Multifractal analysis of the power-2-decaying Gauss-like expansion

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Abstract

Each real number $x \in [0,1]$ admits a unique power-2-decaying Gauss-like expansion (P2GLE for short) as $x = \sum_{i\in\mathbb{N}} 2^{-(d_1(x)+d_2(x)+\cdots+d_i(x))}$, where $d_i(x) \in \mathbb{N}$. For any $x \in (0,1]$, the Khintchine exponent $\gamma(x)$ is defined by $\gamma(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} d_j(x)$ if the limit exists. We investigate the sizes of the level sets $E(\xi) := \{x \in (0,1] : \gamma(x) = \xi\}$ for $\xi \geq 1$. Utilizing the Ruelle operator theory, we obtain the Khintchine spectrum $\xi \mapsto \dim_H E(\xi)$, where $\dim_H$ denotes the Hausdorff dimension. We establish the remarkable fact that the Khintchine spectrum has exactly one inflection point, which was never proved for the corresponding spectrum in continued fractions. As a direct consequence, we also obtain the Lyapunov spectrum. Furthermore, we find the Hausdorff dimensions of the level sets $\{x \in (0,1] : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log(d_j(x)) = \xi\}$ and $\{x \in (0,1] : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} 2^{d_j(x)} = \xi\}$.

1. Introduction and main results

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the set of positive integers. The power-2-decaying Gauss-like expansion (P2GLE for short) of a real number can be generated by the transformation $T : (0,1] \to (0,1]$ defined by

$$Tx := 2^n x - 1 \quad \text{for } x \in (1/2^n, 1/2^{n-1}], \quad n \in \mathbb{N}. \tag{1.1}$$

By [8, Proposition 2.1], we know that every real number x in $(0,1]$ can be written uniquely as the form

$$x = \sum_{i=1}^{\infty} 2^{-(d_1(x)+d_2(x)+\cdots+d_i(x))} \tag{1.2}$$

where $d_1(x) = \lceil -\log_2 x \rceil$ and $d_i(x) = d_1(T^{i-1}x)$ for $i \geq 2$. Here, we use $\lceil x \rceil$ to denote the smallest integer greater than or equal to x. For any $d_1, \cdots, d_n \in \mathbb{N}$, the nth-order cylinder is defined by

$$I_n(d_1, \cdots, d_n) := \{x \in (0,1] : d_1(x) = d_1, \cdots, d_n(x) = d_n\}.$$

Let $\varphi : [0,1] \to \mathbb{R}$ be a real-valued function (usually called potential). For any $n \geq 1$, denote the nth Birkhoff sum of φ by

$$S_n\varphi(x) := \sum_{j=0}^{n-1} \varphi(T^j x), \quad x \in [0,1].$$

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The Birkhoff average of φ is defined as

$$\bar{\varphi}(x) := \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x), \quad x \in (0,1]$$

(if the limit exists). In multifractal analysis, we are interested in the Hausdorff dimension of the level sets

$$E_\varphi(\xi) = \{x \in (0,1] : \bar{\varphi}(x) = \xi\}, \quad \xi \in \mathbb{R}. \tag{1.3}$$

The so-called Birkhoff spectrum is the function $t_\varphi : \mathbb{R} \to [0,1] \cup \{-\infty\}$ defined by

$$t_\varphi(\xi) := \dim_H E_\varphi(\xi)$$

where $\dim_H E_\varphi(\xi)$ denotes the Hausdorff dimension of $E_\varphi(\xi)$. By convention, $\dim_H \emptyset = 0$.

Some metric theorems of P2GLE have been obtained in [7–9]. In particular, in [9] we know that the transformation T is measure-preserving and ergodic with respect to the Lebesgue measure. In [8], the author showed that the Hausdorff dimension of the set of numbers with prescribed digits in $A \subset \mathbb{N}$ in their P2GLE is the unique real solution of $\sum_{n\in A} 2^{-ns} = 1$, and hence the set

$$\{x \in (0,1] : d_i(x) \text{ is bounded}\}$$

has Hausdorff dimension one. It is known to [7] that

$$\dim_H\{x \in (0,1] : \lim_{i\to\infty} d_i(x) = \infty\} = 0.$$

In this paper, we aim at finding the Birkhoff spectrum $\xi \mapsto \dim_H E_\varphi(\xi)$ for different choices of potential φ for P2GLE. For any $x \in (0,1]$ with P2GLE (1.2), we define its Khintchine exponent $\gamma(x)$ and Lyapunov exponent $\lambda(x)$ respectively by

$$\gamma(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} d_j(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} d_1(T^j x),$$
$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log\left|(T^n)'(x)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\left|T'(T^j x)\right|,$$

if the limits exist. By definition, the Khintchine exponent is the Birkhoff average of the potential $x \mapsto d_1(x)$, and the Lyapunov exponent is the Birkhoff average of the potential $x \mapsto \log|T'(x)|$.

Since the transformation T is ergodic with respect to the Lebesgue measure, it follows that for Lebesgue almost all x in $(0,1]$,

$$\gamma(x) = \int \lceil -\log_2 x \rceil \, \mathrm{d}x = \sum_{n=1}^{\infty} n2^{-n} = 2,$$
$$\lambda(x) = \int \log\left|T'(x)\right| \mathrm{d}x = (\log 2) \sum_{n=1}^{\infty} n2^{-n} = 2\log 2.$$

Note that $\gamma(x) \geq 1$ and $\lambda(x) \geq \log 2$ for all x in $(0,1]$. We study the level sets:

$$E(\xi) := \{x \in (0,1] : \gamma(x) = \xi\}, \quad \xi \geq 1,$$

and

$$F(\beta) := \{x \in (0,1] : \lambda(x) = \beta\}, \quad \beta \geq \log 2.$$

The Khintchine spectrum and Lyapunov spectrum are defined by

$$\begin{aligned} t(\xi) &: = \dim_H E(\xi), \quad \xi \geq 1, \\ \tilde{t}(\beta) &: = \dim_H F(\beta), \quad \beta \geq \log 2. \end{aligned}$$

Fan, Liao, Wang and Wu ([2]) obtained complete graphs of the Khintchine spectrum and Lyapunov spectrum of the Gauss map associated to continued fractions. Applying the method of [2], we study the Birkhoff spectra of P2GLE. The major tool is the Ruelle operator theory. For a given potential φ, we define the pressure function by

$$P(t,q) = P_{\varphi}(t,q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1\in\mathbb{N}} \sum_{\omega_2\in\mathbb{N}} \cdots \sum_{\omega_n\in\mathbb{N}} \exp\left(\sup_{\substack{d_i(x)=\omega_i \\ 1\leq i\leq n}} S_n(-t\log\left|T'(x)\right| + q\varphi(x)) \right). \quad (1.4)$$

We will prove (Theorem 3.1) that $\dim_H E_{\varphi}(\xi)$ is exactly the first coordinate $t(\xi)$ of the unique solution $\big(t(\xi), q(\xi)\big)$ of the system

$$\begin{cases} P(t,q) = q\xi, \\ \dfrac{\partial P}{\partial q}(t,q) = \xi. \end{cases} \quad (1.5)$$

The notation $P_{\varphi}(t,q)$ and $E_{\varphi}(\xi)$ will be consistently used through this paper, and their specific forms are determined by the potential φ.

We obtain the following results of Khintchine spectrum for P2GLE.

Theorem 1.1. *We have* $\dim_H E(1) = 0$ *and for any* $\xi > 1$,

$$t(\xi) = \dim_H E(\xi) = \frac{\frac{\log(\xi-1)}{\xi} + \log\xi - \log(\xi-1)}{\log 2}. \quad (1.6)$$

Furthermore, the Khintchine spectrum $t : \xi \mapsto \dim_H E(\xi)$ *has the following properties:*

(1) $t(2) = 1$;

(2) $\lim_{\xi\to 1} t(\xi) = 0$ *and* $\lim_{\xi\to\infty} t(\xi) = 0$;

(3) $t'(\xi) > 0$ *for* $\xi < 2$, $t'(2) = 0$, *and* $t'(\xi) < 0$ *for* $\xi > 2$;

(4) $\lim_{\xi\to 1} t'(\xi) = \infty$ *and* $\lim_{\xi\to\infty} t'(\xi) = 0$;

(5) the function $t(\xi)$ *has exactly one inflection point.*

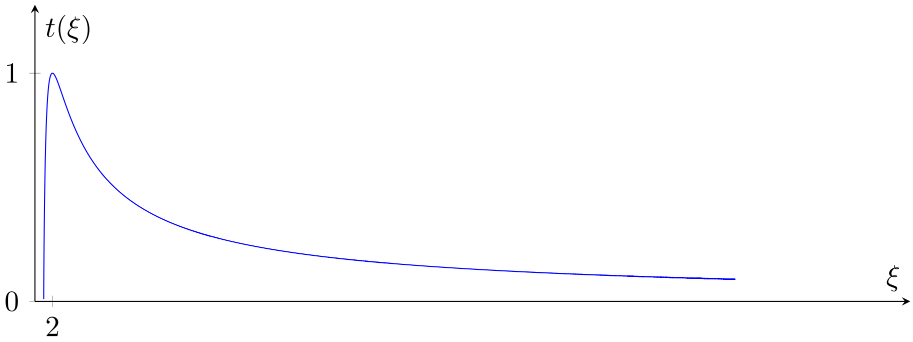

Figure 1: Khintchine spectrum $\big(\varphi(x) = d_1(x)\big)$

We remark that the last point of Theorem 1.1 was not proved for continued fraction expansion in [2]. The reason is that in our P2GLE case, we have an explicit formula for the pressure function $P(t, q)$, while in the continued fraction expansion case such formula does not exist.

The Lyapunov spectrum gains in interest if we realize that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\big|T'(T^j x)\big| = (\log 2) \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} d_j(x).$$

Therefore, replacing ξ by $\frac{\beta}{\log 2}$ in (1.6), we obtain the Lyapunov spectrum.

Corollary 1.1. *We have* $\dim_H F(\log 2) = 0$ *and for any* $\beta > \log 2$*,*

$$\tilde{t}(\beta) = \dim_H F(\beta) = \frac{1}{\beta} \log\left(\frac{\beta}{\log 2} - 1\right) - \frac{1}{\log 2} \log\left(1 - \frac{\log 2}{\beta}\right). \tag{1.7}$$

Furthermore, the Lyapunov spectrum $\tilde{t} : \beta \mapsto \tilde{t}(\beta)$ *has the following properties:*

(1) $\tilde{t}(2\log 2) = 1$*;*

(2) $\lim_{\beta\to\log 2} \tilde{t}(\beta) = 0$ *and* $\lim_{\beta\to\infty} \tilde{t}(\beta) = 0$*;*

(3) $\tilde{t}'(\beta) > 0$ *for* $\beta < 2\log 2$*,* $\tilde{t}'(2\log 2) = 0$*, and* $\tilde{t}'(\beta) < 0$ *for* $\beta > 2\log 2$*;*

(4) $\lim_{\beta\to\log 2} \tilde{t}'(\beta) = \infty$ *and* $\lim_{\beta\to\infty} \tilde{t}'(\beta) = 0$*;*

(5) the function $\tilde{t}(\beta)$ *has exactly one inflection point.*

We illustrate the Birkhoff spectra in Theorem 1.1 and Corollary 1.1 by Figures 1 and 2.

The same method also works for general potential φ. To illustrate some different Birkhoff spectra, we study the special cases of $\varphi(x) = \log(d_1(x))$ and $\varphi(x) = 2^{d_1(x)}$. From the Birkhoff ergodic theorem, we deduce that for Lebesgue almost all x in $(0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\big(d_1(T^j x)\big) = \xi_0 = \int \log\lceil -\log_2 x\rceil \,\mathrm{d}x = \sum_{n=1}^{\infty} 2^{-n} \log n \approx 0.507834,$$

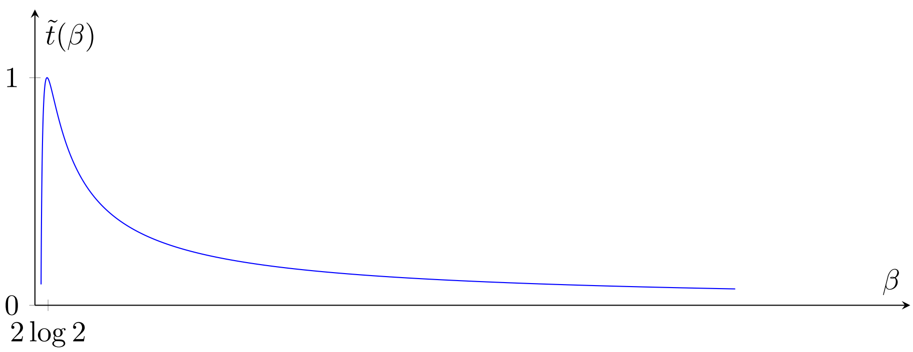

Figure 2: Lyapunov spectrum $\big(\varphi(x) = \log|T'(x)|\big)$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} 2^{d_1(T^j x)} = \int 2^{\lceil -\log_2 x\rceil}\,\mathrm{d}x = \infty.$$

Now let us state our main results for $\varphi(x) = \log(d_1(x))$ and $\varphi(x) = 2^{d_1(x)}$.

Theorem 1.2. *Let $\varphi : (0,1] \to \mathbb{R}$ be the potential defined by $\varphi(x) = \log(d_1(x))$, with $d_1(x) = \lceil -\log_2 x\rceil$, and $E_\varphi(\xi)$ be defined as in (1.3). Then, we have that the pressure function P defined in (1.4) is*

$$P(t,q) = P_\varphi(t,q) = \log\left(\sum_{n=1}^{\infty} 2^{-nt}\cdot n^q\right),$$

and that for any $\xi > 0$, there exists a unique solution $\big(t(\xi), q(\xi)\big)$ to the system (1.5). Moreover, $\dim_H E_\varphi(0) = 0$ and for any $\xi > 0$, $\dim_H E_\varphi(\xi) = t(\xi)$. Let $\xi_0 = \int_0^1 \log\lceil -\log_2 x\rceil\,\mathrm{d}x$. The Birkhoff spectrum $t : \xi \mapsto t(\xi)$ has the following properties:

(1) $t(\xi_0) = 1$;

(2) $t'(\xi) > 0$ for $\xi < \xi_0$, $t'(\xi_0) = 0$, and $t'(\xi) < 0$ for $\xi > \xi_0$;

(3) $\lim_{\xi\to 0} t(\xi) = 0$ and $\lim_{\xi\to\infty} t(\xi) = 0$;

(4) $\lim_{\xi\to 0} t'(\xi) = \infty$;

(5) $t''(\xi_0) < 0$ and there exits $\xi_1 > \xi_0$ such that $t''(\xi_1) > 0$, so the function $t(\xi)$ is neither convex nor concave.

Theorem 1.3. *Let $\varphi(x) = 2^{d_1(x)}$ for $x \in (0,1]$. Then the pressure function P defined in (1.4) is*

$$P(t,q) = P_\varphi(t,q) = \log\left(\sum_{n=1}^{\infty} 2^{-nt} e^{2^n q}\right).$$

We have that $\dim_H E_\varphi(2) = 0$ and for any $\xi > 2$, $\dim_H E_\varphi(\xi) = t(\xi)$, which is the first coordinate of the unique solution of the equation (1.5). The Birkhoff spectrum $t : \xi \mapsto t(\xi)$ has the following properties:

(1) $\dim_H E_\varphi(\infty) = 1$;

(2) $t'(\xi) > 0$ for $\xi > 2$;

(3) $\lim_{\xi\to 2} t(\xi) = 0$ *and* $\lim_{\xi\to\infty} t(\xi) = 1$;

(4) $\lim_{\xi\to 2} t'(\xi) = \infty$ *and* $\lim_{\xi\to\infty} t'(\xi) = 0$;

An outline of this paper is as follows. In Section 2, we review the Ruelle operator theory and the method of pressure function. In Sections 3 and 4, we prove the results for Khintchine and Lyapunov spectra (Theorem 1.1 and Corollary 1.1). Sections 5 and 6 are devoted to proving Theorems 1.2 and 1.3 respectively.

2. Conformal iterated function systems and Ruelle operator theory

2.1. Conformal iterated function systems

From now on, X stands for a non-empty compact connected subset of $\mathbb{R}^d$ equipped with a metric ρ. Let $S = \{\phi_i : X \to X : i \in \mathbb{N}\}$ be an iterated function system for which there exists $0 < s < 1$ such that for each $i \in \mathbb{N}$ and all $x, y \in X$,

$$\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y). \tag{2.1}$$

We follow the notation of [2].

- $\mathbb{N}^n := \{\omega : \omega = (\omega_1, \cdots \omega_n), \omega_k \in \mathbb{N}, 1 \leq k \leq n\}$.
- $\mathbb{N}^\infty := \prod_{i=1}^{\infty} \mathbb{N}$.
- $[\omega|_n] = [\omega_1 \cdots \omega_n] = \{x \in \mathbb{N}^\infty : x_1 = \omega_1, \cdots x_n = \omega_n\}$.
- $|\omega|$ denotes the length of ω.
- $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \cdots \phi_{\omega_n}$ for $\omega = (\omega_1, \cdots \omega_n) \in \mathbb{N}^n$, $n \geq 1$.
- $\|\phi'_\omega\| := \sup_{x\in X} |\phi'_\omega(x)|$ for $\omega \in \bigcup_{n\geq 1} \mathbb{N}^n$.
- $C(X)$ denotes the space of continuous functions on X.
- $\sigma : \mathbb{N}^\infty \to \mathbb{N}^\infty$ is the shift tramsformation.
- $\|\cdot\|_\infty$ denotes the supremum norm on $C(X)$.

Let ∂X denote the boundary of X and Int$X$ denote the interior of X. Our definitions agree with the classical one for conformal iterated function system. For details, see [5] and [2].

An iterated function system $S = \{\phi_i\}_{i\in\mathbb{N}}$ (IFS for short) is said to be conformal if the following conditions are satisfied:

(1) The open set condition is satisfied for Int(X), i.e., $\phi_i(\text{Int}X) \subset \text{Int}(X)$ for each $i \in \mathbb{N}$ and $\phi_i(\text{Int}X) \bigcap \phi_j(\text{Int}X) = \emptyset$ for each $(i, j) \in \mathbb{N}^2, i \neq j$.

(2) There exists an open connected set V with $X \subset V \subset \mathbb{R}^d$ such that all maps ϕ_i, $i \in \mathbb{N}$, extend to C^1 conformal diffemorphisms of V into V.

(3) There exist $h, l > 0$ such that for each $x \in \partial X \subset \mathbb{R}^d$, there exists an open cone Con$(x, h, l) \subset$ Int$X$ with vertex x, central angle of Lebesgue measure h and altitude l.

(4) There exists $K \geq 1$ such that $|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$ for every $\omega \in \cup_{n\geq 1}\mathbb{N}^n$ and every pair of points $x, y \in V$ (Bounded distortion property).

The topological pressure function for the conformal iterated function system $S = \{\phi_i : X \to X : i \in \mathbb{N}\}$ is defined as

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in\mathbb{N}^n} \left\|\phi'_\omega\right\|^t. \tag{2.2}$$

The system S is said to be regular if there exists $t \geq 0$ such that $P(t) = 0$.

Let $X = [0, 1]$. The P2GLE dynamical system is associated with an IFS:

$$S = \{\phi_i(x) = 2^{-i}(x + 1) : i \in \mathbb{N}\}.$$

In fact, the functions ϕ_i are inverse branches of the transformation T defined in (1.1). The coding map $\pi : N^\infty \to X$ is defined by

$$\pi(\omega) = \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X) = \sum_{n=1}^{\infty} 2^{-(\omega_1+\omega_2+\cdots\omega_n)} \quad \text{for any } \omega = (\omega_1, \omega_2 \cdots).$$

Notice that for each $i \in \mathbb{N}$ and for all $x, y \in [0, 1]$,

$$|\phi_i(x) - \phi_i(y)| \leq \frac{1}{2}|x - y|.$$

Lemma 2.1. *The IFS S associated to P2GLE is conformal.*

Proof. (1) The open set condition is satisfied for Int$X = (0, 1)$, since

$$\phi_i(\text{Int}X) = (2^{-i}, 2^{-i+1}) \subset \text{Int}X \quad \text{for each } i \in \mathbb{N},$$

and

$$\phi_i(\text{Int}X) \bigcap \phi_j(\text{Int}X) = \emptyset \quad \text{ for each } (i, j) \in \mathbb{N}^2, i \neq j.$$

(2) Let $V = (-1, 2)$. It follows that $X \subset V$ and all maps $\phi_i, i \in \mathbb{N}$, extend to C^1 conformal diffemorphisms of V into V.
(3) The third condition of conformal IFS was proved by [4, Section 2].
(4) The bounded distortion property holds, since $|\phi'_\omega(y)| = |\phi'_\omega(x)| = 2^{-(\omega_1+\omega_2+\cdots+\omega_n)}$ for every $\omega \in \mathbb{N}^n, n \geq 1$ and every pair of points $x, y \in V$. □

Lemma 2.2. *The iterated function system S is regular.*

Proof. Observe that

$$\left\|\phi'_\omega\right\| = \sup_{x\in X}\left|\phi'_\omega(x)\right| = 2^{-(\omega_1+\omega_2+\cdots+\omega_n)}.$$

By definition, the topological pressure function for S is

$$\begin{aligned} P(t) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in\mathbb{N}^n} \left\|\phi'_\omega\right\|^t \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1\in\mathbb{N}} \sum_{\omega_2\in\mathbb{N}} \cdots \sum_{\omega_n\in\mathbb{N}} 2^{-t(\omega_1+\omega_2+\cdots+\omega_n)} \\ &= -\log\left(2^t - 1\right). \end{aligned}$$

It is easily seen that $P(1) = 0$. Hence, S is regular. □

2.2. Ruelle operator theory

Let $\beta > 0$. A Hölder family of functions of order β is a family of continuous functions

$$G = \{g^{(i)} : X \to \mathbb{R}\}_{i\in\mathbb{N}}$$

such that

$$V_\beta(G) = \sup_{n\geq 1} V_n(G) < \infty,$$

where

$$V_n(G) = \sup_{\omega\in\mathbb{N}^n} \sup_{x,y\in X} \left\{ \left| g^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - g^{(\omega_1)}(\phi_{\sigma(\omega)}(y)) \right| \right\} e^{\beta(n-1)}.$$

A Hölder family of functions $G = \{g^{(i)} : X \to \mathbb{R}\}_{i\in\mathbb{N}}$ is said to be strong if

$$\sum_{i\in\mathbb{N}} \left\| e^{g^{(i)}} \right\|_\infty < \infty. \tag{2.3}$$

The Ruelle operator on $C(X)$ associated with G is defined by

$$\mathcal{L}_G(h)(x) := \sum_{i\in\mathbb{N}} e^{g^{(i)}} h(\phi_i(x)), \quad \forall h \in C(X).$$

Denote by $\mathcal{L}_G^*$ the dual of $\mathcal{L}_G$.

The topological pressure of G is defined by

$$P(G) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\left(\sup_{x\in X} \sum_{j=1}^n g^{(\omega_j)} \circ \phi_{\sigma^j\omega}(x) \right). \tag{2.4}$$

Let $S = \{\phi_i : X \to X\}_{i\in\mathbb{N}}$ be a conformal iterated function system, $\Psi = \{\psi_i : X \to \mathbb{R}, i \in \mathbb{N}\}$, and $F = \{f_i : X \to \mathbb{R}, i \in \mathbb{N}\}$ be two families of real-valued Hölder functions. The amalgamated functions on $\mathbb{N}^\infty$ associated with Ψ and F are defined as

$$\tilde{\psi}(\omega) = \psi_{\omega_1}(\pi(\sigma\omega)), \qquad \tilde{f}(\omega) = f_{\omega_1}(\pi(\sigma\omega)).$$

The following theorems will be useful.

Theorem 2.1. *([3, Theorem 6.4]) Let $\Psi = \{\psi_i\}_{i\in\mathbb{N}}$ and $F = \{f_i\}_{i\in\mathbb{N}}$ be two Hölder families of functions. Suppose the sets $\{i \in \mathbb{N} : \sup_x(\psi_i(x)) > 0\}$ and $\{i \in \mathbb{N} : \sup_x(f_i(x)) > 0\}$ are finite. Then the function $(t, q) \mapsto P(t, q) = P(t\Psi - qF)$ is real analytic with respect to $(t, q) \in Int(D)$, where*

$$D = \left\{ (t, q) : \sum_{i\in\mathbb{N}} \exp\left(\sup_x (t\psi_i(x) - qf_i(x)) \right) < \infty \right\}.$$

Theorem 2.2. *([3, Section 2]) Suppose that $t\Psi - qF$ is a strong Hölder family for any $(t, q) \in D$. For each $(t, q) \in D$, there exists a unique $(t\Psi - qF)$-conformal probability measure $\nu_{t,q}$ on X such that $\mathcal{L}^*_{t\Psi-qF}\nu_{t,q} = e^{P(t,q)}\nu_{t,q}$, and a unique shift invariant ergodic probability measure $\tilde{\mu}_{t,q}$ on $\mathbb{N}^\infty$ such that the measure $\mu_{t,q} := \tilde{\mu}_{t,q} \circ \pi^{-1}$ on X is equivalent to $\nu_{t,q}$ and the Gibbs property is satisfied:*

$$\frac{1}{C} \leq \frac{\tilde{\mu}_{t,q}([\omega|_n])}{\exp\left(\sum_{j=1}^n (t\Psi - qF)^{(\omega_j)}(\pi(\sigma^j\omega)) - nP(t,q) \right)} \leq C \quad \text{for all } \omega \in \mathbb{N}^\infty.$$

Theorem 2.3. *([3]) If $t\Psi - qF$ is a strong Hölder family for any $(t,q) \in D$ and*

$$\int \left(\left|\tilde{\psi}\right| + \left|\tilde{f}\right| \right) \mathrm{d}\tilde{\mu}_{t,q} < \infty,$$

then

$$\frac{\partial P}{\partial t} = \int \tilde{\psi}\, \mathrm{d}\tilde{\mu}_{t,q} \quad \textit{and} \quad \frac{\partial P}{\partial q} = -\int \tilde{f}\, \mathrm{d}\tilde{\mu}_{t,q}. \tag{2.5}$$

If $t\tilde{\psi} - q\tilde{f}$ is not cohomologous to a constant with respect to the transformation T, i.e., there is no function $u \in C(X)$ such that

$$t\tilde{\psi} - q\tilde{f} = u - u \circ T + C,$$

where C is a constant, then $P(t,q)$ is strictly convex and the matrix

$$H(t,q) := \begin{bmatrix} \dfrac{\partial^2 p}{\partial t^2} & \dfrac{\partial^2 p}{\partial t \partial q} \\ \dfrac{\partial^2 p}{\partial t \partial q} & \dfrac{\partial^2 p}{\partial q^2} \end{bmatrix}$$

is positive definite.

Consider the level sets

$$E_\varphi(\xi) = \{x \in (0,1] : \bar{\varphi}(x) = \xi\}, \quad \xi \in \mathbb{R}.$$

Let $P(t,q)$ be the corresponding pressure function of the potential $t\Psi - qF$ and D be the analytic area of $P(t,q)$. By [2, Lemma 4.7], we know that if $\tilde{\psi}(\omega) = -\log|T'(\pi(\omega))|$ and $\tilde{f}(\omega) = -\varphi(\pi(\omega))$. Then

$$-\int \log\left|T'(x)\right| \mathrm{d}\mu_{t,q} = \int \tilde{\psi}\, \mathrm{d}\tilde{\mu}_{t,q} \quad \text{and} \quad \int \varphi(x)\, \mathrm{d}\mu_{t,q} = -\int \tilde{f}\, \mathrm{d}\tilde{\mu}_{t,q}, \tag{2.6}$$

and $t\tilde{\psi} - q\tilde{f}$ is not cohomologous to a constant.

Applying Theorem 2.3, we arrive at the following proposition.

Proposition 2.1. *([2, Proposition 4.8]) The following statements hold on D.*

(1) $P(t,q)$ is analytic and strictly convex.

(2) $P(t,q)$ is strictly decreasing and strictly convex with respect to t, i.e., $(\partial p/\partial t)(t,q) < 0$ and $(\partial^2 p/\partial t^2)(t,q) > 0$. Furthermore,

$$\frac{\partial p}{\partial t}(t,q) = -\int \log\left|T'(x)\right| \mathrm{d}\mu_{t,q}. \tag{2.7}$$

(3) $P(t,q)$ is strictly increasing and strictly convex with respect to q, i.e., $(\partial p/\partial q)(t,q) > 0$ and $(\partial^2 p/\partial q^2)(t,q) > 0$. Furthermore,

$$\frac{\partial p}{\partial q}(t,q) = \int \varphi(x)\, \mathrm{d}\mu_{t,q}. \tag{2.8}$$

(4) The matrix

$$H(t,q) := \begin{bmatrix} \dfrac{\partial^2 p}{\partial t^2} & \dfrac{\partial^2 p}{\partial t \partial q} \\ \dfrac{\partial^2 p}{\partial t \partial q} & \dfrac{\partial^2 p}{\partial q^2} \end{bmatrix}$$

is positive definite.

3. Khintchine spectrum

Recall that the transformation T on $X = (0,1]$ associated to P2GLE is defined as

$$Tx = 2^n x - 1 \quad \text{for } x \in (1/2^n, 1/2^{n-1}], \quad n \in \mathbb{N}$$

and the corresponding conformal iterated function system on X is

$$S = \Big\{ \phi_i(x) = 2^{-i}(x+1) : i \in \mathbb{N} \Big\}.$$

Now we will calculate the Hausdorff dimensions of the level sets

$$E(\xi) = \Big\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} d_j(x) = \xi \Big\} = \Big\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \lceil -\log_2(T^j(x)) \rceil = \xi \Big\}, \quad \xi \geq 1.$$

Let $\Psi = \{\log|\phi_i'| : i \in \mathbb{N}\} = \{-i\log 2 : i \in \mathbb{N}\}$ and $F = \{-i : i \in \mathbb{N}\}$. It can be checked that Ψ and F are two families of real-valued Hölder functions.

Proposition 3.1. *Let $D := \{(t,q) : q - t\log 2 < 0\}$. For any $(t,q) \in D$, we have*

(1) the family $t\Psi - qF$ is Hölder and strong;

(2) the topological pressure P of the potential $t\Psi - qF$ is

$$P(t,q) = \log \frac{e^{-t\log 2 + q}}{1 - e^{-t\log 2 + q}}. \tag{3.1}$$

Proof. (1) We first show that the family $t\Psi - qF$ is Hölder. Since

$$(t\Psi - qF)^{(i)} := t\log\big|\phi_i'\big| + qi = i(-t\log 2 + q),$$

we have $V_n(t\Psi - qF) = 0$. By definition, $t\Psi - qF$ is Hölder.

For $(t,q) \in D$, replacing $g^{(i)}$ by $i(-t\log 2 + q)$ in (2.3), we obtain that

$$\sum_{i\in\mathbb{N}} \Big\| e^{i(-t\log 2 + q)} \Big\|_\infty < \infty.$$

Thus $t\Psi - qF$ is strong.

(2) It suffices to notice that $(t\Psi - qF)^{(i)}$ is a constant function for each $i \in \mathbb{N}$. Thus

$$\sup_{x\in X} \sum_{j=1}^{n} (t\Psi - qF)^{(\omega_j)} \circ \phi_{\sigma^j\omega}(x) = (-t\log 2 + q) \sum_{j=1}^{n} \omega_j.$$

By definition,

$$\begin{aligned} P(t,q) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1,\omega_2\cdots\omega_n \in \mathbb{N}} \exp\left((-t\log 2 + q) \sum_{j=1}^{n} \omega_j \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\frac{e^{-t\log 2 + q}}{1 - e^{-t\log 2 + q}} \right)^n \\ &= \log \frac{e^{-t\log 2 + q}}{1 - e^{-t\log 2 + q}}. \end{aligned}$$

$\square$

3.1. Proof of (1.6) *in Theorem 1.1*

Let us find the solution of the system (1.5).

Proposition 3.2. *For any $\xi \in (1,\infty)$, the unique solution of the system*

$$\begin{cases} P(t,q) = q\xi, \\ \dfrac{\partial P}{\partial q}(t,q) = \xi \end{cases} \tag{3.2}$$

is

$$\begin{cases} t(\xi) = \dfrac{\frac{\log(\xi-1)}{\xi} + \log\xi - \log(\xi-1)}{\log 2}, \\ q(\xi) = \dfrac{\log(\xi-1)}{\xi}. \end{cases}$$

Proof. Since

$$P(t,q) = \log\frac{e^{-t\log 2+q}}{1-e^{-t\log 2+q}}, \tag{3.3}$$

we have

$$\frac{\partial P}{\partial q}(t,q) = \frac{1}{1-e^{-t\log 2+q}}. \tag{3.4}$$

Substituting (3.3) and (3.4) into the above system yields

$$e^{-t\log 2+q} = \frac{e^{q\xi}}{1+e^{q\xi}} = \frac{\xi-1}{\xi}.$$

Hence, $q(\xi) = \dfrac{\log(\xi-1)}{\xi}$ and $t(\xi) = \dfrac{\frac{\log(\xi-1)}{\xi} + \log\xi - \log(\xi-1)}{\log 2}$.

□

Lemma 3.1. *For any $\xi > 1$, let $\big(t(\xi), q(\xi)\big)$ be the solution of* (3.2)*. Then the measure $\mu_{t(\xi),q(\xi)}$ obtained by Theorem 2.2 is supported on $E(\xi)$, i.e., $\mu_{t(\xi),q(\xi)}(E(\xi)) > 0$.*

Proof. By (2.8),

$$\int \lceil -\log_2 x\rceil \,\mathrm{d}\mu_{t,q} = \frac{\partial P}{\partial q}(t,q).$$

Take $(t,q) = \big(t(\xi), q(\xi)\big)$. By the ergodicity of $\mu_{t(\xi),q(\xi)}$ and the second equation in (3.2), we have that for $\mu_{t(\xi),q(\xi)}$-almost every x,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\lceil -\log_2(T^j(x))\rceil = \int \lceil -\log_2 x\rceil \,\mathrm{d}\mu_{t(\xi),q(\xi)} = \frac{\partial P}{\partial q}\big(t(\xi),q(\xi)\big) = \xi.$$

Therefore, $\mu_{t(\xi),q(\xi)}(E(\xi)) = 1 > 0$.

□

The following lemma is a consequence of Theorem 2.2. We use the notation $f \asymp g$ to mean that $\frac{1}{C}f(x) \le g(x) \le Cf(x)$ for some constant $C > 0$ and for all $x \in X$.

Lemma 3.2. *For $x \in X$, let $I_n(x) := \{y \in X : d_i(y) = d_i(x), 1 \le i \le n\}$ and $\omega = \pi^{-1}(x)$. Then*

$$\mu_{t,q}(I_n(x)) \asymp \exp\left(\sum_{j=1}^{n}(t\Psi - qF)^{(\omega_j)}(\pi(\sigma^j\omega)) - nP(t,q)\right). \tag{3.5}$$

Proof. Notice that $\mu_{t,q}(I_n(x)) = \tilde{\mu}_{t,q}([\omega|_n])$. Then $\mu_{t,q}$ is a Gibbs measure on X, i.e., there exists $C > 0$, such that

$$\frac{1}{C} \le \frac{\mu_{t,q}(I_n(x))}{\exp\Big(\sum_{j=1}^{n}(t\Psi - qF)^{(\omega_j)}(\pi(\sigma^j\omega)) - nP(t,q)\Big)} \le C.$$

□

Theorem 3.1. *For any $\xi > 1$, we have*

$$\dim_H E(\xi) = t(\xi) = \frac{\frac{\log(\xi-1)}{\xi} + \log\xi - \log(\xi-1)}{\log 2}. \tag{3.6}$$

Proof. Let $(t,q) \in D$. Since

$$(t\Psi - qF)^{(i)} = i(-t\log 2 + q),$$

we have

$$\exp\left(\sum_{j=1}^{n}(t\Psi - qF)^{(\omega_j)}(\pi(\sigma^j\omega)) - nP(t,q)\right) = \exp\left((-t\log 2 + q)\sum_{j=1}^{n}\omega_j - nP(t,q)\right).$$

By Lemma 3.5,

$$\mu_{t,q}(I_n(x)) \asymp \exp\left((-t\log 2 + q)\sum_{j=1}^{n}d_j - nP(t,q)\right). \tag{3.7}$$

Fix $\xi > 1$. For any $x \in E(\xi)$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}d_j(x) = \xi.$$

Observe that

$$|I_n(x)| = 2^{-d_1(x)+d_2(x)+\cdots d_n(x)}.$$

Take $(t,q) = \big(t(\xi), q(\xi)\big)$. Then $P(t,q) = q\xi$, and

$$\begin{aligned} d_\mu(x) &= \liminf_{n\to\infty}\frac{\log\mu_{t,q}(I_n(x))}{\log|I_n(x)|} \\ &= \liminf_{n\to\infty}\frac{(-t\log 2 + q)\sum_{j=1}^{n}d_j - nP(t,q)}{-(\sum_{j=1}^{n}d_j)\log 2} \\ &= \frac{-t\xi\log 2 + q\xi - q\xi}{-\xi\log 2} \\ &= t. \end{aligned}$$

By Billingsley's lemma (see [1, Lemma 1.4.1]), for $\xi > 1$, we have $\dim_H E(\xi) = t(\xi)$. □

Summarizing, we have proved that the unique solution $t(\xi)$ of (3.2) is the Khintchine spectrum for $\xi > 1$. Now let us discuss the properties of the function $\xi \mapsto t(\xi)$.

3.2. Proofs of Theorem 1.1 (1)-(4)

Recall that the level set

$$E(2) = \{x \in (0,1] : \gamma(x) = 2\}$$

has Lebesgue measure one. This implies $\dim_H E(2) = 1$.

By Theorem 3.1,

$$\dim_H E(\xi) = t(\xi) = \frac{\frac{\log(\xi-1)}{\xi} + \log \xi - \log(\xi - 1)}{\log 2}, \qquad \text{for all } \xi > 1.$$

Letting $\xi \to 1$ and $\xi \to \infty$ respectively, we have

$$\lim_{\xi \to 1} t(\xi) = 0, \qquad \lim_{\xi \to \infty} t(\xi) = 0.$$

This is assertion (2) of Theorem 1.1.

By differentiating the equation in (3.6) with respect to ξ, we arrive at the following proposition.

Proposition 3.3. *For $\xi > 1$,*

$$t'(\xi) = \frac{\log(\xi - 1)}{-\xi^2 \log 2}. \tag{3.8}$$

By (3.8), we see immediately that $t'(\xi) > 0$ for $\xi < 2$, $t'(2) = 0$, and $t'(\xi) < 0$ for $\xi > 2$. Furthermore,

$$\lim_{\xi \to 1} t'(\xi) = \infty, \qquad \lim_{\xi \to \infty} t'(\xi) = 0.$$

We are now in a position to show the boundary case of Theorem 1.1.

Proposition 3.4. *For $\xi = 1$, $\dim_H E(1) = 0$.*

Proof. It is sufficient to show that

$$\dim_H E(1) \le t \log 2, \qquad \text{for all } t > 0. \tag{3.9}$$

For any $t > 0$, from $\lim_{\xi \to 1} t(\xi) = 0$ and $q(\xi) = \dfrac{\log(\xi - 1)}{\xi}$, it follows that there exists $\xi > 1$ such that $0 < t(\xi) < t$ and $q(\xi) < 0$. Since $P(t, q)$ is strictly decreasing with respect to t, we can choose $\varepsilon_0 > 0$ such that

$$\frac{P(t, q(\xi)) - P\big(t(\xi), q(\xi)\big)}{q(\xi)} > \varepsilon_0.$$

For $n \ge 1$, set

$$E_0^n(\varepsilon_0) := \Big\{x \in [0,1) : \frac{1}{n}\sum_{j=1}^{n} d_j(x) < \frac{1}{\log 2}(\xi + \varepsilon_0)\Big\}.$$

Since $\frac{1}{\log 2}(\xi+\varepsilon_0) > 1$,

$$E(1) \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_0^n(\varepsilon_0).$$

Let $\mathcal{I}(n,\xi,\varepsilon_0)$ denote the family of all nth-order cylinders $I_n(d_1,\cdots,d_n)$ such that

$$\frac{1}{n}\sum_{j=1}^{n} d_j(x) < \frac{1}{\log 2}(\xi+\varepsilon_0).$$

Then

$$E_0^n(\varepsilon_0) = \bigcup_{J\in\mathcal{I}(n,\xi,\varepsilon_0)} J.$$

Hence, $\{J : J \in \mathcal{I}(n,\xi,\varepsilon_0), n \geq 1\}$ is a cover of $E(1)$. It remains to prove that

$$\sum_{n=1}^{\infty} \sum_{J\in\mathcal{I}(n,\xi,\varepsilon_0)} |J|^{t\log 2} < \infty.$$

By (3.7) and the fact that $|I_n(x)| = 2^{-(d_1(x)+\cdots+d_n(x))}$, we have

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t,q))2^{q\sum_{j=1}^{n} d_j}|I_n(x)|^{t\log 2}.$$

Thus,

$$\sum_{n=1}^{\infty} \sum_{J\in\mathcal{I}(n,\xi,\varepsilon_0)} |J|^{t\log 2} = \sum_{n=1}^{\infty} \sum_{\sum_{j=1}^{n} d_j<(\xi+\varepsilon_0)n\log_2 e} \frac{e^{nP(t,q(\xi))}}{2^{q(\xi)\sum_{j=1}^{\infty} d_j}} \frac{|J|^{t\log 2}2^{q(\xi)\sum_{j=1}^{\infty} d_j}}{e^{nP(t,q(\xi))}} \tag{3.10}$$

$$\leq C\sum_{n=1}^{\infty} e^{nP(t,q(\xi))-(\xi+\varepsilon_0)q(\xi)} \cdot \sum_{J\in\mathcal{I}(n,\xi,\varepsilon_0)} \mu_{t,q(\xi)}(J) < \infty, \tag{3.11}$$

where C is a constant. Hence, we obtain (3.9). □

3.3. Proof of Theorem 1.1 (5)

The following proposition proves the last assertion of Theorem 1.1, i.e., $t(\xi)$ has only one inflection point.

Proposition 3.5. *We have*

$$t''(\xi) = \frac{2(\xi-1)\log(\xi-1)-\xi}{\xi^3(\xi-1)\log 2}, \quad \forall \xi > 1. \tag{3.12}$$

Further, there exists a point $\tilde{\xi}$ such that $t''(\tilde{\xi}) = 0$ and $t''(\xi) < 0$ for $\xi < \tilde{\xi}$, $t''(\xi) > 0$ for $\xi > \tilde{\xi}$.

Proof. For $\xi > 1$, we have $\xi^3(\xi-1)\log 2 > 0$. Define a function f by $f(\xi) = 2(\xi-1)\log(\xi-1)-\xi$ for all $\xi > 1$. Notice that

$$\lim_{\xi\to 1} f(\xi) = -1 < 0, \qquad \lim_{\xi\to\infty} f(\xi) = \infty.$$

Taking the derivative of f gives $f'(\xi) = 2\log(\xi-1)+1$. We then deduce that f is strictly decreasing on $(1, 1+e^{-1/2})$ and increasing on $(1+e^{-1/2},\infty)$. Observe that $f(3) < 0$, and $f(1+e) > 0$. Therefore, there exists exactly one point $\tilde{\xi} \in (3, 1+e)$ such that $t''(\tilde{\xi}) = 0$. The rest of the statements are evident.

□

4. Birkhoff spectrum for $\varphi : x \mapsto \log(d_1(x))$

Let $\varphi(x) = \log(d_1(x)) = \log(\lceil -\log_2 x \rceil)$. We consider the Hausdorff dimensions of the sets

$$E_\varphi(\xi) = \Big\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\big(d_1(T^j x)\big) = \xi \Big\}, \quad \xi \geq 0.$$

Recall that, for Lebesgue almost all x in $(0,1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\big(d_1(T^j x)\big) = \xi_0 = \int_0^1 \log\lceil -\log_2 x \rceil \, \mathrm{d}x.$$

Thus $\dim_H E_\varphi(\xi_0) = 1$.

Let $\Psi = \{\log|\phi_i'| : i \in \mathbb{N}\} = \{-i\log 2 : i \in \mathbb{N}\}$ and $F = \{-\log i : i \in \mathbb{N}\}$.

Proposition 4.1. *Let $D := \{(t,q) : t > 0,$ or $t = 0, q < -1\}$. For any $(t,q) \in D$:*

(1) the family $t\Psi - qF$ is Hölder and strong;

(2) the topological pressure $P(t,q)$ of the potential $t\Psi - qF$ is given by

$$P(t,q) = \log\left(\sum_{n=1}^{\infty} 2^{-nt} \cdot n^q\right). \tag{4.1}$$

Proof. (1) Since $(t\Psi - qF)^{(i)} = -ti\log 2 + q\log i$, we have $V_n(t\Psi - qF) = 0$. By definition, the family $t\Psi - qF$ is Hölder.

For $(t,q) \in D$, it can be checked that

$$\sum_{i\in\mathbb{N}} e^{-ti\log 2 + q\log i} < \infty.$$

Thus $t\Psi - qF$ is strong.

(2) Observe that

$$\sup_{x\in X} \sum_{j=1}^{n} (t\Psi - qF)^{(\omega_j)} \circ \phi_{\sigma^j\omega}(x) = -t\log 2 \sum_{j=1}^{n} \omega_j + q \sum_{j=1}^{n} \log \omega_j.$$

By definition,

$$\begin{aligned} P(t,q) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1,\omega_2\cdots\omega_n\in\mathbb{N}} \exp\left(-t(\log 2)\sum_{j=1}^{n}\omega_j + q\sum_{j=1}^{n}\log\omega_j\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\omega=1}^{\infty} \exp(-t\omega\log 2 + q\log\omega)\right)^n \\ &= \log\left(\sum_{n=1}^{\infty} 2^{-nt} \cdot n^q\right). \end{aligned}$$

□

By Theorem 2.1 and Proposition 4.1, we know that D is the analytic area of the pressure $P(t,q)$. From Mayer [6] (see also Pollicott and Weiss [10, Proposition 4]), we know that $\mu_{1,0}$ is the Lebesgue measure on $[0,1]$. By (2.8), we obtain

$$\frac{\partial P}{\partial q}(1,0)=\int \log\lceil -\log_2 x\rceil \,\mathrm{d}x=\xi_0. \tag{4.2}$$

4.1. Further sutdy on $P(t,q)$

Now we show some properties of $P(t,q)$ and $\partial P/\partial q$.

Proposition 4.2. *For fixed $t>0$,*

$$\frac{\partial P}{\partial q}(t,q)=\frac{\sum_{n=1}^{\infty}2^{-nt}(\log n)n^q}{\sum_{n=1}^{\infty}2^{-nt}n^q},\quad \forall q\in\mathbb{R}. \tag{4.3}$$

Moreover, we have

$$P(t,0)=-\log\big(2^t-1\big), \tag{4.4}$$

and

$$P(0,q)=\log\zeta(-q), \tag{4.5}$$

where ζ is the Riemann zeta function, defined by

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \forall s>1.$$

Consequently:

(1) $P(1,0)=0$, and

$$\lim_{q\to-\infty}P(0,q)=0;$$

(2) for fixed $t>0$,

$$\lim_{q\to\infty}P(t,q)=\infty;$$

(3) for fixed $t\geq 0$,

$$\lim_{q\to-\infty}\frac{\partial P}{\partial q}(t,q)=0; \tag{4.6}$$

(4) we have

$$\lim_{t\to 0}\frac{\partial P}{\partial q}(t,0)=\infty. \tag{4.7}$$

Proof. Fix $t>0$ and $q_0>0$. Notice that for any $n\in\mathbb{N}$ and $q\leq q_0$,

$$2^{-nt}(\log n)n^q\leq 2^{-nt}(\log n)n^{q_0}.$$

Since the series $\sum_{n=1}^{\infty}2^{-nt}(\log n)n^{q_0}$ is convergent, we arrive at (4.3). Since

$$P(t,0)=\log\left(\sum_{n=1}^{\infty}2^{-nt}\right)=-\log\big(2^t-1\big),$$

we have $P(1,0)=0$. Notice that

$$P(0,q)=\log\left(\sum_{n=1}^{\infty} n^q\right)=\log\zeta(-q),$$

and

$$\lim_{q\to-\infty}\zeta(-q)=1.$$

Then, we have $P(0,q)\to 0$ $(q\to-\infty)$. We have thus proved (4.4), (4.5) and part (1).

Next, let $q\to\infty$. We note that $n^q\to\infty$ for all $n>1$, hence $P(t,q)\to\infty$. This proves part (2).

Now, let $q\to-\infty$. Since $n^q\to 0$, we have

$$\lim_{q\to-\infty}\frac{\partial P}{\partial q}(t,q)=\lim_{q\to-\infty}\frac{\sum_{n=1}^{\infty}2^{-nt}(\log n)n^q}{2^{-t}+\sum_{n=2}^{\infty}2^{-nt}n^q}=0,$$

which is the assertion of part (3).

For the final part, let $q=0$. By (4.3), we have

$$\frac{\partial P}{\partial q}(t,0)=\frac{\sum_{n=1}^{\infty}2^{-nt}(\log n)}{\sum_{n=1}^{\infty}2^{-nt}}.$$

Fix $M>0$. Then there extsts $N\in\mathbb{N}$, such that $\log n>M$ for any $n\geq N$. Hence,

$$\frac{\partial P}{\partial q}(t,0)>\frac{(\log 2)\sum_{n=2}^{k-1}2^{-nt}+M\sum_{n=k}^{\infty}2^{-nt}}{\sum_{n=1}^{\infty}2^{-nt}}=2^{-2t}(\log 2)(1-2^{-(k-2)t})+M2^{-kt}.$$

Notice that

$$\lim_{t\to 0}2^{-2t}(\log 2)(1-2^{-(k-2)t})+M2^{-kt}=M.$$

Thus,

$$\lim_{t\to 0}\frac{\partial P}{\partial q}(t,0)\geq M.$$

Hence, letting $M\to\infty$, we obtain (4.7). $\square$

Recall that $\xi_0=(\partial P/\partial q)(1,0)$. Let $\tilde{D}=\{(t,q)\in D: 0\leq t\leq 1\}$.

Proposition 4.3 ([2]). *For any $\xi\in(0,\infty)$, the system*

$$\begin{cases} P(t,q)=q\xi, \\ \dfrac{\partial P}{\partial q}(t,q)=\xi \end{cases} \tag{4.8}$$

admits a unique solution $\big(t(\xi),q(\xi)\big)\in\tilde{D}$. For $\xi=\xi_0$, the unique solution is $(t(\xi_0),q(\xi_0))=(1,0)$. Further, the functions $t(\xi)$ and $q(\xi)$ are analytic.

By differentiating the first equation in system (4.8) with respect to ξ and applying the second equation in (4.8), we obtain the first derivative of function t with respect to ξ, i.e.,

$$t'(\xi)=\frac{q(\xi)}{(\partial P/\partial t)\big(t(\xi),q(\xi)\big)},\quad \forall\xi>0. \tag{4.9}$$

4.2. Proof of Theorem 1.2

As is widely known, if f is a convex continuously differentiable function on an interval I, then f' is increasing and

$$f'(x) \leq \frac{f(y)-f(x)}{y-x} \leq f'(y) \quad \forall x, y \in I, x < y.$$

Using this fact, we next present some properties on $t(\xi)$.

The proof of the following proposition is similar to that of [2, Proposition 4.14].

Proposition 4.4. *$q(\xi) < 0$ for $\xi < \xi_0$; $q(\xi_0) = 0$; $q(\xi) > 0$ for $\xi > \xi_0$.*

Proof. By the convexity of the function $P(1, q)$ with respect to q, and the fact $P(1, 0) = 0$, we have

$$\frac{P(1,q)-0}{q-0} \geq \frac{\partial P}{\partial q}(1,0) = \xi_0 \quad (q>0); \quad \frac{0-P(1,q)}{0-q} \leq \frac{\partial P}{\partial q}(1,0) = \xi_0 \quad (q<0).$$

Thus

$$P(1, q) \geq \xi_0 q, \quad \forall q \in \mathbb{R}.$$

Proposition 4.3 shows that $t(\xi) = 1$ if and only if $\xi = \xi_0$. Suppose $t \in (0, 1)$. For $\xi > \xi_0$, if $q(\xi) \leq 0$, then

$$P(t, q) > P(1, 0) \geq q\xi_0 \geq q\xi$$

which contradicts to the first equation in (4.8). Thus

$$q(\xi) > 0 \quad (\xi > \xi_0).$$

The rest of the proof runs as before.

∎

Let us use Proposition 4.4 and (4.9) to deduce some further properties of $t(\xi)$.

Proposition 4.5. *We have $t'(\xi) > 0$ for $\xi < \xi_0$, $t'(\xi_0) = 0$, and $t'(\xi) < 0$ for $\xi > \xi_0$. Furthermore,*

$$t(\xi) \to 0 \qquad (\xi \to 0), \tag{4.10}$$
$$t(\xi) \to 0 \qquad (\xi \to \infty). \tag{4.11}$$

Proof. By Proposition 2.1(2), $\partial P/\partial t < 0$. Hence, according to Proposition 4.4 and (4.9), $t(\xi)$ is strictly increasing on $(0, \xi_0)$ and decreasing on (ξ_0, ∞). Then by the analyticity of $t(\xi)$, we can obtain two analytic inverse functions on the two intervals respectively. For the first inverse function, write $\xi_1 = \xi_1(t)$. Then $\xi_1 \in (0, \xi_0)$ and $\xi_1'(t) > 0$. Considering equations (4.8) as equations on t, we have

$$\xi_1(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t)).$$

Since $\xi_1(t) \in (0, \xi_0)$, we conclude that $q(\xi_1(t)) < 0$, and in consequence, $P(t, q(\xi_1(t))) < 0$. By Proposition 4.2(2), $\lim_{q\to\infty} P(t, q) = \infty$. Thus there exits $q_0(t)$ such that $q_0(t) > q(t)$ and $P(t, q_0(t)) = 0$. From $\partial^2 p/\partial q^2 > 0$, we see that

$$\xi_1(t) = \frac{\partial P}{\partial q}(t, q(t)) < \frac{\partial P}{\partial q}(t, q_0(t)).$$

Since $P(0,q)=\log\zeta(-q)$, then $\lim_{t\to 0} q_0(t)=-\infty$. Consequently,

$$\lim_{t\to 0}\frac{\partial P}{\partial q}(t,q_0(t))=\lim_{q\to-\infty}\frac{\partial P}{\partial q}(0,q)=0.$$

From the above, it follows that $\lim_{t\to 0}\xi_1(t)=0$, and so $\lim_{\xi\to 0} t(\xi)=0$.

For the second inverse function, write $\xi_2=\xi_2(t)$. Then $\xi_2\in(\xi_0,\infty)$ and $\xi_2'(t)<0$. An analysis similar to that in the proof of the first inverse function shows that

$$\xi_2(t)=\frac{\partial P}{\partial q}(t,q(t))>\frac{\partial P}{\partial q}(t,0).$$

From (4.7), we deduce that $\lim_{\xi\to\infty} t(\xi)=0$. ☐

As in the proof of [2, Proposition 5.1], we obtain the following proposition.

Proposition 4.6. *We have*

$$\lim_{\xi\to 0} q(\xi)=-\infty. \tag{4.12}$$

Proposition 4.7. *We have the following two properties of $\partial P/\partial t$.*

(1) For $t>0$,

$$\lim_{q\to-\infty}\frac{\partial P}{\partial t}(t,q)=-\log 2.$$

(2) For $q=0$,

$$\lim_{t\to 0}\frac{\partial P}{\partial t}(t,0)=-\infty.$$

Proof. For $t>0$ and $q<-2$,

$$\frac{\partial P}{\partial t}(t,q)=\frac{(-\log 2)\sum_{n=1}^{\infty}2^{-nt}n^{q+1}}{\sum_{n=1}^{\infty}2^{-nt}n^{q}}. \tag{4.13}$$

Therefore,

$$\lim_{q\to-\infty}\frac{\partial P}{\partial t}(t,q)=-\log 2.$$

For $q=0$, $P(t,0)=-\log\big(2^t-1\big)$. Therefore,

$$\frac{\partial P}{\partial t}(t,0)=\frac{-2^t\log 2}{2^t-1},\qquad \lim_{t\to 0}\frac{\partial P}{\partial t}(t,0)=-\infty.$$

☐

Combining Propositions 4.6 and 4.7 yields

$$\lim_{\xi\to 0} t'(\xi)=\infty.$$

We have thus proved (1)-(4) of Theorem 1.2.

Our next goal is to calculate the value of $\dim_H E_\varphi(0)$.

Theorem 4.1. *For $\xi = 0$, $\dim_H E_\varphi(0) = 0$.*

Proof. For any $t > 0$, since $\lim_{\xi\to 0} t(\xi) = 0$, there exists $0 < \xi < \xi_0$ such that $0 < t(\xi) < t$ and $q(\xi) < 0$. Then we can choose $\varepsilon_0 > 0$, such that

$$\frac{P(t, q(\xi)) - P\big(t(\xi), q(\xi)\big)}{q(\xi)} > \varepsilon_0.$$

Let $\mathcal{I}(n, \xi, \varepsilon_0)$ be the family of all nth-order cylinders $I_n(d_1, \cdots, d_n)$ such that

$$\frac{1}{n}\sum_{j=1}^{n} \log(d_j(x)) < \xi + \varepsilon_0.$$

Then $\{J : J \in \mathcal{I}(n, \xi, \varepsilon_0), n \geq 1\}$ is a cover of $E_\varphi(0)$. Since $(t\Psi - qF)^{(i)} = -ti \log 2 + q \log i$, by Lemma 3.5,

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t, q))|I_n(x)|^t (d_1 \cdots d_n)^q.$$

A calculation similar to that in the proof of Proposition 3.4 shows that

$$\sum_{n=1}^{\infty} \sum_{J\in\mathcal{I}(n,\xi,\varepsilon_0)} |J|^t = \sum_{n=1}^{\infty} \sum_{(d_1\cdots d_n)<e^{n(\xi+\varepsilon_0)}} \frac{e^{nP(t,q(\xi))}}{(d_1 \cdots d_n)^{q(\xi)}} \frac{|J|^t (d_1 \cdots d_n)^{q(\xi)}}{e^{nP(t,q(\xi))}} \tag{4.14}$$

$$\leq C \sum_{n=1}^{\infty} e^{nP(t,q(\xi))-(\xi+\varepsilon_0)q(\xi)} \cdot \sum_{J\in\mathcal{I}(n,\xi,\varepsilon_0)} \mu_{t,q(\xi)}(J) < \infty. \tag{4.15}$$

Thus $\dim_H E_\varphi(0) \leq t$. By the arbitrariness of t, $\dim_H E_\varphi(0) = 0$. □

For a rigorous proof of the last assertion of Theorem 1.2, we refer the reader to [2, p.103]. A point that should be stressed is that the number of inflection points is not clear yet.

5. Birkhoff spectrum for $\varphi : x \mapsto 2^{d_1(x)}$

Consider the level sets

$$E_\varphi(\xi) = \Big\{x \in (0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} 2^{d_1(T^j x)} = \xi\Big\}, \quad \xi \geq 2$$

with $\varphi(x) = 2^{d_1(x)}$.

Let $\Psi = \{\log|\phi_i'| : i \in \mathbb{N}\} = \{-i \log 2 : i \in \mathbb{N}\}$ and $F = \{-2^i : i \in \mathbb{N}\}$. It can be checked that Ψ and F are two families of real-valued Hölder functions.

Proposition 5.1. *Let $D := \{(t, q) : t > 0, q = 0$ or $q < 0\}$. For $(t, q) \in D$:*

(1) the family $t\Psi - qF$ is Hölder and strong;

(2) the topological pressure P of the potential $t\Psi - qF$ is

$$P(t, q) = \log\left(\sum_{n=1}^{\infty} 2^{-nt} e^{2^n q}\right). \tag{5.1}$$

Proof. (1) Since $(t\Psi - qF)^{(i)} = -ti\log 2 + q2^i$, $V_n(t\Psi - qF) = 0$. By definition, $t\Psi - qF$ is Hölder.

For $(t,q) \in D$, we have

$$\sum_{i\in\mathbb{N}} e^{-ti\log 2 + q2^i} = \sum_{i\in\mathbb{N}} 2^{-it}e^{2^i q} < \infty.$$

Thus $t\Psi - qF$ is strong.

(2) Observe that

$$\sup_{x\in X}\sum_{j=1}^{n}(t\Psi - qF)^{(\omega_j)} \circ \phi_{\sigma^j\omega}(x) = -t\log 2\sum_{j=1}^{n}\omega_j + q\sum_{j=1}^{n}2^{\omega_j}.$$

By definition,

$$\begin{aligned}
P(t,q) &= \lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega_1,\omega_2\cdots\omega_n\in\mathbb{N}}\exp\left(-t\log 2\sum_{j=1}^{n}\omega_j + q\sum_{j=1}^{n}2^{\omega_j}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\omega=1}^{\infty}2^{-\omega t}e^{2^\omega q}\right)^n\\
&= \log\left(\sum_{n=1}^{\infty}2^{-nt}e^{2^n q}\right).
\end{aligned}$$

□

Let us investigate some properties of $P(t,q)$. Let $D_1 = \{(t,q) : t \geq 0, q < 0\}$.

Proposition 5.2. *For* $(t,q) \in D_1$,

$$\frac{\partial P}{\partial q}(t,q) = \frac{\sum_{n=1}^{\infty}2^{-n(t-1)}e^{2^n q}}{\sum_{n=1}^{\infty}2^{-nt}e^{2^n q}}. \tag{5.2}$$

Consequently,

(1) for fixed $0 < t \leq 1$,

$$\lim_{q\to 0}\frac{\partial P}{\partial q}(t,q) = \infty; \tag{5.3}$$

(2) we have

$$\lim_{(t,q)\to(0,0)}\frac{\partial P}{\partial q}(t,q) = \infty; \tag{5.4}$$

(3) for fixed $t \geq 0$ *and* $q < 0$, *we have* $(\partial P/\partial q)(t,q) > 2$, *and*

$$\lim_{q\to-\infty}\frac{\partial P}{\partial q}(t,q) = 2. \tag{5.5}$$

Proof. Fix $t \geq 0$ and $q_0 < 0$. Then for any $q \leq q_0$, we have

$$2^{-n(t-1)}e^{2^n q} \leq 2^{-n(t-1)}e^{2^n q_0}.$$

Notice that
$$\lim_{n\to\infty} \sqrt[n]{2^{-n(t-1)}e^{2^n q_0}} = 0 < 1.$$
By Weierstrass M-test, we have that the series $\sum_{n=1}^{\infty} 2^{-n(t-1)}e^{2^n q}$ is uniformly convergent on the set $\{q : q < q_0\}$. Hence, we obtain (5.2).

Fix $0 < t \leq 1$. Since $e^{2^n q} \to 1$ $(q \to 0)$, we have
$$\lim_{q\to 0} \frac{\partial P}{\partial q}(t,q) = \frac{\sum_{n=1}^{\infty} 2^{-n(t-1)}}{\sum_{n=1}^{\infty} 2^{-nt}}.$$
Notice that
$$\sum_{n=1}^{\infty} 2^{-n(t-1)} = \infty, \qquad \sum_{n=1}^{\infty} 2^{-nt} = \frac{1}{2^t - 1}.$$
Hence, (5.3) is proved.

In order to prove (2), we need to show
$$\lim_{t\to 0} \frac{\sum_{n=1}^{\infty} 2^{-n(t-1)}}{\sum_{n=1}^{\infty} 2^{-nt}} = \infty.$$
This is analogous to that in (4.7) and is left to the reader.

Observe that
$$\sum_{n=1}^{\infty} 2^{-n(t-1)}e^{2^n q} > 2\sum_{n=1}^{\infty} 2^{-nt}e^{2^n q}.$$
As a result,
$$\frac{\partial P}{\partial q}(t,q) > 2.$$
This proves the first part of (3).

Since $\lim_{q\to-\infty} e^{(2^n-2)q} = 0$ for $n \geq 2$, it follows that
$$\lim_{q\to-\infty} \frac{\sum_{n=1}^{\infty} 2^{-n(t-1)}e^{2^n q}}{\sum_{n=1}^{\infty} 2^{-nt}e^{2^n q}} = \lim_{q\to-\infty} \frac{2^{-(t-1)}e^{2q}(1 + \sum_{n=2}^{\infty} 2^{-(n-1)(t-1)}e^{(2^n-2)q})}{2^{-t}e^{2q}(1 + \sum_{n=2}^{\infty} 2^{-(n-1)t}e^{(2^n-2)q})} = 2,$$
which completes the proof. $\square$

Proposition 5.3. *For* $(t,q) \in D_1$,
$$\frac{\partial P}{\partial t}(t,q) = \frac{(-\log 2)\sum_{n=1}^{\infty} n2^{-nt}e^{2^n q}}{\sum_{n=1}^{\infty} 2^{-nt}e^{2^n q}}. \tag{5.6}$$
Consequently,

(1) for fixed $t \geq 0$,
$$\lim_{q\to-\infty} \frac{\partial P}{\partial t}(t,q) = -\log 2; \tag{5.7}$$
(2) for $t = 1$,
$$\lim_{q\to 0} \frac{\partial P}{\partial t}(1,q) = -\frac{\log 2}{2}. \tag{5.8}$$

Proof. By Abel's test, it follows that the series $\sum_{n=1}^{\infty} n2^{-nt}e^{2^n q}$ converges uniformly with respect to t on the set D_1. Hence, we get (5.6).

Fix $t \geq 0$. Write

$$\frac{\sum_{n=1}^{\infty} n2^{-nt}e^{2^n q}}{\sum_{n=1}^{\infty} 2^{-nt}e^{2^n q}} = \frac{e^{2q}(2^{-t} + \sum_{n=2}^{\infty} n2^{-nt}e^{(2^n-2)q})}{e^{2q}(2^{-t} + \sum_{n=2}^{\infty} 2^{-nt}e^{(2^n-2)q})} = \frac{2^{-t} + \sum_{n=2}^{\infty} n2^{-nt}e^{(2^n-2)q}}{2^{-t} + \sum_{n=2}^{\infty} 2^{-nt}e^{(2^n-2)q}}.$$

Notice that for any $n \geq 2$,

$$\lim_{q\to-\infty} e^{(2^n-2)q} = 0.$$

From the above, it follows that

$$\lim_{q\to-\infty} \frac{\sum_{n=1}^{\infty} n2^{-nt}e^{2^n q}}{\sum_{n=1}^{\infty} 2^{-nt}e^{2^n q}} = 1,$$

which proves (5.7).

By (5.6), we have

$$\frac{\partial P}{\partial t}(1,q) = \frac{(-\log 2)\sum_{n=1}^{\infty} n2^{-n}e^{2^n q}}{\sum_{n=1}^{\infty} 2^{-n}e^{2^n q}}.$$

Since $e^{2^n q} \to 1$ $(q \to 0)$ for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} 2^{-n} = 1$, we have

$$\lim_{q\to 0} \frac{\partial P}{\partial t}(1,q) = (-\log 2)\sum_{n=1}^{\infty} n2^{-n} = -\frac{\log 2}{2}. \tag{5.9}$$

$\square$

Proposition 5.4. *Let $\tilde{D} = \{(t,q) \in D : 0 \leq t \leq 1\}$. For any $\xi \in (2,\infty)$, the system*

$$\begin{cases} P(t,q) = q\xi, \\ \dfrac{\partial P}{\partial q}(t,q) = \xi \end{cases} \tag{5.10}$$

admits a unique solution $\big(t(\xi), q(\xi)\big) \in \tilde{D}$. Further, the functions $t(\xi)$ and $q(\xi)$ are analytic.

5.1. Proofs of Theorem 1.3 (1) and (2)

It follows from the Birkhoff ergodic theorem that for Lebesgue almost all x in $(0,1]$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} 2^{d_j(x)} = \int 2^{\lceil -\log_2 x\rceil}\,\mathrm{d}x = \infty.$$

Hence,

$$\dim_H E_\varphi(\infty) = \dim_H\{x \in (0,1] : \bar{\varphi}(x) = \infty\} = 1.$$

Proposition 5.5. *The function $t(\xi)$ is strictly incresing on $(2,\infty)$.*

Proof. From the analytic area of $P(t,q)$, we know that $q(\xi) \leq 0$. Suppose there exists $\xi_1 < \infty$ such that $q(\xi_1) = 0$. Then $P(t(\xi_1),0) = 0$ by Proposition 5.4. Hence, we have $t(\xi_1) = 1$ and $(\partial P/\partial q)(1,0) = \xi_1$. However, by (5.3), we have

$$\lim_{(t,q)\to(1,0)} \frac{\partial P}{\partial q}(t,q) = \infty.$$

This contradicts to the hypothesis that $\xi_1 < \infty$. Thus $t'(\xi) > 0$. $\square$

5.2. Proofs of Theorem 1.3 (3) and (4)

Proposition 5.6. *We have*

$$\lim_{\xi\to 2} t(\xi) = 0, \qquad \lim_{\xi\to 2} q(\xi) = -\infty. \tag{5.11}$$

Furthermore,

$$\lim_{\xi\to 2} t'(\xi) = \infty.$$

Proof. To show the first formula in (5.11), we use the same method as in the proof of Proposition 4.5. Since $\xi \mapsto t(\xi)$ is increasing and analytic on $(2,\infty)$, we can write its inverse function $\xi = \xi(t)$. Then $\xi'(t) > 0$ and

$$\xi(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t)).$$

We conclude from (5.5) that

$$\lim_{q\to -\infty} \frac{\partial P}{\partial q}(0, q) = 2,$$

hence that $\lim_{t\to 0} \xi(t) = 2$, and finally that $\lim_{\xi\to 2} t(\xi) = 0$.

Now suppose that there exists a subsequence $\{\xi_{n_k}\}_{k\geq 1}$, such that

$$\lim_{k\to\infty} \xi_{n_k} = 2, \qquad \lim_{k\to\infty} q(\xi_{n_k}) = M > -\infty.$$

Since $\lim_{\xi\to 2} t(\xi) = 0$, it follows that $\lim_{k\to\infty} t(\xi_{n_k}) = 0$. By Proposition 5.4,

$$\lim_{k\to\infty} \frac{\partial P}{\partial q}(t(\xi_{n_k}), q(\xi_{n_k})) = \lim_{k\to\infty} \xi_{n_k} = 2. \tag{5.12}$$

If $M = 0$, then by (5.4), we have

$$\lim_{k\to\infty} \frac{\partial P}{\partial q}(t(\xi_{n_k}), q(\xi_{n_k})) = \lim_{(t,q)\to(0,0)} \frac{\partial P}{\partial q}(t, q) = \infty,$$

which contradicts to (5.12). If $M < 0$, then by Proposition 5.2 (3), we have

$$\lim_{k\to\infty} \frac{\partial P}{\partial q}(t(\xi_{n_k}), q(\xi_{n_k})) = \frac{\partial P}{\partial q}(0, M) > 2,$$

which is also a contradiction to (5.12). Hence, $q(\xi) \to -\infty$ $(\xi \to 2)$.

Applying the second formula in (5.11) and combining (4.9) and (5.7), we conclude that

$$\lim_{\xi\to 2} t'(\xi) = \infty.$$

$\square$

Proposition 5.7. *We have*

$$\lim_{\xi\to\infty} t(\xi) = 1, \quad \lim_{\xi\to\infty} q(\xi) = 0. \tag{5.13}$$

Furthermore,

$$\lim_{\xi\to\infty} t'(\xi) = 0.$$

Proof. Recall that for any $\xi > 2$,

$$\begin{cases} P\big(t(\xi), q(\xi)\big) = q(\xi)\xi, \\ \dfrac{\partial P}{\partial q}\big(t(\xi), q(\xi)\big) = \xi. \end{cases}$$

Suppose $\big(t(\xi), q(\xi)\big) \to (t_0, q_0)$ as $\xi \to \infty$. The analyticity of $t(\xi)$ and $q(\xi)$ implies that $0 < t_0 \leq 1$ and $q_0 \leq 0$. Suppose $q_0 < 0$. Then by (5.2) and (5.5), we have

$$\lim_{(t,q)\to(t_0,q_0)} \frac{\partial P}{\partial q}(t, q) < \infty,$$

which is a contradiction to

$$\lim_{\xi\to\infty} \frac{\partial P}{\partial q}(t, q) = \lim_{\xi\to\infty} \xi = \infty.$$

Thus $q_0 = 0$. For any $\xi > 2$, since $q(\xi)\xi < 0$ and $P(t_0, 0) \geq 0$, we have

$$P(t_0, 0) = \lim_{\xi\to\infty} q(\xi)\xi = 0.$$

Since $P(1, 0) = 0$ and $(\partial P/\partial q)(t, q) \to \infty$ as $(t, q) \to (1, 0)$, we have $(t_0, q_0) = (1, 0)$, which completes the proof of (5.13).

By (5.8) and (4.9), we have

$$\lim_{\xi\to\infty} t'(\xi) = 0.$$

□

Applying similar arguments to the proof of Theorem 4.1 yields

$$\dim_H E_\varphi(2) = 0.$$

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