree on some winning alternatives, then those alternatives must be selected when we consider a profile consisting where the members of both electorates cast a vote.

**Axiom 2** (Reinforcement). *For any two profiles  $\succ \in \mathcal{D}^n$  and  $\succ' \in \mathcal{D}^{n'}$  with  $F(\succ) \cap F(\succ') \neq \emptyset$ , it should be the case that  $F(\succ \oplus \succ') = F(\succ) \cap F(\succ')$ .*

The next axiom, called the Archimedean property by Smith (1973) and Young (1975), and Overwhelming Majority by Myerson (1995), stipulates that, although a small number of agents cannot completely over-ride the decision of a large majority, they might still be break ties in favor of some of the alternatives selected by the majority.

**Axiom 3** (Continuity). *For any two profiles  $\succ$  and  $\succ'$  there should exist a bound  $K$  such that, for every natural number  $k > K$ , the following inclusion holds:*

$$F(\underbrace{\succ \oplus \dots \oplus \succ}_{k \text{ times}} \oplus \succ') \subseteq F(\succ)$$

In the sequel, we will write  $k\succ$  as a shorthand for the profile  $\succ \oplus \dots \oplus \succ$  consisting of a concatenation of  $k$  copies of the basic profile  $\succ$ .

The next two axioms are minimal responsiveness requirements for a voting rule. The first one is a version of the *Faithfulness* axiom due to Young (1974), and the second one is a version of *Averseness* introduced by Kurihara (2018). The former states that, for the special case of single-voter profiles, the voting rule must select one of the top alternatives. The latter states that for such profiles it should not select all of the bottom alternatives.

**Axiom 4** (Faithfulness). *For any single-voter profile  $(\succ) \in \mathcal{D}^1$ , it should be the case that  $F(\succ) \subseteq T(\succ)$ .*

**Axiom 5** (Averseness). *For any single-voter profile  $(\succ) \in \mathcal{D}^1$  with  $B(\succ) \neq A$ , it should be the case that  $B(\succ) \not\subseteq F(\succ)$ .*

Note that Averseness is a weaker requirement than Faithfulness, and it is implied by the latter.

The remaining axioms we propose all demand, in one way or another, that a voting rule should limit the amount of information it takes into account when producing a winner. This is motivated by the idea that a voter’s attention is a scarce resource and a voting rule that might suggest a change in winners every time a voter reports a change in her preference that only concerns less salient alternatives, might be judged to be too brittle for use in practice.

The following two axioms are instances of this idea that the outcome should depend only on a minimal amount of information. In one case the set of top alternatives of every agent is the only information required; while in the other case, the set of bottom alternatives of every agent is the only information required.<sup>5</sup>

**Axiom 6** (Tops-Only). *For any two profiles  $\succ \in \mathcal{D}^n$  and  $\succ' \in \mathcal{D}^n$  with  $T(\succ_i) = T(\succ'_i)$  for all  $i \leq n$ , it should be the case that  $F(\succ) = F(\succ')$ .*

**Axiom 7** (Bottoms-Only). *For any two profiles  $\succ \in \mathcal{D}^n$  and  $\succ' \in \mathcal{D}^n$  with  $B(\succ_i) = B(\succ'_i)$  for all  $i \leq n$ , it should be the case that  $F(\succ) = F(\succ')$ .*

For the next two axioms, we require some further notation. For a given profile  $\succ \in \mathcal{D}^n$ , the number of times the alternative  $x$  appears in a voter’s top is denoted by  $t_x(\succ)$ :

$$t_x(\succ) = |\{i \leq n \mid x \in T(\succ_i)\}| \quad \text{for any } \succ \in \mathcal{D} \text{ and } x \in A$$

Analogously,  $b_x(\succ)$  is the number of times alternative  $x$  appears in a voter’s bottom set:

$$b_x(\succ) = |\{i \leq n \mid x \in B(\succ_i)\}| \quad \text{for any } \succ \in \mathcal{D} \text{ and } x \in A$$

The following axioms are stronger versions of Tops-Only and Bottoms-Only. They apply to a wider range of pairs of profiles. For instance, while Tops-Only applies to all profiles (of the same cardinality) inducing the same profile of tops, Strong Tops-Only applies to all pairs of profiles (regardless of the cardinality) that agree on the number of times any one alternative appears in a top set.

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<sup>5</sup>For further motivation regarding these two axioms, we refer to Sekiguchi (2012) and Kurihara (2018).

**Axiom 8** (Strong Tops-Only). *For any two profiles  $\succ$  and  $\succ'$  with  $t_x(\succ) = t_x(\succ')$  for all  $x \in A$ , it should be the case that  $F(\succ) = F(\succ')$ .*

**Axiom 9** (Strong Bottoms-Only). *For any two profiles  $\succ$  and  $\succ'$  with  $b_x(\succ) = b_x(\succ')$  for all  $x \in A$ , it should be the case that  $F(\succ) = F(\succ')$ .*

Our final two axioms are variants of axioms introduced by Barberà and Dutta (1982) and Merlin and Naeve (2000) to obtain characterizations of the Plurality and Anti-Plurality rules for linear orders (known as *Top-Invariance* and *Bottom-Invariance*, respectively). In one case, the selected alternatives are not affected by an order change of alternatives not in the top; while in the other case, the selected alternatives are not affected by an order change of alternatives not in the bottom.

**Axiom 10** (NT-Invariance). *For any profile  $\succ \in \mathcal{D}^n$ , any  $i \leq n$ , and any bijection  $\sigma : A \rightarrow A$  respecting  $\sigma(a) = a$  for all  $a \in T(\succ_i)$ , it should be the case that  $F(\succ) = F(\sigma(\succ_i), \succ_{-i})$ .*

**Axiom 11** (NB-Invariance). *For any profile  $\succ \in \mathcal{D}^n$ , any  $i \leq n$ , and any bijection  $\sigma : A \rightarrow A$  respecting  $\sigma(b) = b$  for all  $b \in B(\succ_i)$ , it should be the case that  $F(\succ) = F(\sigma(\succ_i), \succ_{-i})$ .*

## 4 Results

In this section, we provide axiomatic characterizations of the classes of voting rules defined in Section 2 in terms of the axioms put forward in Section 3.

For the class of positional scoring rules, it is well-understood that they can be characterized in terms of Neutrality, Reinforcement, and Continuity.<sup>6</sup> Smith (1973) and Young (1975) proved this to be the case for the classical model of voting with for linear orders. Myerson (1995) does so in a model without an ordering assumption. Our model is closest to that of Kruger and Terzopoulou (2020), who prove the same kind of characterization for the case of voting with incomplete preferences that are merely assumed to be acyclic (but not necessarily transitive). A careful examination of their proof shows that this result extends to our model as well.

**Theorem 1.** *A voting rule satisfies Neutrality, Reinforcement, and Continuity if, and only if, it is a positional scoring rule.*

This result is very useful, as it allows us to directly focus on voting rules that are positional scoring rules, which have a very well defined structure.

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<sup>6</sup>In addition, if voting rules are not assumed to be blind to the identities of voters from the outset (as is the case in our model), one also requires the axiom of Anonymity.

## 4.1 The Plurality Class

We are now ready to provide axiomatic characterizations of the Plurality Class and its subclasses described earlier.

**Theorem 2.** *A voting rule satisfies Neutrality, Reinforcement, Continuity, Faithfulness, and NT-Invariance if, and only if, it belongs to the Plurality Class.*

*Proof.* The fact that a voting rule that belongs to the Plurality Class satisfies all of the axioms mentioned is straightforward to verify.

For the other direction, let  $F$  be an arbitrary voting rule that satisfies all five axioms. By Theorem 1, we know that  $F$  is a positional scoring rule, so there must be a scoring function  $s$  such that  $F = F_s$ . We are first going to derive constraints on  $s$  for the special case of preferences that are linear orders. This will later allow us to fully characterize  $s$  on all possible inputs.

So let us show that on inputs that are profiles of strict linear orders,  $F_s$  must act exactly like the standard Plurality rule for linear orders. First, observe that, for any strict linear order  $\succ$  with top element  $x \in A$ , by Faithfulness, it must be the case that  $s_\succ(x) > s_\succ(y)$  for all  $y \in A \setminus \{x\}$ . Now consider two linear orders  $\succ_1$  and  $\succ_2$  such that  $x \succ_1 y \succ_1 a_1 \succ_1 \cdots \succ_1 a_{m-2}$  and  $y \succ_2 x \succ_2 a_1 \succ_2 \cdots \succ_2 a_{m-2}$ , for some enumeration of the alternatives in  $A$ . By Neutrality, it must be the case that  $F_s((\succ_1) \oplus (\succ_2)) = \{x, y\}$ , yielding the following constraint on  $s$ :

$$s_{\succ_1}(x) + s_{\succ_2}(x) = s_{\succ_1}(y) + s_{\succ_2}(y)$$

By NT-Invariance, given any bijection  $\sigma : A \rightarrow A$  that keeps fixed the top alternatives, we have that  $F_s((\sigma(\succ_1)) \oplus (\succ_2)) = F_s((\succ_1) \oplus (\succ_2)) = \{x, y\}$ . So we obtain an additional constraint:

$$s_{\sigma(\succ_1)}(x) + s_{\succ_2}(x) = s_{\sigma(\succ_1)}(y) + s_{\succ_2}(y)$$

Now, given that  $s_{\succ_1}(x) = s_{\sigma(\succ_1)}(x)$  by definition of  $\sigma$ , combining the constraints we established, we obtain  $s_{\succ_1}(y) = s_{\sigma(\succ_1)}(y)$ . This is true for any linear order  $\succ_1$ , any permutation  $\sigma$  of the non-top alternatives, and any non-top alternative  $y$ . Thus,  $F_s$  indeed acts like the standard Plurality rule in those cases where the input profile is made up of linear orders.

We now proceed to the general case of profiles consisting of arbitrary partial orders. Let  $\succ$  be a partial order and fix any  $x \in NT(\succ)$  and  $y \in T(\succ)$ . By Faithfulness, we have that the scoring function is such that  $s_\succ(a) \geq s_\succ(b)$  for all  $a \in T(\succ)$  and  $b \in NT(\succ)$ , with at least one case where  $s_\succ(a) > s_\succ(b)$ .

Let  $\mathbf{L} = (\succ^x) \oplus (\succ^y)$  be a profile consisting of two linear orders,  $\succ^x$  and  $\succ^y$ , such that  $x \succ^x y \succ^x a_1 \succ^x \dots \succ^x a_{m-2}$  and  $y \succ^y x \succ^y a_1 \succ^y \dots \succ^y a_{m-2}$ . As  $F_s$ , when applied to profiles of linear orders, acts like the standard Plurality rule, we know that  $F_s(\mathbf{L}) = \{x, y\}$ . By Continuity—used to be sure that the outcome is a subset of  $\{x, y\}$ —and Neutrality—used to be sure that in fact both  $x$  and  $y$  are selected, there exists a  $k \in \mathbb{N}$  such that  $F(k\mathbf{L} \oplus (\succ) \oplus (\succ^{xy})) = \{x, y\}$ . This yields the following constraint:

$$s_{\succ}(x) + s_{\succ^{xy}}(x) = s_{\succ}(y) + s_{\succ^{xy}}(y)$$

By NT-Invariance, given any bijection  $\sigma : A \rightarrow A$  that keeps fixed the top alternatives, we have that  $F(k\mathbf{L} \oplus \sigma(\succ) \oplus (\succ^{xy})) = F(k\mathbf{L} \oplus (\succ) \oplus (\succ^{xy})) = \{x, y\}$ . So we obtain an additional constraint:

$$s_{\sigma(\succ)}(x) + s_{\sigma(\succ^{xy})}(x) = s_{\sigma(\succ)}(y) + s_{\sigma(\succ^{xy})}(y)$$

Now, given that  $s_{\succ}(y) = s_{\sigma(\succ)}(y)$  by definition of  $\sigma$ , combining the constraints we established, we obtain that  $s_{\succ}(x) = s_{\sigma(\succ)}(x)$  holds for any permutation of the non-top alternatives. Thus we can conclude that  $F_s$  indeed belongs to the Plurality Class.  $\square$

An interesting and immediate question that arises in view of Theorem 2 is whether all of the axioms mentioned in fact are necessary for the characterization, that is, whether they are independent. We now show that this is indeed the case, by providing five examples of voting rules that satisfy all but one of the axioms. The most interesting of these is the example of a voting rule that satisfies all of the axioms but Continuity, as Ching (1996) showed that the Continuity axiom is *not* necessary for the characterization of the Plurality rule in the standard model of voting with linear orders, thereby refining Richelson's (1978) original characterization.

**Proposition 3.** *The axioms of Neutrality, Continuity, Reinforcement, Faithfulness, and NT-Invariance are logically independent.*

*Proof.* *Neutrality, Continuity, Reinforcement, and Faithfulness but not NT-Invariance:* the Borda-style rule that assigns to each alternative  $x$  in a partial order a score equal to the number of alternatives that  $x$  is preferred to.

*Neutrality, Continuity, Reinforcement, and NT-Invariance but not Faithfulness:* the rule that always selects the whole set of alternatives.

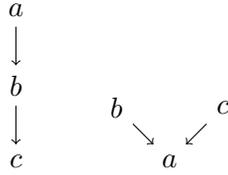
*Continuity, Reinforcement, NT-Invariance, and Faithfulness but not Neutrality:* the rule assigns points to alternatives just like the Uniformly

Simple Plurality rule but that doubles the score of one fixed alternative  $a \in A$  before determining the winners.

*Neutrality, Continuity, NT-Invariance, and Faithfulness but not Reinforcement:* let  $F$  be the rule that for any single-agent profile selects the top of that single agent, and that in all other profiles selects the two alternatives with the most appearances in tops (in case of ties it selects all the tied alternatives).

*Neutrality, Reinforcement, NT-Invariance, and Faithfulness but not Continuity:* the rule defined by means of the following two-step procedure. In the first step, select all the winning alternatives returned by the Uniformly Simple Plurality rule. In the second step, select as winning alternatives from this set those alternatives that appear as a unique top element in at least one of the agents' preferences—in case there are such alternatives; otherwise pick all alternatives selected in the first step.<sup>7</sup>  $\square$

**Example 5.** To exemplify the operation of the rule demonstrating that Continuity is a necessary axiom for our characterization of the Plurality Class, consider the following two preferences,  $\succ_1$  and  $\succ_2$ :



In the two-agent profile  $(\succ_1, \succ_2)$ , in the first step we select  $\{a, b, c\}$ , but as only  $a$  appears as a unique top, in the second step we obtain  $\{a\}$ .

The failure of Continuity can now be observed by noting that the rule instead returns  $\{b, c\}$  for any profile that is composed of  $k > 1$  copies of  $(\succ_1, \succ_2)$  and one further copy of  $(\succ_2)$ . Indeed, for any such profile, we select  $\{b, c\}$  in the first step, after which no further selection will occur in the second step.  $\triangle$

The following result, an immediate corollary of Theorem 2, shows what happens if we require a voting rule to also satisfy Tops-Only, on top of the axioms characterizing the Plurality Class.

**Corollary 4.** *A voting rule satisfies Neutrality, Reinforcement, Continuity, Faithfulness, NT-Invariance, and Tops-Only if, and only if, it is in the Simple Plurality Class.*

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<sup>7</sup>The inspiration for the definition of this voting rule comes from an example used by Myerson (1995).

If we require a voting rule to instead satisfy the stronger version of Tops-Only, then the scoring function must give the same score to all the top alternatives.

**Corollary 5.** *A voting rule satisfies Neutrality, Reinforcement, Continuity, Faithfulness, NT-Invariance, and Strong Tops-Only if, and only if, it is in the Uniformly Simple Plurality Class.*

Recall that the Uniformly Simple Plurality Class consists of just a single rule, so we may also interpret Corollary 5 as a characterization of that rule.

## 4.2 The Anti-Plurality Class

In this section we present the characterization of all the voting rules that belong to the Anti-Plurality Class.

**Theorem 6.** *A voting rule satisfies Neutrality, Reinforcement, Continuity, Averseness, and NB-Invariance if, and only if, it is in the Anti-Plurality Class.*

*Proof.* The fact that a voting rule that belongs to the Anti-Plurality Class satisfies all of the axioms mentioned is straightforward to verify.

For the other direction, let  $F$  be an arbitrary voting rule that satisfies all five axioms. By Theorem 1, we know that  $F$  is a positional scoring rule, so there must be a scoring function  $s$  such that  $F = F_s$ . We are first going to derive constraints on  $s$  for the special case of preferences that are linear orders. This will later allow us to fully characterize  $s$  on all possible inputs.

So let us show that on inputs that are profiles of strict linear orders,  $F_s$  must act exactly like the standard Anti-Plurality rule for linear orders. Let  $\succ$  be a linear order  $a_1 \succ a_2 \succ \dots \succ a_m$ . By Averseness,  $s_{\succ}(a_i) > s_{\succ}(a_m)$  for all  $i \neq m$ . Assume that for an  $a_i \neq a_m$ , it is the case that  $a_i \in F(\succ)$ . By NB-Invariance, given any bijection  $\sigma : A \rightarrow A$  that keeps fixed the bottom alternatives, we have that  $a_i \in F(\sigma(\succ)) = F(\succ)$ . And by Neutrality, for any  $a_i \neq a_m$ , it must be the case that  $\sigma(a_i) \in \sigma(F(\succ)) = F(\succ)$ . This implies that  $A \setminus \{a_m\} = F(\succ)$ , and thus all the alternatives that are not in the bottom get the same score. Thus,  $F_s$  indeed acts like the standard Anti-Plurality rule in those cases where the input profile is made up of linear orders.

We now proceed to the general case of profiles consisting of arbitrary strict partial orders. Let  $\succ$  be a partial order and let  $x \in NB(\succ)$ , and  $y \in B(\succ)$ . Let  $\mathbf{L}_1 = (\succ^{x^1}) \oplus (\succ^{y^1})$ ,  $\mathbf{L}_2 = (\succ^{x^2}) \oplus (\succ^{y^2})$ ,  $\dots$ , and  $\mathbf{L}_{m-2} = (\succ^{x^{(m-2)}}) \oplus (\succ^{y^{(m-2)}})$  be profiles such that  $x \succ^{x^1} y \succ^{x^1} a_1 \succ^{x^1}$

$\dots \succ^{x1} a_{m-2}, y \succ^{y1} x \succ^{y1} a_1 \succ^{y1} \dots \succ^{y1} a_{m-2}, x \succ^{x2} y \succ^{x2} a_2 \succ^{x2}$   
 $\dots \succ^{x2} a_{m-3} \succ^{x2} a_1, y \succ^{y2} x \succ^{y2} a_2 \succ^{y2} \dots \succ^{y2} a_{m-3} \succ^{y2} a_1, \dots,$   
 $x \succ^{x(m-2)} y \succ^{x(m-2)} a_{m-2} \succ^{x(m-2)} a_1 \succ^{x(m-2)} \dots \succ^{x(m-2)} a_{m-3},$  and  
 $y \succ^{y(m-2)} x \succ^{y(m-2)} a_{m-2} \succ^{y(m-2)} a_1 \succ^{y(m-2)} \dots \succ^{y(m-2)} a_{m-3}.$  Let  
 $\mathbf{L} = \mathbf{L}_1 \oplus \dots \oplus \mathbf{L}_{m-2}.$ <sup>8</sup> As  $F_s$ , when applied to profiles of linear orders,  
acts like the standard Anti-Plurality rule, we know that  $F(\mathbf{L}) = \{x, y\}$ . By  
Continuity—used to be sure that the outcome is a subset of  $\{x, y\}$ —and  
Neutrality—used to be sure that in fact both  $x$  and  $y$  are selected, there  
exists a  $k \in \mathbb{N}$  such that  $F(k\mathbf{L} \oplus (\succ) \oplus (\succ^{xy})) = \{x, y\}$ . This yields the  
following constraint:

$$s_{\succ}(x) + s_{\succ^{xy}}(x) = s_{\succ}(y) + s_{\succ^{xy}}(y)$$

By NB-Invariance, given any bijection  $\sigma : A \rightarrow A$  that keeps  
fixed the bottom alternatives, we have that  $F(k\mathbf{L} \oplus (\sigma(\succ)) \oplus$   
 $(\succ^{xy})) = F(k\mathbf{L} \oplus (\succ) \oplus (\succ^{xy})) = \{x, y\}$ . Thus, we obtain the next con-  
straint:

$$s_{\sigma(\succ)}(x) + s_{\succ^{xy}}(x) = s_{\sigma(\succ)}(y) + s_{\succ^{xy}}(y)$$

Now, given that  $s_{\succ}(x) = s_{\sigma(\succ)}(x)$  by definition of  $\sigma$ , combining the con-  
straints we established, we obtain that  $s_{\succ}(y) = s_{\sigma(\succ)}(y)$  is true for any  
permutation of the not-top alternatives. This, together with Averseness,  
implies that  $s_{\succ}(a) \geq s_{\succ}(b)$  for all  $a \in NB(\succ)$  and  $b \in B(\succ)$  with at least  
one case where  $s_{\succ}(a) > s_{\succ}(b)$ , otherwise all the set  $B(\succ)$  is contained in  
 $F(\succ)$ . Thus we can conclude that  $F_s$  indeed belongs to the Anti-Plurality  
Class.  $\square$

Next, as we previously did for Theorem 2, we show that the axioms of  
Theorem 6 are logically independent. Again, it is interesting that Continuity  
is required for the characterization, in contrast to the result by Kurihara  
(2018).

**Proposition 7.** *Neutrality, Continuity, Reinforcement, Averseness, and NB-Invariance are logically independent.*

*Proof.* *Neutrality, Continuity, Reinforcement, and Averseness but not NB-Invariance:* the Borda style rule used in Proposition 3.

*Neutrality, Continuity, Reinforcement, and NB-Invariance but not Averseness:* the rule that always selects the whole set of alternatives.

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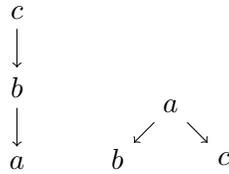
<sup>8</sup>In this profile  $x$  and  $y$  never appear at the bottom while the rest of the alternatives appear at the bottom of a linear order exactly twice.

*Continuity, Reinforcement, NB-Invariance, and Averseness but not Neutrality:* the rule that assigns points to alternatives just like the Uniformly Simple Anti-Plurality rule but that doubles the score of one fixed alternative  $a \in A$  before determining the winners.

*Neutrality, Continuity, NB-Invariance, and Averseness but not Reinforcement:* let  $F$  be the rule that for any single-agent profile selects the alternatives that are not in the bottom of that single agent, and that in all other profiles selects the two alternatives with the least appearances in bottoms (in case of ties it selects all the tied alternatives).

*Neutrality, Reinforcement, NT-Invariance, and Faithfulness but not Continuity:* the rule defined by means of the following two-step procedure. In the first step, select all the winning alternatives returned by the Uniformly Simple Anti-Plurality rule. In the second step, select as winning alternatives from this set those alternatives that do not appear as a unique bottom element in at least one of the agents' preferences—in case there are such alternatives; otherwise pick all alternatives selected in the first step.  $\square$

**Example 6.** To exemplify the operation of the rule demonstrating that Continuity is a necessary axiom for our characterization of the Anti-Plurality Class, consider the following two preferences,  $\succ_1$  and  $\succ_2$ :



In the two-agent profile  $(\succ_1, \succ_2)$ , in the first step we select  $\{a, b, c\}$ , but as only  $a$  appears as a unique bottom, in the second step we obtain  $\{b, c\}$ .

The failure of Continuity can now be observed by noting that the rule instead returns  $\{a\}$  for any profile that is composed of  $k > 1$  copies of  $(\succ_1, \succ_2)$  and one further copy of  $(\succ_2)$ . Indeed, for any such profile, we select  $\{a\}$  in the first step, after which no further selection will occur in the second step.  $\triangle$

As it happens with the Plurality Class, we obtain a narrower class of Anti-Plurality rules when we require voting rules to also satisfy Bottoms-Only.

**Corollary 8.** *A voting rule satisfies Neutrality, Reinforcement, Continuity, Averseness, NB-Invariance, and Bottoms-Only if, and only if, it is in the Simple Anti-Plurality Class.*

If we require the voting rule to satisfy the stronger version of Bottoms-Only, then the scoring function must give the same score to all the bottom alternatives.

**Corollary 9.** *A voting rule satisfies Neutrality, Reinforcement, Continuity, Averseness, NB-Invariance, and Strong Bottoms-Only if, and only if, it is in the Uniformly Simple Anti-Plurality Class.*

Here again, recall that the Uniformly Simple Anti-Plurality Class consists of just a single rule, so we may also interpret Corollary 9 as a characterization of that rule.

## 5 Conclusion

We have provided axiomatic characterizations for generalized versions of both the Plurality and Anti-Plurality rules to scenarios in which preferences take the form of partial orders. In the standard scenario with preferences that are linear orders, the Plurality (Anti-Plurality) rule gives one point (zero points) only to the most (least) preferred alternative and zero points (one point) to the rest. We generalized this defining feature to the case of partial orders in a manner that accounts for the fact that there is more than one way in which an alternative might be most (least) preferred in a partial order, given that two preferences that agree, say, on their most preferred alternatives might still differ in other respects.

An intriguing path for future investigation lies in exploring similar generalizations of other scoring rules to the domain of partial orders, including but not limited to the Borda, Cumulative, or Stepwise scoring rules.<sup>9</sup>

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<sup>9</sup>See Kruger and Terzopoulou (2020) for definitions of the Cumulative and Stepwise scoring rules.

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