Comparison results for Markov tree distributions

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Abstract

We develop comparison results for Markov tree distributions extending ordering results from the literature on discrete time Markov processes and recently studied ordering results for conditionally independent factor models to tree structures. Based on fairly natural positive dependence conditions, our main contribution is a comparison result with respect to the supermodular order. Since this order is a pure dependence order, it has many applications in optimal transport, finance, and insurance. As an illustrative example, we consider hidden Markov models and study distributional robustness for functionals of the random walk under model uncertainty. Further, we show that, surprisingly, more general comparison results via the recently established rearrangement-based Schur order for conditional distributions, which implies an ordering of Chatterjee's rank correlation, do not carry over from star structures to trees. Several examples and a detailed discussion of the assumptions demonstrate the generality of our results and provide further insights into the behavior of multidimensional distributions.

Keywords convex risk measure, distributional robustness, factor model, hidden Markov model, Markov process, optimal transport, positive dependence, supermodular order, stochastically increasing, vine copula model.

1 Introduction

Let T = (N, E) be a tree with finitely or countably many nodes $N = \{0, 1, 2, ...\}$ and edges $E \subset N \times N$. A sequence $(X_n)_{n \in N}$ of random variables is said to follow a Markov tree distribution with respect to T if, for each two finite and disjoint sets $A, B \subset N$, the variables $(X_a)_{a \in A}$ and $(X_b)_{b \in B}$ are conditionally independent given X_i for any node $i \in N$ that separates A and B, see Definition 2.3. This concept extends the Markov property from a chain of nodes to tree structures, noting that for Markov processes future and past events are conditionally independent given the present value. Further subclasses are conditionally independent factor models, hidden Markov models [29, 43], tree-indexed Markov chains [22, 89], and vine copula models truncated after the first level [17, 26], see Figure 1. Every Markov tree distribution can be specified by a sequence $F = (F_n)_{n \in N}$ of univariate marginal distribution functions and a sequence $B = (B_e)_{e \in E}$ of bivariate copulas that describe the dependence structure between each two random variables that are adjacent in T, see Proposition 2.5. We write $\mathcal{M}(F, T, B)$ for the Markov tree distribution with these specifications.

The main contribution of our paper, Theorem 1.3, is a supermodular ordering result for Markov tree distributed random variables $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$, where we establish general conditions on the specifications of X and Y to infer integral inequalities of the form

$$\mathbb{E}[f(X)] \le \mathbb{E}[f(Y)] \tag{1}$$

for all supermodular functions f such that the expectations exist. The class of supermodular functions includes many interesting functions such as componentwise minima and maxima, distribution and survival functions, as well as convex functions of the component sum, see Table 1. In particular, for any law-invariant convex risk measure Ψ , the functional $\Psi(\sum_n X_n)$ is consistent with the supermodular order, which furthermore has the appealing properties that it is a pure dependence order (i.e. $law(X_n) = law(Y_n)$ for all $n \in N$) and invariant under increasing transformations of the components [87, 101, 104]. Hence, it allows applications in various fields, where the marginal distributions are assumed to be fixed, such as

optimal transport or mathematical finance [18, 19, 20, 50, 92, 115, 121]. Integral stochastic orders are useful in various applications, since many parametric models in statistics exhibit monotonicity properties in their parameters for several classes of functionals [82, 85, 101, 120]. The novelty of our approach is that we establish, for the flexible class of Markov tree distributions, general conditions on the marginal and dependence specifications implying integral inequalities as in (1) for several classes of functions, see Section 3. In particular, our results provide simple conditions for constructing and comparing positive supermodular dependent distributions: As building blocks, we can take any set of univariate distribution functions and any set of bivariate copulas that are conditionally increasing and pointwise larger than the product copula. A supermodular comparison based on bivariate tree specifications is then obtained by a pointwise comparison of the bivariate copulas, see Theorem 3.1.

As a consequence, we obtain distributional robustness for various functionals that are consistent with the supermodular order. To this end, consider for a tree T = (N, E), for marginal distribution functions $F = (F_n)_{n \in N}$, and for suitable subclasses $(\mathcal{C}_e)_{e \in E}$ of bivariate copulas, the problem

minimize/maximize
$$\mathbb{E} f(X)$$
 subject to $X = (X_n)_{n \in \mathbb{N}} \sim \mathcal{M}(F, T, B)$, (2)

$$B = (B_e)_{e \in E} \text{ with } B_e \in \mathcal{C}_e \text{ for all } e \in E, \qquad (3)$$

where f is a supermodular function. Condition (2) ensures that the univariate marginal distributions are fixed and that X satisfies the Markov property with respect to the tree T. The copula constraints in (3) relate to the dependence structure of the transition kernels of the Markov tree distributions. Under positive dependence conditions on the classes C_e of bivariate copulas as above, we determine in Corollary 3.2 solutions to the above optimization problem in order to obtain distributional robustness for various functionals. Our comparison results with respect to the lower orthant, upper orthant, and directionally convex order also allow to incorporate distributional robustness in the marginal distributions. In Section 4, we give an application to hidden Markov models, where we determine a dependence uncertainty band for the distribution function of the maximal observations of a random walk under model uncertainty and noise. Note that the above optimization problem can also be interpreted as a multi-marginal Markovian optimal transport problem with dependence constraints. It is related to various optimal transport problems studied in the literature, for instance, to multi-marginal optimal transport, optimal transport with linear constraints, and martingale optimal transport [18, 19, 20, 50, 121]. Our comparison results also allow connections to the literature on similarity of stochastic processes, as studied in the context of adaptive, causal or bicausal optimal transport in [11, 12, 13, 53, 98].

1.1 Main result

While comparison results for Markov processes have been known for a long time [38, 58, 103], a supermodular comparison of star structures has been shown recently in [8]. Both results are based on asymmetric positive dependence conditions, which differ for chain and star structures. Our main comparison result for Markov tree distributions is based on the proof for ordering Markov processes in [58] and incorporates star structures in a technically sophisticated way. To motivate the different and non-intuitive positive dependence assumptions in Theorem 1.3, we first provide the comparison results for chain and star structures.

Regarding the notation, we write $U \uparrow_{st} V$ for random variables U and V, if U is stochastically increasing (SI) in V, i.e., if the conditional distribution U|V = v is increasing in v with respect to the stochastic order, see (13). The following result is a direct extension of the supermodular comparison of stationary discrete-time Markov processes in [58, Theorem 3.2] to a non-stationary setting.

Proposition 1.1 (Supermodular ordering of Markov processes)

Let $X = (X_i)_{i \in \mathbb{N}_0}$ and $Y = (Y_i)_{i \in \mathbb{N}_0}$ be Markov processes in discrete time. Assume for all $i \in \mathbb{N}_0$ that

- (i) $X_{i+1} \uparrow_{st} X_i$,
- (*ii*) $Y_i \uparrow_{st} Y_{i+1}$,
- (*iii*) $(X_i, X_{i+1}) \leq_{sm} (Y_i, Y_{i+1})$ (resp. \geq_{sm}).

Then it follows that $X \leq_{sm} Y$ (resp. \geq_{sm}). In particular, X and Y are positive supermodular dependent.

a)
$$X_0 \xrightarrow{B_{01}} X_1 \xrightarrow{B_{12}} X_2 \xrightarrow{B_{23}} X_3 \cdots$$



Figure 1 Examples of Markov tree distributions: a) A Markov process in discrete time, where the underlying tree is a chain. b) Random variables X_1, \ldots, X_d that are conditionally independent given the common factor variable X_0 , where the underlying tree has a star-like structure. c) A tree-indexed Markov process with a general underlying tree structure. d) A hidden Markov model with hidden nodes $(X_i)_{i \in 2\mathbb{N}_0}$ and observable nodes $(X_i)_{i \in 2\mathbb{N}_0+1}$. Each model is uniquely determined by the univariate marginal distributions specifying the nodes and by the bivariate copulas $(B_{ij})_{(i,j)\in E}$ specifying the edges of the underlying tree T = (N, E), see Proposition 2.5. If Lebesgue-densities exist, a Markov tree distribution is a vine copula model truncated after the first level, where a) corresponds to a D-vine structure and b) to a C-vine structure, see [26].

In terms of Markov tree distributions, the underlying tree structure corresponds to a chain of nodes, see Figure 1a). Assumptions (i)-(iii) of the above proposition only refer to the bivariate distributions of X and Y and are therefore easy to verify. While the first two conditions are positive dependence concepts, the third condition relates to the supermodular ordering of bivariate random variables that are adjacent in the underlying chain. In contrast to higher-dimensional distributions, for bivariate distributions the supermodular order can easily be verified because, for the two-dimensional case, the supermodular order is equivalent to identical marginal distributions and the pointwise order of the associated bivariate distribution functions, see (9). Due to assumptions (i) and (ii), X and Y are SI in 'opposite directions'. As we discuss in Section 5, these rather odd conditions can neither be changed to the weaker notion of positive supermodular dependence nor be replaced by SI in the 'same direction'. Note that, under the assumptions of Proposition 1.1, X and Y are positive supermodular dependent, which is a positive dependence concept and implies, in particular, pairwise non-negative correlations.

For our main result, we make use of the following recently established supermodular ordering result which compares positive supermodular dependent random variables that are conditionally independent given a common factor variable. Such factor models may be interpreted as Markov tree distributions where the underlying tree has a star-like structure, see Figure 1b).

Lemma 1.2 (Supermodular ordering of Markovian star structures, [8, Corollary 4(i)])

Let $X = (X_0, \ldots, X_d)$ and $Y = (Y_0, \ldots, Y_d)$ be random vectors. Assume that X_1, \ldots, X_d are conditionally independent given X_0 and that Y_1, \ldots, Y_d are conditionally independent given Y_0 . Assume for all $i \in \{1, \ldots, d\}$ that

- (i) $X_i \uparrow_{st} X_0$,
- (*ii*) $Y_i \uparrow_{st} Y_0$,
- (*iii*) $(X_0, X_i) \leq_{sm} (Y_0, Y_i)$.

Then it follows that $X \leq_{sm} Y$. In particular, X and Y are positive supermodular dependent.



Figure 2 An example that illustrates the positive dependence conditions (i) and (ii) in Theorem 1.3 for a tree T = (N, E) on 12 nodes with root 0. For $(i, j) \in E$, an arrow $X_i \longrightarrow X_j$ indicates that X_j is stochastically increasing in X_i , i.e., $X_j \uparrow_{st} X_i$. An arrow $X_i \longleftrightarrow X_j$ indicates that $X_j \uparrow_{st} X_i$ and $X_i \uparrow_{st} X_j$. This applies similarly to the variables (Y_i, Y_j) . Note that there is no positive dependence condition between X_0 and X_7 and between Y_3 and Y_4 . The leaves consist of the set $L = \{2, 4, 5, 6, 10, 11\}$. The set $P \subseteq N$ consists of the leaf $\ell = 4$ and the path $p(0, \ell)$ between the root 0 and the leaf ℓ , i.e., $P = \{1, 3, 4\}$. The node k^* is given by $k^* = 7$.

While for chain structures, X and Y must satisfy opposite SI conditions, the above result requires that X and Y fulfil the same SI conditions, see Section 5 and Figures 7 and 8. As our main theoretical contribution in this paper, we extend Proposition 1.1 and Lemma 1.2 to arbitrary Markov tree distributions where the underlying tree T = (N, E) may have finitely or countably many nodes, see Figure 1c). To this end, let T = (N, E) be a directed tree with root $0 \in N$. Let $L \subset N$ be the set of leaves of T, see (4). Let $P = \{\ell_1, \ell_2, \ell_3 \dots\} \subseteq N \setminus \{0\}$ with $(0, \ell_1), (\ell_1, \ell_2), (\ell_2, \ell_3), \dots \in E$ be a path of nodes that starts with a child of the root and either terminates at a leaf node $\ell \in L$ or has infinitely many nodes. Further, let k^* be a child of the root 0 that is not an element in the path P unless it is the only child, i.e., $(0, k^*) \in E$ with $k^* \notin P$ if deg $(0) \geq 2$, where deg(n) denotes the degree of a node $n \in N$, see Definition B.1. Then the following result holds true.

Theorem 1.3 (Supermodular ordering of Markov tree distributions)

Let $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ be sequences of random variables that follow a Markov tree distribution with respect to T. Assume for all $e = (i, j) \in E$ that

- (i) $X_j \uparrow_{st} X_i \text{ if } j \neq k^*$,
- (*ii*) $Y_i \uparrow_{st} Y_j$ if $j \notin L$, and $Y_j \uparrow_{st} Y_i$ if $j \notin P$,
- (iii) $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ (resp. \geq_{sm}).

Then it follows that $X \leq_{sm} Y$ (resp. \geq_{sm}). If additionally (X_0, X_{k^*}) is positive supermodular dependent, then X is positive supermodular dependent. Moreover, if additionally (Y_i, Y_j) is positive supermodular dependent for $(i, j) \in E$ with $j \in P \cap L$, then Y is positive supermodular dependent.

The non-intuitive SI assumptions in Theorem 1.3 are illustrated in Figure 2 and can, like the supermodular comparison of bivariate distributions, be easily verified as the following remark shows. A detailed discussion of the assumptions in Section 5 proves the generality of the above theorem and establishes that none of the SI assumptions can be omitted or weakened to positive supermodular dependence. As a direct consequence of the above result, we give in Theorem 3.1 simple sufficient conditions on the bivariate copula specifications for a supermodular comparison of Markov tree distributions.

Remark 1.4 (a) For random variable U and V, the SI condition $U \uparrow_{st} V$ is a positive dependence property which is equivalent to the conditional survival probability P(U > u|V = v) being for all $u \in \mathbb{R}$ increasing in v outside a V-null set that may depend on u. In particular, this implies that $P(U > u, V > v) \ge P(U > u)P(V > v)$, see Section 2.3. Many well-known families of bivariate distributions are SI, such as extreme value distributions [48], various Archimedean copulas [86], and the bivariate normal distribution for non-negative parameter of correlation [96], see also [5]. Further, the uniquely determined increasing rearrangement of a bivariate copula, recently studied in the context of dependence measures, is by construction SI, see [4, 6, 10, 107]. Since SI random vectors are invariant under increasing transformations of the components (i.e., $U \uparrow_{st} V$ implies $f(U) \uparrow_{st} g(V)$ for all increasing functions f and g) the SI property is a copula-based dependence concept, i.e., it suffices to analyse copulas, see (5) for the notion of copula. Considering directed trees allows to incorporate asymmetric dependencies, noting that, in general, $U \uparrow_{st} V$ does not imply $V \uparrow_{st} U$, see [85, Example 3.4]. Directed trees are also used for modeling causal inference, see, e.g., [31, 93].

- (b) As already mentioned, for bivariate distributions, the supermodular order can easily be verified due to its characterization by the concordance order in (9). However, for dimensions larger than 2, the supermodular order is strictly stronger than the concordance order and a verification is challenging since no small of class of functions generating the supermodular order is known. Therefore, Theorem 1.3 is meaningful because it provides a new method for constructing and comparing multivariate distributions based on bivariate building blocks, using that for bivariate distributions, various ordering result are well-known: For the bivariate normal distribution, the supermodular order corresponds to an ordering of the correlation parameter, which goes back to [106], see [7, 25, 87] for extensions to multivariate normal and elliptical distributions. A characterization of the supermodular order for bivariate Archimedean copulas in terms of their Archimedean generator and for bivariate extreme-value copulas in terms of their Pickands dependence function follows from [88, Theorem 4.4.2] and [5, Theorem 3.4], respectively. Note that it suffices to compare the underlying copulas since the supermodular order is invariant under increasing transformations.
- (c) Theorem 1.3 compares, in particular, positive supermodular dependent random vectors and also covers the extreme cases of positive supermodular dependence, i.e., independence and comonotonicity: Exactly in the case where X_i and X_j are independent for all $e = (i, j) \in E$, we have that $(X_n)_{n \in N}$ is a sequence of independent random variables. In the case of continuous marginal distributions, if (X_i, X_j) is comonotonic for all $e = (i, j) \in E$, then also $(X_n)_{n \in N}$ is comonotonic. Note that comonotonicity models perfect positive dependence and relates to the upper Fréchet bound which is the greatest element with respect to the supermodular order in Frèchet classes (i.e., in classes of distributions with fixed marginals), see (12). For discontinuous marginal distributions, comonotonicity can generally not be modeled by a Markov tree distribution because the marginal distributions can affect the dependence structure in Markovian models, see Example A.2.
- (d) Theorem 1.3 extends comparison results for conditionally independent factor models and for discrete time Markov processes to Markov tree distributions, compare Figure 2. To see this, let T be a chain, i.e., the edges of the tree are given by E = {(0,1), (1,2), (2,3),...}. Then condition (i) in Theorem 1.3 is X_{i+1} ↑_{st} X_i for i ∈ {1,2,3,...} and condition (ii) simplifies to Y_i ↑_{st} Y_{i+1} for i ∈ {0,1,2,...}. Hence, Theorem 1.3 generalizes Proposition 1.1. In particular, we obtain that condition (i) in Proposition 1.1 can be skipped for i = 0. In the case where T is a star on d+1 nodes, i.e., when L = N \{0} (all nodes except the root are leaves), then the set of edges is given by E = {(0,1), (0,2),..., (0,d)}. In this case, conditions (i) and (ii) of Theorem 1.3 simplify to X_j ↑_{st} X₀ for i ∈ {1,...,d} \ {k*} and Y_j ↑_{st} Y₀ for j ∈ {1,...,d} \ {ℓ}. Hence, Theorem 1.3 also generalizes Lemma 1.2 noting that condition (i) can be skipped for i = k* and condition (ii) can be skipped for i = ℓ.

1.2 Related literature

The significance of Theorem 1.3 lies in the fact that it compares positive dependent multivariate distributions with respect to the strong notion of supermodular order. In the context of stochastic processes and time series, random variables in temporal or spatial proximity typically depend positively on each other. In risk management, for example, loan defaults are often positively dependent, or insurance losses exhibit positive dependencies [79]. In order to model positive dependence structures, comparison results are of particular importance, indicating the strength of the positive interrelations. Comparison results and various concepts of positive dependence are studied in the literature on multivariate parametric models for the normal distribution [82, 96, 106], for elliptical distributions [1, 7, 61, 120], for mixtures of elliptical distributions [90, 91], and for Archimedean copula models [86] (which corresponds to l^1 -norm symmetric distributions [78]). For non-parametric distributions, general inequalities for positive dependence, dependence, and variables are given under some structural assumptions, such as conditional independence,

common marginals or exchangeability in [6, 8, 21, 105, 111, 112, 114]. Comparison results for Markov chains and Markov processes which exhibit positive dependencies are studied in [14, 16, 38, 57, 58, 69, 103]. The present paper contributes to the literature by extending various ordering results for discrete time Markov processes to tree structures and thus by providing general inequalities for Markov tree distributions. In particular, we obtain distributional robustness for various functionals, which we illustrate in the special case of hidden Markov models. For an overview of inequalities for multivariate distributions, see, e.g., [76, 87, 104, 113].

From a practical point of view, the supermodular order is of great importance in financial and actuarial mathematics. For instance, numerous payoff functions of financial derivatives are supermodular, see Table 1 and [9, 73, 110]. Further, by (10), the supermodular order is useful in risk management in quantifying portfolio risk and determining portfolio risk bounds under dependence information [87, 102], where the marginals can often be inferred from data; see, e.g., [23, 27, 109] and the references therein. From a theoretical point of view, inequalities for distributions with fixed marginals are studied in the field of optimal transport [92, 115]. In this regard, Theorem 1.3 yields solutions to optimal transport problems with additional structural information, such as the Markov property and positive dependencies. The supermodular order also allows for a clear interpretation because it can be described by simple rearrangements or mass transfers, as studied in [83].

In applications, typically, the one-dimensional, and to some extent, the two-dimensional marginal distributions can be estimated or partially estimated from data. However, due to the curse of dimensionality, the entire dependence structure of a random vector can generally not be determined. Therefore, models are needed that are flexible and robust on the one hand, but also easy to understand on the other hand. A large class of time series and evolution models are Markovian, where future and past events are independent conditionally on the present. Such models are completely specified by the bivariate distributions of adjacent variables, i.e., by the univariate marginal distributions and the bivariate copulas specifying the edges of a chain of variables, cf. Figure 1a). Nested models [62] allow for incorporating higher-order conditional dependencies, see [56, 77] for nested Archimedean copula models and [17, 36, 37] for vine copula models. The latter class of models allows to incorporate dependencies in a flexible way and is used in various applications, for example, in the fields of climate and wind [32, 46, 52, 60, 116], finance and risk management [59, 117, 119], and statistical learning [24, 72, 108]. However, further research on distributional and statistical properties of vine copula models is needed, see, e.g., [2, 40, 51, 81]. Since absolutely continuous Markov tree distributions are vine copula models truncated after the first level [26], the results presented in this paper also provide new insight into distributional properties of regular vine copula models.

The rest of the paper is organized as follows. Section 2 provides the definitions of Markov tree distributions, the stochastic orderings, and positive dependence concepts, which we use in this paper. As a consequence of our main result, we derive in Section 3 various comparison results for the lower orthant, upper orthant, and directionally convex order, which also allow flexibility in the marginal distributions. Section 4 provides distributional robustness of various functionals on classes of hidden Markov models. A detailed discussion of the assumptions of our main result and a special ordering property for star structures in consistency with Chatterjee's rank correlation are provided in Section 5. All proofs and, in particular, the proof of Theorem 1.3, which requires further technical details, are deferred to the appendix.

2 Preliminaries

This section provides the basic notation and concepts used in this paper. It covers the definition of trees and copulas, which serve as the basic elements for constructing Markov tree distributions. Proposition 2.5 provides a simple representation of Markov tree distributions in terms of bivariate tree specifications, which is, on the one hand, a useful tool for constructing Markov tree distributions and, on the other hand, helpful to formalize the proofs of the comparison results studied in this paper. The second and third part of this section outline the definitions and basic relationships of the relevant stochastic orders and concepts of positive dependence. For the proofs of the results in this section, we refer to Appendix C.

2.1 Markov tree distributions

Trees can be used to model simple dependencies between random variables. While each node of a tree represents a random variable, the edges model the dependence structure between adjacent random variables [37, 68]. Markov tree distributions are uniquely determined by specifying univariate distribution functions associated with the nodes and bivariate copulas associated with the edges of the tree.

2.1.1 Trees and Markov realizations

We denote by N an at most countable set of nodes which we label with the integers $N = \{0, 1, ..., d\}$ for $d \in \mathbb{N}$, whenever N has finitely many elements, and with $N = \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ otherwise. We assume $|N| \ge 2$ to avoid cumbersome notation for trivial cases, where |N| denotes the number of elements of N. A graph on N is a tuple (N, E), where $E \subset N \times N$ is a set of oriented edges. By abuse of notation we write $\{i, j\} \in E$ if $(i, j) \in E$ or $(j, i) \in E$.

Definition 2.1 (Directed path, undirected path)

Let (N, E) be a graph and let $i, j \in N, i \neq j$, be two nodes.

- (i) A directed path from i to j is a vector $(i, i_1, \ldots, i_m, j) \subseteq N^{m+2}$ of m+2 nodes, $m \in \mathbb{N}_0$, such that $(i, i_1), (i_1, i_2), \ldots, (i_m, j) \in E$,
- (ii) A (undirected) path between i and j is a set $\{i_1, \ldots, i_m\} \subseteq N$ of m distinct nodes, $m \in \mathbb{N}_0$, such that $\{i, i_1\}, \{i_1, i_2\}, \ldots, \{i_m, j\} \in E$,

In the literature on dependence modeling, trees are often defined as acyclic graphs, where the edges are unordered pairs of nodes [17, 35, 36, 63]. Since we generally allow asymmetric dependence properties, we focus on directed trees (a.k.a. polytrees or oriented trees) with a root that we label without loss of generality as $0 \in N$. Due the following definition, a tree is a graph in which all nodes can be reached from the root by a unique directed path. Such trees are also called arborescences.

Definition 2.2 (Tree)

A directed tree is a graph T = (N, E) such that

- (i) for all $i \in N \setminus \{0\}$ there exists a unique directed path from 0 to i,
- (ii) $(i, j) \in E$ implies $(j, i) \notin E$,

In the following we refer to trees in the context of directed trees in the sense of Definition 2.2. By the definition of a tree, a node may have infinite degree, i.e., an infinite number of adjacent nodes. Due to (i) and (ii) in the above definition, an undirected path between i and j is uniquely determined and may contain the root. We denote this path by $p(i, j) \subseteq N$ (or equivalently by $p(j, i) \subseteq N$). By the definition of an undirected path, the nodes i and j are not included in p(i, j). The *leaves* of a tree are defined as the subset of nodes in $N \setminus \{0\}$ having only one adjacent node, i.e.,

$$L := \{k \in N \setminus \{0\} \mid \deg(k) = 1\} \subset N.$$

$$\tag{4}$$

If $|N| < \infty$, the set of leaves is non-empty. The concept of Markov tree dependence uses a tree to model conditional independence between random variables indexed by the nodes of the tree, see [35, 80]. Special cases are Markov processes in discrete time and conditional independent factor models, where the underlying tree is a chain and a star, respectively, see Figure 1*a*) and *b*). A node $i \in N$ is said to *separate* two disjoint sets $A, B \subset N$ if for every $a \in A$ and $b \in B$ the path between *a* and *b* contains *i*.

Definition 2.3 (Markov tree dependence; [80, Definition 5])

Let T = (N, E) be a tree. A distribution μ on $\mathbb{R}^{|N|}$ (resp. $\mathbb{R}^{\mathbb{N}_0}$ if $N = \mathbb{N}_0$) has Markov tree dependence (or is a Markov tree distribution) with respect to T if there exists a sequence $(X_i)_{i \in N} \sim \mu$ of random variables such that for every two finite disjoint subsets $A, B \subset N$ and for every $i \in N$ that separates A and B, the vectors $X_A = (X_j)_{j \in A}$ and $X_B = (X_j)_{j \in B}$ are conditionally independent given X_i . Weaker (i.e., non-Markovian) concepts of tree dependence can also be found in [17, 80]. Hierarchical tree structures which allow the modeling of higher order conditional dependencies are used in the context of vine copula models [17, 36, 37], noting that vine copula models truncated after the first level tree are Markov tree distributions [26].

2.1.2 Bivariate tree specifications

For various comparison results, we make use of the concept of copulas which is a tool that allows to study dependence structures between random variables. More precisely, a *d*-copula is a *d*-variate distribution function $C: [0, 1]^d \rightarrow [0, 1]$ having uniformly on [0, 1] distributed univariate marginal distributions. Due to Sklar's theorem, every *d*-variate distribution function *F* can be decomposed into its marginal distribution functions F_i , $1 \leq i \leq d$, and a *d*-copula *C* such that the joint distribution function can be expressed as the concatenation of these, i.e.,

$$F(x) = C(F_1(x_1), \dots, F_d(x_d)) \quad \text{for all } x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

$$(5)$$

In this case C is called a copula of F. The copula C is uniquely determined on $\operatorname{Ran}(F_1) \times \cdots \times \operatorname{Ran}(F_d)$, where $\operatorname{Ran}(f)$ denotes the range of a function f. Further, for any copula C and for any marginal distribution functions F_1, \ldots, F_d , the right-hand side of (5) defines a d-variate distribution function. If $X = (X_1, \ldots, X_d)$ has distribution function F, we say that $C = C_X$ is a copula of X. We denote by \mathcal{C}_d the class of d-variate copulas. For an overview of the concept of copulas, see, e.g., [42, 88, 101].

As a consequence of the definition of Markov tree dependence, for any path $p(i, j) = \{i_1, \ldots, i_m\}$ from i to j, the conditional distribution $X_i \mid (X_{i_1}, \ldots, X_{i_m}, X_j)$ depends only on the random variable X_{i_1} , which is adjacent to X_i . This implies that Markov tree distributions are completely specified by bivariate distributions corresponding to the edges of the underlying tree. Due to Sklar's theorem, each such bivariate distribution can be decomposed into two marginal distributions and a bivariate copula which describes the dependence structure. For a fixed tree, a bivariate tree specification assigns a univariate distribution function to each node and a bivariate copula to each edge of the tree as follows.

Definition 2.4 (Bivariate tree specification; [80, Definition 4])

A triple $\mathcal{T} = (F, T, B)$ is a bivariate tree specification if

- (i) T = (N, E) is a tree,
- (ii) $F = (F_n)_{n \in \mathbb{N}}$ is a family of univariate distribution functions,
- (iii) $B = (B_e)_{e \in E}$ is a family of bivariate copulas.

For a probability distribution μ on $\mathbb{R}^{|N|}$, if $d < \infty$, and on $\mathbb{R}^{\mathbb{N}_0}$, if $N = \mathbb{N}_0$, denote by μ_n and μ_{ij} , $n, i, j \in N$, the univariate and bivariate marginal distributions with respect to the components n and (i, j), respectively. Then μ is said to *realize* a bivariate tree specification $\mathcal{T} = (F, T, B)$, if for all $n \in N$ and $e = (i, j) \in E$, F_n is the distribution function of μ_n and B_e is a copula of μ_{ij} .

Due to the following proposition, for every bivariate tree specification there exists a unique realizing Markov tree distribution, see [17] for the case when Lebesgue densities exist.

Proposition 2.5 (Markov realization of bivariate tree specification)

For every bivariate tree specification $\mathcal{T} = (F, T, B)$ there is a unique distribution μ that realizes the bivariate tree specification \mathcal{T} such that μ has Markov tree dependence with respect to T.

We denote the uniquely determined Markov realization of a bivariate tree specification $\mathcal{T} = (F, T, B)$ by $\mathcal{M}(F, T, B)$ or $\mathcal{M}(\mathcal{T})$ and write $X \sim \mathcal{M}(F, T, B)$ for a sequence $X = (X_n)_{n \in N}$ of random variables with Markov tree dependence specified by \mathcal{T} .

In the case that the marginal distributions and the bivariate copulas of a bivariate tree specification admit Lebesgue densities, the corresponding Markov realization has also a Lebesgue-density with a simple representation as follows, see [17, 80].

Proposition 2.6 (Density representation of Markov tree distributions)

Let T = (N, E) be a tree with $d := |N| < \infty$ and let $\mathcal{T} = (F, T, B)$ be a bivariate tree specification. Assume that F_n and B_e have Lebesgue densities f_n and b_e , respectively, for all $n \in N$ and $e \in E$. Then, the Markov tree distribution $\mathcal{M}(F, T, B)$ has a Lebesgue density $g \colon \mathbb{R}^{d+1} \to [0, \infty)$ which is given by

$$g(x) = \prod_{n \in N} f_n(x_n) \prod_{e=(i,j) \in E} b_e(F_i(x_i), F_j(x_j)) \quad \text{for all } x = (x_0, \dots, x_d) \in \mathbb{R}^{d+1}.$$
 (6)

2.2 Stochastic orderings

Our comparison results are formulated in terms of integral stochastic orderings, which compare expectations of functions of two random vectors. Therefore, let $V = (V_1, \ldots, V_d)$ and $W = (W_1, \ldots, W_d)$ be *d*-variate random vectors defined on a probability space (Ω, \mathcal{A}, P) which we assume to be non-atomic. For some class \mathcal{F} of real-valued measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, the integral stochastic ordering

 $V \prec_{\mathcal{F}} W$ is defined by $\mathbb{E}f(V) \leq \mathbb{E}f(W)$ for all $f \in \mathcal{F}$,

where the comparison of expectations is generally restricted to the subclass of functions in \mathcal{F} such that the expectations exist. For $f: \mathbb{R}^d \to \mathbb{R}$, denote by $\Delta_i^{\varepsilon} f(x) := f(x + \varepsilon e_i) - f(x)$ the difference operator of length $\varepsilon > 0$ applied to the *i*th component, where e_i is the *i*th unit vector with respect to the standard base in \mathbb{R}^d . Then f is said to be supermodular respectively directionally convex, if $\Delta_i^{\varepsilon_i} \Delta_j^{\varepsilon_j} f(x) \ge 0$ for all $x \in \mathbb{R}^d$ and for all $1 \le i < j \le d$ and $1 \le i \le j \le d$, respectively. Further, f is said to be Δ monotone respectively Δ -antitone, if $\Delta_{i_1}^{\varepsilon_1} \cdots \Delta_{i_k}^{\varepsilon_k} f(x) \ge 0$ and $(-1)^k \Delta_{i_1}^{\varepsilon_1} \cdots \Delta_{i_k}^{\varepsilon_k} f(x) \ge 0$, respectively, for all $k \in \{1, \ldots, d\}$, $\varepsilon_1, \ldots, \varepsilon_k > 0$, $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$ and $x \in \mathbb{R}^d$. Note that Δ -monotone functions and Δ -antitone functions are supermodular. Further, directionally convex functions are the functions that are supermodular and componentwise convex. If f is sufficiently smooth, then f is supermodular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} f(x) \ge 0$ for all $1 \le i < j \le d$ and for all x. Similar properties hold true for sufficiently smooth directionally convex, Δ -monotone and Δ -antitone functions [87]. For several examples of such functions, see Table 1.

Denote by \mathcal{F}_{\uparrow} , \mathcal{F}_{sm} , \mathcal{F}_{dcx} , the class of componentwise increasing, supermodular and directionally convex functions, respectively, by \mathcal{F}_{\uparrow} , \mathcal{F}_{cx} , and $\mathcal{F}_{icx} := \mathcal{F}_{\uparrow} \cap \mathcal{F}_{cx}$ the class of increasing, convex, and increasing convex functions on \mathbb{R} , respectively. Let F_V and \overline{F}_V be the distribution function and survival function associated with a random vector V. We make use of the following integral stochastic orderings.

Definition 2.7 (Stochastic orderings)

- (a) Let $V = (V_1, \ldots, V_d)$ and $W = (W_1, \ldots, W_d)$ be *d*-variate random vectors. Then V is said to be smaller than W with respect to
 - (i) the lower orthant order, written $V \leq_{lo} W$, if $F_V(x) \leq F_W(x)$ for all $x \in \mathbb{R}^d$,
 - (ii) the upper orthant order, written $V \leq_{uo} W$, if $\overline{F}_V(x) \leq \overline{F}_W(x)$ for all $x \in \mathbb{R}^d$,
 - (iii) the concordance order, written $V \leq_c W$, if $V \leq_{lo} W$ and $V \leq_{uo} W$,
 - (iv) the supermodular order, written $V \leq_{sm} W$, if $V \prec_{\mathcal{F}_{sm}} W$,
 - (v) the directionally convex order, written $V \leq_{dex} W$, if $V \prec_{\mathcal{F}_{dex}} W$,
 - (vi) the stochastic order, written $V \leq_{st} W$, if $V \prec_{\mathcal{F}_{\uparrow}} W$.
- (b) Let $V = (V_n)_{n \in \mathbb{N}}$ and $W = (W_n)_{n \in \mathbb{N}}$ be stochastic processes. Let \prec be one of the orderings in (a). Then V is said to be smaller than W with respect to \prec if for all $m \in \mathbb{N}$ and all $(n_1, \ldots, n_m) \in \mathbb{N}^m$, one has $(V_{n_1}, \ldots, V_{n_m}) \prec (W_{n_1}, \ldots, W_{n_m})$.
- (c) Let S and T be real-valued random variables. Then S is said to be smaller than T with respect to
 - (vii) the convex order, written $S \leq_{cx} T$, if $S \prec_{\mathcal{F}_{cx}} T$,
 - (viiii) the increasing convex order, written $S \leq_{icx} T$, if $S \prec_{\mathcal{F}_{icx}} T$.

Note that the comparison of stochastic processes in Definition 2.7(b) is defined through the comparison of the finite-dimensional marginal distributions, which corresponds to the notion of strong comparison of stochastic processes in [87, Definition 5.1.2]. Further, the lower orthant order and the upper orthant order are also integral stochastic orderings which are generated by the class of Δ -antitone and Δ -monotone functions, i.e.,

$$V \leq_{lo} W \iff \mathbb{E}f(V) \leq \mathbb{E}f(W) \text{ for all } f \in \mathcal{F}_{\Delta}^{-},$$

$$V \leq_{uo} W \iff \mathbb{E}f(V) \leq \mathbb{E}f(W) \text{ for all } f \in \mathcal{F}_{\Delta},$$
(7)

see [95, 101]. Some basic relations between the above considered integral stochastic orderings are

$$V \leq_{dcx} W \qquad \longleftarrow \qquad V \leq_{sm} W \qquad \Longrightarrow \qquad V \leq_c W.$$
 (8)

As a direct consequence of the definition, the concordance order requires that V and W have the same univariate marginal distributions, i.e., $V \leq_c W$ implies $V_i \stackrel{d}{=} W_i$ for all $i \in \{1, \ldots, d\}$. Due to (8), also the supermodular ordering requires equal marginal distributions and, thus, both \leq_c and \leq_{sm} are pure dependence orders. In particular, for bivariate random vectors V and W with the same univariate marginal distributions, the lower orthant, upper orthant, concordance, supermodular and directionally convex order are equivalent, i.e., if d = 2, then

$$V \leq_{lo} W \quad \Longleftrightarrow \quad V \leq_{uo} W \quad \Longleftrightarrow \quad V \leq_c W \quad \Longleftrightarrow \quad V \leq_{sm} W \quad \Longleftrightarrow \quad V \leq_{dcx} W$$
(9)

whenever $V_1 \stackrel{d}{=} W_1$ and $V_2 \stackrel{d}{=} W_2$, see [84, Theorem 2.5]. Further, $V \leq_c W$ and thus $V \leq_{sm} W$ implies $\operatorname{Cor}(V_i, V_j) \leq \operatorname{Cor}(W_i, W_j)$ for all $i \neq j$, where Cor denotes the correlation in the sense of Pearson, Spearman or Kendall, whenever defined, see [101, Remark 6.3]. Since, for $d \geq 3$, $V \leq_c W$ does not imply $\sum_i V_i \leq_{cx} \sum_i W_i$ and thus also not $V \leq_{sm} W$, see [84, Theorem 2.6], we focus on comparison results with respect to the stronger concept of supermodular order. As a consequence, we obtain inequalities for various classes of functionals relevant to many applications such as

$$V \leq_{sm} W$$
 or $V \leq_{dcx} W \implies h(V) \leq_{icx} h(W) \implies \Psi(h(V)) \leq \Psi(h(W)),$ (10)

where h is a componentwise increasing or decreasing supermodular function and Ψ is a convex, lawinvariant risk measure on a proper space such as the space of integrable or the space of bounded random variables, see [101, Corollary 6.16], [47, Chapter 4] and [15, 28, 64, 100]. If $h(V) = \sum_{i=1}^{d} V_i$, then $V \leq_{sm} W$ or $V \leq_{dcx} W$ implies $\sum_{i=1}^{d} V_i \leq_{cx} \sum_{i=1}^{d} W_i$, i.e., the component sums are then ordered with respect to the convex order. Hence, due to (10), supermodular or directionally convex comparison results yield, in particular, various comparison results for risk functionals such as the average-value-at-risk of the aggregated risk vector, which may stand for a portfolio risk in finance or for the risk of total damages in insurance. The supermodular order also has the important property that it is invariant under increasing transformations of the components, i.e., for all increasing functions $k_1, \ldots, k_d \colon \mathbb{R} \to \mathbb{R}$, one has

$$(V_1, \dots, V_d) \leq_{sm} (W_1, \dots, W_d) \implies (k_1(V_1), \dots, k_d(V_d)) \leq_{sm} (k_1(W_1), \dots, k_d(W_d)).$$
 (11)

Further, in the pure marginal model, the *comonotonic* random vector $V^c = (V_1^c, \ldots, V_d^c)$, $V_i^c := F_{V_i}^{-1}(U)$ for all *i* with *U* uniformly distributed on (0, 1), describes the worst case distribution in supermodular order, i.e.,

$$V \leq_{sm} V^c \,, \tag{12}$$

see, e.g., [87, Theorem 3.9.14]. For an overview of stochastic orderings, we refer to [87, 101, 104].

2.3 Positive dependence concepts

For modeling positive dependencies, we make use of several positive dependence concepts. To this end, for a *d*-variate random vector $V = (V_1, \ldots, V_d)$, denote by $V^{\perp} := (V_1^{\perp}, \ldots, V_d^{\perp})$ an independent random

vector with the same marginal distributions as V, i.e., $V_1^{\perp}, \ldots, V_d^{\perp}$ are independent and $V_i^{\perp} \stackrel{d}{=} V_i$ for all $i \in \{1, \ldots, d\}$.

Definition 2.8 (Concepts of positive dependence)

A random vector $V = (V_1, \ldots, V_d)$ is said to be

- (i) positive lower orthant dependent (PLOD) if $V^{\perp} \leq_{lo} V$,
- (ii) positive upper orthant dependent (PLOD) if $V^{\perp} \leq_{uo} V$,
- (iii) positive supermodular dependent (PSMD) if $V^{\perp} \leq_{sm} V$,
- (iv) conditionally increasing (CI) if

$$V_i \uparrow_{st} (V_j, \ j \in J) \tag{13}$$

for all $i \in \{1, \ldots, d\}$ and $J \subseteq \{1, \ldots, d\} \setminus \{i\}$, where (13) means that the conditional distribution $V_i \mid (V_j = x_j, j \in J)$ is \leq_{st} -increasing in x_j for all $j \in J$, P^{V_J} -a.s., i.e., $\mathbb{E}[f(V_i) \mid V_j = x_j, j \in J]$ is increasing in x_j for all $j \in J$ outside a V_J -null set and for all increasing functions $f \colon \mathbb{R} \to \mathbb{R}$ such that the expectations exist,

- (v) conditionally increasing in sequence (CIS) if (13) holds for all $i \in \{2, \ldots, d\}$ and $J \subseteq \{1, \ldots, i-1\}$,
- (vi) multivariate totally positive of order 2 (MTP₂) if V is absolutely continuous with Lebesgue-density f such that $\log(f)$ is supermodular.

The above concepts of positive dependence are defined similarly for probability distributions and distribution functions, and they are invariant under increasing transformations, see [87, Theorem 3.10.19]. Hence, positive dependence is a copula-based property.

Remark 2.9 A simple criterion which is equivalent to condition (13) is that the conditional survival functions are pointwise increasing, i.e., for $J = (j_1, \ldots, j_m)$ and for all $y \in \mathbb{R}$, the conditional probability $P(V_i > y \mid V_{j_1} = x_1, \ldots, V_{j_m} = x_m)$ is increasing in x_1, \ldots, x_m outside a V_J -null set, see, e.g., [87, Example 2.5.2]. Hence, a bivariate random vector (V_1, V_2) is CIS (i.e., $V_2 \uparrow_{st} V_1$) if and only if, for all $x_2 \in \mathbb{R}$, $P(V_2 > x_2 \mid V_1 = x_1)$ is increasing in x_1 outside a V_1 -null set which may depend on x_2 . A sufficient (and in the case of continuous marginal distributions also necessary) criterion for the latter property is that the underlying copula is concave in its first component, which follows from the Sklar-type representation of conditional distribution functions in [6, Theorem 2.2].

The above defined positive dependence concepts are related by

 $MTP_2 \implies CI \implies CIS \implies PSMD \implies PLOD, PUOD,$ (14)

where the first three implications are strict for $d \ge 2$ and the last implication is strict only for $d \ge 3$, see [87, page 146] for an overview of these concepts. Note that PLOD or PUOD imply pairwise non-negative correlations in the sense of Pearson, Kendall, and Spearman, whenever these coefficients are defined, see, e.g., [101, Remark 6.3].

3 Ordering results for Markov tree distributions

In this section, we provide integral inequalities $\mathbb{E}f(X) \leq \mathbb{E}f(Y)$ for Markov tree distributed sequences $X \sim \mathcal{M}(F,T,B)$ and $Y \sim \mathcal{M}(G,T,C)$ with respect to the marginal specifications $F = (F_n)_{n \in N}$, $G = (G_n)_{n \in N}$ and the bivariate dependence specifications $B = (B_e)_{e \in E}$ and $C = (C_e)_{e \in E}$, respectively. First, we give a variant of Theorem 1.3 formulated in terms of bivariate tree specifications. Then, we establish general criteria on the marginal specifications F, G and the bivariate copula families B, C leading to comparison results with respect to the lower orthant, upper orthant, and directionally convex order. Care is required when comparing Markov tree distributions with different and discontinuous marginal distributions since conditional independence is not a pure dependence property, see Example A.2. All proofs of this section are deferred to Appendix D.

3.1 Inequalities for classes of supermodular functions

Due to Proposition 2.5, every Markov tree distribution has a representation by a bivariate tree specification, i.e., by a family of univariate distribution functions and a family of bivariate copulas specifying the nodes and the edges of the underlying directed tree. Vice versa, every bivariate tree specification has a unique Markov tree realization. In this light, Theorem 1.3 can be translated into the notation of bivariate tree specifications, which often proves useful in practice. To this end, assume that $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ follow Markov tree distributions with respect to a tree T = (N, E). Since the supermodular order is a pure dependence order and implies that the respective marginal distributions coincide, we obtain from condition (iii) of Theorem 1.3, i.e., $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ for all $(i, j) \in E$, that

$$F_{X_n} = F_{Y_n} =: F_n \quad \text{for all } n \in N$$

Further, due to identical marginal distributions, there exist by (9) bivariate copulas for (X_i, X_j) and (Y_i, Y_j) which are pointwise ordered, i.e.,

$$C_{X_i,X_j} \leq_{lo} C_{Y_i,Y_j}$$
 for all $e = (i,j) \in E$,

Condition (i) of Theorem 1.3, i.e., $X_j \uparrow_{st} X_i$, $(i, j) \in E$, means that the bivariate random vector (X_i, X_j) is conditionally increasing in sequence (CIS), see Remark 2.9. Using that the CIS property is invariant under increasing transformations, it follows that there exists a bivariate copula B_e such that

 B_e is CIS and $B_e = C_{X_i, X_i}$ on $\mathsf{Ran}(F_i) \times \mathsf{Ran}(F_j)$ for $e = (i, j) \in E$.

Recall that a bivariate copula is CIS if and only if it is concave in the first component. Denote by D^{\top} the transpose of a bivariate copula D, i.e., $D^{\top}(u,v) := D(v,u)$ for all $(u,v) \in [0,1]^2$. Then, condition (ii) of Theorem 1.3, i.e., $Y_i \uparrow_{st} Y_j$ (and $Y_j \uparrow_{st} Y_i$), $(i,j) \in E$, means that there exists a bivariate copula C_e such that

$$C_e^+$$
 is CIS (C_e is CI) and $C_e = C_{Y_i,Y_i}$ on $\mathsf{Ran}(F_i) \times \mathsf{Ran}(F_j)$.

Finally, we have decomposed X and Y into the bivariate tree specification with marginal distribution functions $F = (F_n)_{n \in \mathbb{N}}$ and pointwise ordered CIS (or CI) copulas $B = (B_e)_{e \in E}$ and $C = (C_e)_{e \in E}$. Since members of many well-known bivariate copula families are CI, see [5], which implies CIS, we formulate the following variant of Theorem 1.3 under slightly stronger assumptions on the bivariate specifications of X and Y.

Theorem 3.1 (Supermodular ordering based on bivariate tree specifications)

Let $X \sim \mathcal{M}(F,T,B)$ and $Y \sim \mathcal{M}(F,T,C)$ be Markov tree distributions. Assume for all $e \in E$ that

- (i) B_e is CIS,
- (*ii*) C_e is CI,
- (iii) $B_e \leq_{lo} C_e$ (resp. \geq_{lo}).

Then it follows that $X \leq_{sm} Y$ (resp. \geq_{sm}). In particular, X and Y are positive supermodular dependent.

As a direct consequence of the above theorem, we can determine solutions to Markovian optimal transport problems with a supermodular cost function as follows.

Corollary 3.2 (Solutions to Markovian optimal transport problems)

For $e \in E$ and for bivariate CI copulas \underline{C}_e and \overline{C}_e with $\underline{C}_e \leq_{lo} \overline{C}_e$, consider the classes

$$\mathcal{C}_e = \{ C \in \mathcal{C}_2 \mid C \text{ is } CI, \underline{C}_e \leq_{lo} C \leq_{lo} \overline{C}_e \}$$

$$\tag{15}$$

of bivariate CI copulas. Then, $\underline{X} \sim \mathcal{M}(F, T, (\underline{C}_e)_{e \in E})$ and $\overline{X} \sim \mathcal{M}(F, T, (\overline{C}_e)_{e \in E})$ solve the minimization/maximization problem (2)–(3) for a supermodular function f.

Remark 3.3 Theorem 3.1 provides a simple method to construct PSMD Markov tree distributions. By Proposition 2.5, any set of univariate distribution functions corresponding to the nodes and any set of bivariate copulas corresponding to the edges of the tree specifies a Markov tree distribution. If the bivariate copulas are CIS and pointwise larger than the product copulas $\Pi(u, v) = uv$, then the implied Markov tree distribution is PSMD. Further, Theorem 3.1 implies a new, flexible method to construct multivariate parametric distributions that are increasing in all of their parameters with respect to the supermodular order. For fixed F and T, this construction relies solely on the family of pointwise increasing CI copulas specifying the edges of the tree T. While construction methods known from the literature use that the supermodular order is closed under mixtures or under independent or comonotonic concatenations [104, Theorem 9.A.3], our construction relies on conditional independence. Corollary 3.2 allows to model distributional robustness based on stochastic orderings. We refer to Section 4 for an application to distributional bounds under model uncertainty.

3.2 Inequalities for classes of directionally convex functions

So far, we have compared Markov tree distributed sequences $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ of random variables with respect to the supermodular order, which requires identical marginal distributions, i.e., $X_n \stackrel{d}{=} Y_n$ for all $n \in N$. However, if the marginal distributions cannot uniquely be determined or are only partially known, some flexibility in the choice of the marginal specifications is desirable. When the marginals are in convex order, distributional robustness with respect to the directionally convex order can be obtained if the underlying multivariate copula is CI. This is the content of the following lemma.

Lemma 3.4 (Common CI copula; [85, Theorem 4.5])

Let $U = (U_0, \ldots, U_d)$ and $V = (V_0, \ldots, V_d)$ be random vector having the same (d+1)-dimensional copula $C = C_U = C_V$. If C is CI, then $U_i \leq_{cx} V_i$ for all $i \in \{0, \ldots, d\}$ implies $U \leq_{dcx} V$.

As we show in Example A.1, bivariate CI specifications do generally not lead to a Markov tree distribution having a CI copula. As a sufficient condition for the underlying copula being CI, the following special case of [44, Proposition 7.1] states that a Markov tree distributed random vector is MTP_2 , whenever the bivariate dependence specifications are MTP_2 .

Lemma 3.5 (MTP₂ specifications)

Let $Y = (Y_0, \ldots, Y_d) \sim \mathcal{M}(G, T, C)$ be a Markov tree distributed random vector such that (Y_i, Y_j) is MTP_2 for all $e = (i, j) \in E$. Then Y is MTP_2 .

Combining the above lemmas and Theorem 1.3 we obtain the following \leq_{dcx} -comparison result for Markov tree distributions, which additionally allows a comparison of the marginal distributions in convex order. Note that the bivariate specifications of the process Y are now assumed to satisfy the stronger positive dependence concept MTP₂.

Theorem 3.6 (Directionally convex ordering of Markov tree distributions)

For a tree T = (N, E), let $X = (X_n)_{n \in N} \sim \mathcal{M}(F, T, B)$ and $Y = (Y_n)_{n \in N} \sim \mathcal{M}(G, T, C)$ be Markov realizations of bivariate tree specifications. Assume for all $n \in N$ that the marginal distribution functions F_n and G_n are continuous. If for all $e = (i, j) \in E$,

- (i) B_e is CIS if $j \neq k^*$ for some fixed child $k^* \in c_0$ of the root 0,
- (ii) C_e is MTP_2 ,
- (*iii*) $B_e \leq_{lo} C_e (resp. \geq_{lo}),$

then $F_n \leq_{cx} G_n$ (resp. \geq_{cx}) for all $n \in N$ implies $X \leq_{dcx} Y$ (resp. \geq_{dcx}).

	function $f(x_0,\ldots,x_d)$	Properties
(1) (2)	$egin{array}{c} 1_{\{x\leq t\}} \ 1_{\{x\geq s\}} \end{array}$	Δ -antitone, supermodular Δ -monotone, supermodular; $-f \Delta$ -antitone
(3) (4)	$P(X_0 \le x_0, \dots, X_d \le x_d) P(X_0 \ge x_0, \dots, X_d \ge x_d)$	Δ -monotone, supermodular Δ -antitone, supermodular
(5) (6)	$\min\{x_0,\ldots,x_0\}\\\max\{x_0,\ldots,x_0\}$	Δ -monotone, supermodular supermodular, $-f \Delta$ -antitone
(7) (8)	$\mathbb{1}_{\{\min\{x_0,,x_d\} \ge K\}}$ $\mathbb{1}_{\{\max\{x_0,,x_d\} \le K\}}$	Δ -monotone, supermodular Δ -antitone, supermodular
$(9) \\ (10) \\ (11) \\ (12)$	$(K - \max\{x_0, \dots, x_d\})_+ (\max\{x_0, \dots, x_d\} - K)_+ (K - \min\{x_0, \dots, x_d\})_+ (\min\{x_0, \dots, x_d\} - K)_+$	Δ -antitone, supermodular, directionally convex supermodular, directionally convex; $-f \Delta$ -antitone supermodular, directionally convex; $-f \Delta$ -monotone Δ -monotone, supermodular, directionally convex
(13)	$\varphi(\sum_{n=0}^{d} \alpha_n x_n), \varphi \text{ convex}$	supermodular, directionally convex

Table 1 The table shows important examples and classifications of functions relevant to the integral stochastic orders considered in Section 3, where (X_1, \ldots, X_d) is a random vector on a probability space (Ω, \mathcal{A}, P) and where $s, t \in \mathbb{R}^d$, $K \in \mathbb{R}$, $(y)_+ := \max\{y, 0\}$, $\alpha_n \geq 0$. Since the integral stochastic orders satisfy various invariance properties, the above examples also apply to the respective transformations of the functions, for example to componentwise increasing/decreasing transformation in the case of the supermodular order, see (11).

3.3 Inequalities for classes of lower orthant and upper orthant functions

In the previous subsection, we have shown a comparison result for Markov tree distributions that allows flexibility of the bivariate dependence specifications in the pointwise order and flexibility of the marginal distributions in convex order. Now, we are interested in the case where the marginal distribution functions (survival functions) are not convex but pointwise ordered. We only consider the case of continuous marginal distribution functions, noting that the univariate marginal distributions generally affect the dependence structure under the Markov property, see Example A.2,

Theorem 3.7 (Orthant orderings of Markov tree distributions)

Let $X = (X_n)_{n \in \mathbb{N}} \sim \mathcal{M}(F,T,B)$ and $Y = (Y_n)_{n \in \mathbb{N}} \sim \mathcal{M}(G,T,C)$ be Markov tree distributions with respect to a tree T = (N,E) which satisfy the positive dependence conditions (i) and (ii) of Theorem 1.3. Assume for all $n \in \mathbb{N}$ that the marginal distribution functions F_n and G_n are continuous. Then the following statements hold true:

- (i) $B_e \leq_{lo} C_e$ (resp. \geq_{lo}) for all $e \in E$ and $F_n \leq_{lo} G_n$ (resp. \geq_{lo}) for all $n \in N$ implies $X \leq_{lo} Y$ (resp. \geq_{lo}).
- (ii) $B_e \leq_{uo} C_e$ (resp. \geq_{uo}) for all $e \in E$ and $F_n \leq_{uo} G_n$ (resp. \geq_{uo}) for all $n \in N$ implies $X \leq_{uo} Y$ (resp. \geq_{uo}).

Remark 3.8 Due to the characterization of the lower/upper orthant order in (7), Theorem 3.7 yields a comparison of integrals of Δ -antitone/-monotone functions in terms of the bivariate tree specifications. Note that, for bivariate copulas, $B_e \leq_{lo} C_e$ and $B_e \leq_{uo} C_e$ are equivalent due to (9), while, for univariate distribution functions, $F_i \geq_{lo} G_i$, $F_i \leq_{st} G_i$, and $F_i \leq_{uo} G_i$ are equivalent, which is a direct consequence of the definition of these orders.

4 Distributional robustness in hidden Markov models

In this section, we apply our comparison results to hidden Markov models (HMMs), which are a subclass of Markov tree distributions. We begin this section with the definition of HMMs and explain how these models can be embedded into our setting. Subsequently, we discuss its interpretation and applications. By employing Theorem 1.3 and the findings presented in Section 3, we can extract valuable results for this model class. The presented insights will then be illustrated by deducing uncertainty bounds for the distribution function of the maximum of the noisy observations of a random walk. For a comprehensive overview of the theory of hidden Markov models, including the subsequent definitions, we refer to [29, 43], along with the literature referenced therein.

A HMM is a bivariate Markov process $(X, X^*) = (X_n, X_n^*)_{n \in \mathbb{N}_0}$ consisting of the hidden (latent) Markov process $X = (X_n)_{n \in \mathbb{N}_0}$ and the observations process $X^* = (X_n^*)_{n \in \mathbb{N}_0}$ which, conditional on X, is a sequence of independent random variables such that the conditional distribution of X_n^* under X only depends on X_n . Any HMM (X, X^*) has a functional representation, known as a (general) state-space model, by

$$X_n = f_n(X_{n-1}, \delta_n) \quad P\text{-almost surely for } n \in \mathbb{N},$$

$$X_n^* = f_n^*(X_n, \varepsilon_n) \quad P\text{-almost surely for } n \in \mathbb{N}_0,$$
(16)

for some measurable functions $f_n, f_n^* \colon \mathbb{R}^2 \to \mathbb{R}$ and i.i.d. uniformly on [0, 1] distributed random variables $\{\delta_n\}_{n \in \mathbb{N}}, \{\varepsilon_n\}_{n \in \mathbb{N}_0}$, that are independent of the initial random variable X_0 . In the language of Markov tree distributions, a HMM can equivalently be written as a sequence $Z = (Z_n)_{n \in \mathbb{N}_0}$ of random variables defined by $Z_{2n} = X_n$ and $Z_{2n+1} = X_n^*$, for all $n \in \mathbb{N}_0$, which admits a Markov tree distribution with respect to the tree $T = (\mathbb{N}_0, E)$ with edges E given by

$$E = \{(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0 \,|\, j - i \in \{1,2\}\},\tag{17}$$

see Figure 1 d) and Figure 3 for an illustration of the underlying tree structure.

There are two perspectives of the interpretation and application of HMMs. Firstly, in fields such as communication theory, one can view the hidden process X as a signal transmitted via a communications channel. Given the inherent noise in the channel, the receiver perceives the distortion X^* of the original signal and aims to reconstruct the original signal, see, e.g., [43, 65]. Conversely, and in many models like in mathematical finance, one is directly interested in the observable process X^* , which is driven by an external factor process X. For example, X^* may describe the market price of a stock, with X representing an economic factor process influencing the stock price fluctuations. Various economic applications of this kind are studied, for instance, in [49, 54, 71, 70, 75, 94]. Note that any ARMA process has a representation of the form (16) with linear functions f and f^* , see [3]. In the sequel, we follow the latter approach and make inferences on the distorted observations of the hidden process.

The following result compares hidden Markov models in supermodular, lower, and upper orthant order and is a direct consequence of Theorems 1.3 and 3.7.

Corollary 4.1 (Comparison results for hidden Markov models)

Let (X, X^*) and (Y, Y^*) be HMMs and assume that $X_n^* \uparrow_{st} X_n$ for all $n \in \mathbb{N}$ and that $X_{n+1} \uparrow_{st} X_n$, $Y_n \uparrow_{st} Y_{n+1}$ as well as $Y_n^* \uparrow_{st} Y_n$ for all $n \in \mathbb{N}_0$.

(i) If $(X_n, X_{n+1}) \leq_{sm} (Y_n, Y_{n+1})$ and $(X_n, X_n^*) \leq_{sm} (Y_n, Y_n^*)$ for all $n \in \mathbb{N}_0$, then $(X, X^*) \leq_{sm} (Y, Y^*)$.

For the following, let all marginal distribution functions be continuous.

- (b) If $C_{X_n,X_{n+1}} \leq_{lo} C_{Y_n,Y_{n+1}}$, $C_{X_n,X_n^*} \leq_{lo} C_{Y_n,Y_n^*}$, $X_n \leq_{lo} Y_n$, and $X_n^* \leq_{lo} Y_n^*$ for all $n \in \mathbb{N}_0$, then $(X,X^*) \leq_{lo} (Y,Y^*)$.
- (c) If $C_{X_n,X_{n+1}} \leq_{uo} C_{Y_n,Y_{n+1}}$, $C_{X_n,X_n^*} \leq_{uo} C_{Y_n,Y_n^*}$, $X_n \leq_{uo} Y_n$, and $X_n^* \leq_{uo} Y_n^*$ for all $n \in \mathbb{N}_0$, then $(X,X^*) \leq_{uo} (Y,Y^*)$.
- (d) If $C_{X_n,X_{n+1}} \leq_{lo} C_{Y_n,Y_{n+1}}$ and $C_{X_n,X_n^*} \leq_{lo} C_{Y_n,Y_n^*}$ with $C_{Y_n,Y_{n+1}}$ and C_{Y_n,Y_n^*} MTP₂ and if $X_n \leq_{cx} Y_n$ and $X_n^* \leq_{cx} Y_n^*$ for all $n \in \mathbb{N}_0$, then $(X,X^*) \leq_{dcx} (Y,Y^*)$.

Typically, there is a rather strong positive dependence between the latent variable X_n and its distorted observation X_n^* . Hence, it is natural to assume that $X_n^* \uparrow_{st} X_n$, which equivalently means that $f_n^*(x, z)$ in (16) can be chosen to be increasing in x for all z, see [87, Lemma 3.10.10]. Further, as with many

Figure 3 The graphs illustrate the SI conditions on the hidden Markov processes $(X, X^*) = (X_n, X_n^*)_{n \in \mathbb{N}}$ and $(Y, Y^*) = (Y_n, Y_n^*)_{n \in \mathbb{N}}$ which lead to the comparison results in Theorem 4.1, where an arrow $U \to V$ indicates $V \uparrow_{st} U$. The root of the underlying tree corresponds to the variable X_0 and Y_0 , respectively. Note that only the SI condition between X_0 and X_0^* can be dropped, see Theorem 1.3 and Proposition 5.2.

models for stochastic processes, the outcomes in close temporal distance are typically strongly positive dependent. Hence, also the assumptions $X_{n+1} \uparrow_{st} X_n$ or $X_n \uparrow_{st} X_{n+1}$ are fairly natural.

To illustrate Corollary 4.1, we incorporate model uncertainty into the hidden Markov model and make inferences on distributional robustness. More precisely, for a random walk X, considered as hidden process, we determine uncertainty bands for the distribution function $F_{\max_{n\leq d}\{X_n^*\}}$ of the maximum of the first d+1 noisy observation X^* of X, where both the hidden process and the observations are subject to some dependence uncertainty and systematic error. Thereby, one crucial point is that the function

$$(x_0, \dots, x_n) \mapsto \mathbb{1}_{\{\max\{x_0, \dots, x_d\} \le t\}}$$

$$(18)$$

is Δ -antitone and, in particular, supermodular.

4.1 A noisy random walk

As a starting point we model the hidden process by a classical random walk where the independent increments are standard normal, i.e.,

$$X_n = \sum_{i=1}^n \xi_i \,, \quad X_0 := 0 \,, \tag{19}$$

for i.i.d. standard normally distributed random variables $\{\xi_i\}_{i\in\mathbb{N}}$. Then $X = (X_n)_{n\in\mathbb{N}_0}$ is a discrete time Gaussian Markov process specified by the means $\mathbb{E}X_n = 0$ and the covariances $\operatorname{Cov}(X_i, X_j) = \min\{i, j\}$ for all $i, j, n \in \mathbb{N}$. Equivalently, by Proposition 2.5, X can be considered as a discrete time Markov process with marginal specifications $X_n \sim \mathcal{N}(0, n)$ and Gaussian bivariate copula specifications $C_{X_n, X_{n+1}} = C_{\rho_{n, n+1}}^{\operatorname{Ga}}$, where $C_{\rho_{n, n+1}}^{\operatorname{Ga}}$ denotes the Gaussian copula with correlation parameter

$$\rho_{n,n+1} = \operatorname{Cor}(X_n, X_{n+1}) = \sqrt{n/(n+1)}, \quad n \in \mathbb{N}_0.$$
(20)

We assume that the observed process $X^* = (X_n^*)_{n \in \mathbb{N}}$ is modeled by a distortion $X_n^* = X_n + \varepsilon_n$ of the hidden process with i.i.d. standard normal errors $\{\varepsilon_n\}_{n \in \mathbb{N}_0}$. Consequently, the observations X_n^* follow a normal distribution with mean $\mathbb{E}X_n^* = 0$ and variance $\operatorname{Var}(X_n^*) = n + 1$. The joint distribution of (X_n, X_n^*) is bivariate normal with correlation ρ_n^* given by

$$\rho_n^* = \operatorname{Cor}(X_n, X_n^*) = \sqrt{n/(n+1)}, \quad \text{for } n \in \mathbb{N}_0.$$
(21)

An illustration of the hidden Markov model (X, X^*) in terms of its bivariate tree specification is given in Figure 4 (by setting $\theta_n = \rho_{n,n+1}$ and $\theta_n^* = \rho_n^*$). The distribution function $F_{\max\{X_0^*,\ldots,X_d^*\}}$ of the maximal observation is plotted with a dashed line in Figure 6.

4.2 A noisy random walk under model uncertainty

In the following, we incorporate model uncertainty into the above defined HMM allowing now (slightly) dependent increments of the hidden process X and errors that may depend on the hidden variables as well as systematic errors in the observations X^* . To this end, we consider a class of hidden Markov process

 $(X, X^*) = (X_n, X_n^*)_{n \in \mathbb{N}_0}$ where the marginal distributions are (partially) specified as

$$X_n \sim \mathcal{N}(0,n) \quad \text{and} \quad X_n^* \sim \mathcal{N}(\mu_n, n), \quad \text{for } \mu_n \in [\underline{\mu}_n, \overline{\mu}_n] \subset \mathbb{R}.$$
 (22)

The parameter μ_n can be interpreted as an unknown systematic observation error which is assumed to be bounded by some constants $\underline{\mu}_n \leq \overline{\mu}_n$. We assume that the dependence structure between the hidden variable X_n and its observation \overline{X}_n^* is partially specified by a bivariate Gaussian copula

$$C_{X_n,X_n^*} \in \left\{ C_{\rho}^{\mathrm{Ga}} \mid \rho \in [\underline{\rho}_n^*, \overline{\rho}_n^*] \right\},$$
(23)

where the lower and upper correlation bounds are given by

$$\underline{\rho}_n^* := \left((1 - \alpha^*) \sqrt{n/(n+1)} \right) \vee 0 \quad \text{and} \quad \overline{\rho}_n^* := \left((1 + \alpha^*) \sqrt{n/(n+1)} \right) \wedge 1 = 0$$

for some dependence uncertainty parameter $\alpha^* \geq 0$. Here, \vee and \wedge denote the maximum and minimum of two numbers, respectively. Hence, compared to (20), the joint distribution (X_n, X_n^*) is still bivariate normal, but there is only partial knowledge about the correlation parameter which is assumed to be in the interval $[\rho_n^*, \overline{\rho}_n^*]$. For $\alpha^* = 0$ we are back in the completely specified setting (21).

It remains to model the dependence structure of the hidden process X. To this end, we assume that the copulas between X_n and X_{n+1} are partially specified where we consider three different cases due to the following examples. In the first case, we consider Gaussian dependence specifications, in the second case Clayton copula specifications (which have lower tail dependencies), and in the third case survival Clayton copula specifications (which have upper tail dependencies). In each example, we determine sharp model uncertainty bounds for the distribution function $F_{\max\{X_0^*,\ldots,X_d^*\}}$ of the maximum of the first d+1observations. Figure 5 shows samples from the Gaussian, Clayton, and survival Clayton copula each having a parameter such that Kendall's τ equals 0.975. As we will see in the sequel, the tail dependencies have a strong effect on the distribution of the maximal observation, even under model uncertainty.

Example 4.2 (Gaussian dependencies in the hidden process)

We assume that the dependence structure of the hidden Markov process X is partially specified by the Gaussian copulas

$$C_{X_n,X_{n+1}} \in \left\{ C_{\rho}^{\text{Ga}} \mid \rho \in [\underline{\rho}_{n,n+1}, \overline{\rho}_{n,n+1}] \right\},$$
(24)

where the lower and upper correlation bounds are given by

$$\underline{\rho}_{n,n+1} := \left((1-\alpha)\sqrt{n/(n+1)} \right) \vee 0 \quad and \quad \overline{\rho}_{n,n+1} := \left((1+\alpha)\sqrt{n/(n+1)} \right) \wedge 1 \, .$$

for some small $\alpha \geq 0$ which serves as a dependence uncertainty parameter. For $\alpha = 0$, we are back in the setting of (20). Concerning the model uncertainty bounds, let $(\underline{X}, \underline{X}^*)$ and $(\overline{X}, \overline{X}^*)$ be hidden Markov processes specified by

$$\underline{X}_n \sim \mathcal{N}(0, n), \qquad \underline{X}_n^* \sim \mathcal{N}(\overline{\mu}_n, n), \qquad \overline{X}_n \sim \mathcal{N}(0, n), \qquad \overline{X}_n^* \sim \mathcal{N}(\underline{\mu}_n, n), \qquad (25)$$

$$C_{\underline{X}_{n},\underline{X}_{n}^{*}} = C_{\underline{\rho}_{n}}^{\mathrm{Ga}}, \qquad C_{\overline{X}_{n},\overline{X}_{n}^{*}} = C_{\overline{\rho}_{n}}^{\mathrm{Ga}}, \qquad (26)$$

$$C_{\underline{X}_{n},\underline{X}_{n+1}} = C_{\underline{\rho}_{n,n+1}}^{\mathrm{Ga}}, \qquad C_{\overline{X}_{n},\overline{X}_{n+1}} = C_{\overline{\rho}_{n,n+1}}^{\mathrm{Ga}},$$

where $\underline{\mu}_n$ and $\overline{\mu}_n$ are the systematic observation error bounds, see (22). It is well-known that the Gaussian copula family is \leq_{lo} -increasing in ρ and that C_{ρ}^{Ga} is CI for $\rho \geq 0$. Further, the normal distribution $\mathcal{N}(\mu, \sigma^2)$ is decreasing in μ with respect to the lower orthant order. Hence, using that the function in (18) is Δ -antitone, we obtain from Corollary 4.1(b) for the distribution function of the maximal observations the following sharp model uncertainty bounds:

$$P\big(\max\{\underline{X}_0^*,\ldots,\underline{X}_d^*\} \le t\big) \le P\big(\max\{X_0^*,\ldots,X_d^*\} \le t\big) \le P\big(\max\{\overline{X}_0^*,\ldots,\overline{X}_d^*\} \le t\big),$$

Figure 4 Illustration of the hidden Markov model with an underlying random walk and noisy observations. For $\theta_n = \sqrt{n/(n+1)}$, the hidden process equals a random walk with independent normally distributed increments. By allowing the dependence parameter θ_n and θ_n^* to be within a predefined interval, for all $n \in \mathbb{N}_0$, we obtain robustness for the evaluation of supermodular functionals. Moreover, by additionally adjusting the systematic error μ additional robustness can be achieved for the evaluation of upper orthant functionals.

for all $t \in \mathbb{R}$. Figure 6 illustrates the uncertainty bands for the distribution function of $\max\{X_0^*, \ldots, X_d^*\}$, where the model uncertainty parameters are specified in the caption.

In the above example, we have modeled the dependencies in the hidden process with a Gaussian copula allowing slightly state-dependent increments of the random walk. In the following, we analyze the behaviour of the maximal observations when incorporating copula families with tail-dependencies into the hidden process. As an example, we consider Clayton and survival Clayton copulas specifying the dependencies of the hidden states where

$$C_{\theta}^{\text{Cl}}(u,v) := (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \vee 0 \quad \text{and} \quad C_{\theta}^{\text{SCl}}(u,v) := 1 - u - v + C_{\theta}^{\text{Cl}}(u,v), \tag{27}$$

denotes the Clayton and survival Clayton copula with parameter $\theta \in [0, \infty)$. For $\theta = 0$ the Clayton copula models independence and for $\theta \to \infty$ it models comonotonicity. While Clayton copulas exhibit lower tail dependencies, their associated survival copulas have upper tail dependencies, see, e.g., [5] and Figure 5. To compare these models with the Gaussian dependence specifications, we consider Kendall's τ as a measure that describes the degree of positive dependence between two random variables and we adjust the parameters accordingly for the Clayton and survival Clayton copulas. Hereby, Kendall's τ of a Gaussian copula with parameter ρ and Kendall's τ of a Clayton copula with parameter θ are given by $\tau(C_{\rho}^{\text{Ga}}) = (\pi/2) \operatorname{arcsin}(\rho)$ and $\tau(C_{\theta}^{\text{Cl}}) = \tau(C_{\theta}^{\text{SCl}}) = \theta/(\theta + 2)$, see, e.g., [5, Table 6]. We use the transformation

$$\theta(\rho) := \frac{\pi \arcsin(\rho)}{2 - \pi \arcsin(\rho)}.$$
(28)

to obtain the parameter θ of the Clayton copula as a function of the Gaussian copula parameter ρ , so that both copulas exhibit the same degree of dependence in the sense of Kendall's τ .

Example 4.3 (Clayton copula dependencies in the hidden process)

We now model the dependencies between the states of the hidden process with Clayton copulas, while leaving the marginal distributions, the dependencies between the hidden variables and the observations, and the dependence uncertainty parameters as before. More precisely, we consider a class of hidden Markov process (X, X^*) which are partially specified by (22), (23), and with dependence uncertainty sets for the hidden process given by the transformations

$$C_{X_n,X_{n+1}} \in \left\{ C_{\theta}^{\text{Cl}} \mid \theta \in [\underline{\theta}_{n,n+1}, \overline{\theta}_{n,n+1}] \right\},$$
(29)

where the lower and upper parameter bounds for the Clayton copulas are given by

$$\underline{\theta}_{n,n+1} := \theta(\underline{\rho}_{n,n+1}) \quad and \quad \theta_{n,n+1} := \theta(\overline{\rho}_{n,n+1}) \,.$$

Concerning the model uncertainty bounds, let $(\underline{X}, \underline{X}^*)$ and $(\overline{X}, \overline{X}^*)$ be hidden Markov processes with

distributions uniquely specified by the marginal distributions in (25), the Gaussian copulas in (26) and the Clayton copulas

$$C_{\underline{X}_n,\underline{X}_{n+1}} = C_{\underline{\theta}_{n,n+1}}^{\text{Cl}}, \quad C_{\overline{X}_n,\overline{X}_{n+1}} = C_{\overline{\theta}_{n,n+1}}^{\text{Cl}}, \tag{30}$$

specifying the dependencies of the hidden process. Using that also the Clayton copula family is \leq_{lo} -increasing in θ and that C_{θ}^{Cl} is CI for $\theta \geq 1$, we obtain, similar to Example 4.2, from Corollary 4.1(b) for the distribution function of the maximal observations the model uncertainty bounds

$$P\left(\max\{\underline{X}_0^*,\ldots,\underline{X}_d^*\}\le t\right)\le P\left(\max\{X_0^*,\ldots,X_d^*\}\le t\right)\le P\left(\max\{\overline{X}_0^*,\ldots,\overline{X}_d^*\}\le t\right),$$

for all $t \in \mathbb{R}$, see Figure 6.

Example 4.4 (Survival Clayton copula dependencies in the hidden process)

To analyze the influence of tail dependencies, we now consider survival Clayton copulas as dependence specifications of the hidden process. The setting is similar to Example 4.3, but now (29) and (30) are replaced by

$$C_{X_n,X_{n+1}} \in \left\{ C_{\theta_{n,n+1}}^{\text{SC1}} \mid \theta \in [\underline{\theta}_{n,n+1}, \overline{\theta}_{n,n+1}] \right\}, \\ C_{\underline{X}_n,\underline{X}_{n+1}} = C_{\underline{\theta}_{n,n+1}}^{\text{SC1}}, \quad C_{\overline{X}_n,\overline{X}_{n+1}} = C_{\overline{\theta}_{n,n+1}}^{\text{SC1}}$$

By definition of a survival copula, also the survival Clayton copula family in (27) is \leq_{lo} -increasing in θ and CI for $\theta \geq 1$. Hence, similar as before, we obtain for the maximal observations the model uncertainty bounds

$$P\left(\max\{\underline{X}_0^*,\ldots,\underline{X}_d^*\}\le t\right)\le P\left(\max\{X_0^*,\ldots,X_d^*\}\le t\right)\le P\left(\max\{\overline{X}_0^*,\ldots,\overline{X}_d^*\}\le t\right),$$

for all $t \in \mathbb{R}$.

Figure 6 illustrates the dependence uncertainty bands for the distribution function of the maximal observations in the classes of hidden Markov models considered in Examples 4.2 - 4.4. The dependence uncertainty parameters are chosen as $\alpha = \alpha^* = 0.015$, i.e., the Pearson correlations in (20) and (21) are assumed to have some uncertainty and can vary up and down by 1.5%. Setting d = 14, we consider the maximum of the first 15 observations. In the left plot, we illustrate the setting without systematic error, i.e., $\mu_n = \overline{\mu}_n = 0$ for all $n \in \{0, \ldots, d\}$. In the right plot, we allow a systematic observation error between $\mu_n = -1$ and $\overline{\mu}_n = +1$ for all $n \in \{0, \ldots, d\}$. As we see, the systematic errors directly affect the width of the uncertainty bands for the distribution function $F_{\max\{X_0^*,\ldots,X_d^*\}}$ in the respective model. For comparison purposes, we include the distribution function of the maximum of $\max\{X_0^*,\ldots,X_d^*\}$ equals the distribution of $\max\{Z, \sqrt{d+1}Z\} - \mu$ for $Z \sim \mathcal{N}(0, 1)$, where we set $\mu = 0$ in the left figure and $\mu = 1$ in the right figure.

While the Gaussian copulas have no tail-dependence for $\rho \in [0, 1)$, the Clayton copulas exhibit lower tail-dependencies. In connection with this lower tail dependence, we observe that for small t, the band is closer to the comonotonic distribution function compared to the case of the survival Clayton copula, which has upper tail dependence. However, this phenomenon reverses for large values of t. In this case, the uncertainty band for the Clayton copula specifications is closer to the distribution function of the independent random variables. To analyze this behaviour in more detail, we write

$$P(\max\{X_0^*, \dots, X_d^*\} \le t) = P(X_n^* \le t \text{ for all } n \le d),$$
(31)

$$P(\max\{X_0^*, \dots, X_d^*\} \ge t) = P(X_n^* \ge t \text{ for some } n \le d).$$
(32)

Due to Figure 6, the distribution function $F_{\max\{X_0^{*\perp},...,X_d^{*\perp}\}}$ of the independent observations, denoted by $X_0^{*\perp},\ldots,X_d^{*\perp}$, is pointwise smaller than the distribution function $F_{\max\{X_0^{*c},\ldots,X_d^{*c}\}}$ of the comono-



Figure 5 Samples of a Gaussian copula (left), Clayton copula (mid), and survival Clayton copula (right), each having Kendall's tau value $\tau = 0.795$ which corresponds to the parameter $\rho = \sqrt{9/10}$ for the Gaussian copula and $\theta = 7.764$ for the Clayton and survival Clayton copula. The plots indicate that the Clayton copula exhibits lower tail-dependencies, while the survival Clayton copula has upper tail-dependencies.



Figure 6 Uncertainty bands of the mapping $t \mapsto E[\mathbb{1}_{\max\{X_0^*,...,X_d^*\} \leq t\}}] = F_{\max\{X_0^*,...,X_d^*\}}(t)$ for the classes of hidden Markov models considered in Examples 4.2 – 4.4 where d = 14. We choose the dependence uncertainty parameter $\alpha = \alpha^* = 0.015$. In the left plot, we assume no systematic error of the observations, i.e., $\underline{\mu}_n = \overline{\mu}_n = 0$ for all n. In the right plot, we assume a systematic error that is lower and upper bounded by $\underline{\mu}_n = -1$ and $\overline{\mu}_n = +1$. The lower and upper black line are the distribution function of the maximum of independent and comonotonic random variables without systematic observation error (left plot) and with systematic error $\mu_n = +1$ for independence and $\mu_n = -1$ for comonotonicity, respectively (right plot). The dashed line is the distribution function $F_{\max\{X_0^*,...,X_d^*\}}$ of the noisy random walk without model uncertainty as defined in Section 4.1.

tonic observations $X_0^{*c}, \ldots, X_d^{*c}$. This is a direct consequence of (12) but also very intuitive: It is more likely that X_n^* in (31) is not greater than t for all $n \leq d$, whenever X_0^*, \ldots, X_d^* are perfectly positive depend rather than independent. Similarly, we can explain the behaviour for the class of models where the hidden process has Clayton copula specifications: Due to lower tail-dependencies, for small t, the event $\{\max\{X_0^*, \ldots, X_d^*\} \leq t\}$ tends to be more likely than in the Gaussian or survival Clayton copula case. Hence, the distribution function of the maximal observation is supposed to be larger and closer to comonotonicity. Vice versa, due to less positive dependence, for large t, the probability for some X_n^* being larger than t is higher compared to the other models, for which large observations occur more simultaneously. Hence, by (32), the distribution function of the maximum is supposed to be smaller in the Clayton copula model, which is confirmed by the plots. In contrast to the Clayton copula specifications, the class of hidden Markov models in Example 4.4 with survival Clayton copula specifications exhibits upper tail dependencies but no lower tail dependencies, see Figure 5. Hence, for large t, the probability that some X_n^* exceeds t is smaller and thus the distribution function of $\max_n\{X_0^*, \ldots, X_d^*\}$ evaluated at t is larger and closer to the comonotonic case. Further, for small t, the probability that all X_n^* do not exceed t is slightly smaller and closer to the independence case compared to the other models.

5 Discussion of the assumptions of Theorem 1.3

In this section, we discuss the generality of our main result, Theorem 1.3. First, we prove that none of the SI assumptions (i) and (ii) of Theorem 1.3 on X and Y can be skipped or weakened to PSMD. Then, we show that a recently establish supermodular comparison result for star structures in [8], which allows to compare general dependencies with positive dependencies, cannot be extended to Markov processes and consequently also not to Markov tree distributions.

5.1 Discussion of the SI assumptions

Our main result, Theorem 1.3, provides simple conditions for a supermodular comparison of multivariate Markov tree distributions. At first glance, the SI conditions on X and Y look rather unintuitive. However, as we show in the sequel, under ordering assumption (iii), they cannot be dropped or weakened. Figure 7 illustrates sufficient SI conditions in the three-dimensional case which, together with assumption (iii), leads to the supermodular comparison of Markov tree distributions. Whenever an SI condition in Theorem 1.3 is skipped, it implies the existence of subvectors exhibiting a dependence structure that aligns (or is even weaker) with SI conditions depicted by the graphs in Figure 8 a),b), or c). For these settings, we provide examples that do not imply a lower orthant comparison result and thus do not imply a supermodular comparison result neither. Proposition 5.2 summarizes the results of this section regarding the necessity of the SI conditions. In essence, for a Markov tree distribution on 3 nodes, it is precisely these two SI conditions in Figure 7 c) that lead to a supermodular comparison result.

The following remark summarizes Examples A.3 - A.5 showing that the SI conditions in Figure 8 do not lead to comparison results as in Proposition 1.1 and Lemma 1.2.

- **Remark 5.1** (a) Condition (ii) in Proposition 1.1 cannot be replaced by $Y_{i+1} \uparrow_{st} Y_i$: As we show in Example A.3, there exist Markov tree distributed random vectors $X = (X_0, X_1, X_2)$ and $Y = (Y_0, Y_1, Y_2)$ for T satisfying the SI conditions in Figure 8a) such that $(X_i, X_{i+1}) \leq_{sm} (Y_i, Y_{i+1})$ but $X \not\leq_{sm} Y$. Hence, the SI conditions in Figure 8a) do not imply a supermodular comparison result.
- (b) Conditions (i)-(ii) in Lemma 1.2 cannot be replaced by (X_0, X_i) being CI and $Y_0 \uparrow_{st} Y_i$: As we show in Example A.4 there exist Markov tree distributed random vectors $X = (X_0, X_1, X_2)$ and $Y = (Y_0, Y_1, Y_2)$ for T satisfying the SI conditions in Figure 8 b) such that $(X_0, X_i) \leq_{sm} (Y_0, Y_i)$ for $i \in \{1, 2\}$ but $X \not\leq_{sm} Y$.
- (c) As an extension of the setting in Figure 8 b), also the SI conditions in Figure 8 c) are not sufficient for a supermodular comparison result like in Proposition 1.1, see Example A.5.

Using Examples A.3 – A.5 summarized in the preceding remark, we show in the following proposition that none of the SI conditions in assumptions (i) and (ii) of Theorem 1.3 can be dropped or weakened to PSMD. Since the supermodular order is closed under marginalization, see [104, Theorem 9.A.9(c)], the proof can be reduced to the cases considered above.

To this end, let T = (N, E) be the tree, P the path, and k^* the specific child of the root as considered in Theorem 1.3. For a node $j^* \in N \setminus \{0\}$ that will be specified in Proposition 5.2, we relax the SI conditions in (i) and (ii) of Theorem 1.3 regarding only one edge: For $(i, j) \in E$ let

$$(i^*) \quad X_j \uparrow_{st} X_i, \text{ if } j \notin \{k^*, j^*\}, \text{ and } (X_{p_{j^*}}^\perp, X_{j^*}^\perp) \leq_{sm} (X_{p_{j^*}}, X_{j^*}), \tag{33}$$

$$(ii^*) \quad Y_i \uparrow_{st} Y_j \text{ if } j \notin L \cup \{j^*\}, \text{ and } Y_j \uparrow_{st} Y_i \text{ if } j \notin P, \text{ and } (X_{p_{j^*}}^{\perp}, X_{j^*}^{\perp}) \leq_{sm} (X_{p_{j^*}}, X_{j^*}),$$
(34)

$$(ii^{**}) \quad Y_i \uparrow_{st} Y_j \text{ if } j \notin L, \text{ and } Y_j \uparrow_{st} Y_i, \text{ if } j \notin P \cup \{j^*\}, \text{ and } (X_{p_{j^*}}^\perp, X_{j^*}^\perp) \leq_{sm} (X_{p_{j^*}}, X_{j^*}).$$
(35)

Note that $p_{j^*} \in N$ denotes the parent node of $j^* \in N$, see Definition B.1. We distinguish between three cases. The first one concerns the SI conditions on X in (i). The second case relates to the SI conditions (ii) on Y, which states that $Y_i \uparrow_{st} Y_j$ for $(i, j) \in E$ whenever $j \notin L \cup \{0\}$, where $N \setminus (L \cup \{0\})$ is a nonempty set if and only if T is not a star. In the third case we consider the SI conditions of the second part of (ii), which asserts that $Y_j \uparrow_{st} Y_i$ for $(i, j) \in E$, whenever $j \notin P \cup \{0\}$, where $N \setminus (P \cup \{0\})$ is a nonempty set if and only if T is not a chain. The following result states that none of the SI conditions in Theorem 1.3 can be dropped. a) $X_0 \longrightarrow X_1 \longrightarrow X_2$ b) $X_1 \longleftarrow X_0 \longrightarrow X_2$ c) $X_0 \longrightarrow X_1 \longrightarrow X_2$ $Y_0 \longleftarrow Y_1 \longleftarrow Y_2$ $Y_1 \longleftarrow Y_0 \longrightarrow Y_2$ $Y_0 \longleftarrow Y_1 \longrightarrow Y_2$

Figure 7 The graphs illustrate sufficient SI conditions which imply $(X_0, X_1, X_2) \leq_{sm} (Y_0, Y_1, Y_2)$ whenever $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ for $(i, j) \in E$, where an arrow $U \to V$ indicates $V \uparrow_{st} U$. The graph in a) corresponds to the setting in Proposition 1.1, while the graph in b) corresponds to the setting in Lemma 1.2. The graph in c) generalises the two previous cases and corresponds to the setting in Theorem 1.3 noting that there is no positive dependence condition on (X_0, X_1) and (Y_1, Y_2) . Similar results are obtained when replacing \leq_{sm} with \geq_{sm} .

- a) $X_0 \longrightarrow X_1 \longleftrightarrow X_2$ c) $X_0 \longleftrightarrow X_1 \longleftrightarrow \cdots \longleftrightarrow X_{d-1} \longleftrightarrow X_d$ $Y_0 \longrightarrow Y_1 \longleftrightarrow Y_2$ $Y_0 \longrightarrow Y_1 \longleftrightarrow \cdots \longleftrightarrow Y_{d-1} \hookleftarrow Y_d$
- b) $X_1 \longleftrightarrow X_0 \longleftrightarrow X_2$ d) $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3$ $Y_1 \longrightarrow Y_0 \longleftarrow Y_2$ $Y_0 \xleftarrow{^{\mathrm{MTP}_2}} Y_1 \xleftarrow{^{\mathrm{MTP}_2}} Y_2 \xleftarrow{^{\mathrm{MTP}_2}} Y_3$

Figure 8 The graphs illustrate the non-sufficient SI conditions in a) Example A.3, b) Example A.4, c) Example A.5 and d) Example A.6, where an arrow $U \to V$ indicates $V \uparrow_{st} U$ and where $U \xleftarrow{\text{MTP}_2} V$ indicates that (U, V) is MTP₂ which implies, in particular, that $U \uparrow_{st} V$ and $V \uparrow_{st} U$. As shown in the respective examples, none of the SI conditions on $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ are sufficient for $X \leq_{sm} Y$ provided that $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ for all $(i, j) \in E$.

Proposition 5.2 (Generality of the SI assumptions of Theorem 1.3)

Assume that $|N| \ge 3$. None of the SI assumptions in Theorem 1.3 can be replaced by the weaker concept of positive supermodular dependence, i.e., for the tree T = (N, E), the following statements hold true:

- (a) For any $j^* \in N \setminus \{0, k^*\}$ there are Markov tree distributed sequences $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ with $X \not\leq_{lo} Y$ such that X and Y fulfill for all $e = (i, j) \in E$ condition (i^*) defined in (33) and conditions (ii) and (iii) of Theorem 1.3.
- (b) If T is not a star, then for any $j^* \in N \setminus (L \cup \{0\})$, there are Markov tree distributed sequences $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ with $X \not\leq_{lo} Y$, such that X and Y fulfill for all $e = (i, j) \in E$ condition (ii*) defined in (34) and conditions (i) and (iii) of Theorem 1.3.
- (c) If T is not a chain, then for any $j^* \in N \setminus (P \cup \{0\})$, there are Markov tree distributed sequences $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ with $X \not\leq_{lo} Y$, such that X and Y fulfill for all $e = (i, j) \in E$ condition (ii^{**}) defined in (35) and conditions (i) and (iii) of Theorem 1.3.

In particular, in each of the statements (a)-(c) it holds that $X \not\leq_{sm} Y$.

5.2 A special property of star structures

In this section, we discuss a considerable extension of the comparison results for star structures in Lemma 1.2 to a comparison of general dependencies, see [8]. The extension is based on the recently established Schur order for conditional distributions which compares conditional distribution functions of bivariate random vectors with respect to their strength of dependence in terms of their variability in the conditioning variable. The Schur order for conditional distributions has the fundamental properties that minimal elements characterize independence (i.e., no variability in the conditioning variable) and maximal elements characterize perfect dependence (i.e., maximal variability of the decreasing rearrangements), see [4, 5], where a random variable Y is said to *completely* or *perfectly depend* on X if there exists a Borel measurable function f (which is not necessarily increasing or decreasing) such that Y = f(X) almost surely. Note that perfect dependence is not a symmetric concept, i.e., perfect dependence of Y on X does not imply perfect dependence of X on Y. As we know from Lemma 1.2 on Markovian star structures, strong positive dependence between Y_n and Y_0 for all $n \in \{1, \ldots, n\}$ leads to strong positive dependence among (Y_1, \ldots, Y_d) in the sense of the supermodular order. As it can easily be verified, also strong negative dependence between each Y_n and Y_0 implies strong positive dependence among (Y_1, \ldots, Y_d) .

A fairly intuitive result for Markovian star structures (Proposition 5.4) states that (Y_1, \ldots, Y_d) exhibits stronger positive dependence than (X_1, \ldots, X_d) whenever, for every $n \in \{1, \ldots, d\}$, X_n is less dependent on the common factor variable X_0 than Y_n on Y_0 in the sense of the Schur order for conditional distributions, where only (Y_n, Y_0) is assumed to exhibit positive dependence. Similarly, one might expect for Markov chains that stronger dependence (in the sense of the Schur order) among (Y_n, Y_{n+1}) than on (X_n, X_{n+1}) for all $n \in \mathbb{N}$ would lead to stronger dependence among $(Y_n)_{n \in \mathbb{N}}$ compared to $(X_n)_{n \in \mathbb{N}}$, at least when all (Y_n, Y_{n+1}) are conditionally increasing. Surprisingly, as we show below, comparison results with respect to the Schur order for conditional distributions cannot be extended from star structures to chain structures and thus neither to Markov tree distributions. In other words, only for star structures, more variability in the conditioning variable in the sense of the Schur order increases the strength of dependence of the whole vector in the supermodular order. Hence, Theorem 1.3 is also general in the sense that there is no extension to the Schur order for conditional distributions, which we formally define as follows, see [4].

Consider for integrable functions $f, g: (0,1) \to \mathbb{R}$ the Schur order $f \prec_S g$ defined by

$$f \prec_{S} g \quad : \iff \quad \int_{0}^{x} f^{*}(t) d\lambda(t) \leq \int_{0}^{x} g^{*}(t) d\lambda(t) \quad \text{for all } x \in (0, 1) \text{ and}$$

$$\int_{0}^{1} f^{*}(t) d\lambda(t) = \int_{0}^{1} g^{*}(t) d\lambda(t) ,$$
(36)

where h^* denotes the decreasing rearrangement of an integrable function $h: (0, 1) \to \mathbb{R}$, i.e., the essentially uniquely determined decreasing function h^* such that $\lambda(h^* \ge w) = \lambda(h \ge w)$ for all $w \in \mathbb{R}$, where λ denotes the Lebesgue measure on (0, 1), see, e.g., [101]; for an overview of rearrangements, see [33, 34, 39, 55, 74, 97]. Roughly speaking, the decreasing rearrangement of a (piecewise constant) function can be obtained by rearranging the graph of the function in descending order. It is immediately clear that minimal elements in the Schur order are constant functions while maximal elements do in general not exist. The Schur order for conditional distributions is defined by comparing conditional distribution functions in their conditioning variable with respect to the Schur order for functions as defined in (36). We denote by q_W the (generalized) quantile function of a random variable W, i.e., $q_W(t) := \inf\{x \in \mathbb{R} \mid F_W(x) \ge t\}$, $t \in (0, 1)$.

Definition 5.3 (Schur order for conditional distributions)

Let (U, V) and (U', V') be bivariate random vectors with $V \stackrel{d}{=} V'$. Then the Schur order for conditional distributions is defined by

$$(V|U) \leq_S (V'|U') \quad : \iff \quad F_{V|U=q_U(\ \cdot\)}(v) \prec_S F_{V'|U'=q_{U'}(\ \cdot\)}(v) \quad \text{for all } v \in [0,1],$$
(37)

By definition, the Schur order for conditional distributions is invariant under rearrangements of the conditional variable and compares the variability of conditional distribution functions in the conditional variables in the sense of the Schur order for functions. Minimal elements characterize independence and maximal elements characterize complete directed dependence, see [4, Theorem 3.5]. Further, the Schur order for conditional distributions has the property that, for $U \stackrel{d}{=} U'$ and $V' \uparrow_{st} U'$,

$$(V|U) \leq_S (V'|U') \implies (U,V) \leq_{sm} (U',V'), \tag{38}$$

i.e., less variability of the conditional distribution function $u \mapsto F_{V|U=u}(v)$ in the conditioning variable than $u \mapsto F_{V'|U'=u}(v)$ in sense of (36) for all v implies that (U, V) is smaller or equal in the supermodular order than (U', V') whenever V' is stochastically increasing in U', see [6, Proposition 3.17], cf. [4, Proposition 3.4]. Note that in (38), there is no positive dependence assumption on (U, V). If additionally $V \uparrow_{st} U$, then also the reverse direction in (38) holds true, see [4, Proposition 3.4], cf. [6, 107]. In this case, the Schur order is equivalent to the supermodular order and we are back in the setting of Lemma 1.2. The following result is a version of [8, Corollary 4(i)] and extends (38) to a vector of conditionally independent random variables. It states that a strengthening of the supermodular ordering condition to the Schur order allows to skip the positive dependence assumption (i) of Lemma 1.2 on (X_i, X_0) .

Proposition 5.4 (Comparison of star structure based on Schur order)

Let $X = (X_0, \ldots, X_d)$ and $Y = (Y_0, \ldots, Y_d)$ be random vectors such that X_1, \ldots, X_d are conditionally independent given X_0 and such that Y_1, \ldots, Y_d are conditionally independent given Y_0 with $X_0 \stackrel{d}{=} Y_0$. Assume for all $i \in \{1, \ldots, d\}$ that

- (i) $Y_i \uparrow_{st} Y_0$,
- (*ii*) $(X_i|X_0) \leq_S (Y_i|Y_0)$.

Then it follows that $X \leq_{sm} Y$. In particular, Y is positive supermodular dependent.

In the following remark, we discuss the special properties of star structures which allow a more general supermodular comparison result based on the Schur order for conditional distributions.

- **Remark 5.5** (a) Since $(V|U) \leq_S (V'|U')$ is equivalent to $(U,V) \leq_{sm} (U',V')$ whenever $U \stackrel{d}{=} U'$, $V \uparrow_{st} U$ and $V' \uparrow_{st} U'$, Proposition 5.4 extends Lemma 1.2 to a large class of conditionally independent distributions that allow a non-positive dependence structure of X. Surprisingly, as we show in Example A.6, Proposition 5.4 cannot be extended from star structures to Markov tree distributions: To be precise, let T = (N, E) for $N = \{0, 1, 2, 3\}$ and $E = \{(0, 1), (1, 2), (2, 3)\}$ be a chain of 4 nodes. Then, we can construct Markov tree distributed random variables $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ such that $(X_i, X_{i+1}) \leq_{sm} (Y_i, Y_{i+1})$ and (Y_i, Y_{i+1}) is MTP_2 for all $i \in \{0, 1, 2\}$. Further, it holds that $(X_i|X_j) \leq_S (Y_i|Y_j)$ and $(X_j|X_i) \leq_S (Y_j|Y_i)$ for all $(i, j) \in E$, as well as $X \not\leq_{lo} Y$. Hence, less variability of the bivariate specifications $\{(X_i, X_j)\}_{(i,j)\in E}$ than $\{(Y_i, Y_j)\}_{(i,j)\in E}$ with respect to the Schur order as in (37) does not imply that the entire vector $X = (X_n)_{n \in N}$ is smaller than $Y = (Y_n)_{n \in N}$ with respect to the lower orthant order and thus neither with respect to the supermodular order, even if the specifications $(Y_i, Y_j)_{(i,j)\in E}$ satisfy the strong positive dependence concept MTP_2 , see also Figure 8 d). Hence, Proposition 5.4 on the rearrangement-based Schur order is not extendable to Markov processes.
- (b) The Schur order for conditional distributions has the interesting property that it implies an ordering for various well-known dependence measures, where a dependence measure is a functional (X, Y) → κ(Y|X) such that (i) κ(Y|X) ∈ [0,1], (ii) κ(Y|X) = 0 if and only if X and Y are independent, and (iii) κ(Y|X) = 1 if and only if Y is completely dependent on X. For example, Chatterjee's rank correlation, which has recently attracted a lot of attention in the statistical literature [10, 30, 41], is consistent with the Schur order for conditional distributions [4]. Due to Proposition 5.4, the Schur order also implies general comparison results with respect to the supermodular order. Roughly speaking, large elements in the Schur order for conditional distributions lead to strong dependencies and, if the dependencies are positive, to large elements in supermodular order.

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Appendix

A Examples

The following example justifies the MTP_2 assumption in Theorem 3.6 by showing that bivariate CI specifications do generally not lead to a Markov realization that is CI.

Example A.1 (A non-CI Markov tree distribution with bivariate CI marginals) For $i \in \{1, 2\}$, consider the doubly stochastic matrices $a^i = (a_{k\ell}^i)_{1 \le k, \ell \le 3} \in \mathbb{R}^{3 \times 3}$ given by

$$a^{1} = \frac{1}{30} \begin{pmatrix} 8 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 7 \end{pmatrix}, \quad a^{2} = \frac{1}{30} \begin{pmatrix} 4 & 4 & 2 \\ 4 & 3 & 3 \\ 2 & 3 & 5 \end{pmatrix}.$$

Let $X = (X_0, X_1, X_2)$ be a random vector that follows a Markov tree distribution with respect to the tree $T = (N, E), N = \{0, 1, 2\}, E = \{(0, 1), (1, 2)\}$, specified by the univariate and bivariate distributions through

$$P[X_i = k] = 1/3 \quad \text{for } i, k \in \{0, 1, 2\},$$
(39)

$$P[X_i = k, X_{i+1} = \ell] = a_{k\ell}^i \quad \text{for } i \in \{0, 1\} \text{ and } k, \ell \in \{0, 1, 2\}.$$

$$\tag{40}$$

Note that the univariate distributions in (39) are the marginals of the bivariate distributions in (40). From the definition of the matrices a^1 and a^2 it can be seen that the subvectors (X_0, X_1) and (X_1, X_2) are CI. It even follows that (X_0, X_2) is CI, because, as a consequence of Lemma B.3, $X_0 \uparrow_{st} (X_1, X_2)$ and $X_2 \uparrow_{st} (X_1, X_0)$. However, the vector (X_0, X_1, X_2) is not CI because

$$P[X_1 \ge 1 | X_0 = 0, X_2 = k] = \begin{cases} 1 - 16/20, & \text{if } k = 0, \\ 1 - 16/19, & \text{if } k = 1, \\ 1 - 16/22, & \text{if } k = 2, \end{cases}$$

is not increasing in k. Hence, if the bivariate specifications are CI, the implied Markov tree distribution is generally not CI.

The next example shows that Theorem 3.7 fails to hold when the continuity assumption on the marginal distribution functions is not imposed. In addition, the example emphasizes that the dependence structure of a random vector determined by a Markov realization of a bivariate tree specification is not only determined by the set of bivariate copulas and the conditional independence structure. It also depends on the choice of the marginal distributions.

Example A.2 (General marginals are not sufficient for Theorem 3.7)

Let T be a tree with nodes $N = \{0, 1, 2\}$ and edges $E = \{(0, 1), (0, 2)\}$. Denote by $F_{U(0,1)}$ the distribution function of the uniform distribution on (0, 1). Consider the marginal specifications F_i and G_i , $i \in \{0, 1, 2\}$, given by

$$F_0 = F_1 = F_2 = G_1 = G_2 = F_{U(0,1)}$$
 and $G_0 = \mathbb{1}_{[1,\infty)}$.

Then F_0, F_1, F_2, G_1 and G_2 are continuous distribution functions and G_0 is the distribution function of the Dirac distribution in 1, which is not continuous. Denote by M^2 the bivariate upper Fréchet copula, i.e., $M^2(u,v) = \min\{u,v\}$ for $(u,v) \in [0,1]$, and consider the bivariate specifications $B_e := C_e := M^2$ for $e \in E$. Let U and V be independent and uniformly on (0,1) distributed random variables. Then, for $F = (F_0, F_1, F_2), G = (G_0, G_1, G_2), B = (B_e)_{e \in E}$, and $C = (C_e)_{e \in E}$, the random vectors $X = (X_0, X_1, X_2) := (U, U, U)$ and $Y = (Y_0, Y_1, Y_2) := (1, U, V)$ are Markov realizations of (F, T, B) and (G, T, C), respectively, i.e.,

$$X \sim \mathcal{M}(F, T, B)$$
 and $Y \sim \mathcal{M}(G, T, B)$.

It holds that $F_i \leq_{lo} G_i$ for all $i \in \{0, 1, 2\}$ and $B_e = C_e$ (which trivially implies $B_e \leq_{lo} C_e$) for $e \in E$. Further, the bivariate copulas B_e and C_e , $e \in E$, are CI. Hence, all assumptions of Corollary 3.7*i* except of continuity of the marginal distributions are satisfied. However, it holds for all $(u, v) \in (0, 1)$ that

$$F_{X_1,X_2}(u,v) = \min\{u,v\} > uv = F_{Y_1,Y_2}(u,v),$$

which implies $(X_1, X_2) \not\leq_{lo} (Y_1, Y_2)$. Since the lower orthant order is closed with respect to marginalization, see, e.g., [87, Theorem 3.3.19], it follows that $X \not\leq_{lo} Y$.

Replacing in the above setting G_0 by $G_0 = \mathbb{1}_{[0,\infty)}$ and considering Y = (0, U, V), it follows similarly that $(X_1, X_2) \not\leq_{uo} (Y_1, Y_2)$ and thus $X \not\leq_{uo} Y$. Hence, we conclude that the continuity assumption for the marginal specifications in Corollary 3.7 cannot be omitted.

Note that in this example, X_1 and X_2 are comonotonic while Y_1 and Y_2 are independent. Since the bivariate tree specifications (F,T,B) and (G,T,C) only differ in the first component of the marginal specifications (i.e., $F_0 \neq G_0$, $F_1 = G_1$, $F_2 = G_2$, and B = C), we conclude that also the marginal distributions can affect the dependence structure of Markov tree distributions. In other words, conditional independence is not only a copula-dependent property.

The following example shows that the SI assumptions on X and Y in opposite order in Proposition 1.1 cannot be replaced by SI in the same order, i.e., there is no version of Proposition 1.1 under the assumptions that $X_{i+1} \uparrow_{st} X_i$ and $Y_{i+1} \uparrow_{st} Y_i$, see also [45, Example 4.4]. We even show that, under two additional SI assumptions, see Figure 8 a) there is no supermodular comparison result.

Example A.3 (Condition (ii) in Proposition 1.1 cannot be replaced by $Y_{i+1} \uparrow_{st} Y_i$) Let T = (N, E), $N = \{0, 1, 2\}$, $E = \{(0, 1), (1, 2)\}$, be a tree on 3 nodes. Consider the doubly stochastic matrices $a^{01}, a^{12}, b^{01}, b^{12} \in \mathbb{R}^{3\times 3}$ given by

$$a^{01} = b^{01} = \frac{1}{30} \begin{pmatrix} 4 & 4 & 2 \\ 3 & 4 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \quad a^{12} = \frac{1}{30} \begin{pmatrix} 4 & 4 & 2 \\ 4 & 3 & 3 \\ 2 & 3 & 5 \end{pmatrix}, \quad b^{12} = \frac{1}{30} \begin{pmatrix} 5 & 4 & 1 \\ 3 & 3 & 4 \\ 2 & 3 & 5 \end{pmatrix}$$

Let $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ be random vectors which follow a Markov tree distribution with respect to the tree T specified by the bivariate distributions given by

$$P[X_i = k, X_{i+1} = l] = a_{kl}^{ii+1} \text{ and } P[Y_i = k, Y_{i+1} = l] = b_{kl}^{ii+1} \text{ for } k, l \in \{0, 1, 2\} \text{ and } i \in \{0, 1\}.$$

All marginal distributions of the vectors X and Y are uniform on $\{0, 1, 2\}$, i.e., $X_i \stackrel{d}{=} Y_i \sim U(\{0, 1, 2\})$ for $i \in \{0, 1, 2\}$. Further, for $i \in \{0, 1\}$, it holds that $(X_i, X_{i+1}) \leq_{lo} (Y_i, Y_{i+1})$ and thus, due to identical marginals, $(X_i, X_{i+1}) \leq_{sm} (Y_i, Y_{i+1})$, see (9). Moreover,

$$\begin{array}{ll} X_1 \uparrow_{st} X_0 \,, & X_2 \uparrow_{st} X_1 \,, & X_1 \uparrow_{st} X_2 \,, \\ Y_1 \uparrow_{st} Y_0 \,, & Y_2 \uparrow_{st} Y_1 \,, & Y_1 \uparrow_{st} Y_2 \,, \end{array}$$

i.e., X and Y are CIS, see Lemma B.3, and the bivariate subvectors (X_1, X_2) and (Y_1, Y_2) are CI. However, Y violates condition (ii) of Proposition 1.1 as well as condition (ii) of Theorem 1.3 since $Y_0 \not\gamma_{st} Y_1$. For the lower orthant set $\Theta := (-\infty, 1]^3$, we obtain

$$P^{X}(\Theta) = 3\sum_{i,j,k=1}^{2} a_{ij}^{01} a_{jk}^{12} = \frac{1.12}{3} > \frac{1.11}{3} = 3\sum_{i,j,k=1}^{2} b_{ij}^{01} b_{jk}^{12} = P^{Y}(\Theta).$$

This shows that $X \not\leq_{lo} Y$ and thus, due to (8), $X \not\leq_{sm} Y$. Hence, condition (ii) of Proposition 1.1 cannot be replaced by $Y_{i+1} \uparrow_{st} Y_i$. Similarly, this example highlights the importance of the assumption $Y_i \uparrow_{st} Y_j$ for all $(i, j) \in E$ with $j \notin L$ in Theorem 1.3.

The following example shows that there is no version of Lemma 1.2 when assumptions (i) and (ii) are replaced by (X_0, X_i) is CI and $Y_0 \uparrow_{st} Y_i$ for $i \in \{1, \ldots, d\}$. These SI conditions correspond to the graph in Figure 8 b).

Example A.4 (Conditions (i)-(ii) in Lemma 1.2 cannot be replaced by (X_0, X_i) CI and $Y_0 \uparrow_{st} Y_i$) Let T = (N, E), $N = \{0, 1, 2\}$, $E = \{(0, 1), (0, 2)\}$, be a tree on three nodes. Consider the doubly stochastic matrices $a^{01}, a^{12}, b^{01}, b^{12} \in \mathbb{R}^{4 \times 4}$ defined by

$$a^{01} = \frac{1}{40} \begin{pmatrix} 3 & 3 & 3 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 1 & 1 & 7 \end{pmatrix}, \quad b^{01} = \frac{1}{40} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 5 & 4 & 1 & 0 \\ 1 & 2 & 5 & 2 \\ 0 & 1 & 2 & 7 \end{pmatrix},$$

$$a^{02} = \frac{1}{40} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 3 & 3 & 2 \\ 1 & 1 & 3 & 5 \end{pmatrix}, \quad b^{02} = \frac{1}{40} \begin{pmatrix} 6 & 2 & 2 & 0 \\ 3 & 3 & 1 & 3 \\ 1 & 3 & 4 & 2 \\ 0 & 2 & 3 & 5 \end{pmatrix}.$$
(41)

Let $X = (X_0, X_1, X_2)$ and $Y = (Y_0, Y_1, Y_2)$ be random vectors which follow a Markov tree distribution with respect to the tree T specified by the bivariate distributions given by

 $P[X_0 = k, X_i = l] = a_{kl}^{0i} \text{ and } P[Y_0 = k, Y_i = l] = b_{kl}^{0i} \text{ for } k, l \in \{0, 1, 2, 3\} \text{ and } i \in \{0, 1\}.$

All marginal distributions of the vectors X and Y are uniform on $\{0, 1, 2, 3\}$, i.e., $X_i \stackrel{d}{=} Y_i \sim U(\{0, 1, 2, 3\})$ for $i \in \{0, 1, 2\}$. Further, for $i \in \{1, 2\}$, it holds that $(X_0, X_i) \leq_{lo} (Y_0, Y_i)$ and thus, due to identical marginals, $(X_0, X_i) \leq_{sm} (Y_0, Y_i)$, see (9). Moreover,

 $\begin{aligned} X_0 \uparrow_{st} X_1, & X_0 \uparrow_{st} X_2, & X_1 \uparrow_{st} X_0, & X_2 \uparrow_{st} X_0, \\ Y_0 \uparrow_{st} Y_1, & Y_0 \uparrow_{st} Y_2, \end{aligned}$

i.e., for $i \in \{1,2\}$, the common factor variable Y_0 is stochastically increasing Y_i and (X_0, X_1) as well as (X_0, X_2) are CI. For the lower orthant set $\Theta := (-\infty, 2]^3$, it holds that

$$P^{X}(\Theta) = 4\sum_{i,j,k=1}^{3} a_{ji}^{01} a_{jk}^{12} = \frac{2.25}{4} > \frac{2.24}{4} = 4\sum_{i,j,k=1}^{3} b_{ji}^{01} b_{jk}^{12} = P^{Y}(\Theta).$$
(42)

This shows that $X \not\leq_{lo} Y$ and thus, due to (8), $X \not\leq_{sm} Y$. Hence, condition (ii) of Lemma 1.2 cannot be replaced by $Y_0 \uparrow_{st} Y_i$. Similarly, this example highlights the importance of the assumption $Y_j \uparrow_{st} Y_i$ for all $(i, j) \in E$ with $j \notin P$ in Theorem 1.3.

The SI conditions in the following example correspond to the graph in Figure 8 c). This example is a comonotonic extension of Example A.4 and will also be used for the proof of Proposition 5.2 which provides the generality of the SI assumptions in Theorem 1.3.

Example A.5 (A commonotonic extension of Example A.4)

Let (X_0, X_1, X_2) and (Y_0, Y_1, Y_2) be the random vectors in Example A.4. Define the (d+1)-dimensional random vectors X' and Y' by

$$X' = (X'_0, \dots, X'_d) \coloneqq (X_1, X_0, \dots, X_0, X_2) \quad and \quad Y' = (Y'_0, \dots, Y'_d) \coloneqq (Y_1, Y_0, \dots, Y_0, Y_2).$$

Then X' and Y' follow a Markov tree distribution with respect to the chain T = (N, E), where $N = \{0, \ldots, d\}$ and $E = \{(0, 1), \ldots, (d - 1, d)\}$. We have that

$$\begin{array}{ll} X'_{i+1}\uparrow_{st}X'_{i}, & X'_{i}\uparrow_{st}X'_{i+1}, & \text{for all } i \in \{0,\ldots,d-1\}, \\ Y'_{i+1}\uparrow_{st}Y'_{i}, & Y'_{j}\uparrow_{st}Y'_{j+1}, & \text{for all } i \in \{0,\ldots,d-2\} \text{ and } j \in \{1,\ldots,d-1\}. \end{array}$$

Moreover, it holds that $(X'_i, X'_{i+1}) \leq_{lo} (Y'_i, Y'_{i+1})$ for all $i = 0, \ldots, d-1$. Using (42) we obtain for the lower orthant set $\Theta \coloneqq (-\infty, 2]^{d+1}$ that $P^{X'}(\Theta) > P^{Y'}(\Theta)$. This shows that $X' \not\leq_{lo} Y'$ and thus $X' \not\leq_{sm} Y'$. This example highlights the importance of the assumption $Y_j \uparrow_{st} Y_i$ for all $(i, j) \in E$ with $j \notin P$ in Theorem 1.3 by extending the usage of Example A.4 to more complex cases, see also Proposition 5.2.

The following example shows that Proposition 5.4 based on the Schur order for conditional distributions cannot be extended to Markov processes and neither to Markov tree distributions.

Example A.6 (Schur order is not applicable to Markov processes) Define the matrices $a, b \in \mathbb{R}^{3 \times 3}$ by

$$a \coloneqq \frac{1}{30} \begin{pmatrix} 5 & 2 & 3\\ 3 & 7 & 0\\ 2 & 1 & 7 \end{pmatrix} \quad and \quad b \coloneqq \frac{1}{30} \begin{pmatrix} 6 & 4 & 0\\ 3 & 4 & 3\\ 1 & 2 & 7 \end{pmatrix}.$$
(43)

Let $X = (X_0, \ldots, X_3)$ and $Y = (Y_0, \ldots, Y_3)$ be random vectors which follow a Markov tree distribution with respect to the tree T = (N, E) for $N = \{0, 1, 2, 3\}$ and $E = \{(0, 1), (1, 2), (2, 3)\}$ with bivariate distributions specified by

$$P[X_i \in [k, k+s), X_{i+1} \in [\ell, \ell+t)] = sta_{k\ell}, P[Y_i \in [k, k+s), Y_{i+1} \in [\ell, \ell+t)] = stb_{k\ell},$$

for all $k, l \in \{0, 1, 2\}$, $i \in \{0, 1, 2, 3\}$ and $s, t \in [0, 1]$. All marginal distributions of the vectors X and Y are uniform on [0, 3), i.e., $X_i \stackrel{d}{=} Y_i \sim U([0, 3))$ for all $i \in \{0, 1, 2, 3\}$. It holds that (Y_i, Y_{i+1}) is MTP_2 (thus, in particular, $Y_i \uparrow_{st} Y_{i+1}$ and $Y_{i+1} \uparrow_{st} Y_i$, see (14)) and $(X_i, X_{i+1}) \leq_{lo} (Y_i, Y_{i+1})$ for all $i \in \{0, 1, 2\}$. Further, the bivariate distributions fulfill the Schur order criteria

$$(X_i|X_j) \leq_S (Y_i|Y_j) \quad and \quad (X_j|X_i) \leq_S (Y_j|Y_i) \quad for \ all \ (i,j) \in E ,$$

$$(44)$$

i.e., for all edges $(i,j) \in E$ and for all $z \in \mathbb{R}$, both $w \mapsto F_{X_i|X_j=q_{X_j}(w)}(z)$ has less variability than $w \mapsto F_{Y_i|Y_j=q_{Y_j}(w)}(z)$ and $w \mapsto F_{X_i|X_j=q_{X_j}(w)}(z)$ has less variability than $w \mapsto F_{Y_i|Y_j=q_{Y_j}(w)}(z)$. Since (Y_i, Y_{i+1}) exhibits positive dependence (with respect to the strong notion MTP_2) one might expect that Y exhibits more positive dependence than X and thus $X \leq_{l_0} Y$. However, considering the lower orthant set $\Theta \coloneqq (-\infty, 2)^4$, we obtain

$$P^{X}(\Theta) = 9 \sum_{i_{1},\dots,i_{4}=1}^{2} a_{i_{1}i_{2}}a_{i_{2}i_{3}}a_{i_{3}i_{4}} = \frac{1.259}{3} > \frac{1.256}{3} = 9 \sum_{i_{1},\dots,i_{4}=1}^{2} b_{i_{1}i_{2}}b_{i_{2}i_{3}}b_{i_{3}i_{4}} = P^{Y}(\Theta),$$

which shows that $X \not\leq_{lo} Y$ and consequently $X \not\leq_{sm} Y$. Note that Theorem 1.3 cannot be applied because $X_i \not\gamma_{st} X_j$ so that the vector X violates assumption (i).

Hence, the random vector $X = (X_0, \ldots, X_3)$ is not smaller in supermodular order than $Y = (Y_0, \ldots, Y_3)$ even though, for all edges $(i, j) \in E$, on the one hand (X_i, X_j) has less variability and is closer to independence than (Y_i, Y_j) with respect to the Schur order for conditional distributions, see (44). On the other hand, (X_i, X_j) is PSMD and (Y_i, Y_j) is MTP_2 , i.e., all bivariate specifications exhibit positive dependencies.

To summarize, this example shows the importance of the SI assumptions in Theorem 1.3 on both $(X_n)_{n \in N}$ and $(Y_n)_{n \in N}$ noting that the SI assumptions on $(X_n)_{n \in N}$ cannot be skipped or weakened to PSMD even under the stronger comparison of the bivariate dependence specifications with respect to the Schur order for conditional distributions. This example also emphasises the special properties of star structures which allow a more general supermodular comparison result based on the Schur order for conditional, see Proposition 5.4.

B Proofs of Section 1

In this section, we provide the proof of Theorem 1.3 which is based on combining Lemma 1.2 with an extension of the conditioning argument in [58, Theorem 3.2] to a non-stationary setting and to Markovian tree structures. For working with trees, this requires additional tools, which we introduce in the sequel. It is worth noting that Proposition 1.1 serves as an extension of [58, Theorem 3.2] to a non-stationary context. This proposition emerges as a direct consequence of Theorem 1.3.



Figure 9 A level-order traversal $\pi = (1, 6, 3, 5, 7, 2, 4, 8)$ for the tree T = (N, E) with nodes $N = \{1, \ldots, 8\}$ and edges $E = \{(1, 6), (1, 3), (6, 5), (6, 7), (3, 2), (3, 4), (3, 8)\}.$

In order to navigate through a tree, we define the commonly used notions of a parent, child, descendant, and ancestor relationship as follows.

Definition B.1 (Parent, child, descendant, ancestor, degree)

Let T = (N, E) be a tree with root $0 \in N$.

- (i) The parent of $i \in N \setminus \{0\}$ is defined as the unique element $p_i \in N$ such that $(p_i, i) \in E$.
- (ii) The set of *children* of $i \in N$ is defined by $c_i \coloneqq \{j \in N | i = p_j\} \subset N$.
- (iii) The set of descendants of $i \in N$ is defined by $d_i := \{j \in N | \text{there is a directed path from } i \text{ to } j\} \subset N$.
- (iv) The set of ancestors of $i \in N \setminus \{0\}$ is defined by $a_i := \{j \in N | i \in d_j\} \subset N$.
- (v) The degree of a node $i \in N \setminus \{0\}$ is defined by $\deg(i) \coloneqq |c_i| + 1$ and $\deg(0) \coloneqq |c_0|$.

According to its definition, a node $i \in N$ is not included in the set of its children c_i , descendants d_i , or ancestors a_i . Note that the root $0 \in N$ is the only element having no parent node.

The proof of Theorem 1.3 is based on an induction, which requires a specific enumeration of the nodes of the tree. To this end, we consider the level-order traversal where the nodes of the tree are visited level by level, starting from the root. This is also known as a breadth-first search. To be precise, we say that $\pi = (\pi_n)_{n \in N}$ is an *enumeration of* N if it is a one-to-one map from N to N. Further, π is a *level-order* traversal if it is an enumeration such that for any pair of nodes i and j with i < j (with respect to the order on \mathbb{N}), the node π_i is in a lower level of the tree compared to π_j , i.e., for all $i, j \in N$,

$$i < j$$
 implies $|p[0, \pi_i]| \le |p[0, \pi_j]|.$ (45)

Note that this enumeration is not uniquely determined. We examplify the level order traversal in Figure 9 within a tree that comprises 8 nodes distributed across 3 levels.

To prove Theorem 1.3, we rely on several lemmas presented in the sequel. The subsequent lemma shows that if any random variable is SI in its ancestor random variables in a Markov tree distributed sequence, then any random vector corresponding to descendant nodes of this node is SI in the random variable associated to this current node. In the context of Markov processes, this means that if at each time step, the present random variable is SI in the preceding variable, then all future variables are SI in the present variable, see [45, Theorem 3.2]. We define, in comparison to Definition 2.1, the undirected paths

$$p[i,j) \coloneqq \{i\} \cup p(i,j), \quad p(i,j] \coloneqq \{j\} \cup p(i,j), \quad p[i,j] \coloneqq \{i,j\} \cup p(i,j).$$

$$(46)$$

Moreover, for a subset of nodes $J \subseteq N$, we write $x_J \coloneqq (x_j)_{j \in J}$ and $X_J \coloneqq (X_j)_{j \in J}$ when referencing to the vector and random vector associated with these components.

Lemma B.2 Let X be a sequence of random variables which follow a Markov tree distribution with respect to a tree T = (N, E). Assume that $X_i \uparrow_{st} (X_j, j \in a_i)$ for all $i \in N \setminus \{0\}$. Then, for all $i \in N$ and for any finite subset $J \subseteq d_i$, it follows that $(X_j, j \in J) \uparrow_{st} X_i$, i.e., for any $\phi \colon \mathbb{R}^{|J|} \to \mathbb{R}$ increasing and bounded, the mapping

$$x_i \mapsto \int \phi(x_J) P^{X_J | X_i = x_i}(dx_J) \tag{47}$$

is increasing.

Proof: We prove the result by induction over |J|. Let |J| = 1, $J = \{j_1\} \subseteq d_i$ for $i \in N$. Then, by assumption, $X_k \uparrow_{st} X_{p_k}$ for all $k \in p(i, j_1]$, see (46). We conclude by [45, Theorem 3.2] that $X_{j_1} \uparrow_{st} X_i$. For the induction step, assume that the statement is valid for all $J \subseteq d_i$ such that |J| = n - 1. Suppose that J has the form $J = \{j_1, \ldots, j_n\} \subseteq d_i$. The following part of the proof is divided in two cases. First, we assume that there are two nodes in J which are not separated by i, i.e.,

$$J \cap \bigcup_{j \in J} p(i,j) \neq \emptyset$$

Then there is a node $j_{\ell} \in J$ such that $J \cap p(i, j_{\ell}) \neq \emptyset$. In particular, the node j_{ℓ} can be choosen such that $j_{\ell} \notin p(i, j)$ for all $j \in J$. Let $j_k \in J$ be the last node in J in the directed path from i to j_{ℓ} , i.e.,

$$j_m \coloneqq \arg \max \left\{ |p(i,j)| | j \in J \cap p(i,j_\ell) \right\}.$$

For $K := J \setminus \{j_\ell\}$ we have for any $\phi \colon \mathbb{R}^n \to \mathbb{R}$ increasing and bounded that

$$\int \phi(x_J) P^{X_J | X_i = x_i}(dx_J) = \int \int (x_{j_\ell}, x_K) P^{X_{j_\ell} | X_K = x_K, X_i = x_i}(dx_{j_\ell}) P^{X_K | X_i = x_i}(dx_K) = \int \underbrace{\int \phi(x_{j_\ell}, x_K) P^{X_{j_\ell} | X_{j_m} = x_{j_m}}(dx_{j_\ell})}_{:= \psi(x_{j_m}, x_K \setminus \{j_m\})} P^{X_K | X_i = x_i}(dx_K),$$
(48)

where we use the Markov tree dependence for the second equality. Applying [45, Theorem 3.2] once more, we can conclude that ψ is an increasing and bounded function in x_{j_m} for all $x_{K\setminus\{j_m\}} \in \mathbb{R}^{n-2}$. Moreover, since ϕ is increasing, also the function ψ is increasing in $x_{K\setminus\{j_m\}}$ for all $x_{j_m} \in \mathbb{R}$. Consequently, ψ is increasing in each argument. By the induction hypothesis, we have that $X_K \uparrow_{st} X_i$ since |K| = n - 1 and $K \subseteq d_i$. Hence, by the definition of the stochastic order, the function in (48) is increasing in x_i , proving that $X_J \uparrow_{st} X_i$.

In the second case we assume that $J \cap p(i,j) = \emptyset$ for all $j \in J = \{j_1, \ldots, j_n\}$. For $K \coloneqq J \setminus \{j_n\}$ we have for any $\phi \colon \mathbb{R}^n \to \mathbb{R}$ increasing and bounded that

$$P^{X_J|X_i=x_i} = P^{X_K|X_i=x_i} \otimes P^{X_{j_n}|X_i=x_i}$$

$$\tag{49}$$

by the Markov property. It follows that

$$\int \phi(x_J) P^{X_J | X_i = x_i} (dx_J) = \int \int \phi(x_J) P^{X_K | X_i = x_i} (dx_K) P^{X_{j_n} | X_i = x_i} (dx_{j_n}).$$
(50)

Since $X_{j_n} \uparrow_{st} X_i$ by assumption and, by the induction hypothesis, $X_K \uparrow_{st} X_i$, we obtain that $X_J \uparrow_{st} X_i$ and the result is proven.

The following lemma, which we were unable to locate in the literature, shows that if all components $X_i, i \in N \setminus \{0\}$ of a Markov tree distributed random vector in a random vector X are SI in its parents, then X_i is SI in all of its ancestors.

Lemma B.3 Let $X = (X_n)_{n \in N}$ be a random vector that has Markov tree dependence with respect to a tree T = (N, E). If $X_i \uparrow_{st} X_i$ for all edges $(i, j) \in E$, then

$$X_i \uparrow_{st} (X_j, \ j \in a_i) \quad \text{for all nodes } i \in N \setminus \{0\}.$$

$$(51)$$

Proof of Lemma B.3 In order to prove the statement, it is sufficient to prove that a Markov process (X_0, \ldots, X_n) is CIS if $X_{i+1} \uparrow_{st} X_i$ for all $i = 0, \ldots, n-1$. For n = 2, the statement is trivial. By induction,

we have to prove that (X_0, \ldots, X_{n+1}) is CIS if (X_0, \ldots, X_n) is CIS and $X_{n+1} \uparrow_{st} X_n$. This follows if $X_{n+1} \uparrow_{st} (X_j, j \in \{1, \ldots, n\})$. By the Markov property it holds that

$$X_{n+1}\uparrow_{st}(X_j, \ j\in\{0,\dots,n\})\Leftrightarrow X_{n+1}\uparrow_{st}X_n,\tag{52}$$

and the result is proven.

Note that property (51) is in general not a positive dependence concept unless X follows a Markov tree distribution. For example, let T = (N, E), $N = \{(1, 2), (1, 3)\}$. For some independet and uniformly on [0, 1] distributed random variables U and V we define $X_1 = U$, $X_2 = V$, $X_3 = 1 - V$. Then $X = (X_1, X_2, X_3)$ meets (51), but is not positive lower (upper) orthant dependent, see Definition 2.8.

However, in the context of chains, property (51) is equivalent to the CIS property, and thus is a positive dependence concept in the sense of (14).

The following lemma extends [58, Theorem 3.1] from Markov processes to Markov tree distributions. It provides sufficient condition such that an integral of a supermodular function, under a conditional distribution, yields a supermodular function dependent on both the conditioning variable and the variables that remain unintegrated. Denote by $p[i, j] := p(i, j) \cup \{i\} \cup \{j\}$ the undirected path from *i* to *j* including *i* and *j*.

Lemma B.4 (Supermodular conditional integration)

(i) Let X be a random vector that follows a Markov tree distribution with respect to a tree T = (N, E). Let $i \in N$ and $J \subseteq d_i$ be a finite set. Assume that $X_{j'} \uparrow_{st} X_j$ for all $(j, j') \in E$. Moreover let K be an arbitrary finite index set and ϕ be a supermodular function on $\mathbb{R}^{1+|K|+|J|}$. Then the function $\varphi \colon \mathbb{R}^{|K|+1} \to \mathbb{R}$ given by

$$\varphi(z_i, z_K) := \int \phi(z_i, z_K, z_J) P^{X_J | X_i = z_i}(dz_J),$$
(53)

is supermodular.

(ii) The assumption $X_{j'} \uparrow_{st} X_j$ for all $(j, j') \in E$ in (i) can be weakened to $X_{j'} \uparrow_{st} X_j$ for all edges $(j, j') \in \{(j, j') \in E \mid j, j' \in \bigcup_{u \in J} p[i, u]\}$ of the subtree that consists of all paths from node *i* to its descendants in $J \subseteq d_i$.

Proof: (i) By the supermodularity of ϕ we obtain for $k_1, k_2 \in K, k_1 \neq k_2$, that the map

$$(z_{k_1}, z_{k_2}) \mapsto \varphi(z_i, z_{k_1}, z_{k_2}, z_{K \setminus \{k_1, k_2\}})$$
(54)

is supermodular for all $(z_i, z_{K \setminus \{k_1, k_2\}}) \in \mathbb{R}^{1+|K|-2}$. In the following we show for $k \in K$ that the map

$$(z_i, z_k) \mapsto \varphi(z_i, z_k, z_{K \setminus \{k\}}) \tag{55}$$

is supermodular for all $z_{K\setminus\{k\}} \in \mathbb{R}^{|K|-1}$. We fix $z_k < z'_k$. By the supermodularity of ϕ it follows for all $j \in J$ that the map

$$z_j \mapsto \phi(z_i, z'_k, z_{K \setminus \{k\}}, z_j, z_{J \setminus \{j\}}) - \phi(z_i, z_k, z_{K \setminus \{k\}}, z_j, z_{J \setminus \{j\}})$$

$$(56)$$

is increasing for all $z_{J\setminus\{j\}} \in \mathbb{R}^{|J|-1}$ and $z_{K\setminus\{k\}} \in \mathbb{R}^{|K|-1}$. Similarly, the map

$$z_i \mapsto \phi(z_i, z'_k, z_{K \setminus \{k\}}, z_J) - \phi(z_i, z_k, z_{K \setminus \{k\}}, z_J)$$

$$(57)$$

is increasing for all $z_J \in \mathbb{R}^{|J|}$ and $z_{K \setminus \{k\}} \in \mathbb{R}^{|K|-1}$. By Lemma B.3 the assumption $X_{j'} \uparrow_{st} X_j$ for all $(j, j') \in E$ implies that $X_j \uparrow_{st} (X_{j'}, j' \in a_j)$ for all $j \in N \setminus \{0\}$. Thus, we can conclude from Lemma B.2

that $(X_j, j \in J) \uparrow_{st} X_i$. Now, for $z_i < z'_i$, we obtain that

$$\varphi(z_i, z'_k, z_{K\setminus\{k\}}) - \varphi(z_i, z_k, z_{K\setminus\{k\}})
\leq \int \phi(z_i, z'_k, z_{K\setminus\{k\}}, z_J) P^{X_J | X_i = z'_i}(dz_J) - \int \phi(z_i, z_k, z_{K\setminus\{k\}}, z_J) P^{X_J | X_i = z'_i}(dz_J)
\leq \varphi(z'_i, z'_k, z_{K\setminus\{k\}}) - \varphi(z'_i, z_k, z_{K\setminus\{k\}}),$$
(58)

where the first inequality follows by $(X_j, j \in J) \uparrow_{st} X_i$ and the increasingness of (56) for $j \in J$. The second inequality holds true because the function in (57) is increasing. Now, (58) implies that

$$\varphi(z_i', z_k', z_{K\setminus\{k\}}) + \varphi(z_i, z_k, z_{K\setminus\{k\}}) \ge \varphi(z_i', z_k, z_{K\setminus\{k\}}) + \varphi(z_i, z_k', z_{K\setminus\{k\}})$$

and we obtain the supermodularity of (55). Finally, the supermodularity in the cases (54) and (55) yields the supermodularity of (53), which proves the statement.

(ii) The second statement follows by applying (i) for the subtree T' = (N', E') with root $i \in N$, nodes $N' = \bigcup_{u \in J} p[i, u]$ and edges $E' = \{(j, j') \in E \mid j, j' \in N'\}$.

The following proposition is a version of Theorem 1.3 under additional SI assumptions. This result is used for the proof of Theorem 1.3 where we show that several SI conditions along a specified path can be omitted. We briefly outline the idea behind the proof of this proposition.

We initiate with the distribution of $X = (X_n)_{n \in N}$ and replace the dependence specifications of X for the specifications of Y iteratively over all star structures within the tree T = (N, E) using a conditioning argument and applying Lemma 1.2 on the supermodular comparison of star structures. More precisely, for the induction, we navigate through the tree using a level-order traversal enumeration as described in (45). A disintegration argument allows us to reduce the proof of the statement in each step to the setting of Lemma 1.2. Finally, by the transitivity of the supermodular order, we obtain a supermodular comparison between X and Y.

Proposition B.5 Let $X = (X_n)_{n \in N}$ and $Y = (Y_n)_{n \in N}$ be sequences of random variables that follow a Markov tree distribution with respect to a tree T = (N, E). Assume that

- (i) $X_j \uparrow_{st} X_i$ for all $(i, j) \in E$,
- (*ii*) $Y_i \uparrow_{st} Y_j$ for all $(i, j) \in E, j \notin L$, and $Y_j \uparrow_{st} Y_i$ for all $(i, j) \in E$,
- (iii) $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ (resp. \geq_{sm}) for all $(i, j) \in E$.
- Then $X \leq_{sm} Y$ (resp. \geq_{sm}).

Proof: By the definitions of the stochastic orderings for stochastic processes (see Definition 2.7), we can assume that $d + 1 := |N| < \infty$ and $N = \{0, \ldots, d\}$. We only show the case that $X \leq_{sm} Y$ if $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ for all $(i, j) \in E$. In the case where $(X_i, X_j) \geq_{sm} (Y_i, Y_j)$ for all $(i, j) \in E$, it follows analogously that $X \geq_{sm} Y$ by reversing all inequality signs.

Denote by $F = (F_0, \ldots, F_d)$ the vector of univariate marginal distribution functions of X and Y. Note that $F_{X_n} = F_{Y_n}$ for all $n \in N$ because $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ implies $X_i \stackrel{d}{=} Y_i$ and $X_j \stackrel{d}{=} Y_j$. Moreover, for $(i, j) \in E$, denote by $B_{(i,j)}$ and $C_{(i,j)}$ a copula for (X_i, X_j) and (Y_i, Y_j) , respectively, and define $B = (B_e)_{e \in E}$ and $C = (C_e)_{e \in E}$. It follows that $X \sim \mathcal{M}(F, B, T)$ and $Y \sim \mathcal{M}(F, C, T)$, see Proposition 2.5. Let $(\pi_n)_{n=1,\ldots,d}$ be a level-order traversal of T as defined in (45). For $i \in N$, define the set of edges connecting the node i with its children by $[i, c_i] \coloneqq \{(i, j) | j \in c_i\} \subseteq E$. Then $[i, c_i]$ has a star-like structure. Note that if $i \in L$, then $c_i = \emptyset$ and thus $[i, c_i] = \emptyset$. For $k = 0, \ldots, d-1$, let $D^k = (D_e^k)_{e \in E}$ be the set of bivariate copulas defined by

$$D_e^k \coloneqq \begin{cases} C_e & \text{if } e \in [\pi_n, c_{\pi_n}] \text{ for } 1 \le n \le k, \\ B_e & \text{if } e \in [\pi_n, c_{\pi_n}] \text{ for } k < n \le d. \end{cases}$$

$$(59)$$

i.e., the specifications of the edges in the first k stars in level-order traversal are replaced by the bivariate copula specifications of Y and the remaining edges are specified by the bivariate copulas associated to

X. For $k \in \{0, \ldots, d-1\}$, consider the random vector $Z^{(k)} \sim \mathcal{M}(F, T, D^k)$. Note that $D^0 = B$ and $D^{d-1} = C$, which implies that $Z^{(0)} \stackrel{d}{=} X$ and $Z^{(d)} \stackrel{d}{=} Y$. We show that

$$Z^{(k-1)} \leq_{sm} Z^{(k)}$$
 for all $k = 1, \dots, d-1$. (60)

Then the statement follows from the transitivity of the supermodular order, see, e.g., [87]. In order to prove (60), fix $k \in \{1, \ldots, d-1\}$. If $\pi_k \in L$ we have that $D^{k-1} = D^k$ and thus $Z^{(k-1)} \stackrel{d}{=} Z^{(k)}$ which trivially implies $Z^{(k-1)} \leq_{sm} Z^{(k)}$. If $\pi_k \notin L$, define $i \coloneqq \pi_k \in N$ and denote by $\{j_1, j_2, \ldots, j_m\} \coloneqq c_i$ an enumeration of the children c_i of node i. Note that, by the definition of the bivariate copula sequences in (59), D^{k-1} and D^k can only differ in the copulas which correspond to the edges $[i, c_i]$ between node i and its children, i.e., $D_e^{k-1} = D_e^k$ for all $e \in E \setminus [i, c_i]$. For proving (60), it is sufficient to show for any bounded supermodular function $\phi \colon \mathbb{R}^{d+1} \to \mathbb{R}$ that

$$E[\phi(Z^{(k-1)})] \le E[\phi(Z^{(k)})], \tag{61}$$

see [87, Theorem 3.9.14]. Recall that d_i denotes the descendants of node i, see Definition B.1. Then we define the function $\varphi^{(0)}$ on $\mathbb{R}^{|d_i|+1}$ by

$$\varphi^{(0)}(z_i, z_{d_i}) \coloneqq \int \phi(z_i, z_{d_i}, z_{N \setminus (\{i\} \cup d_i)}) P^{Z_{N \setminus (\{i\} \cup d_i)}^{(k)} | Z_i^{(k)} = z_i} (dz_{N \setminus (\{i\} \cup d_i)})$$

i.e., we integrate over all variables conditional on z_i except of the variables associated to node i and its descendants d_i . Then, we define recursively over the children of node i for $\ell = 1, \ldots, m$ the functions $\varphi^{(l)} : \mathbb{R}^{1+m+\sum_{r=\ell+1}^{m} |d_{j_r}|} \to \mathbb{R}$ by

$$\varphi^{(\ell)}(z_i, z_{c_i}, z_{d_{j_{\ell+1}}}, \dots, z_{d_{j_m}})$$

$$\coloneqq \int \varphi^{(\ell-1)}(z_i, z_{c_i}, z_{d_{j_\ell}}, \dots, z_{d_{j_m}}) P^{Z_{d_{j_\ell}}^{(k)} | Z_{j_\ell}^{(k)} = z_{j_\ell}}(dz_{d_{j_\ell}}),$$

i.e., the function $\varphi^{(\ell)}$ is obtained through $\varphi^{(\ell-1)}$ by integrating over the variables associated to all descendants $d_{j_{\ell}}$ of the ℓ 'th child of *i* conditioned on the variable associated with j_{ℓ} . Finally, $\varphi^{(m)}$ is a function in $(z_i, z_{j_1}, \ldots, z_{j_m}) = (z_i, z_{c_i})$.

To prove supermodularity of $\varphi^{(0)}$, we will apply Lemma B.4. Recall that p[0, i] denotes the path from 0 up to and including node *i*. From the assumptions (ii) and (i) we know, in particular, that

$$Z_{j'}^{(k)} \uparrow_{st} Z_{j''}^{(k)} \text{ for all } (j', j'') \in E \cap (p[0, i])^2 \quad \text{and} \\ Z_{j''}^{(k)} \uparrow_{st} Z_{j'}^{(k)} \text{ for all } (j', j'') \in E \setminus ((p[0, i])^2 \cup (\{i\} \cup d_i)^2),$$

$$(62)$$

where, for a set M, the Cartesian product $M \times M$ is denoted by M^2 . Consider the subtree $T_0 = (N_0, E_0)$ with nodes $N_0 := N \setminus d_i$ and edges $E_0 := E \cap N_0^2$. The nodes N_0 can be relabeled such that i becomes the new root in T_0 . By (62) the Markov tree distributed sequence $(Z_n^{(k)})_{n \in N_0}$ (with respect to the tree T_0) satisfies the assumptions of Lemma B.4 (ii) and by setting $K = d_i$ and $J = N_0 \setminus \{i\}$ (here d_i still denotes the set of descendents of i with respect to T) we obtain the supermodularity of $\varphi^{(0)}$.

To prove the supermodularity of $\varphi^{(\ell)}$, $\ell \in \{1, \ldots, m\}$, we assume by induction that $\varphi^{(\ell-1)}$ is supermodular. By definition of $Z^{(k)}$ and assumption (i) we have that

$$Z_{j''}^{(k)} \uparrow_{st} Z_{j'}^{(k)} \text{ for all } (j', j'') \in E \cap (d_{j_{\ell}})^2.$$
(63)

Hence, we obtain supermodularity of $\varphi^{(\ell)}$ by Lemma B.4 (ii) by setting $K = \{i\} \cup (c_i \setminus \{j_\ell\}) \cup d_{j_{\ell+1}} \cup \cdots \cup d_{j_m}$ and $J = d_{j_\ell}$. and thus obtain the supermodularity of $\varphi^{(\ell)}$ by Lemma B.4 (ii). In particular, we obtain that $\varphi^{(m)}$ is supermodular. Next, we show that

$$E[\phi(Z^{(k-1)})] = E[\varphi^{(m)}(Z_i^{(k-1)}, Z_{c_i}^{(k-1)})] \quad \text{and}$$

$$(64)$$

$$E[\phi(Z^{(\kappa)})] = E[\varphi^{(m)}(Z_i^{(\kappa)}, Z_{c_i}^{(\kappa)})].$$
(65)

From Markov tree dependence of $Z^{(k)}$ and $Z^{(k-1)}$ and since the sets of bivariate copulas D^{k-1} and D^k are identical for all edges $E \setminus [i, c_i]$, we have

$$\varphi^{(0)}(z_i, z_{d_i}) = \int \phi(z_i, z_{d_i}, z_{N \setminus \{i\} \cup d_i\}}) P^{Z_{N \setminus \{i\} \cup d_i\}}^{(k)} | Z_i^{(k)} = z_i, Z_{d_i}^{(k)} = z_{d_i} \left(dz_{N \setminus \{i\} \cup d_i\}} \right)$$

$$= \int \phi(z_i, z_{d_i}, z_{N \setminus \{i\} \cup d_i\}}) P^{Z_{N \setminus \{i\} \cup d_i\}}^{(k-1)} | Z_i^{(k-1)} = z_i, Z_{d_i}^{(k-1)} = z_{d_i} \left(dz_{N \setminus \{i\} \cup d_i\}} \right)$$
(66)

and

$$\varphi^{(\ell)}(z_{i}, z_{c_{i}}, z_{d_{j_{l+1}}}, \dots, z_{d_{j_{m}}}) = \int \varphi^{(\ell-1)}(z_{i}, z_{c_{i}}, z_{d_{j_{\ell}}}, \dots, z_{d_{j_{m}}}) P^{Z_{d_{j_{\ell}}}^{(k)} | Z_{i}^{(k)} = z_{i}, Z_{c_{i}}^{(k)} = z_{c_{i}}, Z_{d_{j_{\ell+1}}}^{(k)} = z_{d_{j_{\ell+1}}}, \dots, Z_{d_{j_{m}}}^{(k)} = z_{d_{j_{m}}} (dz_{d_{j_{\ell}}})$$

$$= \int \varphi^{(\ell-1)}(z_{i}, z_{c_{i}}, z_{d_{j_{\ell}}}, \dots, z_{d_{j_{m}}}) P^{Z_{d_{j_{\ell}}}^{(k-1)} | Z_{i}^{(k-1)} = z_{i}, Z_{c_{i}}^{(k-1)} = z_{c_{i}}, Z_{d_{j_{\ell+1}}}^{(k-1)} = z_{d_{y_{\ell+1}}}, \dots, Z_{d_{j_{m}}}^{(k)} = z_{d_{j_{m}}} (dz_{d_{j_{\ell}}})$$

$$(67)$$

for all $\ell = 1, ..., m$. From (66), (67), and the disintegration theorem we obtain

$$E[\phi(Z^{(k-1)})] = E[\varphi^{(0)}(Z_i^{(k-1)}, Z_{d_i}^{(k-1)})]$$

$$= E[\varphi^{(1)}(Z_i^{(k-1)}, Z_{c_i}^{(k-1)}, Z_{d_{j_2}}^{(k-1)}, \dots, Z_{d_{j_m}}^{(k-1)})]$$

$$= \cdots$$

$$= E[\varphi^{(m-1)}(Z_i^{(k-1)}, Z_{c_i}^{(k-1)}, Z_{d_{j_m}}^{(k-1)})]$$

$$= E[\varphi^{(m)}(Z_i^{(k-1)}, Z_{c_i}^{(k-1)})],$$
(68)

which shows (64). The proof for Equation (65) follows similarly replacing k - 1 by k in (68). Hence, we obtain from (64) and (65) that

$$E[\phi(Z^{(k-1)})] = E[\varphi^{(m)}(Z_i^{(k-1)}, Z_{c_i}^{(k-1)})]$$

= $E[\varphi^{(m)}(X_i, X_{c_i})]$
 $\leq E[\varphi^{(m)}(Y_i, Y_{c_i})]$
= $E[\varphi^{(m)}(Z_i^{(k)}, Z_{c_i}^{(k)})]$
= $E[\phi(Z^{(k)})],$

For the second and third equality, we use that $(Z_i^{(k-1)}, Z_{c_i}^{(k-1)}) \stackrel{d}{=} (X_i, X_{c_i})$ and $(Z_i^{(k)}, Z_{c_i}^{(k)}) \stackrel{d}{=} (Y_i, Y_{c_i})$, which follows from the definition of $Z^{(k-1)}$ and $Z^{(k)}$, respectively. For the inequality, we apply Lemma 1.2 using on the one hand that, by Markov tree dependence, the random variables $X_{c_i} = (X_{j_1}, \ldots, X_{j_m})$ are conditionally independent given X_i and the random variables $Y_{c_i} = (Y_{j_1}, \ldots, Y_{j_m})$ are conditionally independent given Y_i . On the other hand, $X_{j_\ell} \uparrow_{st} X_i$ by assumption (i), $Y_{j_\ell} \uparrow_{st} Y_i$ by assumption (ii), and $(X_i, X_{j_\ell}) \leq_{sm} (Y_i, Y_{j_\ell})$ for $\ell \in \{1, \ldots, m\}$.

This finally shows (61) and thus (60) for fixed $k \in \{1, ..., d-1\}$. As we can repeat this procedure for each $k \in \{1, ..., d-1\}$, the result is proven.

The comparison result formulated in Proposition B.5 is an extension of Lemma 1.2 to tree structures. However, it does not generalize Proposition 1.1 since the SI assumptions in Proposition B.5 are more stringent when applied to chains. It turns out that these SI assumptions on Y can be omitted in one direction along a specified path P such that an generalization of Proposition 1.1 for trees can be formulated. This result is presented in Theorem 1.3 which can be proven by employing Proposition B.5 and an alternative adjustment of the dependencies, as described for the proof of Proposition B.5. As we show in the following proof of Theorem 1.3, we can omit the SI assumption on X for one of the edges connected with the root and we can omit the SI assumptions on Y along a path P through the tree starting with the root node and terminating at a leaf node whenever it has finite length. The proof of Theorem 1.3 is presented in the following.

Proof of Theorem 1.3 Similar to the proof of Proposition B.5, we can assume that $d + 1 := |N| < \infty$. We show that $X \leq_{sm} Y$. The proof for $X \geq_{sm} Y$ follows analogously. Denote by $F = (F_n)_{n \in N}$ for $F_n = F_{X_n} = F_{Y_n}$ the marginal distribution functions of X and Y. Moreover, for $e = (i, j) \in E$, denote by B_e and C_e any copula for (X_i, X_j) and (Y_i, Y_j) , respectively, and define $B = (B_e)_{e \in E}$ and $C = (C_e)_{e \in E}$. Then $X \sim \mathcal{M}(F, B, T)$ and $Y \sim \mathcal{M}(F, C, T)$, see Proposition 2.5. We enumerate the path $P = p(0, \ell] =:$ $\{\ell_1, \ldots, \ell_m\}$ such that $(\ell_j, \ell_{j+1}) \in E$ for all $j = 0, \ldots, m-1$, where $\ell_0 := 0$ and $\ell_m = \ell$.

In the following, we partition the tree T along the nodes $k^*, 0, \ell_1, \ldots, \ell_m$ into m + 1 disjoint subtrees, denoted as T_0, \ldots, T_m , with corresponding disjoint sets of nodes $N_0, \ldots, N_m \subseteq N$ and edges $E_0, \ldots, E_m \subseteq E$. As we will see, due to the selection of the subtrees, the subvectors $(X_n)_{n \in N_k}$ and $(Y_n)_{n \in N_k}$ fulfill the assumptions of Proposition B.5 for all $k = 0, \ldots, m$. We start with the definition of the set of nodes N_0, \ldots, N_m by

$$N_0 := \begin{cases} \{k^*\} \cup d_{k^*} & \text{if } \deg(0) > 1, \\ \emptyset & \text{if } \deg(0) = 1, \end{cases} \text{ and}$$
$$N_k := N \setminus \left(\{\ell_k\} \cup d_{\ell_k} \cup \bigcup_{j=0}^{k-1} N_j\right) \text{ for } k = 1, \dots, m.$$

Recall that k^* is due to the assumptions of Theorem 1.3 a child node of the root 0 and lies in P if and only if deg(0) = 1. Note also that, for a node i, the set d_i is defined as the set of descendants of i. Each set of nodes N_k forms a tree $T_k = (N_k, E_k)$ along with its corresponding set of edges E_k defined by

$$E_k \coloneqq \{(i,j) \in E \mid i,j \in N_k\}, \text{ for } k = 0,\ldots,m.$$

Note that some set of nodes N_k may be empty or a singleton, depending on the structure of the tree T. It holds that

$$\bigcup_{k=0}^{m} N_k = \bigoplus_{k=0}^{m} N_k = N \text{ and } \bigcup_{k=0}^{m} E_k = \bigoplus_{k=0}^{m} E_k = E \setminus \Big(\{ (0, k^*) \cup \bigcup_{j=1}^{m} \{ (\ell_{j-1}, \ell_j) \} \Big),$$

i.e., $\{N_k\}_{k=0,\ldots,m}$ is a partition of N and $\{E_k\}_{k=0,\ldots,m}$ is a partition of the set of edges E without the edges that are contained in the specified path P and without the edge between the root node and k^* . In the upcoming steps, we will successively replace the bivariate copulas of X with those of Y. For $e \in E$, consider the sequences $A^1 = (A_e^1)_{e \in E}$ and $A^2 = (A_e^2)_{e \in E}$ of bivariate copulas given by

$$\begin{split} A_e^1 &\coloneqq \begin{cases} C_e & \text{if } e = (0, k^*), \\ B_e & \text{else}, \end{cases} \\ A_e^2 &\coloneqq \begin{cases} C_e & \text{if } e \in \{(0, k^*)\} \cup E_0 \\ B_e & \text{else}. \end{cases} \end{split}$$

Furthermore, we define the sequence of bivariate copulas $D^0 = (D^0_e)_{e \in E}$ by

$$D_e^0 \coloneqq \begin{cases} A_e^2 & \text{if } \deg(0) > 1, \\ B_e & \text{if } \deg(0) = 1, \end{cases}$$

and for k = 1, ..., m the sequences $D^{2k-1} = (D_e^{2k-1})_{e \in E}$ and $D^k = (D_e^k)_{e \in E}$ of bivariate copulas by

$$D_{e}^{2k-1} \coloneqq \begin{cases} B_{e} & \text{if } e \in \bigcup_{u=k+1}^{m} E_{u} \cup \bigcup_{u=k}^{m} \{(\ell_{u-1}, \ell_{u})\}, \\ C_{e} & \text{else}, \end{cases}$$
$$D_{e}^{2k} \coloneqq \begin{cases} B_{e} & \text{if } e \in \bigcup_{u=k+1}^{m} E_{u} \cup \bigcup_{u=k+1}^{m} \{(\ell_{u-1}, \ell_{u})\}, \\ C_{e} & \text{else}. \end{cases}$$

Note that by definition $D^{2m} = C$. Keep in mind that our aim is to change all dependencies described by $(B_e)_{e \in E}$ with the dependencies described by $(C_e)_{e \in E}$. The difference between the copula sets D^{2k-1} and D^{2k} lies in the replacement of the copula $B_{(\ell_{k-1},\ell_k)}$ by $C_{(\ell_{k-1},\ell_k)}$. On the other hand, the difference between D^{2k} and D^{2k+1} is given by the replacement of the set of copulas $(B_e)_{e \in E_{k+1}}$ by $(C_e)_{e \in E_{k+1}}$. Let

$$W^{(u)} \sim \mathcal{M}(F, T, A^u) \quad \text{for } u = 1, 2 \quad \text{and}$$

$$\tag{69}$$

$$Z^{(v)} \sim \mathcal{M}(F, T, D^v) \quad \text{for } v = 0, \dots, 2m.$$

$$\tag{70}$$

be random vectors on \mathbb{R}^{d+1} . The following part of the proof is divided into two steps. In the first step we show that $X \leq_{sm} Z^{(0)}$ and in the second step we show that $Z^{(0)} \leq_{sm} Y$.

Step 1: If deg(0) < 2, then $X \stackrel{d}{=} Z^{(0)}$ which trivially implies $X \leq_{sm} Z^{(0)}$. For the other case, assume that deg(0) ≥ 2 . We prove the supermodular comparison

$$X \leq_{sm} W^{(1)} \leq_{sm} W^{(2)} \stackrel{d}{=} Z^{(0)}.$$
(71)

To show the first inequality in (71), let $\phi \colon \mathbb{R}^{d+1} \to \mathbb{R}$ be a bounded supermodular function. Since, by assumption (i), $X_j \uparrow_{st} X_i$ for all $(i, j) \in E_0$ and $X_j \uparrow_{st} X_i$ for all $(i, j) \in E \setminus \{(E_0 \cup \{(0, k^*)\})\}$, we can apply Lemma B.4(ii) twice and find that the function $\tilde{\varphi}^{(1)} \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\widetilde{\varphi}^{(1)}(x_0, x_{k^*}) \coloneqq \int \int \phi(x) P^{X_{d_{k^*}} | X_{k^*} = x_{k^*}} (dx_{d_{k^*}}) P^{X_{N \setminus (N_0 \cup \{0\})} | X_0 = x_0} (dx_{N \setminus (N_0 \cup \{0\})})$$
(72)

is supermodular. Note that we do not need any SI assumption on (X_0, X_{k^*}) . By the disintegration theorem and the Markov property, we have that

$$E[\phi(X)] = E[\widetilde{\varphi}^{(1)}(X_0, X_{k^*})], \text{ and}$$

$$E[\phi(W^{(1)})] = E[\widetilde{\varphi}^{(1)}(W_0^{(1)}, W_{k^*}^{(1)})] = E[\widetilde{\varphi}^{(1)}(Y_0, Y_{k^*})],$$
(73)

where the last equality follows from $(W_0^{(1)}, W_{k^*}^{(1)}) \stackrel{d}{=} (Y_0, Y_{k^*})$. Then the assumption $(X_0, X_{k^*}) \leq_{sm} (Y_0, Y_{k^*})$ together with (73) implies $X \leq_{sm} W^{(1)}$.

To show the second inequality in (71), again, let $\phi \colon \mathbb{R}^{d+1} \to \mathbb{R}$ be a bounded supermodular function. Since, by assumption (i), $X_j \uparrow_{st} X_i$ for all $(i, j) \in E \setminus (E_0 \cup \{(0, k^*)\})$ and by assumption (ii), $Y_0 \uparrow_{st} Y_{k^*}$, we have that $W_j^{(1)} \uparrow_{st} W_i^{(1)}$ for all $(i, j) \in E \setminus (E_0 \cup \{(0, k^*)\})$ and $W_0^{(1)} \uparrow_{st} W_{k^*}^{(1)}$. Due to Lemma B.2, we conclude that the function $\tilde{\varphi}^{(2)}$ given by

$$\widetilde{\varphi}^{(2)}(x_{N_0}) \coloneqq \int \phi(x) P^{W_{N\setminus N_0}^{(1)}|W_{k^*}^{(1)} = x_{k^*}}(dx_{N\setminus N_0})$$
(74)

is supermodular. By the disintegration theorem and the Markov property, we obtain

$$E[\phi(W^{(1)})] = E[\widetilde{\varphi}^{(2)}(W_{N_0}^{(1)})] = E[\widetilde{\varphi}^{(2)}(X_{N_0})],$$

$$E[\phi(W^{(2)})] = E[\widetilde{\varphi}^{(2)}(W_{N_0}^{(2)})] = E[\widetilde{\varphi}^{(2)}(Y_{N_0})],$$
(75)

where the second equality in the first and second line follow from $W_{N_0}^{(1)} \stackrel{d}{=} X_{N_0}$ and $W_{N_0}^{(2)} \stackrel{d}{=} Y_{N_0}$, respectively. Assumptions (i)- (iii), imply that the subvectors $(X_n)_{n \in N_0}$ and $(Y_n)_{n \in N_0}$ fulfill the assumptions of Proposition B.5. Thus, (75) implies $W^{(1)} \leq_{sm} W^{(2)}$, which proves (71) using $W^{(2)} \stackrel{d}{=} Z^{(0)}$ by the definition of $W^{(2)}$ and $Z^{(0)}$.

Step 2: We aim to show that

$$Z^{(0)} \leq_{sm} Z^{(1)} \leq_{sm} \dots \leq_{sm} Z^{(2m)} = Y,$$
(76)

The proof of (76) is devided into two parts. First, we show that

$$Z^{(2k-2)} \leq_{sm} Z^{(2k-1)}, \quad \text{for all } k = 1, \dots, m,$$
(77)

and then show that

$$Z^{(2k-1)} \leq_{sm} Z^{(2k)}, \quad \text{for all } k = 1, \dots, m.$$
 (78)

To prove (77) let $\phi : \mathbb{R}^{d+1} \to \mathbb{R}$ be a bounded and supermodular function and $k \in \{1, \ldots, m\}$. If $|N_k| \leq 1$ we have that $Z^{(2k-2)} \stackrel{d}{=} Z^{(2k-1)}$ and thus $Z^{(2k-2)} \leq_{sm} Z^{(2k-1)}$. So assume that $|N_k| > 1$. Since, by assumption (ii), $Y_j \uparrow_{st} Y_i$ for all $(i, j) \in \bigcup_{u=0}^{k-1} E_u$ and $Y_{k^*} \uparrow_{st} Y_0$ if deg(0) > 1 and $Y_i \uparrow_{st} Y_j$ for all $(i, j) \in \bigcup_{u=1}^{k-1} \{(\ell_{u-1}, \ell_u)\}$ we have that $Z_j^{(2k-2)} \uparrow_{st} Z_i^{(2k-2)}$ for all $(i, j) \in \bigcup_{u=0}^{k-1} E_u$ and $Z_{k^*}^{(2k-2)} \uparrow_{st} Z_0^{(2k-2)}$ if deg(0) > 1 and $Z_i^{(2k-2)} \uparrow_{st} Z_j^{(2k-2)}$ for all $(i, j) \in \bigcup_{u=1}^{k-1} \{(\ell_{u-1}, \ell_u)\}$. Further, by assumption (i), $X_j \uparrow_{st} X_i$ for all $(i, j) \in \bigcup_{u=k+1}^m E_u \cup \bigcup_{u=k+1}^m \{(\ell_{u-1}, \ell_u)\}$ and consequently $Z_j^{(2k-2)} \uparrow_{st} Z_i^{(2k-2)}$ for all $(i, j) \in \bigcup_{u=k+1}^m E_u \cup \bigcup_{u=k}^m \{(\ell_{u-1}, \ell_u)\}$ and consequently $Z_j^{(2k-2)} \uparrow_{st} Z_i^{(2k-2)}$ for all $(i, j) \in \bigcup_{u=k+1}^m E_u \cup \bigcup_{u=k}^m \{(\ell_{u-1}, \ell_u)\}$. Note that if k = 1 and deg(0) = 1, it holds that $|N_1| = \{0\}$, which belongs to the previous case. Now, we obtain from Lemma B.4 (ii), applied on the subtree with nodes $(N \setminus N_k) \cup \{\ell_{k-1}\}$ and root ℓ_{k-1} by setting $K = N_k \setminus \{\ell_{k-1}\}$ and $J = N \setminus N_k$, that the function $\varphi^{(2k-1)}$ given by

$$\varphi^{(2k-1)}(x_{N_k}) \coloneqq \int \phi(x) P^{Z_{N\setminus N_k}^{(2k-2)} | Z_{\ell_{k-1}}^{(2k-2)} = x_{\ell_{k-1}}}(dx_{N\setminus N_k})$$
(79)

is supermodular. Then we obtain

$$E[\phi(Z^{(2k-2)})] = E[\varphi^{(2k-1)}(Z_{N_k}^{(2k-2)})] = E[\varphi^{(2k-1)}(X_{N_k})], \text{ and}$$

$$E[\phi(Z^{(2k-1)})] = E[\varphi^{(2^{k-1})}(Z_{N_k}^{(2k-1)})] = E[\varphi^{(2k-1)}(Y_{N_k})],$$
(80)

where the first equality in each line follows by the disintegration theorem and the Markov property and the second ones since $X_{N_k} \stackrel{d}{=} Z_{N_k}^{(2k-2)}$ and $Y_{N_k} \stackrel{d}{=} Z_{N_k}^{(2k-1)}$ by definition of $Z^{(2k-2)}$ and $Z^{(2k-1)}$ in (70). Assumptions (i)- (iii) of Theorem 1.3, imply that the subvectors $(X_n)_{n \in N_k}$ and $(Y_n)_{n \in N_k}$ fulfill the assumptions of Proposition B.5 which implies $E[\varphi^{(2k-1)}(X_{N_k})] \leq E[\varphi^{(2k-1)}(Y_{N_k})]$. Together with (80), we obtain (77).

To prove (78), let again $\phi \colon \mathbb{R}^{d+1} \to \mathbb{R}$ be a bounded and supermodular function and $k \in \{1, \ldots, m\}$. Since, by assumption (ii), $Y_j \uparrow_{st} Y_i$ for all $(i, j) \in \bigcup_{u=0}^k E_u$ and $Y_{k^*} \uparrow_{st} Y_0$ if deg(0) > 1 and $Y_i \uparrow_{st} Y_j$ for all $(i, j) \in \bigcup_{u=1}^{k-1} \{(\ell_{u-1}, \ell_u)\}$, we have that $Z_j^{(2k-1)} \uparrow_{st} Z_i^{(2k-1)}$ for all $(i, j) \in \bigcup_{u=0}^k E_u$ and $Z_{k^*}^{(2k-1)} \uparrow_{st} Z_0^{(2k-1)}$ if deg(0) > 1 and $Z_i^{(2k-1)} \uparrow_{st} Z_j^{(2k-1)}$ for all $(i, j) \in \bigcup_{u=1}^{k-1} \{(\ell_{u-1}, \ell_u)\}$. Further, by assumption (i), $X_j \uparrow_{st} X_i$ for all $(i, j) \in \bigcup_{u=k+1}^m E_u \cup \bigcup_{u=k+1}^m \{(\ell_{u-1}, \ell_u)\}$ and consequently $Z_j^{(2k-1)} \uparrow_{st} Z_i^{(2k-1)}$ for all $(i, j) \in \bigcup_{u=k+1}^m E_u \cup \bigcup_{u=k+1}^m \{(\ell_{u-1}, \ell_u)\}$. We apply Lemma B.4 (ii) twice and find that the function $\varphi^{(2k)}$ given by

$$\varphi^{(2k)}(x_{\ell_{k-1}}, x_{\ell_k}) = \int \int \phi(x) P^{Z_{N \setminus \{\{\ell_{k-1}, \ell_k\} \cup d_{\ell_k}\}}^{(2k-1)} |Z_{\ell_{k-1}}^{(2k-1)} = x_{\ell_{k-1}}} (dx_{N \setminus \{\{\ell_{k-1}, \ell_k\} \cup d_{\ell_k}\}}) P^{Z_{\ell_k}^{(2k-1)} |Z_{\ell_k}^{(2k-1)} = x_{\ell_k}} (dx_{d_{\ell_k}})$$

$$(81)$$

is supermodular. Again, we obtain

$$E[\phi(Z^{(2k-1)})] = E[\varphi^{(2k)}(Z^{(2k-1)}_{\ell_{k-1}}, Z^{(2k-1)}_{\ell_{k}})] = E[\varphi^{(2k)}(X_{\ell_{k-1}}, X_{\ell_{k}})], \text{ and}$$

$$E[\phi(Z^{(2k)})] = E[\varphi^{(2k)}(Z^{(2k)}_{\ell_{k-1}}, Z^{(2k)}_{\ell_{k}})] = E[\varphi^{(2k)}(Y_{\ell_{k-1}}, Y_{\ell_{k}})].$$
(82)

By assumption (iii) it holds that $(X_{\ell_{k-1}}, X_{\ell_k}) \leq_{sm} (Y_{\ell_{k-1}}, Y_{\ell_k})$, which together with (82) implies (78).

By (77) and (78) we conclude (76). Finally, $X \leq_{sm} Z^{(0)}$, (76) and the transitivity of the supermodular order imply that $X \leq_{sm} Y$, which proves the first statement.

To show positive supermodular dependence of X, we apply the first statement on X and $Y := X^{\perp}$. Note that the random vector X^{\perp} fulfills assumption (ii) trivially. Since X fulfills assumption (i), we have for $(i, j) \in E \setminus \{(0, k^*)\}$ that $X_j \uparrow_{st} X_i$ which implies $(X_i, X_j) \ge_{sm} (X_i^{\perp}, X_j^{\perp}) = (Y_i, Y_j)$, see (14). Hence, with the additional assumption of positive supermodular dependence of (X_0, X_{k^*}) , we obtain $(X_i, X_j) \ge_{sm} (Y_i, Y_j)$ for all $(i, j) \in E$, i.e., assumption (iii) holds for all $(i, j) \in E$. From the first part, we now conclude that $X \ge_{sm} Y = X^{\perp}$.

A similar argument yields positive supermodular dependence of Y under assumption (ii) and positive supermodular dependence of (Y_i, Y_j) for $j \in P \cap L$.

C Proofs of Section 2

Proof of Proposition 2.5 For $|N| < \infty$, we prove the statement by an induction over the number of nodes $N = \{0, \ldots, d\}$. Starting with |N| = 2, the statement is trivial. For the induction step, assume that the statement is true for each tree T with nodes $N = \{0, \ldots, n\}$ for a fixed $n \in \{1, \ldots, d-1\}$. We need to show the statement for $N = \{0, \ldots, n+1\}$. By re-enumeration we may assume that the node n + 1 is a leaf in N with parent node n, i.e., $c_{n+1} = \emptyset$ and $p_{n+1} = n$, see Definition B.1. We define the subtree T' = (N', E') with nodes $N' = \{0, \ldots, n\}$ and edges $E' = \{(j, k) \in E \mid j, k \neq n+1\}$, which is a tree with |N'| - 1 = n nodes. By the induction hypothesis, there is a Markov tree distribution $\mu_{\mathcal{T}'}$ specified through $((F_j)_{j \in N'}, T', (B_e)_{e \in E'})$. Let (X_n, X_{n+1}) be a bivariate random vector having distribution function $F_{X_n, X_{n+1}} := B_{n,n+1}(F_n, F_{n+1})$ defined through Sklar's theorem. For Borel sets $A \in \mathscr{B}(\mathbb{R}^{n+1}), B \in \mathscr{B}(\mathbb{R})$ we define the distribution $\mu_{\mathcal{T}}$ on $\mathcal{B}(\mathbb{R}^{n+2})$ by

$$\mu_{\mathcal{T}}(A \times B) := \int_{A} P^{X_{n+1}|X_n}(B|x_n) \mu_{\mathcal{T}'}(dx_0, \dots, dx_n).$$
(83)

Similar to the proof of [80, Theorem 2], it follows that $\mu_{\mathcal{T}}$ is the unique distribution with the given specifications that has Markov tree dependence. For $|N| = \infty$, the distribution $\mu_{\mathcal{T}}$ can be obtained by the Kolmogorov extension Theorem, see [66, Theorem 6.16], where the assumption of projective families is satisfied as a consequence of the disintegration theorem.

D Proofs of Section 3

For a random vector $X = (X_0, \ldots, X_d)$ with distribution function F_X , we denote by \overline{F}_X the survival function of X which is given by

$$\overline{F}_X(x_0,\ldots,x_d) := P(X_0 > x_0,\ldots,X_d > x_d).$$

There is also an analogous version of Sklar's Theorem for survival functions, cf. [42, Theorem 2.2.13], which states that for any survival function \overline{F}_X with survival marginals $\overline{F}_{X_n}(x) \coloneqq 1 - F_{X_n}(x), x \in \mathbb{R}$, there is a copula \widehat{C}_X , such that

$$\overline{F}_X(x_0,\dots,x_d) = \widehat{C}_X(\overline{F}_{X_0}(x_0),\dots,\overline{F}_{X_d}(x_d)).$$
(84)

The copula \widehat{C}_X is called *survival copula* of X. The following lemma shows that the copula (survival copula) of a Markov tree distribution with respect to a tree T = (N, E), specified by continuous marginal distribution functions F_i , $i \in N$, and bivariate copulas B_e , $e \in E$, is given by the distribution function (the survival copula) of the Markov tree distribution specified by uniform marginals on [0, 1] and the bivariate copulas B_e , $e \in E$. This is especially useful for comparing bivariate tree specification with the same bivariate copula structure but different marginal distributions.

Lemma D.1 Let $X = (X_0, \ldots, X_d) \sim \mathcal{M}((F_0, \ldots, F_d), T, C)$ be a random vector with continuous marginal specifications F_0, \ldots, F_d . Moreover, let $U = (U_0, \ldots, U_d) \sim \mathcal{M}((G, \ldots, G), T, C)$ be a random vector with

distribution function F_U , where G denotes the distribution function of the uniform distribution on [0, 1]. Then the following hold true:

- (i) F_U is the unique copula for X, i.e., $F_X(x_1, \ldots, x_d) = F_U(F_1(x_1), \ldots, F_d(x_d))$.
- (ii) The survival function and the survival copula of U are related by

$$\widetilde{C}_U(u_0, \dots, u_d) = \overline{F}_U(1 - u_0, \dots, 1 - u_d).$$
(85)

Further $\widehat{\mathcal{C}}_U$ is the unique survival copula of X, i.e., $\widehat{\mathcal{C}}_U$ is the unique copula such that

$$\widehat{\mathcal{C}}_U(\overline{F_0}(x_0), \dots, \overline{F_d}(x_d)) = P(X_0 > x_0, \dots, X_d > x_d),$$
(86)

where $\overline{F}_n = \overline{F}_{X_n}$ is the survival function of X_n , $n \in \{0, \ldots, n\}$.

Proof of Lemma D.1 We prove statement (i) by an induction over the number of nodes $N = \{0, \ldots, d\}$. Starting with |N| = 2, the statement is trivial. For the induction step, assume that the statement is true for each tree T with nodes $N = \{0, \ldots, n\}$ for a fixed $n \in \{1, \ldots, d-1\}$. We need to show the statement for $N = \{0, \ldots, n+1\}$. By re-enumeration we may assume that $n+1 \in N$ is a leaf in N with parent node n, i.e., $c_{n+1} = \emptyset$ and $p_{n+1} = n$, see Definition B.1. We define the subtree T' = (N', E') with nodes $N' = \{0, \ldots, n\}$ and edges $E' = \{(j, k) \in E | j, k \neq n+1\}$ which is a tree with number of nodes |N'| - 1 = n. By (83) and the transformation formula, using that $F_i^{-1} \circ F_i = \text{id } P^{X_i}$ -a.s., cf. [101, Theorem 1.2], we have that

$$F_{(X_0,\dots,X_{n+1})}(x_0,\dots,x_n) = \int_{(-\infty,x_0]\times\dots\times(-\infty,x_n]} F_{X_{n+1}|X_n=y_n}(x_{n+1}) P^{(X_0,\dots,X_n)}(dy_0,\dots,dy_n)$$

$$= \int_{(-\infty,F_0(x_0)]\times\dots\times(-\infty,F(x_n)]} F_{X_{n+1}|X_n=F_n^{-1}(u_n)}(x_{n+1}) P^{(F_0(X_0),\dots,F_n(X_n))}(du_0,\dots,du_n).$$
(87)

Since the marginal distribution functions are continuous, the distribution function of the transformed random vector $(F_0(X_0), \ldots, F_n(X_n))$ is the unique copula of (X_0, \ldots, X_n) , see, e.g., [99, Remark 2.2]. Moreover, by the induction hypothesis the distribution function F_{U_0,\ldots,U_n} is also a copula for (X_0, \ldots, X_n) . By uniqueness of the copula, we obtain that $(F_0(X_0), \ldots, F_n(X_n)) \stackrel{d}{=} (U_0, \ldots, U_n)$. By [6, Theorem 2.2], we have

$$F_{X_{n+1}|X_n=F_n^{-1}(u_n)}(x_{n+1}) = \partial_1 C_{n,n+1}(u_n, F_{n+1}(x_{n+1})) = F_{U_{n+1}|U_n=u_n}(F_{n+1}(x_{n+1})),$$

where ∂_1 denotes the operator that takes the partial derivative with respect to the first component. Hence, we obtain that (87) equals

$$\int_{(-\infty,F_0(x_0)]\times\cdots\times(-\infty,F(x_n)]} F_{U_{n+1}|U_n=u_n}(F_{n+1}(x_{n+1}))P^{(U_0,\dots,U_n)}(du_0,\dots,du_n)$$

= $F_{(U_0,\dots,U_{n+1})}(F_1(x_1),\dots,F_d(x_{n+1})),$

which proves the first result. We continue with statement (ii). Equation (85) follows directly by

$$\overline{F}_U(1 - \overline{F}_{U_0}(u_0), \dots, 1 - \overline{F}_{U_d}(u_d)) = \overline{F}_U(u_0, \dots, u_d).$$
(88)

By [42, Theorem 2.2.13] and the continuity of F_0, \ldots, F_d the survival copula for X is unique and it remains to show equation (86). By the inclusion-exclusion principle we obtain that

$$\widehat{\mathcal{C}}_{U}(u_{0},\ldots,u_{d}) = 1 - \sum_{k=1}^{d+1} \left((-1)^{k+1} \sum_{I \subseteq N, |I|=k} F_{U_{I}} \left((1-u_{i})_{i \in I} \right) \right),$$
(89)

see also [42, Equation 1.7.5]. Moreover, by the first statement (i) we obtain that

$$P(X_{0} > x_{0}, \dots, X_{d} > x_{d}) = 1 - P\left(\bigcup_{n=0}^{d} \{X_{n} \le x_{n}\}\right)$$

$$= 1 - \sum_{k=1}^{d+1} \left((-1)^{k+1} \sum_{I \subseteq N, |I|=k} F_{X_{I}}\left((x_{i})_{i \in I}\right)\right)$$

$$= 1 - \sum_{k=1}^{d+1} \left((-1)^{k+1} \sum_{I \subseteq N, |I|=k} F_{U_{I}}\left((1 - \overline{F}_{i}(x_{i}))_{i \in I}\right)\right).$$

(90)

Finally, (89) and (90) shows (86) and the second statement is proven.

Proof of Theorem 3.6 By the definition of the orders for stochastic processes, Definition 2.7 (b), we may assume that $|N| < \infty$. Let $Z = (Z_n)_{n \in N} \sim \mathcal{M}(F, T, C)$ be a random vector having the same marginals as X and the same bivariate dependence specifications as Y. By assumption (ii) and (14) the copulas C_e , $e \in E$, are CI and thus (Z_i, Z_j) is CI for all $(i, j) \in E$. We deduce from Theorem 1.3 that $X \leq_{sm} Z$ and consequently $X \leq_{dcx} Z$. When verifying the assumptions for Theorem 1.3, it is worth noting that distinguishing between the cases where $\deg(0) \geq 2$ and $\deg(0) < 2$ is not necessary since (Y_i, Y_j) is CI for all $(i, j) \in E$ and consequently in the case where $\deg(0) \geq 2$, it is always possible to select the path P in the assumptions of Theorem 1.3 in such a way that $k^* \notin P$. According to [44, Proposition 7.1], the vectors Z and Y are MTP₂ and thus CI. By the continuity of F_n and G_n for all $n \in N$ and Lemma D.1, the vectors Z and Y have the same copula which is CI. Hence, we obtain from [85, Theorem 4.5] using that $F_n \leq_{cx} G_n$ for all $n \in N$, that $Z \leq_{dcx} Y$.

Proof of Theorem 3.7 Consider a sequence $Z = (Z_n)_{n \in N} \sim \mathcal{M}(F, T, C)$ of Markov tree distributed random variables. Since Y fulfills assumption (ii) of Theorem 1.3 this is also the case for Z. Moreover X and Z have the same marginal distributions. Hence, X and Z fulfill the assumptions of Theorem 1.3 and we obtain that $X \leq_{sm} Z$. Due to (8), it holds that $X \leq_{lo} Z$ and $X \leq_{uo} Z$. By the transitivity of the lower and upper orthant order it remains to show that $Z \leq_{lo} Y$ (resp. $Z \leq_{uo} Y$). Since the marginal distribution functions F_n and G_n are assumed to be continuous and since Z and Y have the same bivariate specifications, Lemma D.1 (i) implies that Z and Y have the same copula $\mathcal{C} \in \mathcal{C}_{d+1}$. Since, by assumption, $F_n(x) \leq G_n(x)$ for all $x \in \mathbb{R}$, we obtain from Sklar's Theorem that

$$F_Z(x) = \mathcal{C}(F_0(x_0), \dots, F_d(x_d)) \le \mathcal{C}(G_0(x_0), \dots, G_d(x_d)) = F_Y(x)$$

for all $x = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1}$. This proves $Z \leq_{lo} Y$. To show $Z \leq_{uo} Y$, let $\widehat{\mathcal{C}} \in \mathcal{C}$ be the common survival copula of Z and Y, see Lemma D.1 (ii). Since, by assumption, $\overline{F}_n(x) \leq \overline{G}_n(x)$ for all $x \in \mathbb{R}$, we obtain

$$\overline{F}_Z(x) = \widehat{\mathcal{C}}(\overline{F}_0(x_0), \dots, \overline{F}_d(x_d)) \le \widehat{\mathcal{C}}(\overline{G}_0(x_0), \dots, \overline{G}_d(x_d)) = \overline{F}_Y(x)$$

for all $x = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1}$, which concludes the proof.

E Proofs of Section 4

Proof of Corollary 4.1 Hidden Markov models follow a Markov tree distribution with respect to the tree $T = (\mathbb{N}_0, E)$, where E is defined by Equation (17). The first statement then follows by Theorem 1.3. The second and third statement are a direct consequence of Theorem 3.7. The fourth statement follows from Theorem 3.6.

F Proofs of Section 5

Proof of Proposition 5.2 (a) For the first statement, we consider two cases. If $j^* \in d_{k^*}$, i.e., j^* is an element of the decedents of k^* , we define the vectors by $(X_0, X_{k^*}, X_{p(k^*, j^*)}, X_{j^*}) \coloneqq -Y'$ and $(Y_0, Y_{k^*}, Y_{p(k^*, j^*)}, Y_{j^*}) \coloneqq -X'$, where X' and Y' are given by Example A.5. Then we extend these vectors by independent random variables (note that the independence copula is CI) to the Markov tree distributed sequences $X = (X_i)_{i \in N}$ and $Y = (Y_i)_{i \in N}$. These vectors fulfill assumption (i^*) in (33), and the assumptions (ii) and (iii) of Theorem 1.3. Since $(X_0, X_{k^*}, X_{p(k^*, j^*)}, X_{j^*}) \not\leq_{lo} (Y_0, Y_{k^*}, Y_{p(k^*, j^*)}, Y_{j^*})$, we obtain from the closure of the lower orthant order under marginalization that $X \not\leq_{lo} Y$ and thus $X \not\leq_{sm} Y$. In the case where $j^* \notin d_{k^*}$, which implies $\deg(0) \geq 2$, we define the vectors $(X_{k^*}, X_0, X_{p(0, j^*)}, X_{j^*}) \coloneqq -Y'$ and $(Y_{k^*}, Y_0, Y_{p(0, j^*)}, Y_{j^*}) \coloneqq -X'$, and proceed as in the first case.

(b) Since T is not a star it holds $N \setminus (L \cup \{0\}) \neq \emptyset$. The node $j^* \in N \setminus (L \cup \{0\})$, has a parent node $i \coloneqq p_{j^*}$ and has at least one child node $k \in c_{j^*}$. Define the subvectors (X_i, X_{j^*}, X_k) and (Y_i, Y_{j^*}, Y_k) as in Example A.3 and extend these 3-dimensional vectors by independent random variables (note that the independence copula is CI) to the random vectors $X = (X_i)_{i \in N}$ and $Y = (Y_i)_{i \in N}$. Then X and Y fulfill assumption (ii^*) in (34), and the assumptions (i) and (iii) of Theorem 1.3. We obtain from Example A.3 that $(X_i, X_{j^*}, X_k) \not\leq_{lo} (Y_i, Y_{j^*}, Y_k)$ and thus $X \not\leq_{lo} Y$ and $X \not\leq_{sm} Y$. This shows the second statement.

(c) Since T is not a chain, we have $N \setminus (P \cup \{0\}) \neq \emptyset$. For $j^* \in N \setminus (P \cup \{0\})$ we consider the directed path $(0, \ldots, j^*)$ from the root 0 to j^* . Let *i* be the last node in $(0, \ldots, j^*)$ such that $i \in P \cup \{0\}$, i.e., the unique node $i \in \{0, \ldots, j^*\} \cap (P \cup \{0\})$ such that $c_i \cap \{0, \ldots, j^*\} \cap P = \emptyset$. Since $i \in P$ and $j^* \notin P$ the node *i* has a child node $k \in P \cap c_i$. Define the subvectors $(X_k, X_{p(k,j^*)}, X_{j^*})$ and $(Y_k, Y_{p(k,j^*)}, Y_{j^*})$ as in Example A.5 and extend this 3-dimensional vectors by independent random variables (note that the independence copula is CI) to the random vectors $X = (X_i)_{i \in N}$ and $Y = (Y_i)_{i \in N}$. Then X and Y fulfill assumption (ii^{**}) in (35), and the assumptions (i) and (ii) of Theorem 1.3. Due to Example A.5, we obtain $(X_k, X_{p(k,j^*)}, X_{j^*}) \not\leq_{lo} (Y_k, Y_{p(k,j^*)}, Y_{j^*})$ and consequently $X \not\leq_{lo} Y$ and $X \not\leq_{sm} Y$. This shows the third statement and the result is proven.

Proof of Proposition 5.4 Using the notation in [8], there exists for all $x \in \mathbb{R}$ a Lebesgue-null set N_x such that $F_{X_i|X_0=q_{X_0}(u)}(x) = \partial_2^{F_{X_0}}C_{X_i,X_0}(F_{X_i}(x), u)$ and $F_{Y_i|Y_0=q_{Y_0}(u)}(x) = \partial_2^{F_{Y_0}}C_{Y_i,Y_0}(F_{Y_i}(x), u)$ for all $u \in (0,1) \setminus N_x$, see [6, Theorem 2.2]. Hence, the statement follows from [8, Corollary 4(i)] because $(X_i|X_0) \leq_S (Y_i,Y_0)$ implies $\partial_2^{F_{X_0}}C_{X_i,X_0}(F_{X_i}(x), \cdot) \prec_S \partial_2^{F_{Y_0}}C_{Y_i,Y_0}(F_{Y_i}(x), \cdot)$ for all $x \in \mathbb{R}$.

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