Shellability of Kohnert posets

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April 29, 2024

Abstract

In this paper, we are concerned with identifying among the family of posets associated with Kohnert polynomials, those whose order complex has a certain combinatorial property. In particular, for numerous families of Kohnert polynomials, including key polynomials, we determine when the associated Kohnert posets are (EL-)shellable. Interestingly, under certain diagram restrictions, (EL-)shellability of a Kohnert poset is equivalent to multiplcity freeness of the associated Kohnert polynomial.

1 Introduction

In his thesis ([**12**], 1990), Kohnert showed that Demazure characters (a.k.a. key polynomials) encode certain collections of diagrams consisting of cells distributed in the first quadrant. Moreover, it was conjectured, and subsequently proven [**3, 14, 15**], that a similar result applied to Schubert polynomials. Motivated by this, Assaf and Searles ([**2**], 2022) applied the corresponding polynomial construction paradigm more generally and defined the notion of a "Kohnert polynomial". Such polynomials encode certain collections of diagrams consisting of cells distributed in the first quadrant that are related to a "seed" diagram by a sequence of moves, called "Kohnert moves", that change the position of at most one cell (see also [**4**]). In ([**4**], 2022), the author notes that one can define a natural poset structure on the collection of diagrams encoded by the terms of a Kohnert polynomial. Moreover, the author of [**4**] illustrates that such "Kohnert posets" arising from Kohnert polynomials are not generally well-behaved, noting that, in general, they are not lattices, ranked, nor do they have a unique minimal element. In recent work by L. Colmenarjo, et al. ([**10**], 2023), the authors initiate an investigation into the "not-so-well-behaved" structure of such posets, focusing on identifying when they are ranked and/or bounded. Here, we consider when Kohnert posets are (EL-)shellable and the consequences regarding the associated polynomial.

Starting with modest restrictions, we first consider the Kohnert posets associated with diagrams for which either (1) there is at most one cell per column or (2) the first two rows are empty. Under these restrictions, we are able to find a complete characterization of when the associated Kohnert poset is (EL-)shellable. In fact, for both cases, the indexing diagram (modulo cells in the first row) must be what we call a "hook diagram" (see Theorem 20). Moreover, for diagrams of the form (1) or (2), we find that (EL-)shellability of the associated Kohnert poset is equivalent to the corresponding Kohnert polynomial being multiplicity free (see Theorem 32). With these results, it remains to consider the case of diagrams that contain at least one cell within the first two rows and at least one column with more than one cell.

Based on computational evidence, the case of Kohnert posets associated with diagrams containing at least one cell within the first two rows and at least one column with more than one cell seems much more complicated than that previously considered. Consequently, in this direction we focus on a special case of historical significance: key diagrams. The Kohnert polynomials of key diagrams are Demazure characters; this result forms the motivation for [**12**]. In the case of key diagrams, we are able to characterize when the associated Kohnert posets are graded and EL-shellable (see Theorem 41). Our characterization is in terms of an associated weak composition avoiding three different patterns. Similar to the families of diagrams discussed above, we find that there is a relationship between (EL-)shellability of the Kohnert poset and the Kohnert polynomial being multiplicity-free, though, in this case, the relationship is not as strong. In particular, we find that for key diagrams, if the Kohnert poset is graded and EL-shellable, then the associated Kohnert polynomial is multiplicity-free (see Theorem 49). On the other hand, we are also able to find an example of a key diagram for which the Kohnert poset is not shellable, but the Kohnert polynomial is multiplicity-free.

The remainder of the paper is organized as follows. In Section 2 we cover the requisite background from the theory of posets and formally define Kohnert posets and polynomials. Following this, in Section 3 we establish some structural results relevant to identifying when Kohnert posets are shellable. Then in Sections 4 and 5, we apply the aforementioned structural results to give complete characterizations of those diagrams belonging to three different families which generate (EL-)shellable Kohnert posets. Section 4 focuses on those diagrams with at most one cell per column as well as those for which the first two rows are empty. In the more complicated case of diagrams containing cells within their first two rows and at least one column with more than one cell, we consider the special case of key diagrams. For such diagrams, in Section 5 we find a complete characterization of those which generate (EL-)shellable Kohnert posets in the case that the poset is graded. In addition to the characterizations of (EL-)shellability, Sections 4 and 5 also contain results concerning the polynomial consequences of (EL-)shellability for the associated families of diagrams. Finally, in Section 6, we discuss directions for future research.

2 Preliminaries

In this section, we give the requisite preliminaries from the theory of posets and define our posets of interest.

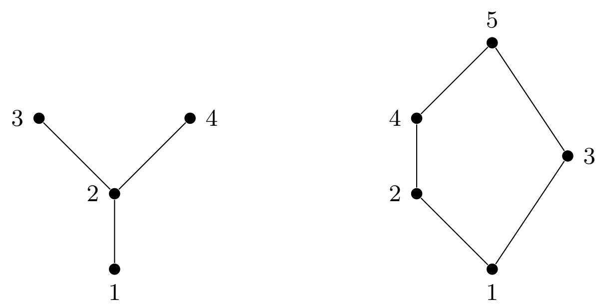
2.1 Posets

Recall that a poset $(\mathcal{P}, \preceq)$ consists of a set $\mathcal{P}$ along with a binary relation $\preceq$ between the elements of $\mathcal{P}$ which is reflexive, anti-symmetric, and transitive. When no confusion will arise, we simply denote a poset $(\mathcal{P}, \preceq)$ by $\mathcal{P}$. Two posets $\mathcal{P}$ and $\mathcal{Q}$ are *isomorphic*, denoted $\mathcal{P} \cong \mathcal{Q}$, if there exists an order-preserving bijection $\mathcal{P} \to \mathcal{Q}$. Ongoing, we assume that all posets are finite.

Let $\mathcal{P}$ be a poset and take $x, y \in \mathcal{P}$. If $x \preceq y$ and $x \neq y$, then we call $x \preceq y$ a **strict relation** and write $x \prec y$. Ongoing, we let $\leq$ and $<$ denote the relation and strict relation, respectively, corresponding to the natural ordering on $\mathbb{Z}$. For $x, y \in \mathcal{P}$ satisfying $x \preceq y$, we set $[x, y] = \{z \in \mathcal{P} \mid x \preceq z \preceq y\}$ and treat $[x, y]$ as a poset with the ordering inherited from $\mathcal{P}$; that is, for $z_1, z_2 \in [x, y]$, $z_1 \prec_{[x,y]} z_2$ if and only if $z_1 \prec_{\mathcal{P}} z_2$. If $x \prec y$ and there exists no $z \in \mathcal{P}$ satisfying $x \prec z \prec y$, then $x \prec y$ is a **covering relation**, denoted $x \prec\!\!\cdot\, y$. Covering relations are used to define a visual representation of $\mathcal{P}$ called the **Hasse diagram** – a graph whose vertices correspond to elements of $\mathcal{P}$ and whose edges correspond to covering relations (see Figure 1). We say that $x \in \mathcal{P}$ is a minimal element (resp., maximal element) if there exists no $z \in \mathcal{P}$ such that $z \prec x$ (resp., $z \succ x$). If $\mathcal{P}$ has a unique minimal and maximal element, then we say that $\mathcal{P}$ is **bounded**. The poset $\mathcal{P}$ is called **ranked** if there exists a *rank function*, i.e., a function $\rho : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ such that

1. if $x \prec y$, then $\rho(x) < \rho(y)$, and
2. if $x \prec y$ is a covering relation, then $\rho(y) = \rho(x) + 1$.

Example 1. *Let $\mathcal{P}_1 = \{1, 2, 3, 4\}$ be the poset with $1 \prec 2 \prec 3, 4$, and let $\mathcal{P}_2 = \{1, 2, 3, 4, 5\}$ be the poset with $1 \prec 2 \prec 4 \prec 5$ and $1 \prec 3 \prec 5$. The Hasse diagrams of $\mathcal{P}_1$ and $\mathcal{P}_2$ are illustrated in Figure 1. Notice that $\mathcal{P}_1$ is ranked but not bounded, while $\mathcal{P}_2$ is bounded but not ranked.*

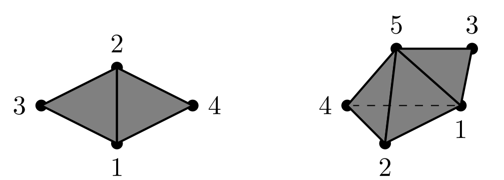


(a) (b)

Figure 1: Hasse diagrams of (a) $\mathcal{P}_1$ and (b) $\mathcal{P}_2$

A totally ordered subset of a poset $\mathcal{P}$ is called a **chain**. We call a chain $\mathcal{C}$ of $\mathcal{P}$ **maximal** if it is contained in no larger chains of $\mathcal{P}$, and we call $\mathcal{C}$ **saturated** if there does not exist $u \in \mathcal{P} \backslash \mathcal{C}$ and $s, t \in \mathcal{C}$ such that $s \prec u \prec t$ and $\mathcal{C} \cup \{u\}$ is a chain. Using the chains of a poset $\mathcal{P}$, one can define a simplicial complex $\Delta(\mathcal{P})$ associated with $\mathcal{P}$. Recall that a **(abstract) simplicial complex** Δ on a vertex set V, is a finite collection of subsets of V, called **faces**, such that $\tau \subseteq \sigma \in \Delta$ implies $\tau \in \Delta$. To define the simplicial complex $\Delta(\mathcal{P})$, we set $V = \mathcal{P}$ and $\Delta(\mathcal{P}) = \{\text{chains of } \mathcal{P}\}$.

Example 2. *The simplicial complexes associated with the posets $\mathcal{P}_1$ and $\mathcal{P}_2$ of Example 1 are illustarted in Figure 2* (a) *and* (b)*, respectively. Note that the vertices of the simplices are labelled by the corresponding elements of the posets.*



(a) (b)

Figure 2: $\Delta(\mathcal{P})$

Given a simplicial complex Δ, the dimension of a face $\sigma \in \Delta$ is defined as $\dim \sigma = |\sigma| - 1$ and the dimension of Δ is defined by $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$. If $\sigma \in \Delta$ satisfies $\dim \sigma = k$, then we refer to σ as a **k-face**; similarly, if $\dim \Delta = k$, we call Δ a **k-complex**. Any face $\sigma \in \Delta$ generates a simplicial complex $\bar{\sigma}$ consisting of σ and all of its subsets; simplicial complexes of this form are called **simplices**. A face $\sigma \in \Delta$ is called a **facet** if it is contained in no other face of Δ. We say that Δ is **pure** if all facets of Δ have the same dimension.

Definition 3. *A simplicial complex Δ is called **shellable** if its facets can be arranged into a total order $F_1, \ldots, F_t$ in such a way that the subcomplex $\left(\bigcup_{i=1}^{k-1} \bar{F}_i\right) \cap \bar{F}_k$ is pure and* $(\dim F_k - 1)$*-dimensional for $2 \leq k \leq t$. Such an ordering of facets is called a **shelling**.*

We call a poset $\mathcal{P}$ **shellable** (resp., **pure**) if $\Delta(\mathcal{P})$ is shellable (resp., pure). If a poset is finite, bounded, and pure, then we say that it is **graded**. As a consequence of the following result, one nice way to identify when a poset $\mathcal{P}$ is not shellable, i.e., when $\Delta(\mathcal{P})$ is not shellable, is by finding a non-shellable interval.

Theorem 4 (Björner and Wachs [**9**])**.** *Every interval of a shellable poset is shellable.*

Extending the work of [**5, 6, 7**], the authors of [**8, 9**] introduce a way of identifying if a bounded poset is shellable without referencing $\Delta(\mathcal{P})$. Given a poset $\mathcal{P}$, let $\mathcal{E}(\mathcal{P}) = \{(x, y) \in \mathcal{P} \times \mathcal{P} \mid x \prec y\}$, i.e., $\mathcal{E}(\mathcal{P})$ is the set of edges in the Hasse diagram of $\mathcal{P}$. An **edge labeling** of $\mathcal{P}$ is a map $\lambda : \mathcal{E}(\mathcal{P}) \to \Lambda$, where Λ is some poset. Given a saturated chain $\mathcal{C} = \{x_0 \prec x_1 \prec \cdots \prec x_n\}$ of $\mathcal{P}$ with an edge labelling λ, we define the vector

$$v(\mathcal{C}, \lambda) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \cdots, \lambda(x_{n-1}, x_n)) \in \mathbb{Z}^n$$

and call $\mathcal{C}$ **rising** if $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{n-1}, x_n)$.

Definition 5. *A bounded poset is* ***EL-shellable*** *if it admits an edge labeling $\lambda : \mathcal{E}(\mathcal{P}) \to \Lambda$ such that for every interval $[x, y]$ of $\mathcal{P}$*

1) *there is a unique rising unrefinable chain $\mathcal{C}_{[x,y]} = \{x = x_0 \prec x_1 \prec \cdots \prec x_n = y\}$ and*

2) *if $\widetilde{\mathcal{C}}$ is any other unrefinable chain between x and y, then $v(\mathcal{C}_{[x,y]})$ is lexicographically less than $v\left(\widetilde{\mathcal{C}}\right)$.*

Theorem 6 (Björner and Wachs [**8**])**.** *Let $\mathcal{P}$ be bounded. If $\mathcal{P}$ is a EL-shellable poset, then $\mathcal{P}$ is shellable.*

Considering the definition, the following EL version of Theorem 4 is immediate.

Theorem 7. *Let $\mathcal{P}$ be bounded. Every interval of an EL-shellable poset is EL-shellable.*

Having covered the necessary preliminaries of posets, we now move to defining our posets of interest.

2.2 Kohnert posets

As mentioned in the introduction, the underlying sets of Kohnert posets are certain collections of diagrams. Formally, a **diagram** is an array of finitely many cells in $\mathbb{N} \times \mathbb{N}$. An example diagram is illustrated in Figure 3 below.

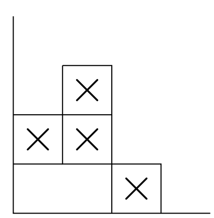


Figure 3: Diagram

We may also think of a diagram as the set of row/column coordinates of the cells defining it, where rows are labeled from bottom to top and columns from left to right. For example, if D is the diagram of Figure 3, then $D = \{(1, 3), (2, 1), (2, 2), (3, 2)\}$. Consequently, if a diagram D contains a cell in position (r, c), then we write $(r, c) \in D$; otherwise, $(r, c) \notin D$.

Remark 8. *Ongoing, when illustrating diagrams with a particular form, it will prove helpful to decorate regions to indicate a particular structure. Other than describing the properties of regions in words (usually in parentheses), regions shaded gray represent empty regions containing no cells, and regions shaded with diagonal lines will represent regions that are arbitrary, i.e., the placement of cells can be arbitrary.*

Now, to any diagram D we can apply what are called “Kohnert moves” defined as follows. For $r > 0$, applying a **Kohnert move** at row r of D results in the rightmost cell in row r of D moving to the first empty position below in the same column (if such a position exists), jumping over other cells as needed. If applying a Kohnert move at row $r > 0$ of D causes the cell in position $(r, c) \in D$ to move down to position (r', c), forming the diagram D', then we write

$$D' = D\Big\downarrow^{(r,c)}_{(r',c)}.$$

We let $KD(D)$ denote the set of all diagrams that can be obtained from D by applying a sequence of Kohnert moves. For example, in Figure 4 we illustrate the diagrams of $KD(D)$ for the diagram D of Figure 3.

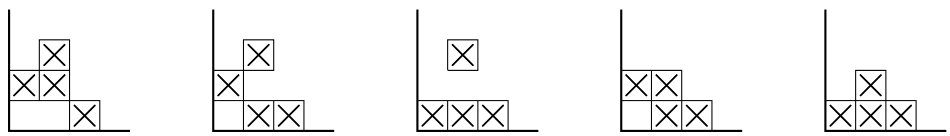


Figure 4: $KD(D)$.

The sets $KD(D)$ form the underlying sets of Kohnert posets.

Given a diagram D, the authors of **[4]** define an ordering on the elements of $KD(D)$ as follows (see also **[1]**). For $D_1, D_2 \in KD(D)$, we say $D_2 \prec D_1$ if D_2 can be obtained from D_1 by applying some sequence of Kohnert moves. For a diagram D, we denote the corresponding poset on $KD(D)$ by $\mathcal{P}(D)$ and refer to it as the **Kohnert poset** associated to D.

In the sections that follow, we study $\mathcal{P}(D)$ for various collections of diagrams D with our main concern being finding restrictions under which $\mathcal{P}(D)$ is shellable. One important family of diagrams considered below is the family of “key” diagrams.

A diagram D whose cells are left-justified is called a **key diagram**. Note that key diagrams are uniquely identified by the weak compositions corresponding to the sequences enumerating the number of cells in each row. Consequently, we denote a key diagram by $\mathbb{D}(\mathbf{a})$, where $\mathbf{a}$ is the aforementioned weak composition. For example, the diagram in Figure 5 is the key diagram $\mathbb{D}(1, 0, 3, 1, 2)$.

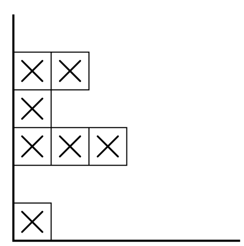


Figure 5: Key diagram

Now, since in certain cases we will see that shellability of Kohnert posets has polynomial consequences, we briefly recall the definition of “Kohnert polynomial”. Given a diagram D, setting

$$wt(D) = \prod_{i \geq 0} x_i^{\#\text{cells in row } i \text{ of D}},$$

we define the **Kohnert polynomial** associated to D as $\mathfrak{K}_D = \sum_{T \in KD(D)} wt(T)$.

3 Structural Results

In this section, we identify two necessary conditions for a diagram to generate a shellable Kohnert poset. Both conditions are in terms of avoiding certain subdiagrams that generate non-shellable intervals in the associated poset. Specifically, given a diagram that does not avoid one of the aforementioned subdiagrams, we are able to identify an interval that is isomorphic to a poset of the form described in Lemma 9 below.

Lemma 9. *Let $n_1, n_2 > 1$. If $\mathcal{P} = \{p_i^1\}_{i=1}^{n_1} \cup \{p_i^2\}_{i=1}^{n_2} \cup \{\hat{0}, \hat{1}\}$ with*

- $\hat{0} \prec p_1^1, p_1^2$,
- $p_i^1 \prec p_{i+1}^1$ *for* $1 \leq i < n_1$,
- $p_i^2 \prec p_{i+1}^2$ *for* $1 \leq i < n_2$, *and*
- $p_{n_1}^1, p_{n_2}^2 \prec \hat{1}$,

then $\mathcal{P}$ is not shellable.

Proof. Note that $\Delta(\mathcal{P})$ contains exactly two facets

$$F_1 = \{\hat{0}, p_1^1, \ldots, p_{n_1}^1, \hat{1}\}$$

and

$$F_2 = \{\hat{0}, p_1^2, \ldots, p_{n_2}^2, \hat{1}\}.$$

Since

$$\dim F_1 - 1 = n_1 \neq \dim \bar{F}_1 \cap \bar{F}_2 = 1 \neq n_2 = \dim F_2 - 1,$$

by definition $\Delta(\mathcal{P})$ is not shellable. □

Example 10. *Consider the poset $\mathcal{P} = \{1, 2, 3, 4, 5, 6\}$ with $1 \prec 2 \prec 4 \prec 6$ and $1 \prec 3 \prec 5 \prec 6$. The Hasse diagram and simplicial complex of $\mathcal{P}$ are illustrated in Figure 6 (a) and (b), respectively. It follows from Lemma 9 (with $n_1 = n_2 = 2$) that $\mathcal{P}$ is not shellable.*

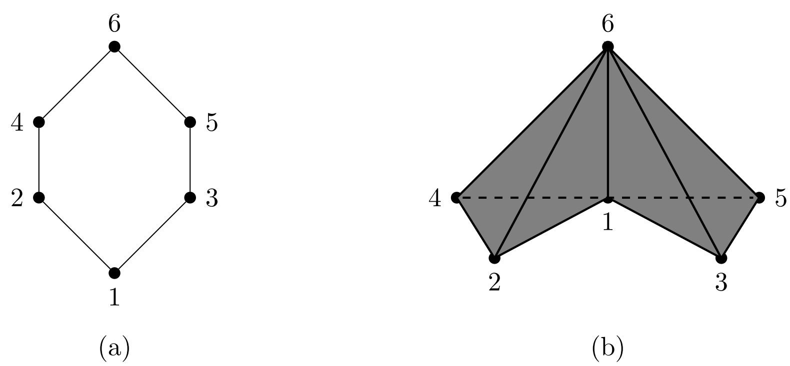


Figure 6: (a) Hasse diagram of $\mathcal{P}$ and (b) $\Delta(\mathcal{P})$

Now, to aid in identifying the non-shellable intervals in the Kohnert posets of Propositions 14 and 15 below, we require the following lemma.

Lemma 11. *Let D be a diagram and $D_1, D_2 \in \mathcal{P}(D)$ satisfy $D_2 \prec D_1$. If there exist $0 < c_1 < c_2 < \cdots < c_n$ and $0 < r_1^i < r_2^i$ for $1 \leq i \leq n$ such that*

$$S = D_1 \backslash \bigcup_{i=1}^{n} \{(\tilde{r}, c_i) \mid r_1^i \leq \tilde{r} \leq r_2^i\} = D_2 \backslash \bigcup_{i=1}^{n} \{(\tilde{r}, c_i) \mid r_1^i \leq \tilde{r} \leq r_2^i\},$$

then

$$\tilde{D} \backslash \bigcup_{i=1}^{n} \{(\tilde{r}, c_i) \mid r_1^i \leq \tilde{r} \leq r_2^i\} = S$$

for all $\tilde{D} \in [D_2, D_1]$.

Proof. Let $R = \bigcup_{i=1}^{n} \{(\tilde{r}, c_i) \mid r_1^i \leq \tilde{r} \leq r_2^i\}$. Assume for a contradiction that there exists $D^* \in [D_2, D_1]$ such that $D^* \backslash R \neq S$. Then there are two cases.

Case 1: There exists $c^*, r^* > 0$ such that $(r^*, c^*) \in D^* \backslash R$ and $(r^*, c^*) \notin S$. Since $D^* \preceq D_1$, $D_1 \backslash R = S$, and nontrivial Kohnert moves result in cells moving to lower rows, it must be the case that

$$|\{(\tilde{r}, c^*) \in D_1 \mid \tilde{r} > r^*\}| > |\{(\tilde{r}, c^*) \in D^* \mid \tilde{r} > r^*\}|.$$

Moreover, since $D_2 \prec D_1$, $D_1 \backslash R = D_2 \backslash R$, and $(r^*, c^*) \notin R$, it follows that

$$|\{(\tilde{r}, c^*) \in D_2 \mid \tilde{r} > r^*\}| = |\{(\tilde{r}, c^*) \in D_1 \mid \tilde{r} > r^*\}| > |\{(\tilde{r}, c^*) \in D^* \mid \tilde{r} > r^*\}|,$$

i.e., there are more cells strictly above row r^* in column c^* of D_2 than in D^*. As the number of cells in a given column above a given row can only decrease upon applying Kohnert moves, it follows that $D_2 \npreceq D^*$, a contradiction.

Case 2: There exists $c^*, r^* > 0$ such that $(r^*, c^*) \notin D^* \backslash R$ and $(r^*, c^*) \in S$. Since $D^* \preceq D_1$, $D_1 \backslash R = S$, and nontrivial Kohnert moves result in cells moving to lower rows, it must be the case that

$$|\{(\tilde{r}, c^*) \in D_1 \mid \tilde{r} < r^*\}| < |\{(\tilde{r}, c^*) \in D^* \mid \tilde{r} < r^*\}|.$$

As in Case 1, we may conclude further that

$$|\{(\tilde{r}, c^*) \in D_2 \mid \tilde{r} < r^*\}| = |\{(\tilde{r}, c^*) \in D_1 \mid \tilde{r} < r^*\}| < |\{(\tilde{r}, c^*) \in D^* \mid \tilde{r} < r^*\}|,$$

i.e., there are less cells strictly below row r^* in column c^* of D_2 than in D^*. As the number of cells in a given column below a given row can only increase upon applying Kohnert moves, it follows that $D_2 \npreceq D^*$, a contradiction. □

Remark 12. *Lemma 11 is a slight strengthening of Lemma 3.3 in* **[10]**.

Corollary 13. *Let D be a diagram. Suppose that for $D_1, D_2 \in \mathcal{P}(D)$ there exists $1 \leq j < r$ and $c > 0$ such that $D_2 \prec D_1$ and $D_2 = D_1 \Big\downarrow^{(r,c)}_{(r-j,c)}$. Then $D_2 \prec\!\!\cdot\, D_1$ if and only if for each $\tilde{r}$ satisfying $r - j < \tilde{r} < r$ there exists $\tilde{c} > c$ such that $(\tilde{r}, \tilde{c}) \in D_1$.*

Proof. First, assume that $D_2 \prec\!\!\cdot\, D_1$. Then D_2 must be formed from D_1 by applying a single Kohnert move at row r so that $(\tilde{r}, c) \in D_1$ for $r - j + 1 \leq \tilde{r} \leq r$. Now, if there exists $\widehat{r}$ satisfying $r - j < \widehat{r} < r$ and $(\widehat{r}, \tilde{c}) \notin D_1$ for all $\tilde{c} > c$, then applying a single Kohnert move at row $\widehat{r}$ of D_1 results in $D_{1.5} = D_1 \Big\downarrow^{(\widehat{r},c)}_{(r-j,c)} \in \mathcal{P}(D)$. Moreover, applying a Kohnert move at row r of $D_{1.5}$ results in $D_{1.5} \Big\downarrow^{(r,c)}_{(\widehat{r},c)} = D_2$, i.e., $D_2 \prec D_{1.5} \prec D_1$, contradicting our assumption that $D_2 \prec\!\!\cdot\, D_1$. Thus, for each $\tilde{r}$ satisfying $r - j < \tilde{r} < r$, there exists $\tilde{c} > c$ such that $(\tilde{r}, \tilde{c}) \in D_1$.

Now, assume that, for each $\tilde{r}$ satisfying $r-j<\tilde{r}<r$, there exists $\tilde{c}>c$ such that $(\tilde{r},\tilde{c})\in D_1$. Take $D_{1.5}\in[D_2,D_1]$ such that $D_{1.5}\prec D_1$. Assume that $D_{1.5}$ can be formed from D_1 by applying a Kohnert move at row $\widehat{r}$. Note that D_1 and D_2 differ only in positions (r,c) and $(r-j,c)$. Thus, if $R=\{(\tilde{r},c)\mid r-j\leq\tilde{r}\leq r\}$, then $D_1\backslash R=D_2\backslash R=D_{1.5}\backslash R$ by Lemma 11. Since, for each $\tilde{r}$ satisfying $r-j<\tilde{r}<r$, there exists $\tilde{c}>c$ such that $(\tilde{r},\tilde{c})\in D_1$, it follows that $\widehat{r}=r$ or $r-j$; but applying a Kohnert move at row r of D_1 results in D_2, while applying a Kohnert move at row $r-j$ either does nothing or affects a cell in a column $\tilde{c}<c$, contradicting $D_1\backslash R=D_{1.5}\backslash R$. Therefore, we may conclude that $D_{1.5}=D_2$, i.e., $D_2\prec D_1$. The result follows. □

With Lemma 11 in hand, we can now prove Propositions 14 and 15 which provide necessary conditions for a diagram to generate a shellable Kohnert poset.

Proposition 14. *Let D be a diagram. Suppose that there exists $D^*\in\mathcal{P}(D)$ and $r,c_1,c_2\in\mathbb{N}$ such that*

(i) $c_1<c_2$,

(ii) $(r+1,c_1),(r+2,c_2)\in D^*$,

(iii) $(r+2,\tilde{c}),(r,c_2)\notin D^*$ *for* $\tilde{c}>c_2$, *and*

(iv) $(r+1,\tilde{c}),(r,c_1)\notin D^*$ *for* $\tilde{c}>c_1$.

Then $\mathcal{P}(D)$ is not shellable. (See Figure 7 for an illustration of D^*.)

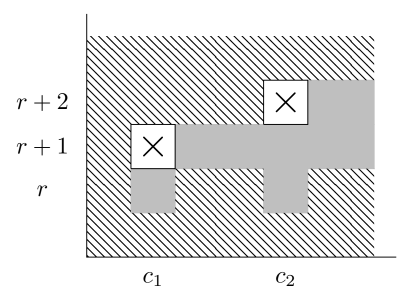


Figure 7: Subdiagrams described in Proposition 14

Proof. Considering Theorem 4, it suffices to show that $\mathcal{P}(D)$ contains an interval which is not shellable. Let

$$\widehat{D}=(D^*\setminus\{(r+1,c_1),(r+2,c_2)\})\cup\{(r,c_1),(r,c_2)\}.$$

Note that $\widehat{D}\in\mathcal{P}(D)$ since $\widehat{D}$ can be formed from D^* by applying a Kohnert move at row $r+2$ of D^* followed by two Kohnert moves at row $r+1$. We claim that the interval $[\widehat{D},D^*]$ is not shellable. To establish the claim, we first determine the elements of $[\widehat{D},D^*]$.

Note that D^* and $\widehat{D}$ differ only in positions $\{(r+1,c_1),(r+2,c_2),(r,c_1),(r,c_2)\}$; that is, letting $R=\{(\tilde{r},c_1)\mid r\leq\tilde{r}\leq r+1\}\cup\{(\tilde{r},c_2)\mid r\leq\tilde{r}\leq r+2\}$, we have that $D^*\backslash R=\widehat{D}\backslash R=S$. Thus, applying Lemma 11, it follows that

$$\tilde{D}\backslash R=S$$

for all $\tilde{D}\in[\widehat{D},D^*]$. Note that this implies that all diagrams $\tilde{D}\in[\widehat{D},D^*]$ must be formed from D^* by applying sequences of Kohnert moves at rows $r+1$ or $r+2$. In particular, it is straightforward to verify, keeping Lemma 11 in mind, that the only diagrams contained in $[\widehat{D},D^*]$ are

$$D^*,\qquad D_1^1=D^*\Big\downarrow_{(r+1,c_2)}^{(r+2,c_2)},\qquad D_2^1=D_1^1\Big\downarrow_{(r,c_2)}^{(r+1,c_2)},\qquad D_1^2=D^*\Big\downarrow_{(r,c_1)}^{(r+1,c_1)},$$

$$D_2^2 = D_1^2\Big\downarrow_{(r+1,c_2)}^{(r+2,c_2)}, \qquad \text{and} \qquad \widehat{D}.$$

Now, to determine how the diagrams of $\left[\widehat{D}, D^*\right]$ are related, note that we can form $\widehat{D}$ from D^* by either applying in succession

1) a Kohnert move at row $r+2$ followed by two Kohnert moves at row $r+1$, or

2) a Kohnert move at row $r+1$, a Kohnert move at row $r+2$, then a Kohnert move at row $r+1$.

The first sequence of Kohnert moves described above corresponds to the chain

$$\widehat{D} \prec D_2^1 \prec D_1^1 \prec D^*$$

while the second sequence corresponds to

$$\widehat{D} \prec D_2^2 \prec D_1^2 \prec D^*.$$

The fact that all relations in the two chains are covering relations follows from Corollary 13. We claim that no other relations exist between the elements of $[\widehat{D}, D^*]$. To see this, first note that since $D_1^2, D_1^1 \prec D^*$ and $D_1^2 \neq D_1^1$, it follows that D_1^1 and D_1^2 are unrelated in $[\widehat{D}, D^*]$. Similarly, since $\hat{D} \prec D_2^1, D_2^2$ and $D_2^1 \neq D_2^2$, it follows that D_2^1 and D_2^2 are unrelated in $[\widehat{D}, D^*]$. It remains to consider relations between the elements D_1^1 and D_2^2, and the elements D_1^2 and D_2^1. Starting with D_1^1 and D_2^2, note that since $D_1^1 \succ \widehat{D} \prec D_2^2$, if D_1^1 and D_2^2 are related, then $D_2^2 \prec D_1^1$. Now, since $D_1^1\backslash\{(r+1,c_1),(r,c_1)\} = D_2^2\backslash\{(r+1,c_1),(r,c_1)\}$, if $D_2^2 \prec D_1^1$, then applying Lemma 11 it follows that there exists a sequence of Kohnert moves at rows r and $r+1$ which takes D_1^1 to D_2^2. Evidently, such a sequence of Kohnert moves does not exist. Consequently, D_1^1 and D_2^2 are unrelated in $[\widehat{D}, D^*]$. Moving to D_1^2 and D_2^1, since $D_1^2 \succ \widehat{D} \prec D_2^1$, if D_1^2 and D_2^1 are related, then $D_2^1 \prec D_1^2$; but

$$D_1^2\backslash\{(r+2,c_2),(r+1,c_2),(r,c_2)\} = \widehat{D}\backslash\{(r+2,c_2),(r+1,c_2),(r,c_2)\} \neq D_2^1\backslash\{(r+2,c_2),(r+1,c_2),(r,c_2)\},$$

so that applying Lemma 11 we have $D_2^1 \notin [\widehat{D}, D_1^2]$. Thus, since $\widehat{D} \prec D_2^1$, it follows that $D_2^1 \not\prec D_1^2$, i.e., D_1^2 and D_2^1 are unrelated in $[\widehat{D}, D^*]$, establishing the claim.

Now, let $\mathcal{S}$ be the poset of Example 10, and define the map $f : [\widehat{D}, D^*] \to \mathcal{S}$ by $f(\widehat{D}) = 1$, $f(D_2^1) = 2$, $f(D_1^1) = 3$, $f(D_2^2) = 4$, $f(D_1^2) = 5$, and $f(D^*) = 6$. Considering our work above, it follows that f forms an order-preserving bijection. Therefore, $[\widehat{D}, D^*]$ is not shellable. The result follows. $\square$

Proposition 15. *Let D be a diagram. Suppose that there exists $D^* \in \mathcal{P}(D)$ such that for some $r, r^*, c, c^* \in \mathbb{N}$ satisfying $c < c^*-1$ and $r < r^*-1$ we have*

(i) *$(r^*, c^*) \in D^*$ and $(r^*, \tilde{c}) \notin D^*$ for all $\tilde{c} > c^*$;*

(ii) *$|\{(r^*, \tilde{c}) \in D^* \mid c < \tilde{c} < c^*\}| > 0$;*

(iii) *for $r < \tilde{r} \leq r^*$, $(\tilde{r}, c) \in D^*$;*

(iv) *$(r, c), (r^*-1, \tilde{c}) \notin D^*$ for $\tilde{c} > c$; and*

(v) *for each $\tilde{r}$ satisfying $r < \tilde{r} < r^*-1$ there exists $\tilde{c} > c$ such that $(\tilde{r}, \tilde{c}) \in D^*$.*

Then $\mathcal{P}(D)$ is not shellable. (See Figure 8 for an illustration of D^*.)

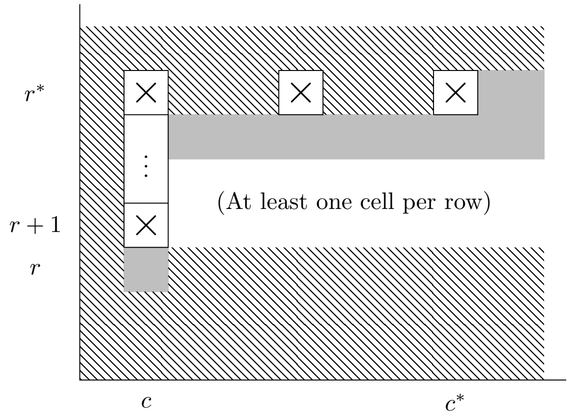


Figure 8: Subdiagrams described in Proposition 15

Proof. As in the proof of Proposition 14, we show that there exists a diagram $\widehat{D} \in \mathcal{P}(D)$ such that $[\widehat{D}, D^*]$ is isomorphic to a poset of the form described in Lemma 9 so that, by Theorem 4, $\mathcal{P}(D)$ is not shellable. To define $\widehat{D}$, let

$$C = \{\tilde{c} \mid c \leq \tilde{c} \leq c^* \text{ and } (r^*, \tilde{c}) \in D_{max}\} = \{c = c_0 < c_1 < \cdots < c_{m-1} = c^*\},$$

where, by assumption, $m \geq 3$. Then $\widehat{D}$ is the diagram obtained from D^* by applying m Kohnert moves at row r^*; that is, $\widehat{D}$ is the diagram obtained from D^* by moving the rightmost $m-1$ cells in row r^* down to row r^*-1, and moving the m^{th} cell from right to left in row r^* down to row r.

First, to determine the elements of $[\widehat{D}, D^*]$, consider the following two chains from $\widehat{D}$ to D^*, both defined by the sequences of Kohnert moves applied to form $\widehat{D}$ from D^*.

1) Form the chain $\mathcal{C}_1$ by applying m Kohnert moves at row r^*, i.e.,

$$\mathcal{C}_1 : \qquad D^* \succ D_1^1 \succ D_2^1 \succ \cdots \succ D_{m-1}^1 \succ \widehat{D},$$

where $D_1^1 := D^* \Big\downarrow_{(r^*-1, c_{m-1})}^{(r^*, c_{m-1})}$, $D_{i+1}^1 := D_i^1 \Big\downarrow_{(r^*-1, c_{m-i-1})}^{(r^*, c_{m-i-1})}$ for $1 \leq i \leq m-2$, and $\widehat{D} = D_{m-1}^1 \Big\downarrow_{(r, c_0)}^{(r^*, c_0)}$.

2) Form the chain $\mathcal{C}_2$ by applying one Kohnert move at row r^*-1 followed by m Kohnert moves at row r^*, i.e.,

$$\mathcal{C}_2 : \qquad D^* \succ D_1^2 \succ D_2^2 \succ \cdots \succ D_m^2 \succ \widehat{D},$$

where $D_1^2 := D^* \Big\downarrow_{(r, c_0)}^{(r^*-1, c_0)}$, $D_{i+1}^2 := D_i^2 \Big\downarrow_{(r^*-1, c_{m-i})}^{(r^*, c_{m-i})}$ for $1 \leq i \leq m-1$, and $\widehat{D} = D_m^2 \Big\downarrow_{(r^*-1, c_0)}^{(r^*, c_0)}$.

We claim that the elements contained in $\mathcal{C}_1$ and $\mathcal{C}_2$ constitute the elements of $[\widehat{D}, D^*]$. To see this, set $R = \{(\tilde{r}, c) \mid r \leq \tilde{r} \leq r^*\} \cup \bigcup_{i=1}^{m-1} \{(r^*, c_i), (r^*-1, c_i)\}$ and note that, applying Lemma 11, we have $D^* \backslash R = \widehat{D} \backslash R = \tilde{D} \backslash R$ for all $\tilde{D} \in [\widehat{D}, D^*]$. Thus, every $\tilde{D} \in [\widehat{D}, D^*]$ must satisfy property (v) of D^*. Consequently, applying a nontrivial Kohnert move to $\tilde{D} \in [\widehat{D}, D^*]$ at any row other than r^*-1 or r^* must affect the position of a cell outside of R; but, since $D^* \backslash R = \widehat{D} \backslash R = \tilde{D} \backslash R$, this implies that $\tilde{D} \in [\widehat{D}, D^*]$ must be formed from D^* by a sequence of Kohnert moves at rows r^*-1 and r^*. Under these restrictions, it is straightforward to verify the claim.

Now, we determine the relations defining the poset $[\widehat{D}, D^*]$. Applying Corollary 13, it follows that

$$\widehat{D} \prec D_{m-1}^1 \prec \cdots \prec D_1^1 \prec D^*$$

and

$$\widehat{D} \prec D_m^2 \prec \cdots \prec D_1^2 \prec D^*.$$

It remains to consider relations between D_i^1 and D_j^2 for $1 \leq i \leq m-1$ and $1 \leq j \leq m$. We show that there are no such relations. Let

$$R_i^1 = \{(r^*, c_k), (r^*-1, c_k) \mid m-i \leq k \leq m-1\}$$

for $1 \leq i \leq m-1$. Since

$$(r,c) \notin D^* \backslash R_i^1 = D_i^1 \backslash R_i^1 \neq D_j^2 \backslash R_i^1 \ni (r,c)$$

for $1 \leq i \leq m-1$ and $1 \leq j \leq m$, it follows from Lemma 11 that $D_j^2 \notin [D_i^1, D^*]$ for $1 \leq i \leq m-1$ and $1 \leq j \leq m$; that is, $D_i^1 \not\prec D_j^2$ for $1 \leq i \leq m-1$ and $1 \leq j \leq m$. Consequently, if D_j^2 and D_i^1 are related for some choice of $1 \leq i \leq m-1$ and $1 \leq j \leq m$, then $D_j^2 \prec D_i^1$. For a contradiction, assume that $D_j^2 \prec D_i^1$ for some choice of $1 \leq i \leq m-1$ and $1 \leq j \leq m$. Then $D_m^2 \prec D_i^1$. Letting

$$R_i^2 = \{(r^*, c_k), (r^*-1, c_k) \mid 1 \leq k < m-i\} \cup \{(\tilde{r}, c) \mid r \leq \tilde{r} \leq r^*-1\}$$

for $1 \leq i \leq m-1$, we have that

$$D_m^2 \backslash R_i^2 = D_i^1 \backslash R_i^2$$

for $1 \leq i \leq m-1$. Thus, since D_i^1 satisfies property (v) of D^* as noted above, it follows from Lemma 11 that all diagrams in $[D_m^2, D_i^1]$ – in particular, D_m^2 – must be formed from D_i^1 by applying sequences of Kohnert moves at rows r^* or r; but applying a Kohnert move at row r^* results in D_{min} which is not contained in $[D_m^2, D_i^1]$, while applying a Kohnert move at row r either does nothing or affects a cell in some column $\tilde{c} < c$, resulting in a diagram not contained in $[D_m^2, D_i^1]$, by Lemma 11. Therefore, $D_m^2 \not\prec D_i^1$ and we conclude that there are no relations between D_i^1 and D_j^2 for $1 \leq i \leq m-1$ and $1 \leq j \leq m$.

Finally, above we showed that the poset $[\widehat{D}, D^*]$ is completely defined by the relations

$$D^* \succ D_1^1 \succ D_2^1 \succ \cdots \succ D_{m-1}^1 \succ \widehat{D}$$

and

$$D^* \succ D_1^2 \succ D_2^2 \succ \cdots \succ D_m^2 \succ \widehat{D}.$$

Let $\mathcal{P}_m$ be a poset of the form described in Lemma 9 with $n_1 = m-1$ and $n_2 = m$, i.e., $\mathcal{P}_m = \{p_i^1\}_{i=1}^{m-1} \cup \{p_i^2\}_{i=1}^{m} \cup \{\hat{0}, \hat{1}\}$ with

- $\hat{0} \prec p_1^1, p_1^2$,
- $p_i^1 \prec p_{i+1}^1$ for $1 \leq i < m-1$,
- $p_i^2 \prec p_{i+1}^2$ for $1 \leq i < m$, and
- $p_{m-1}^1, p_m^2 \prec \hat{1}$.

Define the map $f : [\widehat{D}, D^*] \to \mathcal{P}_m$ by $f(\widehat{D}) = \hat{0}$, $f(D_i^1) = p_{m-i}^1$ for $1 \leq i \leq m-1$, $f(D_j^2) = p_{m-j+1}^2$ for $1 \leq j \leq m$, and $f(D^*) = \hat{1}$. Considering our work above, it follows that f is an order-preserving bijection. Therefore, $[\widehat{D}, D^*]$ is not shellable. The result follows. □

Ongoing, we will not require the full strength of Proposition 15, but instead, we make use of the following special case.

Corollary 16. *Let D be a diagram. Suppose that there exists $D^* \in \mathcal{P}(D)$ such that for some $r, c_1, c_2 \in \mathbb{N}$ satisfying $c_1 < c_2$ we have*

(i) $(r+1, c_1), (r+2, c_1), (r+2, c_2) \in D^*$,

(ii) $|\{(r+2,\tilde{c}) \in D^* \mid c_1 < c < c_2\}| > 0$

(iii) $(r+2,\tilde{c}) \notin D^*$ *for* $\tilde{c} > c_2$,

(iv) $(r+1,\tilde{c}) \notin D^*$ *for* $\tilde{c} > c_1$, *and*

(v) $(r, c_1) \notin D^*$.

Then $\mathcal{P}(D)$ *is not shellable.* (*See Figure 9 for an illustration of* D^*.)

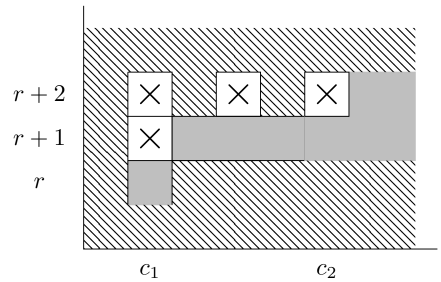


Figure 9: Subdiagrams described in Corollary 16

Proof. This result corresponds to taking $r^* = r+2$ in Proposition 15. □

Remark 17. *The families of subdiagrams considered in Proposition 15 and Corollary 16 form a subset of those considered in Theorem 3.5 and Corollary 3.6, respectively, of* **[10]**.

In the sections that follow, we utilize the above results to characterize when diagrams belonging to restricted families are associated with (EL-)shellable Kohnert posets.

4 Hook diagrams

In this section, we give a complete classification of diagrams D for which $\mathcal{P}(D)$ is (EL-)shellable and either

1) each nonempty column of D contains exactly one cell or

2) the first two rows of D are empty.

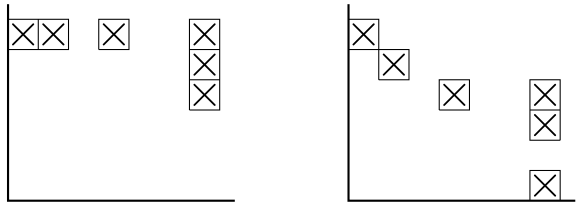
In both cases, the classification is given in terms of certain diagrams that we call "hook diagrams".

For $r_1 \leq r_2 \in \mathbb{Z}_{>0}$ and $C = \{c_1, \cdots, c_m\} \subset \mathbb{Z}_{>0}$, if $c_m = \max C$, then

$$H(r_1, r_2; C) = \{(r_2, c_i) \mid 1 \leq i \leq m\} \cup \{(j, c_m) \mid r_1 \leq j \leq r_2\}.$$

A **hook diagram** D is a diagram for which there exists $r_1, r_2 \in \mathbb{Z}_{>0}$ and $C \subset \mathbb{Z}_{>0}$ such that $D \in KD(H(r_1, r_2; C))$.

Example 18. *In Figure 10* (a) *we illustrate* $H(4, 6; \{1, 2, 4, 7\})$ *and in Figure 10* (b) *a hook diagram* $D \in KD(H(4, 6; \{1, 2, 4, 7\}))$.



(a) (b)

Figure 10: (a) $H(4,6;\{1,2,4,7\})$ and (b) a hook diagram $D \in KD(H(4,6;\{1,2,4,7\}))$

The following lemma provides a characterization of hook diagrams in terms of the distribution of their cells.

Proposition 19. *Let D be a diagram. Then D is a hook diagram if and only if there exists $c^* > 0$ such that*

(i) *column c^* of D is nonempty and for all $\tilde{c} > c^*$, column $\tilde{c}$ of D is empty;*

(ii) *if D contains more than one cell in column c, then $c = c^*$;*

(iii) *if r is maximal such that $(r, c^*) \in D$ and $\tilde{c} < c$, then $(\tilde{r}, \tilde{c}) \in D$ implies that $\tilde{r} \geq r$; and*

(iv) *if $(\tilde{r}_1, \tilde{c}_1), (\tilde{r}_2, \tilde{c}_2) \in D$ with $\tilde{c}_1 < \tilde{c}_2 < c^*$, then $\tilde{r}_1 \geq \tilde{r}_2$.*

Proof. Assume that D is a hook diagram. Then there exists $r_1 \leq r_2 \in \mathbb{Z}_{>0}$ and $C = \{c_1, \cdots, c_m\} \subset \mathbb{Z}_{>0}$ such that $D \in KD(H(r_1, r_2; C))$. Evidently, if $c^* = \max C$, then properties (i) and (ii) hold in D. It remains to show that properties (iii) and (iv) also hold in D. We show that D satisfies (iii), as (iv) follows via a similar argument. For a contradiction, assume that D does not satisfy property (iii). Ongoing, for $\tilde{D} \in KD(H(r_1, r_2; C))$, we let $r^*(\tilde{D}) = \max\{r \mid (r, c^*) \in \tilde{D}\}$. Since D does not satisfy property (iii), there exists $(r, c) \in D$ such that $c < c^*$ and $r < r^*(D)$. Consequently, since $(r_2, c) \in H(r_1, r_2; C)$ and $r^*(H(r_1, r_2; C)) = r_2$, it follows that there must exist $D_1, D_2 \in KD(H(r_1, r_2; C))$ such that $(\tilde{r}_1, c) \in D_1$ with $\tilde{r}_1 \geq r^*(D_1)$, $(\tilde{r}_2, c) \in D_2$ with $\tilde{r}_2 < r^*(D_2)$, and D_2 can be formed from D_1 by applying a single Kohnert move; that is, in forming D from $H(r_1, r_2; C)$, there must be a point at which a diagram that does not satisfy property (iii) in column c, namely D_2, is formed from one that does, namely D_1. Now, because $\tilde{r}_1 \geq r^*(D_1) \geq r^*(D_2) > \tilde{r}_2$ and D_2 is obtained from D_1 by applying a single Kohnert move, it follows that $D_2 = D_1\Big\downarrow^{(\tilde{r}_1, c)}_{(\tilde{r}_2, c)}$. Further, it must be the case that $\tilde{r}_2 = \tilde{r}_1 - 1$ since all diagrams in $KD(H(r_1, r_2; C))$ contain a single cell in column c. However, this implies that $\tilde{r}_1 = r^*(D_1) = r^*(D_2)$; that is, $(\tilde{r}_1, c)$ is not the rightmost cell in row $\tilde{r}_1$ of D_1, contradicting that $D_2 = D_1\Big\downarrow^{(\tilde{r}_1, c)}_{(\tilde{r}_2, c)}$. Thus, D must satisfy property (iii). For property (iv), one can use almost the exact same argument as that given above, replacing $(\tilde{r}_1, c), (r^*(D_1), c^*) \in D_1$ and $(\tilde{r}_2, c), (r^*(D_2), c^*) \in D_2$ with $(\tilde{r}_1, \tilde{c}_1), (\tilde{r}'_1, \tilde{c}_2) \in D_1$ and $(\tilde{r}_2, \tilde{c}_1), (\tilde{r}'_2, \tilde{c}_2) \in D_2$, respectively, where $\tilde{c}_1 < \tilde{c}_2$, $\tilde{r}_1 \geq \tilde{r}'_1$, and $\tilde{r}_2 < \tilde{r}'_2$.

Now, for the backward direction, assume that D contains $m > 1$ nonempty columns and that column c^* of D contains more than one cell; the cases where $m = 1$ and/or nonempty columns of D contain exactly one cell follow via similar – but simpler – arguments. Let $r^1_1 \leq \cdots \leq r^1_{n_1}$ be the nonempty rows of columns $\tilde{c} < c^*$ of D and $r^2_1 < \cdots < r^2_{n_2}$ be the nonempty rows of D in column c^*. Note that $r^1_1 \geq r^2_{n_2}$ by condition (iii). If C denotes the set of nonempty columns of D, then we claim that $D \in KD(H(r^1_{n_1} - n_2 + 1, r^1_{n_1}; C))$. To see this, note that we can form D from $H(r^1_{n_1} - n_2 + 1, r^1_{n_1}; C)$ as follows.

1. For $1 \leq i \leq n_2$ in increasing order, if $r^1_{n_1} - n_2 + i \neq r^2_i$, then apply in succession one Kohnert move at rows $r^1_{n_1} - n_2 + i$ down to $r^2_i + 1$ in decreasing order; otherwise, apply no Kohnert moves.

2. For $1 \leq i \leq n_1$ in increasing order, if $r^1_{n_1} \neq r^1_i$, then apply in succession one Kohnert move at rows $r^1_{n_1}$ down to $r^1_i + 1$ in decreasing order; otherwise apply no Kohnert moves.

Thus, $D \in KD(H(r^1_{n_1} - n_2 + 1, r^1_{n_1}; C))$ and the result follows. □

Theorem 20 below is the main result of this section, and its proof – along with some noteworthy consequences regarding the corresponding Kohnert polynomials – constitutes the remainder of this section.

Theorem 20.

(a) *Let D be a diagram for which all nonempty columns contain exactly one cell. Then $\mathcal{P}(D)$ is (EL-)shellable if and only if $D \backslash \{(1, \tilde{c}) \mid \tilde{c} > 0\}$ is a hook diagram.*

(b) *Let D be a diagram for which the first two rows are empty. Then $\mathcal{P}(D)$ is (EL-)shellable if and only if D is a hook diagram.*

Before proceeding with the proof of Theorem 20, we include the following lemma which relates the Kohnert posets of the two diagrams occurring in Theorem 20 (a).

Lemma 21. *If D is a diagram for which all nonempty columns contain exactly one cell, then $\mathcal{P}(D) \cong \mathcal{P}(D \backslash \{(1, \tilde{c}) \mid \tilde{c} > 0\})$.*

Proof. Immediate from the definitions of Kohnert move and Kohnert poset. □

4.1 Sufficiency

In this section, we establish the backward directions of Theorem 20 (a) and (b). Considering Lemma 21, it suffices to show that if D is a hook diagram, then $\mathcal{P}(D)$ is EL-shellable.

To show that hook diagrams generate EL-shellable Kohnert posets, we first establish that such posets are bounded.

Lemma 22. *If D is a hook diagram, then $\mathcal{P}(D)$ is bounded.*

Proof. It suffices to establish the result for $D = H(r_1, r_2; C)$ with $r_1 \leq r_2 \in \mathbb{Z}_{>0}$ and $C = \{c_1, \cdots, c_m\} \subset \mathbb{Z}_{>0}$. We claim that $D_{min} = H(1, r_2 - r_1 + 1; C)$ is the unique minimal element of $\mathcal{P}(D)$. To see this, take $\tilde{D} \in \mathcal{P}(H(r_1, r_2; C))$. If $|C| = 1$ or $r_1 = r_2$, then the result follows by Corollary 6.2 of **[10]**. So, assume that $|C| = m > 1$, $r_2 > r_1$, and $c_1 < \cdots < c_m$. Considering Proposition 19, there exists $\tilde{r}_{m-1} \geq \cdots \geq \tilde{r}_1$ and $r^*_1 < \cdots < r^*_{r_2 - r_1 + 1}$ such that

$$\tilde{D} = \{(\tilde{r}_{m-i}, c_i) \mid 1 \leq i \leq m - 1\} \cup \{(r^*_i, c_m) \mid 1 \leq i \leq r_2 - r_1 + 1\}.$$

Note that one can form D_{min} from $\tilde{D}$ by applying successively

1) for $1 \leq i \leq r_2 - r_1 + 1$ in increasing order, one Kohnert move at rows r^*_i through $i + 1$ in decreasing order, followed by

2) for $1 \leq i \leq m - 1$ in increasing order, one Kohnert move at rows $\tilde{r}_i$ through $r_2 - r_1 + 2$ in decreasing order.

Consequently, $D_{min} \preceq \tilde{D}$. As $\tilde{D}$ was arbitrary, the claim and, hence, the result follows. □

Example 23. *In Figure 11 below we illustrate the unique minimal element D_{min} of the Kohnert posets associated with the two hook diagrams of Example 18.*

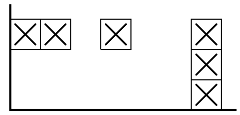


Figure 11: Minimal element associated with a hook diagram

In order to establish that Kohnert posets associated with hook diagrams are EL-shellable, we describe an edge labeling that satisfies Definition 5. First, however, we require the following lemma.

Lemma 24. *Let D be a hook diagram. If $D' = D\Big\downarrow_{(r-k,c)}^{(r,c)}$ and $D' \prec D$, then $k = 1$.*

Proof. Assume otherwise. Without loss of generality, let $D' = D\Big\downarrow_{(r-k,c)}^{(r,c)}$ with $k = 2$. Note that, since $D' \prec D$, it must be the case that D' can only be formed from D by applying exactly one Kohnert move. Consequently, it follows that $(r-1,c) \in D$. Now, since $(r,c),(r-1,c) \in D$, we may conclude that c is the unique column in D with more than one cell. So, by condition (i) of Proposition 19, for all $\tilde{c} > c$, column $\tilde{c}$ of D is empty. Thus, D' can be formed from D by applying a single Kohnert move at row $r-1$ followed by a single Kohnert move at row r; that is

$$D' \prec D\Big\downarrow_{(r-2,c)}^{(r-1,c)} \prec D,$$

a contradiction. The result follows. □

To define our edge labelings for Kohnert posets arising from hook diagrams, we utilize the following labeling of the cells in the associated diagrams. Let D be a hook diagram where $C = \{c_1, \ldots, c_m\}$ is the set of nonempty columns in D with $c_1 < \cdots < c_m$, and let $R = \{r_i \mid (r_i, c_m) \in D \text{ for } 1 \le i \le n\}$ with $r_1 > \cdots > r_n$. Decorate the cell located in column c_j of D with the label j for $1 \le j < m$, then decorate the cell in column c_m and row r_i of D with the label $m - i + 1$ for $1 \le i \le n$. To extend this labeling to remaining diagrams of $\mathcal{P}(D)$, if $D_1, D_2 \in \mathcal{P}(D)$ are such that

$$D_2 = D_1\Big\downarrow_{(r-1,c)}^{(r,c)} \prec D_1 \tag{1}$$

and the label of $(r,c) \in D_1$ is L, then decorate the cell $(r-1,c) \in D_2$ with the label L and maintain the labels of the cells in $D_1 \cap D_2$. Now, we define our edge labeling $\lambda : \mathcal{E}(\mathcal{P}(D)) \to \mathbb{Z}_{>0}$ by $\lambda(D_2, D_1) = L$ if (1) holds and the label of $(r,c) \in D_1$ is L. See Example 25.

Example 25. *In Figure 12 below we illustrate our cell decoration and edge labeling on an interval $[D_2, D_1] \subset \mathcal{P}(D)$, where D is a hook diagram.*

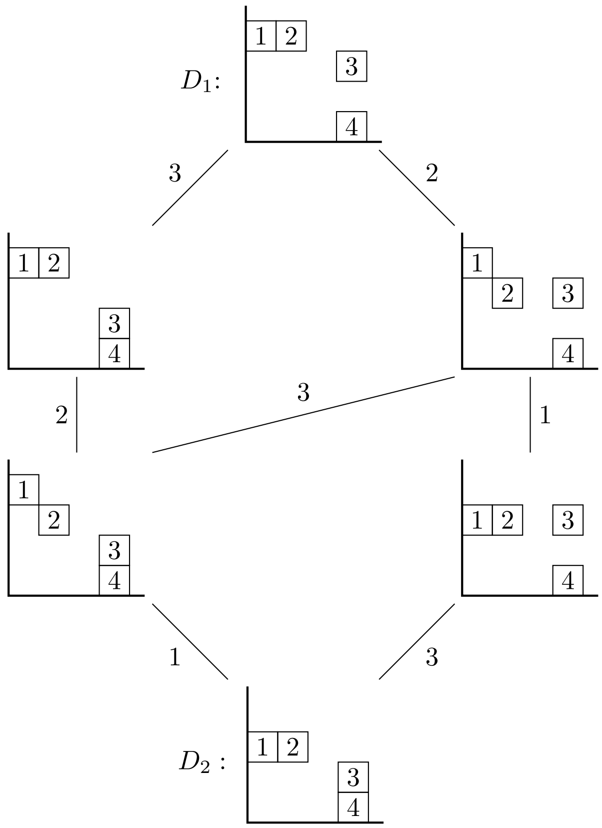


Figure 12: Edge-labeled interval of $\mathcal{P}(D)$

Lemma 26. *Let D be a hook diagram and $I = [D_2, D_1]$ be an interval in $\mathcal{P}(D)$ equipped with the edge labeling described above. If $\mathcal{C}_1, \mathcal{C}_2$ are maximal chains in I, then the multiset of edge labels corresponding to $\mathcal{C}_1$ is equal to that of $\mathcal{C}_2$.*

Proof. Let $\mathcal{C}_1, \mathcal{C}_2$ be two maximal chains in I. If $(r, c) \in D_1$ has label L and is moved, via a sequence of Kohnert moves, to position $(r-k, c) \in D_2$ for some $k \in \mathbb{Z}_{>0}$, then it follows from Lemma 24 that among the edges of $\mathcal{C}_1$ and $\mathcal{C}_2$ in the Hasse diagram of $\mathcal{P}(D)$, exactly k are labeled L. Since $(r, c) \in D_1$ was arbitrary, the result follows. ☐

Remark 27. *As a consequence of Lemma 24, given an interval $I = [D_2, D_1] \subseteq \mathcal{P}(D)$, each maximal chain in I has its own unique ordered list of edge labels. To see this, note that each edge label in a given chain in I is defined by a particular cell being moved down one row. Thus, if two chains have the same ordered list of edge labels, then they correspond to the same sequence of Kohnert moves, i.e., they are equal chains. Combining this observation with Lemma 26, we conclude that if $\mathcal{C}_1 \neq \mathcal{C}_2$ are two maximal chains in I, then the ordered list of edge labels corresponding to $\mathcal{C}_1$ is a nontrivial permutation of the ordered list of edge labels corresponding to $\mathcal{C}_2$.*

Theorem 28. *If D is a hook diagram, then $\mathcal{P}(D)$ is EL-shellable.*

Proof. Let $\mathcal{P} = \mathcal{P}(D)$ and assume that D contains $n > k$ cells. By Lemma 22, $\mathcal{P}$ is bounded. Thus, it remains to show that $\mathcal{P}$ admits an edge labeling that satisfies Definition 5. Decorate the cells of all $\widetilde{D} \in \mathcal{P}$ and label the Hasse diagram of $\mathcal{P}$ as described above. Take $D_1, D_2 \in \mathcal{P}$ satisfying $D_2 \prec D_1$. Assume that for $1 \leq i \leq k$, if the cell in location (r, c) of D_1 is decorated with label L_i, then the cell $(r - \mu_i, c) \in D_2$,

$\mu_i > 0$, is decorated with label L_i; that is, D_2 is formed from D_1 by moving each cell in D_1 with label L_i down μ_i rows. By Lemmas 24 and 26, it follows that each chain in the interval $I = [D_2, D_1]$ has the multiset of edge labels $\{L_1^{\mu_1}, \ldots, L_k^{\mu_k}\}$, where, without loss of generality, we may assume $L_i < L_j$ whenever $i < j$.

We describe the sequence of Kohnert moves that corresponds to the unique rising unrefinable chain $\mathcal{C}^*$ in I. In short, $\mathcal{C}^*$ is the chain obtained by moving each cell individually from its location in D_1 to its position in D_2 in decreasing order of the cells' labels. Explicitly, define $\mathcal{C}^*$ to be the chain of diagrams obtained via the following:

Step 1: If the cell labeled L_k is located in row $\tilde{r}_k$ of D_1, then apply, in succession, a single Kohnert move to D_1 at rows $\tilde{r}_k$ through $\tilde{r}_k - \mu_k + 1$ in decreasing order. Call the resulting diagram D^k.

Step i: If the cell labeled L_{k-i+1} is located in row $\tilde{r}_{k-i+1}$ of D_1, then apply, in succession, a single Kohnert move to D^{k-i+2} at rows $\tilde{r}_{k-i+1}$ through $\tilde{r}_{k-i+1} - \mu_i + 1$ in decreasing order. Call the resulting diagram D^{k-i+1}.

Considering the method of labeling the cells of D_1 along with Proposition 19, it follows that applying the sequence of Kohnert moves outlined above has the desired effect. Note that the ordered list of labels corresponding to $\mathcal{C}^*$ is $(L_1^{\mu_1}, \ldots, L_k^{\mu_k})$. It follows from Remark 27 that this list is lexicographically minimal with respect to all chains in I and that $\mathcal{C}^*$ is the unique rising chain in I. $\square$

4.2 Necessity

In this section, we finish the proof of Theorem 20. We consider each part separately, starting with Theorem 20 (a).

Proof of Theorem 20 (a). The backward direction was established in Section 4.1.

Let $\widehat{D} = D\backslash\{(1,\tilde{c}) \mid \tilde{c} > 0\} = \{(r_1, c_1), \ldots, (r_n, c_n)\}$ with $c_1 < \ldots < c_n$. Note that $r_i > 1$ for $1 \le i \le n$. Assume that $\widehat{D}$ is not a hook diagram. Then, applying Proposition 19, there exists $1 \le i < j \le n$ such that $r_i < r_j$. Let

$$n(k) = \begin{cases} |\{\tilde{c} \mid \tilde{c} > c_i,\ (r_i, \tilde{c}) \in \widehat{D}\}|, & k = i \\ |\{\tilde{c} \mid \tilde{c} > c_j,\ (r_k, \tilde{c}) \in \widehat{D}\}|, & i < k \le j, \end{cases}$$

for k satisfying $i \le k \le j$, and form $D^* \in \mathcal{P}(\widehat{D})$ from $\widehat{D}$ as follows.

1) For k satisfying $i \le k \le j$ in increasing order, successively apply $n(k)$ Kohnert moves at rows r_k through 2 in decreasing order.

2) If $r_i < r_j - 1$, then apply, in succession, one Kohnert move at rows r_j through $r_i + 2$ in decreasing order; otherwise, do nothing.

By our assumptions on D, it follows that

- $(r_i, c_i), (r_i + 1, c_j) \in D^*$,
- $(r_i + 1, \tilde{c}), (r_i - 1, c_j) \notin D^*$ for $\tilde{c} > c_j$, and
- $(r_i, \tilde{c}), (r_i - 1, c_i) \notin D^*$ for $\tilde{c} > c_i$.

Thus, applying Proposition 14 with $r = r_i - 1, c_1 = c_i$, and $c_2 = c_j$, $\mathcal{P}(\widehat{D})$ is not shellable. The result now follows from Lemma 21. $\square$

For the proof of Theorem 20 (b), Lemmas 29 through 31 below identify necessary conditions for diagrams whose first two rows are empty to be associated with shellable Kohnert posets. Utilizing the aforementioned lemmas along with Proposition 19, we will be able to show that the only diagrams that remain are hook diagrams, finishing the proof of Theorem 20 (b).

Lemma 29. *Let D be a diagram for which all cells of D are contained in rows $2 < r_1 < r_2 < \ldots < r_n$ and $\mathcal{P}(D)$ is shellable. If c^* is the rightmost nonempty column of D and r^* is maximal such that $(r^*, c^*) \in D$, then $(\tilde{r}, \tilde{c}) \in D$ for $\tilde{c} < c^*$ implies that $\tilde{r} \geq r^*$.*

Proof. Assume otherwise. Then there exists a maximal $c < c^*$ such that, for some i and j satisfying $1 \leq i < j \leq n$, we have $(r_i, c), (r_j, c^*) \in D$. Assume that j is chosen to be minimal with the aforementioned properties, i.e., (r_j, c^*) is the lowest cell in column c^* for which there exists a cell in column c lying in a strictly lower row. Note that, by the maximality of c, there are no cells below row r_j in columns strictly between c and c^*. Further, by the minimality of j, there are no cells in column c^* strictly between rows r_i and r_j. Now, form $\widehat{D} \in \mathcal{P}(D)$ as follows.

1) Letting $k = |\{\tilde{c} \mid \tilde{c} = c \text{ or } c^*,\ (r_i - 1, \tilde{c}) \in D\}|$, apply k Kohnert moves at row $r_i - 1$.

2) Next, if $(r_i, c^*) \in D$, then apply a single Kohnert move at row r_i followed by a single Kohnert move at row $r_i - 1$; otherwise, do nothing.

3) Finally, if $r_i < r_j - 1$, then apply, in succession, one Kohnert move at rows r_j through $r_i + 2$ in decreasing order; otherwise, do nothing.

Note that since $r_1 > 2$, any Kohnert moves applied in steps 1 and 2 are nontrivial. By our assumptions on D, it follows that

- $(r_i, c), (r_i + 1, c^*) \in \widehat{D}$,
- $(r_i + 1, \tilde{c}), (r_i - 1, c^*) \notin \widehat{D}$ for $\tilde{c} > c^*$, and
- $(r_i, \tilde{c}), (r_i - 1, c) \notin \widehat{D}$ for $\tilde{c} > c$.

Therefore, applying Proposition 14 with $r = r_i - 1$, $c_1 = c$, and $c_2 = c^*$, it follows that $\mathcal{P}(D)$ is not shellable, a contradiction. The result follows. □

Lemma 30. *Let D be a diagram for which all cells of D are contained in rows $2 < r_1 < r_2 < \ldots < r_n$, there exists a column in which D contains more than one cell, and $\mathcal{P}(D)$ is shellable. If c^* is the rightmost column of D containing more than one cell, then all columns $\tilde{c} > c^*$ of D are empty.*

Proof. Assume otherwise. Let $c > c^*$ be minimal such that column c of D is nonempty. Since c^* is the rightmost column of D containing more than one cell, all nonempty columns $\tilde{c} > c^*$ of D must contain exactly one cell. Let D_1 denote the diagram formed by bottom-justifying the cells in columns $\tilde{c} > c$ of D. Evidently, $D_1 \in \mathcal{P}(D)$. Assume that $(r_i, c), (r_j, c^*) \in D_1$ where r_j is minimal, i.e., (r_j, c^*) is the lowest cell in column c^* of D_1. Note that since column c^* contains more than one cell and r_j is the row occupied by the lowest such cell, it follows that $j < n$. There are two cases.

Case 1: $i > j$. In this case, form the diagram D_2 from D_1 as follows. If $r_j < r_i - 1$, then apply, in succession, one Kohnert move at rows r_i through $r_j + 2$ in decreasing order; otherwise, do nothing. By our assumptions on D,

- $(r_j, c^*), (r_j + 1, c) \in D_2$,
- $(r_j + 1, \tilde{c}), (r_j - 1, c) \notin D_2$ for $\tilde{c} > c$, and
- $(r_j, \tilde{c}), (r_j - 1, c^*) \notin D_2$ for $\tilde{c} > c^*$.

Thus, applying Proposition 14 with $r = r_j - 1$, $c_1 = c^*$, and $c_2 = c$, it follows that $\mathcal{P}(D)$ is not shellable, a contradiction.

Case 2: $i \leq j$. Assume that r_k is minimal such that $(r_k, c^*) \in D_1$ and $r_k > r_j$, i.e., r_k is the second lowest nonempty row in column c^* of D. Form the diagram D_2 from D_1 as follows.

1) If $i < j$, then apply, in succession, one Kohnert move at rows r_j through $r_i + 1$ in decreasing order; otherwise, do nothing.

2) Apply, in succession, one Kohnert move at rows r_k through $r_i + 1$ in decreasing order.

By our assumptions on D, it follows that

- $(r_i - 1, c^*), (r_i, c) \in D_2$,
- $(r_i, \tilde{c}), (r_i - 2, c) \notin D_2$ for $\tilde{c} > c$, and
- $(r_i - 1, \tilde{c}), (r_i - 2, c^*) \notin D_2$ for $\tilde{c} > c^*$.

Applying Proposition 14 with $r = r_i - 2$, $c_1 = c^*$, and $c_2 = c$, it follows that $\mathcal{P}(D)$ is not shellable, a contradiction.

The result follows. □

Lemma 31. *Let D be a diagram for which all cells of D are contained in rows $2 < r_1 < r_2 < \ldots < r_n$ and $\mathcal{P}(D)$ is shellable. If there exists a column in which D contains more than one cell, then it is unique.*

Proof. Assume otherwise. Then there exists at least two columns of D each of which contains more than one cell. Let $c_1^* < c_2^*$ be the rightmost two such columns of D and assume that r_i is maximal such that $(r_i, c_2^*) \in D$. Then

(i) for all $\tilde{c}$ satisfying $c_1^* < \tilde{c} < c_2^*$, column $\tilde{c}$ of D contains at most one cell;

(ii) applying Lemma 30, all columns $\tilde{c} > c_2^*$ of D are empty; and

(iii) applying Lemma 29, $(\tilde{r}, \tilde{c}) \in D$ for $\tilde{c} < c_2^*$ implies that $\tilde{r} \geq r_i$.

Now, assume that r_j and r_k are minimal such that $r_j < r_k$ and $(r_j, c_1^*), (r_k, c_1^*) \in D$, i.e., $(r_j, c_1^*), (r_k, c_1^*)$ are the two lowest cells in column c_1^* of D. Note that, considering (iii) above, $r_i \leq r_j < r_k$. Define

$$n(t) = |\{\tilde{c} \mid (r_t, \tilde{c}) \in D,\ c_1^* \leq \tilde{c} < c_2^*\}|$$

for $i < t \leq k$. Form $\widehat{D} \in \mathcal{P}(D)$ as follows.

1) For $i < t \leq k$ in increasing order, apply, in succession, $n(t)$ Kohnert moves at rows r_t through $r_i + 1$ in decreasing order.

2) If $(r_i - 1, c_2^*) \in D$, then apply a single Kohnert move at row $r_i - 1$; otherwise, do nothing.

3) If $(r_i - 2, c_2^*) \in D$, then apply a single Kohnert move at row $r_i - 2$; otherwise, do nothing.

By our assumptions on D, it follows that

- $(r_i - 1, c_1^*), (r_i, c_2^*) \in \widehat{D}$,
- $(r_i, \tilde{c}), (r_i - 2, c_2^*) \notin \widehat{D}$ for $\tilde{c} > c_2^*$, and
- $(r_i - 1, \tilde{c}), (r_i - 2, c_1^*) \notin \widehat{D}$ for $\tilde{c} > c_1^*$.

Therefore, applying Proposition 14 with $r = r_i - 2$, $c_1 = c_1^*$, and $c_2 = c_2^*$, it follows that $\mathcal{P}(D)$ is not shellable, a contradiction. The result follows. □

Combining the results above, we can now finish the proof of Theorem 20.

Proof of Theorem 20 (b). The backward direction was established in Section 4.1.

Let D be a diagram for which all cells are contained in rows $2 < r_1 < r_2 < \ldots < r_n$. Note that if D has no columns with more than one cell, then the result follows by (a). Consequently, we assume that there exists a column in which D contains more than one cell.

Assume that $\mathcal{P}(D)$ is shellable. Applying Lemma 31, there exists a unique column c^* in which D contains more than one cell. By Lemma 30, all columns $\tilde{c} > c^*$ of D are empty. Moreover, by Lemma 29, if r_i is maximal such that $(r_i, c^*) \in D$ and $\tilde{c} < c^*$, then $(\tilde{r}, \tilde{c}) \in D$ implies $\tilde{r} \geq r_i$. Consequently, D satisfies all the properties listed in Proposition 19 except possibly property (iv). For a contradiction, assume that D does not satisfy property (iv) of Proposition 19; that is, assume that there exists $\widehat{c}_1 < \widehat{c}_2 < c^*$ such that $(r_i, \widehat{c}_1), (r_j, \widehat{c}_2) \in D$ with $r_i < r_j$. By Lemma 29, if r^* is maximal such that $(r^*, c^*) \in D$ then $r_j > r_i \geq r^*$, and it follows that all cells in rows r satisfying $r_i < r \leq r_j$ must be unique in their respective columns. If $r_i > r^*$, then it also follows that each cell in row r_i is unique in its respective column; however, if $r_i = r^*$, then each cell in row r_i, excluding the cell (r^*, c^*), is unique in its respective column. Define

$$n(k) = \begin{cases} |\{\tilde{c} \mid (r_k, \tilde{c}),\ \tilde{c} > \widehat{c}_1\}|, & i \leq k < j \\ |\{\tilde{c} \mid (r_k, \tilde{c}),\ \tilde{c} > \widehat{c}_2\}|, & k = j \end{cases}$$

for k satisfying $i \leq k \leq j$. Form $\widehat{D} \in \mathcal{P}(D)$ from D as follows.

1) For k satisfying $i \leq k \leq j$ in increasing order, apply, in succession, $n(k)$ Kohnert moves at rows r_k through r_i in decreasing order; note that since $r_1 > 2$, any such Kohnert moves are nontrivial.

2) If $r_i < r_j - 1$, then apply, in succession, one Kohnert move at rows r_j through $r_i + 2$ in decreasing order; otherwise, do nothing.

By our assumptions on D, it follows that

- $(r_i, \widehat{c}_1), (r_i + 1, \widehat{c}_2) \in \widehat{D}$,
- $(r_i + 1, \tilde{c}), (r_i - 1, \widehat{c}_2) \notin \widehat{D}$ for $\tilde{c} > \widehat{c}_2$, and
- $(r_i, \tilde{c}), (r_i - 1, \widehat{c}_1) \notin \widehat{D}$ for $\tilde{c} > \widehat{c}_1$.

Thus, applying Proposition 14 with $r = r_i - 1$, $c_1 = \widehat{c}_1$, and $c_2 = \widehat{c}_2$, it follows that $\mathcal{P}(D)$ is not shellable, a contradiction. Consequently, D satisfies all of the properties listed in Proposition 19; that is, D is a hook diagram. □

Interestingly, among the families of diagrams considered in Theorem 20, the corresponding Kohnert posets are (EL-)shellable precisely when the associated Kohnert polynomials are multiplicity free.

Theorem 32. *Let D be a diagram for which either there is at most one cell per column or the first two rows are empty. Then $\mathcal{P}(D)$ is (EL-)shellable if and only if $\mathfrak{K}_D$ is multiplicity-free.*

Proof. For the backward direction, assume that D is not (EL-)shellable, i.e., D is not a hook diagram. Note that in establishing each of the backward directions of Theorem 20, we found that if D did not satisfy at least one of the defining properties (i)–(iv) of hook diagrams given in Proposition 19, then there existed $D^* \in \mathcal{P}(D)$ which contained a subdiagram of the form described in Proposition 14. Given such a diagram D^*, let $r, c_1, c_2 \in \mathbb{N}$ be such that

(i) $1 \leq r$ and $c_1 < c_2$,

(ii) $(r + 1, c_1), (r + 2, c_2) \in D^*$,

(iii) $(r + 2, \tilde{c}), (r, c_2) \notin D^*$ for $\tilde{c} > c_2$, and

(iv) $(r + 1, \tilde{c}), (r, c_1) \notin D^*$ for $\tilde{c} > c_1$.

Let D_1 denote the diagram formed by applying a Kohnert move at row $r+2$ of D^* followed by row $r+1$ and D_2 denote the diagram formed by applying a Kohnert move at row $r+1$ of D^* followed by row $r+2$. Then we have that

$$D_1 = (D^* \backslash \{(r+2, c_2)\}) \cup \{(r, c_2)\} \neq (D^* \backslash \{(r+1, c_1), (r+2, c_2)\}) \cup \{(r, c_1), (r+1, c_2)\} = D_2$$

and $wt(D_1) = wt(D_2)$. Thus, $\mathfrak{K}_D$ is not multiplicity free.

Now, for the forward direction, we break the proof into two cases. Assume that $n > 0$ is maximal such that row n of D is nonempty.

Case 1: D contains at most one cell per column. Let

$$C_1 = \{c \mid (1, c) \in D\} = \{c_1^1, \ldots, c_{m_1}^1\}$$

and

$$C_2 = \{c \mid (r, c) \in D \text{ and } c \notin C_1\} = \{c_1^2 < \ldots < c_{m_2}^2\}.$$

Note that for $\tilde{D} \in \mathcal{P}(D)$, $wt(\tilde{D}) = \prod_{i=1}^n x_i^{a_i}$ where $a_1 \geq m_1$ and $\sum_{i=1}^n a_i - m_1 = m_2$. Now, it is straightforward to show that if $\mathcal{P}(D)$ is (EL-)shellable and $(r_i, c_i^2) \in \tilde{D} \in \mathcal{P}(D)$ for $1 \leq i \leq m_2$, then $r_1 \geq r_2 \geq \cdots \geq r_{m_2}$. Thus, if $\prod_{i=1}^n x_i^{a_i}$ is a monomial in $\mathfrak{K}_D$, then it corresponds uniquely to the diagram

$$\tilde{D} = \{(1, c) \mid c \in C_1\} \cup \{(1, c_{m_2-i+1}^2) \mid 1 \leq i \leq a_1 - m_1 + 1\} \cup \bigcup_{j=2}^{n} \left\{ (j, c_{m_2-i+1}^2) \mid \sum_{k=1}^{j-1} a_k < i \leq \sum_{k=1}^{j} a_k \right\}.$$

Consequently, $\mathfrak{K}_D$ is multiplicity free.

Case 2: The first two rows of D are empty. Assume that $D \in H(r_1, r_2; C)$ with $C = \{c_1 < \ldots < c_m\}$. Since we have already considered the situation where D contains at most one cell per column in Case 1, we may assume that $L = r_2 - r_1 > 0$. Take $\tilde{D} \in \mathcal{P}(D)$ and assume that r is maximal such that $(r, c_m) \in \tilde{D}$. Then considering Proposition 19 (i)–(iii), there exists $1 \leq i_1 < \cdots < i_L < n$ such that $wt(\tilde{D}) = \prod_{j=1}^L x_{i_j} \prod_{k=i_L+1}^n x_k^{a_k}$, where $\sum_{k=i_L+1}^n a_k = |C|$. Now, considering Proposition 19 (iv), if $\prod_{j=1}^L x_{i_j} \prod_{k=i_L+1}^n x_k^{a_k}$ is a monomial of $\mathfrak{K}_D$, then it corresponds uniquely to the diagram

$$\begin{aligned} \tilde{D} = &\{(i_j, c_m) \mid 1 \leq j \leq L\} \cup \{(i_L+1, c_{m-i+1}) \mid 0 < i \leq a_{i_L+1}\} \\ &\cup \bigcup_{j=2}^{n-i_L} \left\{ (i_L+j, c_{m-i+1}) \mid \sum_{k=i_L+1}^{i_L+j-1} a_k < i \leq \sum_{k=i_L+1}^{i_L+j} a_k \right\}. \end{aligned}$$

Consequently, $\mathfrak{K}_D$ is multiplicity free. □

Remark 33. *In the following section, we find, in particular, that the equivalence between (EL-)shellability of a Kohnert poset and the associated Kohnert polynomial being multiplicity free does not hold in general. More specifically, for key diagrams we find that if a Kohnert poset is graded and EL-shellable, then the associated Kohnert polynomial is multiplicity free; however, there are examples of key diagrams whose Kohnert polynomials are multiplicity free and whose Kohnert posets are not shellable.*

It remains to consider diagrams that contain at least one cell within the first two rows and for which at least one column contains more than one cell. While we do not obtain a complete classification in this case, in the following section we consider the special case of key diagrams.

5 Key diagrams

In this section, we consider Kohnert posets associated with key diagrams. Recall from Section 2 that key diagrams are defined by weak compositions: given a weak composition $\mathbf{a} = (a_1, \ldots, a_n)$, the associated key diagram is defined by

$$\mathbb{D}(\mathbf{a}) = \bigcup_{i=1}^{n} \{(i, j) \mid 1 \leq j \leq a_i\}.$$

Ongoing, we let $|\mathbf{a}| = \sum_{i=1}^{n} a_i$.

The main result of this section is a characterization of the key diagrams that generate pure, shellable Kohnert posets in terms of their associated compositions. To state the main result, we require the notion of "pure composition" introduced in **[10]**.

Definition 34. *A weak composition* $\mathbf{a} = (a_1, \ldots, a_n)$ *is called a* ***pure composition*** *if there exists no* $1 \leq j_1 < j_2 < j_3 \leq n$ *such that*

- $a_{j_1} < a_{j_2} < a_{j_3}$,
- $a_{j_1} < a_{j_3} < a_{j_2}$, *or*
- $a_{j_1} + 1 < a_{j_2} = a_{j_3}$.

In **[10]**, the following properties of pure compositions and Kohnert posets associated with key diagrams are established. To set the notation, given a weak composition $\mathbf{a} = (a_1, \ldots, a_n)$, ongoing we set $\max(\mathbf{a}) = \max\{a_i \mid 1 \leq i \leq n\}$ and $\min(\mathbf{a}) = \min\{a_i \mid 1 \leq i \leq n\}$, i.e., $\max(\mathbf{a})$ (resp., $\min(\mathbf{a})$) is the maximal (resp., minimal) entry of $\mathbf{a}$.

Lemma 35 (Lemma 6.8, **[10]**)**.** *If* $\mathbf{a} = (a_1, \ldots, a_n)$ *is a pure composition, then there exists* $i_1 = 1 < \ldots < i_m < n = i_{m+1} - 1$ *such that* $\alpha_j = (a_{i_j}, \ldots, a_{i_{j+1}-1})$ *for* $1 \leq j \leq m$ *satisfies* $\min(\alpha_{j-1}) \geq \max(\alpha_j)$ *for* $1 < j \leq m$*. Moreover, each* $\alpha_j = (a_{i_j}, \ldots, a_{i_{j+1}-1})$ *is of one of the following forms:*

(i) $a_{i_j} \geq \ldots \geq a_{i_{j+1}-1}$*; that is,* α_j *is a weakly decreasing sequence.*

(ii) *There exists* $p \in \mathbb{Z}_{\geq 0}$ *such that* $a_{i_j} = p$ *and* $\{a_{i_j}, \ldots, a_{i_{j+1}-1}\} = \{p, p+1\}$*; that is, all entries of* α_j *are* p *or* $p+1$ *for some* $p \in \mathbb{Z}_{\geq 0}$*, the first entry is* p*, and some other entry must be equal to* $p+1$*.*

(iii) $a_{i_j} \geq \ldots \geq a_{i_{j+1}-2} < a_{i_{j+1}-1} - 1$*; that is, the entries of* α_j *are in decreasing order, except the last one which is at least two larger than the penultimate one.*

(iv) *There exist* $p \in \mathbb{Z}_{\geq 0}$ *and* $i_j^* \in \mathbb{Z}_{>0}$ *with* $i_j + 1 < i_j^* < i_{j+1} - 1$ *such that* $a_{i_j} = p$*,* $\{a_{i_j}, \ldots, a_{i_j^*-1}\} = \{p, p+1\}$*,* $p > a_{i_j^*} \geq \ldots \geq a_{i_{j+1}-2}$*, and* $a_{i_{j+1}-1} = p+1$*.*

Theorem 36 (Theorem 6.1, **[10]**)**.** *For all weak compositions* $\mathbf{a}$*, the Kohnert poset* $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *is bounded.*

Theorem 37 (Theorem 6.5, **[10]**)**.** *Let* $\mathbf{a}$ *be a weak composition. Then* $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *is pure if and only if* $\mathbf{a}$ *is pure.*

Remark 38. *To be precise, Theorem 6.5 of* **[10]** *actually establishes that for a weak composition* $\mathbf{a}$*, the Kohnert poset* $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *is ranked if and only if* $\mathbf{a}$ *is pure. However, it is straight-forward to verify that a bounded poset is pure if and only if it is ranked.*

Ongoing, given a pure composition $\mathbf{a}$, we will refer to a decomposition $(\alpha_1, \ldots, \alpha_m)$ of $\mathbf{a}$ as shown to exist in Lemma 35 as a **pure decomposition** of $\mathbf{a}$. Moreover, we refer to the weak compositions of the forms described in $(i) - (iv)$ of Lemma 35, i.e., those weak compositions that form the building blocks of pure compositions, as **basic pure compositions**.

Example 39. *Consider the pure composition*

$$\mathbf{a} = (15, 15, 15, 14, 14, 15, 14, 15, 13, 11, 10, 7, 15, 7, 6, 5, 4, 6, 3, 3, 4, 3, 4, 3, 2, 1, 0, 1).$$

One choice of pure decomposition of $\mathbf{a}$ *is given by*

$$\alpha_1 = (15, 15, 15, 14, 14, 15, 14, 15, 13, 11, 10, 7, 15), \qquad \alpha_2 = (7, 6, 5, 4, 6), \qquad \alpha_3 = (3, 3, 4, 3, 4),$$

$$\alpha_4 = (3, 2, 1), \qquad \text{and} \qquad \alpha_5 = (0, 1).$$

Note that α_1 *is of type* (iv)*,* α_2 *is of type* (iii)*,* α_3 *and* α_5 *are of type* (ii)*, and* α_4 *is of type* (i)*.*

Remark 40. *In* **[10]***, given a pure composition* $\mathbf{a}$*, a procedure for finding a choice of pure decomposition of* $\mathbf{a}$ *is provided within the proof of Lemma 6.8.*

We claim that Theorem 37 can be extended to give a characterization of pure, (EL-)shellable Kohnert posets associated with key diagrams. In particular, in the remainder of this section we establish the following.

Theorem 41. *Let* $\mathbf{a}$ *be a weak composition. Then* $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *is pure and (EL-)shellable if and only if* $\mathbf{a}$ *is pure.*

5.1 Necessity

In this section we find necessary conditions for a key diagram to be associated with a shellable Kohnert poset in terms of the corresponding composition avoiding certain patterns. In particular, we establish the following.

Proposition 42. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *be a weak composition. Suppose that either*

(a) *there exist* $1 \leq i_1 < i_2 < i_3 \leq n$ *for which one of the following holds*

  (i) $a_{i_1} < a_{i_2} < a_{i_3}$

  (ii) $a_{i_1} \leq a_{i_3} - 3 \leq a_{i_2} - 3$*;*

or

(b) *there exist* $1 \leq j_1 < j_2 < j_3 < j_4 \leq n$ *for which one of the following holds*

  (i) $a_{j_1} \leq a_{j_2} < a_{j_3} - 1 \leq a_{j_4} - 1$

  (ii) $a_{j_1} \leq a_{j_2} < a_{j_4} < a_{j_3}$

  (iii) $a_{j_2} < a_{j_1} < a_{j_4} < a_{j_3}$

  (iv) $a_{j_2} < a_{j_1} < a_{j_3} \leq a_{j_4}$.

Then $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *is not shellable.*

Remark 43. *Note that Proposition 42 establishes more than is needed to prove Theorem 41; in particular, the proof of Theorem 41 does not require part (b-iv) of Proposition 42. We include (b-iv) as, altogether, we conjecture that a weak composition* $\mathbf{a}$ *avoiding the patterns of Proposition 42 is equivalent to the Kohnert poset* $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *being (EL-)shellable, with no restrictions involving purity (see Conjecture 51).*

To prove Proposition 42, we first show that the result holds if we assume that the patterns described therein are followed by consecutively occurring terms in the composition.

Lemma 44. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *be a weak composition. Suppose that either*

(a) *there exists* $1 \leq i < n - 2$ *for which one of the following holds*

  (i) $a_i < a_{i+1} < a_{i+2}$

(ii) $a_i \leq a_{i+2} - 3 \leq a_{i+1} - 3$;

or

(b) *there exists* $1 \leq j < n-3$ *for which one of the following holds*

(i) $a_j \leq a_{j+1} < a_{j+2} - 1 \leq a_{j+3} - 1$

(ii) $a_j \leq a_{j+1} < a_{j+3} < a_{j+2}$

(iii) $a_{j+1} < a_j < a_{j+3} < a_{j+2}$

(iv) $a_{j+1} < a_j < a_{j+2} \leq a_{j+3}$.

Then $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *is not shellable.*

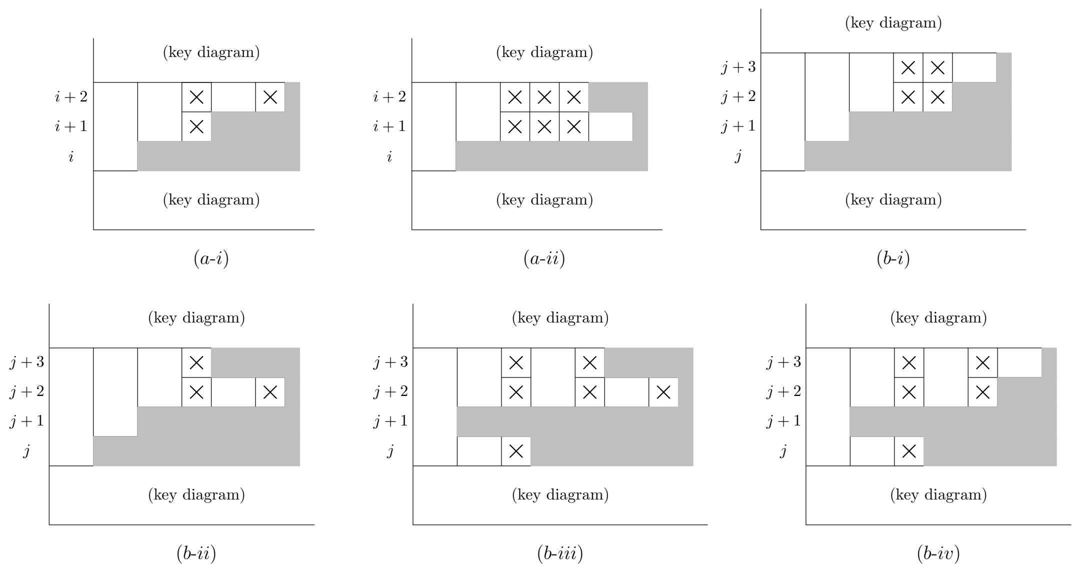


Figure 13: Forms of key diagrams for weak compositions described in Lemma 44

Proof. (*a-i*) By assumption, we have that

- $(i+1, a_{i+1}), (i+2, a_{i+2}) \in \mathbb{D}(\mathbf{a})$ with $a_{i+1} < a_{i+2}$,
- $(i, a_{i+2}), (i+2, \tilde{c}) \notin \mathbb{D}(\mathbf{a})$ for $\tilde{c} > a_{i+2}$, and
- $(i, a_{i+1}), (i+1, \tilde{c}) \notin \mathbb{D}(\mathbf{a})$ for $\tilde{c} > a_{i+1}$.

Thus, applying Proposition 14 with $r = i$, $c_1 = a_{i+1}$, and $c_2 = a_{i+2}$, the result follows.

(*a-ii*) Let D denote the diagram obtained from $\mathbb{D}(\mathbf{a})$ by applying $a_{i+1} - a_{i+2} + 2$ Kohnert moves at row $i+1$ (see Figure 14)

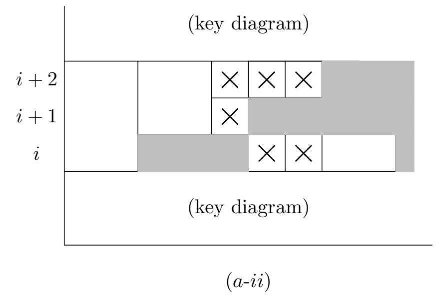


Figure 14: Diagram related to key diagram in case (*a*-*ii*)

Note that

- $(i+1, a_{i+2}-2), (i+2, a_{i+2}-2), (i+2, a_{i+2}) \in D$,
- $|\{(i+2, \tilde{c}) \in D \mid a_{i+2}-2 < \tilde{c} < a_{i+2}\}| = 1 > 0$,
- $(i+2, \tilde{c}) \notin D$ for $\tilde{c} > a_{i+2}$,
- $(i+1, \tilde{c}) \notin D$ for $\tilde{c} > a_{i+2}-2$, and
- $(i, a_{i+2}-2) \notin D$.

Thus, applying Corollary 16 with $r = i$, $c_1 = a_{i+2}-2$, and $c_2 = a_{i+2}$, the result follows.

(*b*-*i*),(*b*-*iv*) If $a_{j+2} < a_{j+3}$, then the result follows as in (*a*-*i*) taking $i = j+1$. So, assume that $a_{j+2} = a_{j+3}$. Let D denote the diagram obtained from $\mathbb{D}(\mathbf{a})$ by applying a single Kohnert move at row $j+2$ followed by a single Kohnert move at row $j+1$ (see Figure 15).

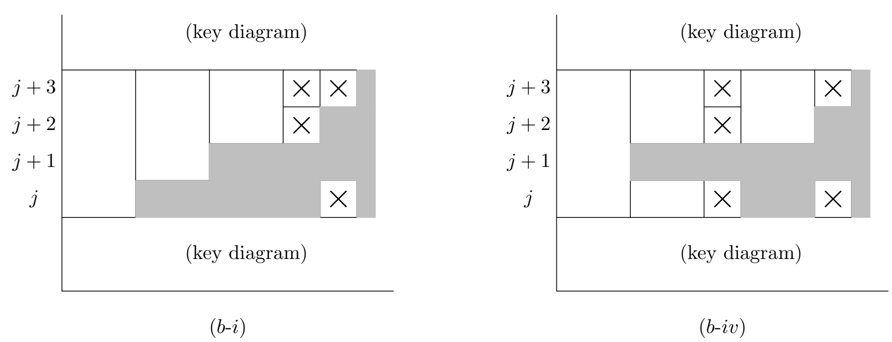


Figure 15: Diagrams related to key diagram in cases (*b*-*i*) (left) and (*b*-*iv*) (right)

By assumption,

- $(j+3, a_{j+3}), (j+2, a_{j+3}-1) \in D$,
- $(j+1, a_{j+3}), (j+3, \tilde{c}) \notin D$ for $\tilde{c} > a_{j+3}$, and
- $(j+1, a_{j+3}-1), (j+2, \tilde{c}) \notin D$ for $\tilde{c} > a_{j+3}-1$.

Thus, applying Proposition 14 with $r = j+1$, $c_1 = a_{j+3} - 1$, and $c_2 = a_{j+3}$, the result follows.

(b-ii),(b-iii) Let D denote the diagram obtained from $\mathbb{D}(\mathbf{a})$ by applying a single Kohnert move at row $j+3$ (see Figure 16).

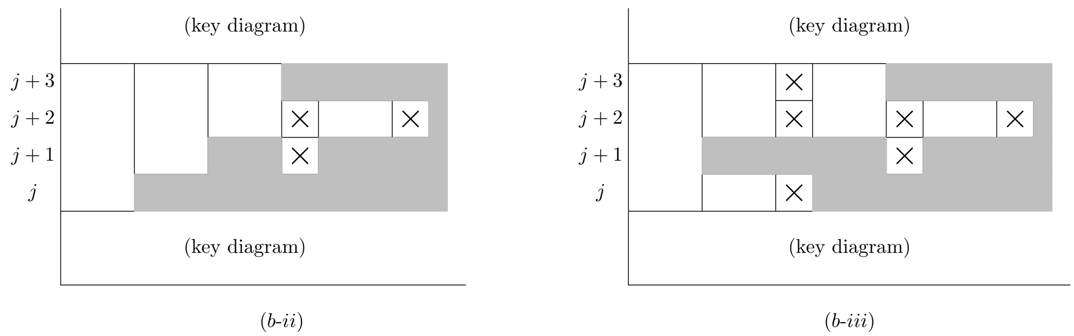


Figure 16: Diagrams related to key diagram in cases (b-ii) (left) and (b-iii) (right)

By assumption,

- $(j+1, a_{j+3}), (j+2, a_{j+2}) \in D$ with $a_{j+3} < a_{j+2}$,
- $(j, a_{j+2}), (j+2, \tilde{c}) \notin D$ for $\tilde{c} > a_{j+2}$, and
- $(j, a_{j+3}), (j+1, \tilde{c}) \notin D$ for $\tilde{c} > a_{j+3}$.

Thus, applying Proposition 14 with $r = j$, $c_1 = a_{j+3}$, and $c_2 = a_{j+2}$, the result follows. $\square$

Now, to prove Proposition 42, we make use of the following result from **[10]**. For notation, given a weak composition $\mathbf{a} = (a_1, \ldots, a_n)$, we denote by $\mathbf{a}s_{i,j}$ the weak composition obtained from $\mathbf{a}$ by exchanging the entries a_i and a_j; that is, $\mathbf{a}s_{i,j} = (a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_{j-1}, a_i, a_{j+1}, \ldots, a_n)$.

Lemma 45 (Lemma 6.15, **[10]**)**.** *Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a weak composition. If there exist $i < j$ such that $a_i < a_j$, then $\mathbb{D}(\mathbf{a}s_{i,j}) \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$.*

Proof of Proposition 42. We include only the proofs for cases (a-i) and (b-i) as the remaining cases following via very similar arguments. In both cases, we show that there exists a key diagram $T \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ such that T has one of the patterns described in Lemma 44. Consequently, applying Lemma 44, it will follow that $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ contains an interval that is not shellable – namely, the interval between T and the unique minimal element of $\mathcal{P}(\mathbb{D}(\mathbf{a}))$. Considering Theorem 4, we may thus conclude that $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is not shellable.

(a-i) Let i_1 be maximal, i_2 be arbitrary given our choice of i_1, and i_3 minimal given our choice of i_2. If $i_1 < i_2 - 1$, then $a_{i_2-1} \geq a_{i_2} > a_{i_1}$ by maximality of i_1. Thus, applying Lemma 45, $\mathbb{D}(\mathbf{a}s_{i_1,i_2-1}) \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$. Now, if $i_3 > i_2 + 1$, then $a_{i_2+1} \leq a_{i_2} < a_{i_3}$ by minimality of i_3. Consequently, applying Lemma 45, we find that

$$\mathbb{D}(\mathbf{a}s_{i_1,i_2-1}s_{i_2+1,i_3}) \in \mathcal{P}(\mathbb{D}(\mathbf{a}s_{i_1,i_2-1})) \subseteq \mathcal{P}(\mathbb{D}(\mathbf{a})),$$

i.e., $\mathbb{D}(\mathbf{a}s_{i_1,i_2-1}s_{i_2+1,i_3}) \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$. As the values $a_{i_1}, a_{i_2}, a_{i_3}$ occur as terms i_2-1, i_2, and i_2+1, respectively, in $\mathbf{a}s_{i_1,i_2-1}s_{i_2+1,i_3}$, the result follows.

(b-i) Let j_1 be maximal, j_2 be arbitrary given our choice of j_1, j_3 be minimal given our choice of j_2, and j_4 be arbitrary given our choice of j_3. Note that if $j_1 < j_2 - 1$, then $a_{j_2-1} > a_{j_2} \geq a_{j_1}$ by maximality of j_1; and if $j_2 < j_3 - 1$, then for $j_2 < i < j_3$ we have $a_i \neq a_{j_3}$ by minimality of j_3. Let j_3^* be minimal such that

$j_2 < j_3^* \leq j_3$ and $a_{j_3^*} \leq a_{j_3}$. By our choice of j_3^*, for $j_2 < j < j_3^*$ we have $a_j > a_{j_3} > a_{j_2}$; in particular, if $j_3^* - 1 \neq j_2$, then $a_{j_3^*-1} > a_{j_2}$. Thus, if we set $\mathbf{b} = \mathbf{a}s_{j_2,j_3^*-1}s_{j_3^*,j_3}$ and apply Lemma 45 twice, it follows that $\mathbb{D}(\mathbf{b}) \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$. Now, by construction, we have that $b_i = a_i > a_{j_1}$ for $j_1 < i < j_2$, $b_{j_2} = a_{j_3^*-1} \geq a_{j_2} \geq a_{j_1}$, and $b_i > a_{j_3} > a_{j_1}$ for $j_2 < i < j_3^* - 1$. Consequently, if we set $\mathbf{c} = \mathbf{b}s_{j_1,j_3^*-2}$, then an application of Lemma 45 shows that $\mathbb{D}(\mathbf{c}) \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$. Finally, if $c_{j_3^*+1} \geq a_{j_4}$, then set $\mathbf{d} = \mathbf{c}$; otherwise, set $\mathbf{d} = \mathbf{c}s_{j_3^*+1,j_4}$ and note that since $j_3^* + 1 < j_4$ and $c_{j_4} = a_{j_4}$, Lemma 45 implies that $\mathbb{D}(\mathbf{d}) \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$. By construction,

$$a_{j_1} = d_{j_3^*-2} \leq a_{j_2} = d_{j_3^*-1} < a_{j_3} - 1 = d_{j_3^*} - 1 \leq a_{j_4} - 1 \leq d_{j_3^*+1} - 1;$$

that is, $d_{j_3^*-2}, d_{j_3^*-1}, d_{j_3^*}$, and $d_{j_3^*+1}$ occur as consecutive terms in $\mathbf{d}$ and form the pattern described in (*b*-*i*). The result follows. ☐

5.2 Sufficiency

In this section, we finish the proof of Theorem 41 by establishing that the key diagrams of pure compositions generate (EL-)shellable Kohnert posets. To do so, given a pure composition $\mathbf{a}$, we show that the Kohnert poset $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ decomposes into a direct product of graded, (EL-)shellable posets. In particular, we start by showing that, for a pure composition $\mathbf{a}$ with pure decomposition $(\alpha_1, \ldots, \alpha_m)$, the poset $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is isomorphic to the direct product $\mathcal{P}(\mathbb{D}(\alpha_1)) \times \cdots \times \mathcal{P}(\mathbb{D}(\alpha_m))$ (see Proposition 46). Then we show that the Kohnert posets $\mathcal{P}(\mathbb{D}(\alpha_i))$ for $1 \leq i \leq m$ are isomorphic to intervals within Kohnert posets of hook diagrams (see Proposition 47) and, consequently, are graded and (EL-)shellable. Since the direct product of graded and (EL-)shellable is graded and (EL-)shellable by Theorem 48 below, it will follow that $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is graded and (EL-)shellable.

Proposition 46. *Let* $\mathbf{a}$ *be a pure composition and assume that* $\mathbf{a} = (\alpha_1, \ldots, \alpha_m)$ *is a pure decomposition of* $\mathbf{a}$*. Then* $\mathcal{P}(\mathbb{D}(\mathbf{a})) \cong \mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m))$.

Proof. Assume that $\alpha_j = (a_1^j, \ldots, a_{m_j}^j)$ for $1 \leq j \leq m$. Let $m_0 = 0$ and define ψ on diagrams $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ by $\psi(D) = (D_1, \ldots, D_m)$, where

$$D_j = \left\{ (r, c) \;\middle|\; 1 + \sum_{i=0}^{j-1} m_i \leq r^* = r + \sum_{i=0}^{j-1} m_i \leq \sum_{i=0}^{j} m_i \text{ and } (r^*, c) \in D \right\}$$

for $1 \leq j \leq m$; that is, D_j is the diagram formed by shifting the cells occupying rows $1 + \sum_{i=0}^{j-1} m_i \leq r \leq \sum_{i=0}^{j} m_i$ of $\mathbb{D}(\mathbf{a})$ so that they occupy rows $1 \leq r \leq m_j$. As an example, in Figure 17 we illustrate $\psi(\mathbb{D}(\mathbf{a}))$ where

$$\mathbf{a} = (6, 5, 4, 5, 4, 3, 5, 3, 1, 3, 1, 0, 1) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

with $\alpha_1 = (6, 5)$, $\alpha_2 = (4, 5, 4, 3, 5)$, $\alpha_3 = (3, 1, 3)$, and $\alpha_4 = (1, 0, 1)$.

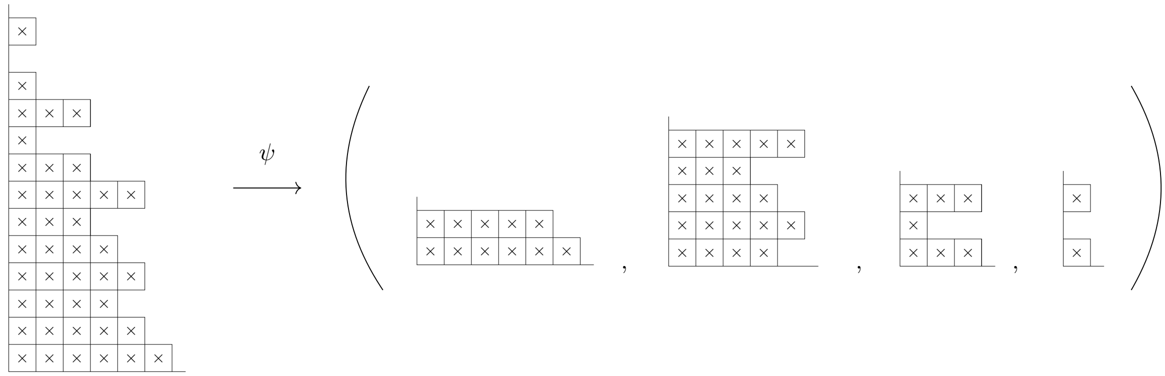


Figure 17: $\psi(\mathbb{D}(\mathbf{a}))$

We claim that ψ defines an isomorphism between the posets $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ and $\mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m))$.

First, we show that

$$\mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m)) \subseteq \text{im } \psi.$$

By definition, $\psi(\mathbb{D}(\mathbf{a})) = (\mathbb{D}(\alpha_1), \ldots, \mathbb{D}(\alpha_m)) \in \text{im } \psi$. Consequently, it suffices to show that if

$$(D_1, \ldots, D_j, \ldots, D_m) \in \text{im } \psi$$

and D_j^* can be formed from D_j by applying a single Kohnert move for $1 \leq j \leq m$, then

$$(D_1, \ldots, D_j^*, \ldots, D_m) \in \text{im } \psi.$$

Let $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ satisfy $\psi(D) = (D_1, \ldots, D_j, \ldots, D_m)$. Assume that D_j^* can be formed from D_j by applying a single Kohnert move at row r with $D_j^* = D_j\Big\downarrow^{(r,c)}_{(r',c)}$. Then (r, c) is rightmost in row r of D_j, $(\tilde{r}, c) \in D_j$ for all $r' < \tilde{r} < r$, and $(r', c) \notin D_j$. Thus, by the definition of ψ, it follows that

1. $(r' + \sum_{i=0}^{j-1} m_i, c) \notin D$,
2. $(\tilde{r} + \sum_{i=0}^{j-1} m_i, c) \in D_j$ for all $r' + \sum_{i=0}^{j-1} m_i < \tilde{r} + \sum_{i=0}^{j-1} m_i < r + \sum_{i=0}^{j-1} m_i$, and
3. $(r + \sum_{i=0}^{j-1} m_i, c) \in D$ is rightmost in row $r + \sum_{i=0}^{j-1} m_i$ of D.

Consequently, applying a Kohnert move at row $r + \sum_{i=0}^{j-1} m_i$ of D results in the diagram

$$D^* = D\Big\downarrow^{(r+\sum_{i=0}^{j-1} m_i, c)}_{(r'+\sum_{i=0}^{j-1} m_i, c)}.$$

Evidently, $\psi(D^*) = (D_1, \ldots, D_j^*, \ldots, D_m)$. Therefore, $\mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m)) \subseteq \text{im } \psi$.

Now, to show that the above inclusion is in fact an equality, considering the definition of ψ, it suffices to show that $(*)$ for $1 \leq j \leq m$, cells in rows $1 + \sum_{i=0}^{j-1} m_i \leq r \leq \sum_{i=0}^{j} m_i$ in $\mathbb{D}(\mathbf{a})$ cannot be moved via Kohnert moves to rows $\tilde{r} < \sum_{i=0}^{j-1} m_i$ in $\mathbb{D}(\mathbf{a})$; that is, no sequence of Kohnert moves can move cells in rows corresponding to α_j in $\mathbb{D}(\mathbf{a})$, for $1 \leq j \leq m$, into rows corresponding to α_i in $\mathbb{D}(\mathbf{a})$ for $1 \leq i < j$. Since $\min(\alpha_{j-1}) \geq \max(\alpha_j)$ for $1 < j \leq m$, it follows that $\min(\alpha_i) \geq \max(\alpha_j)$ for $1 \leq i < j \leq m$. Thus, $\{(r, c) \mid 1 \leq c \leq \max(\alpha_j) \text{ and } 1 \leq r \leq \sum_{i=0}^{j-1} m_i\} \subset \mathbb{D}(\mathbf{a})$. Consequently, it follows that for all $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$, there are no empty positions in columns 1 through $\max(\alpha_j)$ below row $1 + \sum_{i=0}^{j-1} m_i$. Therefore, since the cells in rows $1+\sum_{i=0}^{j-1} m_i \leq r \leq \sum_{i=0}^{j} m_i$ of $\mathbb{D}(\mathbf{a})$ only occupy columns $1 \leq c \leq \max(\alpha_j)$, property $(*)$ holds. Hence,

$$\mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m)) = \text{im } \psi.$$

As ψ is clearly one-to-one, it follows that $\psi : \mathcal{P}(\mathbb{D}(\mathbf{a})) \to \mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m))$ is a bijection.

It remains to show that ψ and ψ^{-1} are order preserving. Since the proofs are similar, we consider only the proof for ψ. For ψ, take $D_1, D_2 \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ such that $D_2 \prec D_1$. Then there exists a sequence of rows $R = \{r_i\}_{i=1}^{n}$ such that if $T_0 = D_1$ and, for $1 \leq i \leq n$, T_i is formed from T_{i-1} by applying a single Kohnert move at row r_i, then $T_{i-1} \neq T_i$ for $1 \leq i \leq n$ and $T_n = D_1$. Let R_j denote the (possibly empty) subsequence of R consisting of all $r_i \in R$ such that $1+\sum_{i=0}^{j-1} m_i \leq r_i \leq \sum_{i=1}^{j} m_i$. For $1 \leq j \leq m$, if R_j is nonempty, then assume $R_j = \{r_i^j\}_{i=1}^{n_j}$. Let $\psi(D_i) = (D_1^i, \ldots, D_m^i)$ for $i = 1, 2$. By construction, if $T_{0,j} = D_1^j$ for $1 \leq j \leq m$ and $T_{i,j}$ is the diagram formed from $T_{i-1,j}$ by applying a Kohnert move at row $r_i^j - \sum_{k=0}^{j-1} m_k$ for $1 \leq i \leq n_j$, then $T_{n_j,j} = D_j^2$. Thus, $D_j^2 \preceq D_j^1$ for all $1 \leq j \leq m$, and $D_k^2 \prec D_k^1$ for at least one $1 \leq k \leq m$; that is,

$$\psi(D_2) = (D_1^2, \ldots, D_m^2) \prec (D_1^1, \ldots, D_m^1) = \psi(D_1)$$

in $\mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m))$ and ψ is order preserving. □

Proposition 47. *If $\mathbf{a} = (a_1, \ldots, a_n)$ is a basic pure composition, then $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is EL-shellable.*

Proof. Recall that if $\mathbf{a} = (a_1, \ldots, a_n)$ is a basic pure composition, then $\mathbf{a}$ is of one of the following forms:

(i) $a_1 \geq \ldots \geq a_n$,

(ii) there exists $p \in \mathbb{N}$ such that $a_1 = p$ and $\{a_1, \ldots, a_n\} = \{p, p+1\}$,

(iii) $a_1 \geq \ldots \geq a_{n-1} < a_n - 1$, or

(iv) there exists $2 < i^* \in \mathbb{N}$ and $p \in \mathbb{N}$ such that $a_1 = p$, $\{a_1, \ldots, a_{i^*-1}\} = \{p, p+1\}$, $p > a_{i^*} \geq \ldots \geq a_{n-1}$, and $a_n = p+1$.

Note that if $\mathbf{a}$ is of type (i) then, there are no empty positions below cells. Consequently, $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is a poset consisting of a single element which is trivially EL-shellable. To establish the result for the remaining cases, we show that the associated Kohnert poset $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is isomorphic to an interval in the Kohnert poset of a hook diagram.

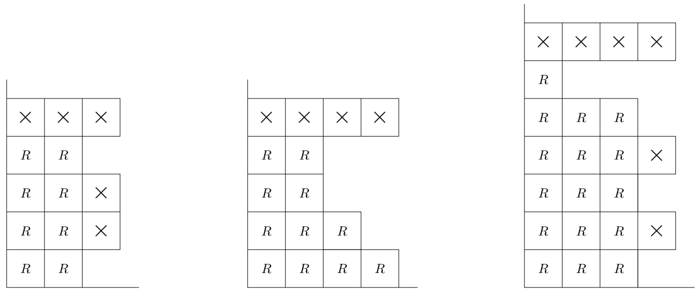
Let R denote the collection of cells $(r, c) \in \mathbb{D}(\mathbf{a})$ for $1 \leq r < n$ such that $(\tilde{r}, c) \in \mathbb{D}(\mathbf{a})$ for $1 \leq \tilde{r} \leq r$. When $\mathbf{a}$ is of type (ii) or (iv) with $\max(\mathbf{a}) = p+1$ we have

$$R = \{(r, c) \mid a_r = p+1,\ 1 \leq r < n,\ 1 \leq c \leq p\} \cup \{(r, c) \mid a_r \leq p, 1 \leq c \leq a_r\}, \tag{2}$$

while when $\mathbf{a}$ is of type (iii) we have that

$$R = \mathbb{D}(a_1, \ldots, a_{n-1}). \tag{3}$$

In Figure 18 below we illustrate R for the key diagrams of the basic pure compositions $(2, 3, 3, 2, 3)$, $(4, 3, 2, 2, 4)$, and $(3, 4, 3, 4, 3, 2, 1, 4)$ of types *(ii)*, *(iii)*, and *(iv)*, respectively, where the cells of R are marked with an "R".



$\mathbb{D}(2,3,3,2,3)$ $\mathbb{D}(4,3,2,2,4)$ $\mathbb{D}(3,4,3,4,3,1,4)$

Figure 18: Cells of R

Note that in all cases, considering (2) and (3), the cells of R are both left and bottom justified. Consequently, $R \subset D$ for all $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$. Now, for any $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$, let

$$Hk(D) = D \backslash R.$$

For $D = \mathbb{D}(\mathbf{a})$, we have the following options for the form of $Hk(D)$ depending on the type of $\mathbf{a}$:

(ii) $Hk(D)$ consists of the a_n cells in row n of D and the cells below in column a_n,

(iii) $Hk(D)$ consists of the a_n cells in row n of D, or

(iv) $Hk(D)$ consists of the a_n cells in row n of D and the cells below in column a_n.

Note that, in each case, $Hk(\mathbb{D}(\mathbf{a}))$ is a hook diagram. We claim that Hk defines a poset isomorphism between $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ and an interval of $\mathcal{P}(H)$, where $H = Hk(\mathbb{D}(\mathbf{a}))$.

First we show that Hk is an order-preserving map from $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ to $\mathcal{P}(H)$. Let

$$\mathcal{P}_1 = \{Hk(D) \mid D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))\},$$

where we define a partial ordering on $\mathcal{P}_1$ by $D' \prec D$ if D' can be formed from D by applying some sequence of Kohnert moves. Since $Hk(\mathbb{D}(\mathbf{a})) = H \in \mathcal{P}_1$ and $D \prec \mathbb{D}(\mathbf{a})$ for all $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$, if we can show that $Hk : \mathcal{P}(\mathbb{D}(\mathbf{a})) \to \mathcal{P}_1$ is order preserving, then it will follow that $\mathcal{P}_1 \subseteq \mathcal{P}(H)$; that is, it will follow that Hk is an order-preserving map from $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ to $\mathcal{P}(H)$. Take $D_1, D_2 \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ such that $D_2 \prec D_1$. Suppose that D_2 is formed from D_1 by applying a single Kohnert move at row r and $D_2 = D_1\Big\downarrow^{(r,c)}_{(r',c)}$. We show that $Hk(D_2)$ can be formed from $Hk(D_1)$ by applying a single Kohnert move at row r and $Hk(D_2) = Hk(D_1)\Big\downarrow^{(r,c)}_{(r',c)}$, i.e., $Hk(D_2) \prec Hk(D_1)$. Since $D_2 = D_1\Big\downarrow^{(r,c)}_{(r',c)}$, it follows that $(r,c) \in D_1$ is rightmost in row r, $(\widehat{r}, c) \in D_1$ for all $r' < \widehat{r} < r$, and $(r', c) \notin D_1$. Thus, since the cells of $R \subset D_1$ are both bottom and left justified, we may conclude that

- $(r,c), (r',c) \notin R$,
- $(r,c) \in D_1 \backslash R = Hk(D_1)$ is rightmost in row r of $Hk(D_1)$, and
- $(r',c) \notin D_1 \backslash R = Hk(D_1)$.

Consequently, if applying a Kohnert at row r of $Hk(D_1)$ does not result in $Hk(D_1)\Big\downarrow^{(r,c)}_{(r',c)}$, then there must exist $\hat{r}$ such that $r' < \widehat{r} < r$, $(\widehat{r}, c) \notin Hk(D_1)$, and $(\widehat{r}, c) \in D_1$; however, this implies $(\widehat{r}, c) \in R$, which is impossible since $(r',c) \notin R$ and the cells of R are bottom justified. Hence, applying a Kohnert move at row r of $Hk(D_1)$ results in

$$Hk(D_1)\Big\downarrow_{(r',c)}^{(r,c)} = (D_1\backslash R)\Big\downarrow_{(r',c)}^{(r,c)} = \left(D_1\Big\downarrow_{(r',c)}^{(r,c)}\right)\backslash R = D_2\backslash R = Hk(D_2),$$

where the second equality follows since $(r',c) \notin R$. Now, since $Hk(D_2)$ can be formed from $Hk(D_1)$ by applying a Kohnert move at row r, it follows that $Hk(D_2) \prec Hk(D_1)$. Therefore, if $D_1, D_2 \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ satisfy $D_2 \prec D_1$, then $Hk(D_2) \prec H_k(D_1)$; that is, the map $Hk : \mathcal{P}(\mathbb{D}(\mathbf{a})) \to \mathcal{P}_1$ is order preserving. As noted above, it follows that Hk is an order-preserving map from $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ to $\mathcal{P}(H)$.

Since Hk is clearly one-to-one, in order to establish the claim, it only remains to show that Hk maps $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ onto an interval of $\mathcal{P}(H)$ and that the inverse of Hk is order preserving. Let $\mathbf{a}'$ be the weak composition formed by sorting the entries of $\mathbf{a}$ into weakly decreasing order, and note that, by Corollary 6.2 of **[10]**, $\mathbf{a}'$ is the unique minimal element of $\mathcal{P}(\mathbb{D}(\mathbf{a}))$. If $H' = Hk(\mathbb{D}(\mathbf{a}'))$, then we have that $Hk : \mathcal{P}(\mathbb{D}(\mathbf{a})) \to [H', H]$ since Hk is order preserving. To show that Hk maps $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ onto $[H', H]$, we investigate the map ϕ, which we define to send $D \in [H', H]$ to $D \cup R$.

Similar to the case of Hk, let

$$\mathcal{P}_2 = \{\phi(D) \mid D \in [H', H]\}$$

and define a partial ordering on $\mathcal{P}_2$ by $D' \prec D$ if D' can be formed from D by applying some sequence of Kohnert moves. We shall show that $\phi : [H', H] \to \mathcal{P}_2$ is order preserving. Since $\phi(H) = \mathbb{D}(\mathbf{a})$ and $\phi(H') = \mathbb{D}(\mathbf{a}')$, it will then follow that

$$\mathbb{D}(\mathbf{a}') = \phi(H') \preceq \phi(\widehat{H}) \preceq \phi(H) = \mathbb{D}(\mathbf{a})$$

for all $\widehat{H} \in [H', H]$, i.e., it will follow that $\mathcal{P}_2 \subseteq \mathcal{P}(\mathbb{D}(\mathbf{a}))$ and ϕ is an order-preserving map from $[H', H]$ to $\mathcal{P}(\mathbb{D}(\mathbf{a}))$. Let $S = [\max(\mathbf{a})]^2\backslash R$. Since the cells of R are both bottom and left justified, it follows that there exists $0 < r_1^i \leq r_2^i$ for $1 \leq i \leq \max(\mathbf{a})$ such that

$$S = \bigcup_{i=1}^{\max(\mathbf{a})} \{(\tilde{r}, i) \mid r_1^i \leq \tilde{r} \leq r_2^i\}.$$

Moreover, since $H = Hk(\mathbb{D}(\mathbf{a})) = \mathbb{D}(\mathbf{a})\backslash R$ and $H' = Hk(\mathbb{D}(\mathbf{a}')) = \mathbb{D}(\mathbf{a}')\backslash R$, we have that $H\backslash S = \emptyset = H'\backslash S$. Consequently, applying Lemma 11, it follows that $\widehat{H}\backslash S = \emptyset$ for all $\widehat{H} \in [H', H]$. In particular,

$$\widehat{H} \cap R = \emptyset \text{ for all } \widehat{H} \in [H', H].$$

Now, take $H_1, H_2 \in [H', H]$ such that $H_2 \prec H_1$. Suppose that H_2 can be formed from H_1 by applying a single Kohnert move at row r and $H_2 = H_1\Big\downarrow_{(r',c)}^{(r,c)}$. We show that $\phi(H_2)$ can be formed from $\phi(H_1)$ by applying a single Kohnert move at row r and $\phi(H_2) = \phi(H_1)\Big\downarrow_{(r',c)}^{(r,c)}$. Since $H_2 = H_1\Big\downarrow_{(r',c)}^{(r,c)}$, it follows that $(r,c) \in H_1$ is rightmost in row r, $(\widehat{r}, c) \in H_1$ for all $r' < \widehat{r} < r$, and $(r',c) \notin H_1$. Thus, since the cells of R are left justified and $H_1 \cap R = \emptyset$, it follows that $(r,c) \notin R$ and $(r,c) \in H_1 \cup R = \phi(H_1)$ is rightmost in row r of $\phi(H_1)$. Moreover, since $H_2 \cap R = \emptyset$, it follows that $(r',c) \notin R$ and, as a result, $(r',c) \notin H_1 \cup R = \phi(H_1)$. Consequently, if applying a single Kohnert move at row r of $\phi(H_1)$ does not result in $\phi(H_1)\Big\downarrow_{(r',c)}^{(r,c)}$, then there must exist $r' < \widehat{r} < r$ such that $(\widehat{r}, c) \notin \phi(H_1)$; but this is impossible since $H_1 \subset H_1 \cup R = \phi(H_1)$. Hence, applying a single Kohnert move at row r of $\phi(H_1)$ results in

$$\phi(H_1)\Big\downarrow_{(r',c)}^{(r,c)} = (H_1 \cup R)\Big\downarrow_{(r',c)}^{(r,c)} = \left(H_1\Big\downarrow_{(r',c)}^{(r,c)}\right) \cup R = H_2 \cup R = \phi(H_2).$$

Since $\phi(H_2)$ can be formed from $\phi(H_1)$ by applying a Kohnert move at row r, it follows that $\phi(H_2) \prec \phi(H_1)$. Therefore, if $H_1, H_2 \in \mathcal{P}(H)$ satisfy $H_2 \prec H_1$, then $\phi(H_2) \prec \phi(H_1)$; that is, $\phi : [H', H] \to \mathcal{P}_2$ is order preserving. As noted above, it follows that ϕ is an order-preserving map from $[H', H]$ to $\mathcal{P}(\mathbb{D}(\mathbf{a}))$.

Now, our work above shows that $Hk : \mathcal{P}(\mathbb{D}(\mathbf{a})) \to [H', H]$ and $\phi : [H', H] \to \mathcal{P}(\mathbb{D}(\mathbf{a}))$ are both order preserving. Since $R \subset D$ for all $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ and $\widehat{H} \cap R = \emptyset$ for all $\widehat{H} \in [H', H]$, it follows that $\phi(Hk(D)) = D$ for all $D \in \mathcal{P}(\mathbb{D}(\mathbf{a}))$ and $Hk(\phi(\widehat{H})) = \widehat{H}$ for all $\widehat{H} \in [H', H]$. Thus, $\phi = Hk^{-1}$ so that Hk defines an isomorphism between $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ and $[H', H]$. Combining Theorem 7 and Theorem 28, the result follows. □

Now, to prove Theorem 41 using the results above, we require the following result of **[5]**.

Theorem 48 (Björner **[5]**)**.** *The product of graded posets is EL-shellable if and only if each of the posets is EL-shellable.*

We are now in a position to complete the proof of Theorem 41.

Proof of Theorem 41. Considering Theorem 37 and Proposition 42, it remains to show that if $\mathbf{a}$ is pure, then $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is EL-shellable. Let $(\alpha_1, \ldots, \alpha_m)$ be a pure decomposition of $\mathbf{a}$. Combining Theorems 36 and 37 along with Proposition 47, it follows that $\mathcal{P}(\mathbb{D}(\alpha_j))$ is graded and EL-shellable for $1 \leq j \leq m$. Thus, since $\mathcal{P}(\mathbb{D}(\mathbf{a})) \cong \mathcal{P}(\mathbb{D}(\alpha_1)) \times \ldots \times \mathcal{P}(\mathbb{D}(\alpha_m))$ by Proposition 46, the result follows from Theorem 48. □

In the theorem below, we consider the polynomial consequences of a key diagram having a pure and EL-shellable Kohnert poset.

Theorem 49. *Let* $\mathbf{a}$ *be a weak composition. If* $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ *is pure and EL-shellable, then* $\mathfrak{K}_{\mathbb{D}(\mathbf{a})}$ *is multiplicity free.*

Proof. It is shown in **[11]** that, given a weak composition $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, the polynomial $\mathfrak{K}_{\mathbb{D}(\mathbf{a})}$ is multiplicity free if and only if there exists no three indices $1 \leq i_1 < i_2 < i_3 \leq n$ such that $a_{i_1} < a_{j_2} < a_{j_3}$ and no four indices $1 \leq j_1 < j_2 < j_3 < j_4 \leq n$ such that

1. $a_{j_1} = a_{j_2} < a_{j_3} - 1 < a_{j_4} - 1$,
2. $a_{j_1} = a_{j_2} < a_{j_4} < a_{j_3}$,
3. $a_{j_2} < a_{j_1} < a_{j_4} < a_{j_3}$, and
4. $a_{j_2} < a_{j_1} < a_{j_3} = a_{j_4}$.

It is straightforward to verify that pure compositions avoid all five of the patterns above, so the result follows from Theorem 41. □

As previously noted, unlike in the case of the diagrams considered in Section 4, a key diagram having a multiplicity-free Kohnert polynomial is not equivalent to the (EL-)shellability of the diagram's Kohnert poset. To see this, note that for $\mathbf{a} = (0, 3, 3)$ we have

$$\mathfrak{K}_{\mathbb{D}(\mathbf{a})} = x_1^3x_2^3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^3x_3^3 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3 + x_2^3x_3^3,$$

which is mulitplicity free. However, considering the Hasse diagram of $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ illustrated in Figure 19, we see that $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ contains an interval isomorphic to the poset $\mathcal{P}$ of Example 10. Thus, $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is not shellable.

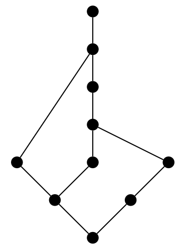


Figure 19: Hasse diagram of $\mathcal{P}(\mathbb{D}(0,3,3))$

6 Epilogue

In this article, our focus was a characterization of (EL-)shellable Kohnert posets. While we were able to establish some general results in Propositions 14 and 15, we were not able to obtain a complete characterization. Instead, we determined characterizations for some particular families of diagrams, including those with at most one cell per column and those whose first two rows are empty. Restricting attention to pure, (EL-)shellable Kohnert posets, we were able to determine characterizations for those diagrams associated with key polynomials. With respect to a general characterization, we make the following conjectures.

Conjecture 50. *There exists a finite number of families of subdiagrams $\mathcal{F}$ such that given any diagram D, $\mathcal{P}(D)$ is shellable if and only if there exists no $\tilde{D} \in \mathcal{P}(D)$ such that $\tilde{D}$ contains a subdiagram from $\mathcal{F}$.*

Conjecture 51. *Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a weak composition. Then $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ is shellable if and only if there exist no $1 \leq i_1 < i_2 < i_3 \leq n$ for which*

- $a_{i_1} < a_{i_2} < a_{i_3}$ *or*
- $a_{i_1} \leq a_{i_3} - 3 \leq a_{i_2} - 3,$

and there exist no $1 \leq j_1 < j_2 < j_3 < j_4 \leq n$ for which

- $a_{j_1} \leq a_{j_2} < a_{j_3} - 1 \leq a_{j_4} - 1,$
- $a_{j_1} \leq a_{j_2} < a_{j_4} < a_{j_3},$
- $a_{j_2} < a_{j_1} < a_{j_4} < a_{j_3},$ *or*
- $a_{j_2} < a_{j_1} < a_{j_3} \leq a_{j_4}.$

Note the similarity of Conjecture 50 with Conjecture 8.1 of **[10]**.

Along with the characterizations for shellability, we also found that, for the families of diagrams considered here, (EL-)shellability of the Kohnert poset had some interesting polynomial consequences. More specifically, for diagrams with either one cell per nonempty column or the first two rows empty, (EL-)shellability of the Kohnert poset was equivalent to the associated Kohnert polynomial being multiplicity free. On the other hand, for key diagrams, we found that the Kohnert poset being pure and (EL-)shellable only implied that the Kohnert polynomial was multiplicity free. Recall that the weak composition $\mathbf{a} = (0, 3, 3)$ generated a key polynomial $\mathfrak{K}_{\mathbb{D}(\mathbf{a})}$ which was mutliplicity free, but $\mathcal{P}(\mathbb{D}(\mathbf{a}))$ was not shellable. The authors wonder if there is a stronger polynomial property equivalent to (EL-)shellability in the case of key diagrams.

In addition to the conjectures listed above, many interesting questions remain concerning Kohnert posets. For example, it is of interest to the authors whether results similar to those contained in this article can be obtained for Kohnert posets arising from Rothe diagrams. Given a permutation $w = [w_1, \ldots, w_n]$, the associated Rothe diagram is defined as $\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\} \subset \mathbb{N} \times \mathbb{N}$, and it was shown in **[3, 14, 15]** that the Kohnert polynomial $\mathfrak{K}_{\mathbb{D}(w)}$ is the Schubert polynomial corresponding to w. Thus, a result analogous to Theorem 49 for Kohnert posets of Rothe diagrams would not only shed light on the relationship between the behaviors of Kohnert posets and Schubert polynomials, but it would also suggest a more general phenomenon. In fact, based on experimental evidence, it appears that EL-shellability of Kohnert posets associated with so-called "southwest" diagrams – a family of diagrams that contains key and Rothe diagrams as subfamilies (see **[2]**) – implies that each corresponding Kohnert polynomial is multiplicity free. In a slightly different direction, the authors are aware that E. Philips is working on identifying those Kohnert posets that are lattices **[13]**.

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