

Chemotaxis-inspired PDE model for airborne infectious disease transmission: analysis and simulations

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Abstract. Partial differential equation (PDE) models for infectious disease have received renewed interest in recent years. Most models of this type extend classical compartmental formulations with additional terms accounting for spatial dynamics, with Fickian diffusion being the most common such term. However, while diffusion may be appropriate for modeling vector-borne diseases, or diseases among plants or wildlife, the spatial propagation of airborne diseases in human populations is heavily dependent on human contact and mobility patterns, which are not necessarily well-described by diffusion. By including an additional chemotaxis-inspired term, in which the infection is propagated along the positive gradient of the susceptible population (from regions of low- to high-density of susceptibles), one may provide a more suitable description of these dynamics. This article introduces and analyzes a mathematical model of infectious disease incorporating a modified chemotaxis-type term. The model is analyzed mathematically and the well-posedness of the resulting PDE system is demonstrated. A series of numerical simulations are provided, demonstrating the ability of the model to naturally capture important phenomena not easily observed in standard diffusion models, including propagation over long spatial distances over short time scales and the emergence of localized infection hotspots.

Keywords: nonlinear parabolic equations, reaction-diffusion systems, chemotaxis, existence and uniqueness of solutions, epidemic models, COVID-19.

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1 Introduction

Mathematical models have long found application in the modeling and study of infectious disease, dating back to Daniel Bernoulli's model of smallpox vaccination in the

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1700s [13]. Over the following centuries, mathematical models have been found numerous applications in in epidemiology and public health. The most commonly employed models are *compartmental models*, based on systems of ordinary differential equations (ODEs), following the framework of the susceptible-infectious-recovered (hereafter abbreviated as *SIR*) model, itself originating from a special case of the more general model introduced by Kermack and McKendrick [24] (see [10] for a more modern treatment, as well as [29, 30]). These models divide the population into different compartments based on disease stage. Such models are easily implemented and analyzed, and have relatively low computational overhead. However, they do not naturally incorporate spatial information.

To overcome this limitation, analogous models based on partial differential equations (PDEs) have been proposed [32, 35]. These models have historically found success, particularly in modeling vector-borne diseases and diseases spread among wildlife [23, 32]. These models are typically of reaction-diffusion type, and model the spatial propagation of disease as a Fickian diffusive process, in which the infection moves along a negative gradient. Reaction-diffusion models have also been used to model infectious diseases in human populations. While much recent work in this area has been applied towards COVID-19 [2, 7, 11, 12, 19], PDE models have also been used for other diseases, or described in more general settings [1, 25, 34].

While PDE models offer the advantage of describing spatial dynamics, their use in modeling infectious disease in human populations remains limited, for reasons both practical and epidemiological. From the practical point of view, PDEs require more computational resources, more data, and more effort to implement and solve compared to their ODE counterparts. Additionally, their mathematical analysis is often complicated, and the definition of important epidemiological quantities (such as the reproduction number) is not straightforward [1, 2].

More seriously, however, there are the issues from the epidemiological point of view: it is not clear that a diffusive process provides a suitable description of infectious disease in a human population, given the complex, multiscale nature of human mobility [5, 15]. Diffusion models do not provide a natural description of nonlocal transmission dynamics. In contrast, ODE models are quite flexible in this regard, and one may model (potentially distant) geographic regions via an additional compartmental stratification. Thus, in addition to their computational and mathematical attractiveness, carefully designed ODE models may also provide more suitable spatial description compared to a PDE model in certain settings. This is particularly well-suited for models of sexually transmitted diseases, where transmission does not depend strongly on local population density [9, 37]. Such models are, however, very data-hungry and their parametrization can become challenging as the spatial resolution increases. This causes difficulties when applying this approach to airborne infectious diseases in human populations, as transmission may depend heavily on local population density patterns [21, 36].

The inability of reaction-diffusion models of infectious disease to incorporate nonlocal mobility is a well-known limitation, and significant effort has been applied towards resolving it. Models incorporating bilaplacian [32] and fractional diffusion [40] terms may naturally reproduce nonlocal behavior; however, both their implementation and analysis are different. Reaction-diffusion models have also been extended by, for ex-

ample, incorporating additional advection terms (possibly endowed with a network structure [1, 7, 34]) or mass-transfer operators [19] to allow for nonlocal effects.

While such efforts have produced promising results, the focus in these instances is to better reproduce the *nonlocal* behavior. However, another equally important question, whether diffusion is in fact an appropriate description of the *local* spatial dynamics of an airborne infectious disease in human populations, has not received similar attention. Indeed, while additional care is occasionally given in defining the diffusive process (for instance, preferential or nonlinear diffusion [19, 38]), such considerations affect only the *rate* and *areas* in which diffusion occurs. The underlying idea that an infectious disease in a human population should *diffuse*, that is, travel from regions of high-infection density to regions of low-infection density, as the limit of a Brownian motion, is discussed in few works (see e.g., [2, 11, 12, 16, 17]).

In the present work, we consider that diffusion-based models of airborne infectious diseases in humans are also limited in their description of *local* dynamics, and that such limitations are of similar, if not greater, importance as their well-known problems with describing *nonlocal* dynamics. In order to overcome these limitations, we introduce a new model to better-capture the spatial propagation observed in real-world human contagion. Specifically, the model does not only consider movement of the disease along the negative gradient of infection density, but also the nonlinear movement along the *positive gradient* of susceptible density; that is, from regions of low-to-high susceptible concentration. While the underlying physical process is not the same, we note that this model bears mathematical similarity to well-known models of *chemotaxis*, a process in which a biological species is attracted or repelled in response to a chemical stimulus [6, 8, 18, 28, 31, 33].

The article is outlined as follows. We first introduce the mathematical model, characterizing transmission, diffusion and a *chemotactic-inspired movement*, as previously explained. We describe its basic properties and explain the underlying intuition. After establishing the necessary formality and notation, we proceed to examine the mathematical well-posedness of the model. Following this analysis, we prove the existence of a solution (see Theorem 3.2) with its properties, as e.g., positiveness. Then, after showing an additional regularity of the solution in Proposition 3.4, we prove also its uniqueness in Theorem 3.5 in the case when the diffusion coefficient for susceptibles is constant. Moreover, in Theorem 4.1 we investigate the asymptotic behavior of our system as the diffusion coefficients for susceptible and removed compartments tend to 0 by proving the convergence to the resulting mixed ODE-PDE system. Then, we briefly discuss some important considerations regarding the numerical solution of the model. We then perform a proof-of-concept simulation and show that the model provides better qualitative agreement with surveillance data compared to a purely diffusive model. Additionally, we show that the chemotaxis model recreates important dynamics not seen in the pure diffusion model, including transmission over long geographic distances over short time scales. Furthermore, we show that these dynamics will occur naturally, and do not require spatiotemporal variation of model parameters. We conclude by summarizing the presented results and suggesting several directions for future work in this area.

2 Mathematical model

Let us begin in a general setting, by taking a bounded connected domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, and let $i(x, t)$ describe the density of infected individuals at a point x in Ω at a time t . Next, we consider a bounded connected, arbitrary control region $\Omega_a \subset \Omega$, with $\partial\Omega_a$ sufficiently smooth. The overall concentration $I_a(t)$ of infection in Ω_a is given by

$$I_a(t) = \int_{\Omega_a} i(x, t) dx. \quad (2.1)$$

Following the standard reasoning from conservation laws, $I_a(x, t)$ can change only due to:

- (A) Internal sources and sinks, or
- (B) Infected population entering (or exiting) Ω_a from across the boundary $\partial\Omega_a$.

Physically, (A) above refers to the generation of new infections from inside Ω_a , also known as *incidence* and denoted as $\lambda(x, t)$, and removals from the infected population, either from recovery from infection or death, within Ω_a . We assume, for simplicity, that all removals are recoveries and occur at a rate ϕ .

Regarding (B), we may denote the movement by infected individuals across the boundaries as a flux \mathbf{j} , assumed generic for the moment. From standard arguments, then we have that

$$\partial_t \int_{\Omega_a} i(x, t) dx = \int_{\Omega_a} (\lambda(x, t) - \phi i(x, t)) dx - \int_{\partial\Omega_a} \mathbf{j} \cdot \mathbf{n}_a d\sigma, \quad (2.2)$$

where \mathbf{n}_a denotes the outward normal unit vector to $\partial\Omega_a$. Applying the divergence theorem, moving the time derivative inside the integral, and rearranging the latter equality implies

$$\int_{\Omega_a} (\partial_t i(x, t) - \lambda(x, t) + \phi i(x, t) + \nabla \cdot \mathbf{j}(x, t)) dx = 0. \quad (2.3)$$

Since this must hold for any choice of Ω_a , we thus have

$$\partial_t i = \lambda - \phi i - \nabla \cdot \mathbf{j} \quad (2.4)$$

over the whole of Ω . We must specify how we define λ and \mathbf{j} , with the latter being the primary focus of the present work. For the former, we will assume a standard frequency-dependent formulation [9], given as:

$$\lambda(x, t) = \beta(x, t) \frac{i}{n}, \quad (2.5)$$

where $s(x, t)$ refers to the susceptible population at (x, t) , $\beta(x, t)$ is the contact rate (units 1/time), and $n(x, t) = s(x, t) + i(x, t) + r(x, t)$ is the total living population at (x, t) . Here, $r(x, t)$ represents the recovered population. We briefly remark that other choices of λ may be more suitable for certain applications, including, among others, density-dependent or Holling-type formulations. We refer the reader to [9] for further discussion of these issues. We now focus our attention on the choice of \mathbf{j} .

2.1 Standard flux choice: Fickian diffusion

For \mathbf{j} , the most common choice is to assume the fl *Fickian diffusion* flux – that is, the flux is proportional to the negative gradient of infected density. Put simply, this assumption states that the infected population will tend towards a uniform concentration, moving from regions of high-infection density to low-infection density. Mathematically, we may describe this as

$$\mathbf{j} = -\nu_i \nabla i, \quad (2.6)$$

with $\nu_i(x, t)$ being the *diffusion coefficient*.

Following this reasoning for additional compartments, and taking advantage of a standard *frequency-dependent* formulation of λ , one may derive the *susceptible-infected-removed (SIR)* system for susceptible, infected, and removed population densities $s(x, t)$, $i(x, t)$, $r(x, t)$, respectively:

$$\partial_t s = -\frac{\beta i}{s + i + r} s + \nabla \cdot (\nu_s \nabla s), \text{ in } Q := \Omega \times (0, T), \quad (2.7)$$

$$\partial_t i = \frac{\beta i}{s + i + r} s - \phi i + \nabla \cdot (\nu_i \nabla i), \text{ in } Q, \quad (2.8)$$

$$\partial_t r = \phi i + \nabla \cdot (\nu_r \nabla r), \text{ in } Q, \quad (2.9)$$

$$\nabla s \cdot \mathbf{n} = \nabla i \cdot \mathbf{n} = \nabla r \cdot \mathbf{n} = 0, \text{ on } \Sigma := \partial\Omega \times (0, T), \quad (2.10)$$

$$s(0) = s_0, i(0) = i_0, r(0) = r_0, \text{ in } \Omega. \quad (2.11)$$

We recall that Ω denotes a bounded domain of \mathbb{R}^d , $d = 1, 2, 3$, with a smooth boundary $\partial\Omega$. Here, \mathbf{n} stands for the outward normal unit vector to the boundary $\partial\Omega$ and $T > 0$ represents a final time.

2.2 Flux definition 2: Chemotaxis-inspired

Let us consider an alternative choice for the flux. We assume that the spatial evolution of the infectious disease depends primarily on two processes:

1. An undirected component, modeled as a random walk. As mentioned the previous example, in the continuous limit, this is described by a diffusion operator.
2. A *directed* component, which corresponds to mobility which is predictable, and not random.

To model this second process, we assume the following:

- The spatial propagation of an infectious disease depends on human mobility, in particular how this mobility relates to *contacts*. In general, contacts will increase moving from regions of lower to higher population density.
- Higher levels of infection should, in general, serve to *increase*, not decrease, the spatial propagation of the disease *provided sufficient availability of susceptible individuals*.

Combining the two points, we therefore postulate that *an airborne infectious disease should propagate according to the density of susceptible individuals*, and in particular *will move from regions with a lower concentration of susceptible individuals to regions with a higher concentration of susceptible individuals*. The rate of this movement should be also governed in part by the concentration of infected individuals in the area.

Putting these pieces together suggests the following, alternative definition for the flux:

$$\mathbf{j} = -\nu_i \nabla i + \chi(i) \nabla s, \quad (2.12)$$

where $\chi(i)$ is a strictly positive function of i . A possible specification for χ is

$$\chi(x, t, i) = \mu_i(x, t) \frac{i}{1 + i/C_0}, \quad (2.13)$$

where $\mu_i > 0$ is a chemotactic coefficient and $C_0 > 0$ is a *capacity term* defining a level of population density at which saturation occurs. Observe that when $i \ll C_0$, $\chi(i) \approx \mu_i i$; hence, χ increases in proportion to i for smaller i . However, as i increases, this begins to change; note that for $i \approx C_0$, $\chi(i)$ will increase in proportion to $\mu_i i/2$. Finally, for $i \gg C_0$ we see that

$$\lim_{i \rightarrow \infty} \chi(i) = \lim_{i \rightarrow \infty} \frac{\mu_i i}{1 + i/C_0} = C_0 \mu_i. \quad (2.14)$$

Hence, the parameter C_0 defines a level of infection density at which additional increases in infection density result in a reduced effect on the spatial dynamics. Such terms may be explained physically by, for example, natural limitations on human mobility.

Proceeding as before, the general modified model reads as follows:

$$\partial_t s - \nabla \cdot (\nu_s \nabla s) + \frac{\beta i}{s + i + r} s = 0, \text{ in } Q, \quad (2.15)$$

$$\partial_t i - \nabla \cdot (\nu_i \nabla i - \chi(i) \nabla s) + \phi i - \frac{\beta s}{s + i + r} i = 0, \text{ in } Q, \quad (2.16)$$

$$\partial_t r - \nabla \cdot (\nu_r \nabla r) - \phi i = 0, \text{ in } Q, \quad (2.17)$$

$$\nabla s \cdot \mathbf{n} = (\nu_i \nabla i - \chi(i) \nabla s) \cdot \mathbf{n} = \nabla r \cdot \mathbf{n} = 0, \text{ on } \Sigma, \quad (2.18)$$

$$s(0) = s_0, i(0) = i_0, r(0) = r_0, \text{ in } \Omega. \quad (2.19)$$

Looking at the boundary conditions (2.18) we deduce in particular that

$$\nabla i \cdot \mathbf{n} = 0 \text{ on } \Sigma \quad (2.20)$$

as well. From here on out we will regard (2.7)-(2.11) as simply a particular case of (2.15)-(2.19) with $\mu_i = 0$; note the capacity term C_0 no longer serves any purpose in such a case. A full list of the model parameters in (2.15)-(2.19), their description, and their units are provided in Table 1.

Parameter	Name	Unit
β	Contact rate	Days ⁻¹
ϕ	Removal rate	Days ⁻¹
ν_j	Diffusion coefficient, compartment j	km ² · Days ⁻¹
μ_i	Chemotactic coefficient	km ² · Persons · Days ⁻¹
C_0	Capacity/saturation term	Persons

Table 1: Description of model parameters for (2.15)-(2.19) with the choice (2.13). Note that (2.7)-(2.11) corresponds to the case of $\mu_i = 0$.

3 Mathematical analysis of the chemotaxis-inspired model

In this section, we present the mathematical results regarding the system (2.15)-(2.19). In order to allow more generality to the system parameters, which in principle can be different in space and vary in time we assume that they are space and time dependent.

We begin by specifying some notation. Let

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad W = \{v \in H^2(\Omega); \nabla v \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

As usual, we identify the space H with its dual space H' and we note that $W \subset V \subset H \equiv H' \subset V'$ with dense and compact embeddings. We denote by $\langle \cdot, \cdot \rangle$ the pairing between V' and V .

We assume the following hypotheses on the system parameters:

$$(\nu_s, \nu_i, \nu_r) \in (L^\infty(Q))^3, \quad 0 < \nu_m \leq \nu_s, \nu_i, \nu_r \leq \nu_M \text{ a.e. in } Q, \quad (3.1)$$

$$\beta \in L^\infty(Q), \quad 0 \leq \beta \leq \beta_M \text{ a.e. in } Q, \quad (3.2)$$

with ν_m, ν_M, β_M constants,

$$\phi \text{ is a positive coefficient,} \quad (3.3)$$

$$\chi : Q \times [0, +\infty) \rightarrow \mathbb{R} \text{ is a Carathéodory function,} \quad (3.4)$$

that is, $\chi(\cdot, \cdot, v)$ is measurable in Q for all $v \geq 0$ and $\chi(x, t, \cdot)$ is continuous in $[0, +\infty)$ for a.e. $(x, t) \in Q$.

Moreover, there exist two positive constants χ_M and $\chi_{1,M}$ such that

$$|\chi(x, t, v)| \leq \chi_M, \quad \frac{|\chi(x, t, v)|}{v} \leq \chi_{1,M} \quad \text{for all } v > 0 \text{ and a.e. } (x, t) \in Q. \quad (3.5)$$

An example of such a function may be (cf. (2.13))

$$\chi(x, t, v) = \chi_0(x, t) \frac{v}{v + c_0}, \quad \text{with } \chi_0 \in L^\infty(Q), \quad \chi_0 \geq 0 \text{ a.e. in } Q, \quad c_0 > 0.$$

Finally, for the initial data we assume that

$$(s_0, i_0, r_0) \in (L^2(\Omega))^3, \quad 0 \leq s_0 \leq s_M, \quad i_0 \geq 0, \quad r_0 \geq 0 \text{ a.e. in } \Omega, \quad (3.6)$$

with s_M constant.

The aim is to prove that the initial-boundary value problem (2.15)-(2.19) has a weak solution. To this end, we introduce an approximating system depending on a parameter $\varepsilon > 0$, namely,

$$\partial_t s - \nabla \cdot (\nu_s \nabla s) + \frac{\beta i^+}{s^+ + i^+ + r^+ + \varepsilon} s = 0, \quad \text{in } Q, \quad (3.7)$$

$$\partial_t i - \nabla \cdot (\nu_i \nabla i - \chi(i^+) \nabla s) + \phi i - \frac{\beta s^+}{s^+ + i^+ + r^+ + \varepsilon} i = 0, \quad \text{in } Q, \quad (3.8)$$

$$\partial_t r - \nabla \cdot (\nu_r \nabla r) - \phi i = 0, \quad \text{in } Q, \quad (3.9)$$

$$\nabla s \cdot \mathbf{n} = \nabla i \cdot \mathbf{n} = \nabla r \cdot \mathbf{n} = 0, \quad \text{on } \Sigma, \quad (3.10)$$

$$s(0) = s_0, \quad i(0) = i_0, \quad r(0) = r_0, \quad \text{in } \Omega, \quad (3.11)$$

where v^+ represents the positive part of $v = s, i, r$. We note that the last terms on the left-hand sides of the first two equations make sense now because the denominators are positive. Similarly, the argument i^+ of χ is nonnegative.

First, we shall prove that system (3.7)-(3.11) has a weak solution (s, i, r) , with s, i, r all nonnegative, so that (s, i, r) solves also the system

$$\partial_t s - \nabla \cdot (\nu_s \nabla s) + \frac{\beta i}{s + i + r + \varepsilon} s = 0, \quad \text{in } Q, \quad (3.12)$$

$$\partial_t i - \nabla \cdot (\nu_i \nabla i - \chi(i) \nabla s) + \phi i - \frac{\beta s}{s + i + r + \varepsilon} i = 0, \quad \text{in } Q, \quad (3.13)$$

$$\partial_t r - \nabla \cdot (\nu_r \nabla r) - \phi i = 0, \quad \text{in } Q, \quad (3.14)$$

along with the boundary conditions (3.10) and the initial conditions (3.11).

Next, relying on appropriate a priori estimates we shall pass to the limit as $\varepsilon \rightarrow 0$ in (3.12)-(3.14) and prove that the limit is a weak solution to the original system (2.15)-(2.19). We note that the fraction $\frac{\beta i s}{s+i+r}$ is well defined whenever $s + i + r > 0$ and it will be well specified when $s + i + r = 0$, as we shall show in the existence theorem.

We start with the auxiliary approximating system (3.7)-(3.11).

Theorem 3.1 *Under the assumptions (3.1)-(3.6) there exists a triplet*

$$(s_\varepsilon, i_\varepsilon, r_\varepsilon) \in (H^1(0, T; V') \cap C([0, T]; H) \cap L^2(0, T; V))^3 \quad (3.15)$$

satisfying the following notion of weak solution to system (3.7)-(3.11):

$$\int_0^T \langle \partial_t s_\varepsilon(t), v(t) \rangle dt + \int_Q \nu_s \nabla s_\varepsilon \cdot \nabla v dx dt + \int_Q \frac{\beta i_\varepsilon^+}{s_\varepsilon^+ + i_\varepsilon^+ + r_\varepsilon^+ + \varepsilon} s_\varepsilon v dx dt = 0,$$

$$\begin{aligned}
& \int_0^T \langle \partial_t i_\varepsilon(t), v(t) \rangle dt + \int_Q \nu_i \nabla i_\varepsilon \cdot \nabla v dx dt + \int_Q \phi i_\varepsilon v dx dt \\
& + \int_Q \frac{\beta s_\varepsilon^+}{s_\varepsilon^+ + i_\varepsilon^+ + r_\varepsilon^+ + \varepsilon} i_\varepsilon v dx dt = \int_Q \chi(i_\varepsilon^+) \nabla s_\varepsilon \cdot \nabla v dx dt, \\
& \int_0^T \langle \partial_t r_\varepsilon(t), v(t) \rangle dt + \int_Q \nu_r \nabla r_\varepsilon \cdot \nabla v dx dt - \int_Q \phi i_\varepsilon v dx dt = 0,
\end{aligned}$$

for all $v \in L^2(0, T; V)$. Moreover, it has the properties

$$0 \leq s_\varepsilon \leq s_M, \quad i_\varepsilon \geq 0, \quad r_\varepsilon \geq 0 \quad \text{a.e. in } Q, \quad (3.16)$$

whence it turns out that the solution $(s_\varepsilon, i_\varepsilon, r_\varepsilon)$ solves also a weak formulation of equations (3.12)-(3.14) with the boundary conditions (3.10) and the initial conditions (3.11), that is, for all $v \in L^2(0, T; V)$,

$$\begin{aligned}
& \int_0^T \langle \partial_t s(t), v(t) \rangle dt + \int_Q \nu_s \nabla s \cdot \nabla v dx dt + \int_Q \frac{\beta i}{s + i + r} s v dx dt = 0, \\
& \int_0^T \langle \partial_t i(t), v(t) \rangle dt + \int_Q \nu_i \nabla i \cdot \nabla v dx dt + \int_Q \phi i v dx dt \\
& + \int_Q \frac{\beta s}{s + i + r} i v dx dt = \int_Q \chi(i) \nabla s \cdot \nabla v dx dt, \\
& \int_0^T \langle \partial_t r(t), v(t) \rangle dt + \int_Q \nu_r \nabla r \cdot \nabla v dx dt - \int_Q \phi i v dx dt = 0.
\end{aligned}$$

Proof. We apply the Schauder fixed point theorem, by introducing the space

$$X = (L^2(Q))^3 \quad (3.17)$$

and the ball

$$M = \{(s, i, r) \in X; \quad \|s\|_{L^2(Q)} + \|i\|_{L^2(Q)} + \|r\|_{L^2(Q)} \leq R\}, \quad (3.18)$$

with R denoting a large enough positive constant, which will be specified later.

We fix $(\bar{s}, \bar{i}, \bar{r}) \in M$ and consider the system

$$\partial_t s - \nabla \cdot (\nu_s \nabla s) + \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} s = 0, \quad \text{in } Q, \quad (3.19)$$

$$\partial_t i - \nabla \cdot (\nu_i \nabla i - \chi(\bar{i}^+) \nabla s) + \phi i - \frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} i = 0, \quad \text{in } Q, \quad (3.20)$$

$$\partial_t r - \nabla \cdot (\nu_r \nabla r) - \phi i = 0, \quad \text{in } Q, \quad (3.21)$$

with the boundary and initial conditions (3.10), (3.11) and prove below that it has a solution (s, i, r) .

Then, we define the mapping

$$\Phi : M \rightarrow X, \quad \Phi(\bar{s}, \bar{i}, \bar{r}) := (s, i, r) \quad (3.22)$$

and show that $\Phi(M) \subset M$, $\Phi(M)$ is a compact set of X and that Φ is continuous.

Now we present the arguments that prove that Φ is well defined. Since, due to (3.2), the coefficient of s in (3.19) satisfies

$$0 \leq \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} \leq \beta_M \quad \text{a.e. in } Q, \quad (3.23)$$

it follows from the classical results concerning the well-posedness of parabolic problems (see, e.g., [26]) that there exists a unique weak solution

$$s \in H^1(0, T; V') \cap C([0, T]; H) \cap L^2(0, T; V) \quad (3.24)$$

to (3.19), coupled with $\nabla s \cdot \mathbf{n} = 0$ on Σ and $s(0) = s_0$.

Moreover, by testing (3.19) by $-s^-(t)$, where s^- denotes the negative part of s , and integrating over $(0, t)$ we obtain

$$\begin{aligned} & \frac{1}{2} \|s^-(t)\|_H^2 + \int_0^t \int_\Omega \nu_s |\nabla s^-|^2 dx d\tau \\ &= \frac{1}{2} \|s_0^-\|_H^2 - \int_0^t \int_\Omega \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} |s^-|^2 dx d\tau. \end{aligned}$$

Recalling that $s_0 \geq 0$ a.e. in Ω and using (3.23), we realize that the right-hand side is nonpositive and so we conclude that $s^-(t) = 0$ for all $t \in [0, T]$, whence $s \geq 0$ a.e. in Q . We proceed similarly by testing (3.19) by the positive part $(s(t) - s_M)^+$, integrating over $(0, t)$ and recalling again (3.6) and (3.23). Then, we obtain that

$$\begin{aligned} & \frac{1}{2} \|(s(t) - s_M)^+\|_H^2 + \int_0^t \int_\Omega \nu_s |\nabla (s - s_M)^+|^2 dx d\tau \\ &= \frac{1}{2} \|(s_0 - s_M)^+\|_H^2 - \int_0^t \int_\Omega \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} s(s - s_M)^+ dx d\tau \leq 0 \end{aligned}$$

and can infer that $s \leq s_M$ a.e. in Q . Thus, we have proved that

$$0 \leq s \leq s_M \quad \text{a.e. in } Q. \quad (3.25)$$

Next, we deal with (3.20) rewritten as

$$\partial_t i - \nabla \cdot (\nu_i \nabla i) + \phi i - \frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} i = -\nabla \cdot (\chi(\bar{i}^+) \nabla s), \quad (3.26)$$

together with $\nabla i \cdot \mathbf{n} = 0$ on Σ and $i(0) = i_0$. We note that

$$0 \leq \frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} \leq \beta_M \quad \text{a.e. in } Q \quad (3.27)$$

and that the right-hand side of (3.26) belongs to $L^2(0, T; V')$ thanks to (3.5). Relying again on the existence results for parabolic problems, we deduce that there exists a unique solution

$$i \in H^1(0, T; V') \cap C([0, T]; H) \cap L^2(0, T; V) \quad (3.28)$$

to the problem (3.26) along with $\nabla i \cdot \mathbf{n} = 0$ on Σ and $i(0) = i_0$.

As i is now fixed, applying the same argument as before yields that (3.21), together with $\nabla r \cdot \mathbf{n} = 0$ on Σ and $r(0) = r_0$, has a unique solution

$$r \in H^1(0, T; V') \cap C([0, T]; H) \cap L^2(0, T; V). \quad (3.29)$$

Next, we deduce some uniform estimates. In the sequel, we shall denote by C several positive constants (that may differ from a line to the other) depending on the problem parameters, but independent of ε .

First, we test (3.19) by $s(t)$, integrate over $(0, t)$, taking (3.1) and (3.2) into account, and obtain

$$\frac{1}{2} \|s(t)\|_H^2 + \nu_m \int_0^t \|\nabla s(\tau)\|_H^2 d\tau \leq \frac{1}{2} \|s_0\|_H^2 + \beta_M \int_0^t \|s(\tau)\|_H^2 d\tau,$$

whence, by the Gronwall lemma, we deduce the estimate

$$\|s\|_{L^\infty(0, T; H) \cap L^2(0, T; V)} \leq C, \quad (3.30)$$

where C depends only on $\|s_0\|_H$, ν_m , β_M and T . Further, by the comparison of the terms in (3.19) we also infer that

$$\|\partial_t s\|_{L^2(0, T; V')} \leq C. \quad (3.31)$$

Next, by testing (3.26) by $i(t)$ and integrating over $(0, t)$ we have that

$$\begin{aligned} & \frac{1}{2} \|i(t)\|_H^2 + \nu_m \int_0^t \|\nabla i(\tau)\|_H^2 d\tau + \phi \int_0^t \|i(\tau)\|_H^2 d\tau \\ & \leq \frac{1}{2} \|i_0\|_H^2 + \beta_M \int_0^t \|i(\tau)\|_H^2 d\tau + \chi_M \int_0^t \|\nabla s(\tau)\|_H \|\nabla i(\tau)\|_H d\tau. \end{aligned} \quad (3.32)$$

For the last term on the right-hand side, we apply the Young inequality

$$\chi_M \int_0^t \|\nabla s(\tau)\|_H \|\nabla i(\tau)\|_H d\tau \leq \frac{\nu_m}{2} \int_0^t \|\nabla i(\tau)\|_H^2 d\tau + \frac{|\chi_M|^2}{2\nu_m} \int_0^t \|\nabla s(\tau)\|_H^2 d\tau,$$

(3.30) and the Gronwall lemma in order to obtain

$$\|i\|_{L^\infty(0, T; H) \cap L^2(0, T; V)} \leq C. \quad (3.33)$$

Now, by comparing the terms in (3.26) we easily arrive at

$$\|\partial_t i\|_{L^2(0, T; V')} \leq C. \quad (3.34)$$

Proceeding in the same way in (3.21) with $\nabla r \cdot \mathbf{n} = 0$ on Σ and $r(0) = r_0$, and using (3.33) we infer that

$$\|r\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|\partial_t r\|_{L^2(0,T;V')} \leq C. \quad (3.35)$$

Due to the estimates (3.30), (3.33), (3.35) it turns out that there exists a constant R such that

$$\|s\|_{L^2(Q)} + \|i\|_{L^2(Q)} + \|r\|_{L^2(Q)} \leq R$$

and we employ exactly this constant in the definition (3.18) of M .

Therefore, it follows that $\Phi(M) \subset M$. Recalling also (3.31) and (3.34), due to the compact embedding of $H^1(0,T;V') \cap L^2(0,T;V)$ into $L^2(0,T;H) \equiv L^2(Q)$ (see, e.g., [27, p. 58]), we conclude that $\Phi(M)$ is a compact set of X .

It remains to prove that Φ is continuous. We consider a sequence $(\bar{s}_n, \bar{i}_n, \bar{r}_n) \in M$ such that $(\bar{s}_n, \bar{i}_n, \bar{r}_n) \rightarrow (\bar{s}, \bar{i}, \bar{r})$ strongly in X , as $n \rightarrow \infty$. Letting now $(s_n, i_n, r_n) = \Phi(\bar{s}_n, \bar{i}_n, \bar{r}_n)$ and $(s, i, r) = \Phi(\bar{s}, \bar{i}, \bar{r})$, we want to prove that $(s_n, i_n, r_n) \rightarrow (s, i, r)$ strongly in X , as $n \rightarrow \infty$.

Relying on the previous estimates for the solution to (3.19)-(3.21) we have that

$$\begin{aligned} & \|s_n\|_{H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)} + \|i_n\|_{H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)} \\ & + \|r_n\|_{H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C. \end{aligned}$$

Hence, by compactness, there is a subsequence of n , still denoted by n , such that

$$\begin{aligned} & s_n \rightarrow s, \quad i_n \rightarrow i, \quad r_n \rightarrow r \\ & \text{weakly in } H^1(0,T;V') \cap L^2(0,T;V), \text{ weak}^* \text{ in } L^\infty(0,T;H), \\ & \text{strongly in } L^2(0,T;H) \text{ and a.e. in } Q, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.36)$$

We have to prove that (s, i, r) is exactly $\Phi(\bar{s}, \bar{i}, \bar{r})$ and that the whole sequence (s_n, i_n, r_n) converges in X . For this aim, we should be able to pass to the limit as $n \rightarrow \infty$ in the variational formulation of the problem. Namely, we have that

$$\int_0^T \langle \partial_t s_n(t), v(t) \rangle dt + \int_Q \nu_s \nabla s_n \cdot \nabla v dx dt + \int_Q \frac{\beta \bar{i}_n^+}{\bar{s}_n^+ + \bar{i}_n^+ + \bar{r}_n^+ + \varepsilon} s_n v dx dt = 0, \quad (3.37)$$

$$\begin{aligned} & \int_0^T \langle \partial_t i_n(t), v(t) \rangle dt + \int_Q \nu_i \nabla i_n \cdot \nabla v dx dt + \int_Q \phi i_n v dx dt \\ & + \int_Q \frac{\beta \bar{s}_n^+}{\bar{s}_n^+ + \bar{i}_n^+ + \bar{r}_n^+ + \varepsilon} i_n v dx dt = \int_Q \chi(\bar{i}_n^+) \nabla s_n \cdot \nabla v dx dt, \end{aligned} \quad (3.38)$$

$$\int_0^T \langle \partial_t r_n(t), v(t) \rangle dt + \int_Q \nu_r \nabla r_n \cdot \nabla v dx dt - \int_Q \phi i_n v dx dt = 0, \quad (3.39)$$

for all $v \in L^2(0,T;V)$.

First, we discuss the convergence of some terms. We have

$$\frac{\beta \bar{i}_n^+}{\bar{s}_n^+ + \bar{i}_n^+ + \bar{r}_n^+ + \varepsilon} v \rightarrow \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} v \quad \text{a.e. in } Q$$

and

$$\left| \frac{\beta \bar{i}_n^+}{\bar{s}_n^+ + \bar{i}_n^+ + \bar{r}_n^+ + \varepsilon} v \right| \leq \beta_M |v|,$$

whence, by the Lebesgue dominated convergence theorem, we deduce that

$$\frac{\beta \bar{i}_n^+}{\bar{s}_n^+ + \bar{i}_n^+ + \bar{r}_n^+ + \varepsilon} v \rightarrow \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} v \text{ strongly in } L^2(Q).$$

Recalling (3.36), as $s_n \rightarrow s$ strongly in $L^2(Q)$, we obtain

$$\int_Q \frac{\beta \bar{i}_n^+}{\bar{s}_n^+ + \bar{i}_n^+ + \bar{r}_n^+ + \varepsilon} s_n v dx dt \rightarrow \int_Q \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} s v dx dt \text{ as } n \rightarrow \infty.$$

Similarly, we deduce that

$$\int_Q \frac{\beta \bar{s}_n^+}{\bar{s}_n^+ + \bar{i}_n^+ + \bar{r}_n^+ + \varepsilon} i_n v dx dt \rightarrow \int_Q \frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} i v dx dt \text{ as } n \rightarrow \infty.$$

Still in the second equation we have the convergence of the term on the right-hand side

$$\int_Q \chi(\bar{i}_n^+) \nabla s_n \cdot \nabla v dx dt \rightarrow \int_Q \chi(\bar{i}^+) \nabla s \cdot \nabla v dx dt \text{ as } n \rightarrow \infty$$

because of the same argument: indeed, we have

$$\chi(\bar{i}_n^+) \nabla v \rightarrow \chi(\bar{i}^+) \nabla v \text{ strongly in } (L^2(Q))^d,$$

due to (3.4)-(3.5) and the Lebesgue dominated convergence theorem, and

$$\nabla s_n \rightarrow \nabla s \text{ weakly in } (L^2(Q))^d,$$

due to (3.36). Then, by passing to the limit in (3.37)-(3.39) and using again (3.36) we finally deduce that (for all $v \in L^2(0, T; V)$)

$$\int_0^T \langle \partial_t s(t), v(t) \rangle dt + \int_Q \nu_s \nabla s \cdot \nabla v dx dt + \int_Q \frac{\beta \bar{i}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} s v dx dt = 0, \quad (3.40)$$

$$\begin{aligned} & \int_0^T \langle \partial_t i(t), v(t) \rangle dt + \int_Q \nu_i \nabla i \cdot \nabla v dx dt + \int_Q \phi i v dx dt \\ & + \int_Q \frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} i v dx dt = \int_Q \chi(\bar{i}^+) \nabla s \cdot \nabla v dx dt, \end{aligned} \quad (3.41)$$

$$\int_0^T \langle \partial_t r(t), v(t) \rangle dt + \int_Q \nu_r \nabla r \cdot \nabla v dx dt - \int_Q \phi i v dx dt = 0, \quad (3.42)$$

whence we see that (s, i, r) is the unique solution to the initial boundary value problem for (3.19)-(3.21). Moreover, owing to the lower semicontinuity of norms, we note that (s, i, r) satisfies the estimate

$$\begin{aligned} & \|s\|_{H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)} + \|i\|_{H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)} \\ & + \|r\|_{H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C. \end{aligned} \quad (3.43)$$

As the limit is uniquely determined, it turns out that the whole sequence (s_n, i_n, r_n) converges strongly in X . In fact, by contradiction let us assume that there is a subsequence n_k such that $(s_{n_k}, i_{n_k}, r_{n_k})$ does not converge. Hence, by arguing as in (3.36) there exists another subsequence $n_{k,j}$ such that $(s_{n_{k,j}}, i_{n_{k,j}}, r_{n_{k,j}})$ satisfies (3.36) and consequently converges to the unique solution (s, i, r) to the initial boundary value problem for (3.19)-(3.21). Then we arrive at a contradiction.

Therefore, by the Schauder fixed point theorem, we have proved that there exists a triplet $(s_\varepsilon, i_\varepsilon, r_\varepsilon)$ that is a weak solution to system (3.7)-(3.11).

At this point, as s_ε clearly satisfies (3.25), we have to show that i_ε and r_ε are nonnegative in order to complete the proof of (3.16). To this end we first introduce a Lipschitz continuous approximation of the logarithm function. Namely, let $\delta \in (0, 1)$ and define

$$j_\delta(v) := \begin{cases} \ln \delta, & \text{if } v < \delta \\ \ln v, & \text{if } \delta \leq v \leq 1/\delta \\ \ln \frac{1}{\delta}, & \text{if } v > 1/\delta \end{cases} \quad \text{for } v \in \mathbb{R}.$$

We note that j_δ is the derivative of the convex and nonnegative function

$$\widehat{j}_\delta(v) := \begin{cases} v \ln \delta - \delta + 1, & \text{if } v < \delta \\ v \ln v - v + 1, & \text{if } \delta \leq v \leq 1/\delta \\ v \ln \frac{1}{\delta} - \frac{1}{\delta} + 1, & \text{if } v > 1/\delta \end{cases} \quad \text{for } v \in \mathbb{R},$$

with \widehat{j}_δ taking the minimum value 0 at $v = 1$.

In the next computations we shall use, for simplicity, the notation i and r instead of i_ε and r_ε . We test (3.20) by $j_\delta(i)$ and integrate with respect to time. Then, with the help of (3.1) we easily have that

$$\begin{aligned} & \int_{\Omega} \widehat{j}_\delta(i(t)) dx + \int_{Q_t \cap \{\delta \leq i \leq 1/\delta\}} \nu_m \frac{|\nabla i|^2}{i} dx d\tau \\ & \leq \int_{\Omega} \widehat{j}_\delta(i_0) dx + \int_{Q_t} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) i j_\delta(i) dx d\tau \\ & \quad + \int_{Q_t \cap \{\delta \leq i \leq 1/\delta\}} \frac{\chi(i^+)}{i} \nabla s \cdot \nabla i dx d\tau, \end{aligned} \quad (3.44)$$

where $Q_t := \Omega \times (0, t)$, $t \in (0, T)$. Now, we treat in detail the terms on the right-hand side. We note that

$$\begin{aligned} & \int_{\Omega} \widehat{j}_\delta(i_0) dx = \int_{\Omega \cap \{i_0 < \delta\}} (i_0 \ln \delta - \delta + 1) dx \\ & \quad + \int_{\Omega \cap \{\delta \leq i_0 \leq 1/\delta\}} (i_0 \ln i_0 - i_0 + 1) dx + \int_{\Omega \cap \{i_0 > 1/\delta\}} \left(i_0 \ln \frac{1}{\delta} - \frac{1}{\delta} + 1 \right) dx. \end{aligned}$$

As $i_0 \geq 0$ a.e. in Ω and $\delta < 1$, it is clear that

$$\int_{\Omega \cap \{i_0 < \delta\}} (i_0 \ln \delta - \delta + 1) dx \leq \int_{\Omega} 1 dx.$$

For the second integral on the right-hand side above, the reader can easily check that

$$\int_{\Omega \cap \{\delta \leq i_0 \leq 1/\delta\}} (i_0 \ln i_0 - i_0 + 1) dx \leq \int_{\Omega} (|i_0|^2 + 1) dx \leq C,$$

since $i_0 \in H$. For the third integral we have that

$$\int_{\Omega \cap \{i_0 > 1/\delta\}} \left(i_0 \ln \frac{1}{\delta} - \frac{1}{\delta} + 1 \right) dx \leq \int_{\Omega \cap \{i_0 > 1/\delta\}} i_0 \ln i_0 dx \leq C.$$

About the second term on the right-hand side of (3.44), we split the integral in three parts as for the previous one and discuss each of them separately. Taking into account the sign of factors and using (3.2), we infer that

$$\begin{aligned} & \int_{Q_t \cap \{i < \delta\}} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) i j_{\delta}(i) dx d\tau \\ &= \int_{Q_t \cap \{i < \delta\}} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) i \ln \delta dx d\tau \\ &= \int_{Q_t \cap \{i < \delta\}} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) (\hat{j}_{\delta}(i) + \delta - 1) dx d\tau \\ &\leq \beta_M \int_0^t \int_{\Omega} \hat{j}_{\delta}(i(\tau)) dx d\tau + \phi \int_{Q_t} 1 dx d\tau. \end{aligned}$$

Next, we have that

$$\begin{aligned} & \int_{Q_t \cap \{\delta \leq i \leq 1/\delta\}} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) i j_{\delta}(i) dx d\tau \\ &= \int_{Q_t \cap \{\delta \leq i \leq 1/\delta\}} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) (\hat{j}_{\delta}(i) + i - 1) dx d\tau \\ &\leq \beta_M \int_0^t \int_{\Omega} \hat{j}_{\delta}(i(\tau)) dx d\tau + (\beta_M + \phi) \int_{Q_t} (|i| + 1) dx d\tau. \end{aligned}$$

Finally, it turns out that

$$\begin{aligned} & \int_{Q_t \cap \{i > 1/\delta\}} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) i j_{\delta}(i) dx d\tau \\ &= \int_{Q_t \cap \{i > 1/\delta\}} \left(\frac{\beta \bar{s}^+}{\bar{s}^+ + \bar{i}^+ + \bar{r}^+ + \varepsilon} - \phi \right) \left(\hat{j}_{\delta}(i) + \frac{1}{\delta} - 1 \right) dx d\tau \\ &\leq \beta_M \int_0^t \int_{\Omega} \hat{j}_{\delta}(i(\tau)) dx d\tau + \beta_M \int_{Q_t} (|i| + 1) dx d\tau, \end{aligned}$$

due to $1/\delta < i$ in the set of integration.

Now we discuss the last term on the right-hand side of (3.44). As $\chi(i^+) = \chi(i)$ whenever $i \geq 0$, using (3.5) and (3.43) we deduce that

$$\int_{Q_t \cap \{\delta \leq i \leq 1/\delta\}} \frac{\chi(i^+)}{i} \nabla s \cdot \nabla i dx d\tau \leq \chi_{1,M} \int_Q |\nabla s| |\nabla i| dx d\tau \leq C.$$

Then, collecting the previous estimates to control the right hand side of (3.44), we can apply the Gronwall lemma in order to obtain

$$\int_{\Omega} \widehat{j}_{\delta}(i(t)) dx + \int_{Q_t \cap \{\delta \leq i \leq 1/\delta\}} \nu_m \frac{|\nabla i|^2}{i} dx d\tau \leq C, \quad (3.45)$$

with C independent of δ and ε as well. Our aim is now to pass to the limit as $\delta \rightarrow 0$. Since it holds that

$$\widehat{j}_{\delta}(i) \rightarrow \widehat{j}(i) := \begin{cases} +\infty & \text{if } i < 0 \\ 1 & \text{if } i = 0 \\ i \ln i - i + 1 & \text{if } i > 0 \end{cases}$$

pointwise, by the Fatou lemma we recover that

$$\int_{\Omega} \widehat{j}(i(t)) dx \leq C \quad \text{for all } t \in [0, T]$$

(recall that $i \in C([0, T]; H)$), whence necessarily $i \geq 0$ a.e. in Q . Moreover, still by virtue of (3.45), we infer that

$$\int_Q 4\nu_m |\nabla i^{1/2}|^2 dx d\tau \leq C,$$

which offers the additional estimate (cf. (3.43))

$$\|i^{1/2}\|_{L^{\infty}(0,T;L^4(\Omega)) \cap L^2(0,T;V)} \leq C. \quad (3.46)$$

At this point, it remains to prove that r is non negative. For this aim it suffices to test (3.9) by the negative part $r^-(t)$, integrate and use (3.6) and the sign of the term $-\phi i$ in the equation. Therefore, we can drop out the positive parts in (3.7)-(3.8) and conclude the proof of Theorem 3.1. \square

Theorem 3.2 *Under the assumptions (3.1)-(3.6) there exists a triplet*

$$(s, i, r) \in (H^1(0, T; V') \cap C([0, T]; H) \cap L^2(0, T; V))^3 \quad (3.47)$$

that is a weak solution to system (2.15)-(2.19), with the properties

$$0 \leq s \leq s_M, \quad i \geq 0, \quad r \geq 0 \quad \text{a.e. in } Q \quad (3.48)$$

and where the term

$$\frac{\beta si}{s+i+r} \text{ in equations (2.15) and (2.16)}$$

has to be understood as 0 whenever $s+i+r=0$. (3.49)

Moreover, if we set $n = s + i + r$ the conservation property

$$\int_{\Omega} n(x, t) dx = \int_{\Omega} n_0(x) dx, \text{ for all } t \in [0, T], \quad (3.50)$$

holds, where $n_0 := s_0 + i_0 + r_0$.

Proof. We recall that the weak solution $(s_{\varepsilon}, i_{\varepsilon}, r_{\varepsilon})$ to equations (3.12)-(3.14), with the boundary conditions (3.10) and the initial conditions (3.11), satisfies the uniform estimates (3.43), (3.46) in addition to the properties (3.16). Then, there exist a triplet (s, i, r) and a subsequence, still denoted by ε , such that

$$\begin{aligned} s_{\varepsilon} &\rightarrow s, \quad i_{\varepsilon} \rightarrow i, \quad r_{\varepsilon} \rightarrow r \\ &\text{weakly in } H^1(0, T; V') \cap L^2(0, T; V), \text{ weak}^* \text{ in } L^{\infty}(0, T; H), \\ &\text{strongly in } L^2(0, T; H) \text{ and a.e. in } Q, \text{ as } \varepsilon \rightarrow 0, \end{aligned} \quad (3.51)$$

with (s, i, r) satisfying (3.48). Then, we aim to pass to the limit in the variational form of (3.12)-(3.14), that is,

$$\int_0^T \langle \partial_t s_{\varepsilon}(t), v(t) \rangle dt + \int_Q \nu_s \nabla s_{\varepsilon} \cdot \nabla v dx dt + \int_Q \frac{\beta i_{\varepsilon}}{s_{\varepsilon} + i_{\varepsilon} + r_{\varepsilon} + \varepsilon} s_{\varepsilon} v dx dt = 0,$$

$$\begin{aligned} &\int_0^T \langle \partial_t i_{\varepsilon}(t), v(t) \rangle dt + \int_Q \nu_i \nabla i_{\varepsilon} \cdot \nabla v dx dt + \int_Q \phi i_{\varepsilon} v dx dt \\ &+ \int_Q \frac{\beta s_{\varepsilon}}{s_{\varepsilon} + i_{\varepsilon} + r_{\varepsilon} + \varepsilon} i_{\varepsilon} v dx dt = \int_Q \chi(i_{\varepsilon}) \nabla s_{\varepsilon} \cdot \nabla v dx dt, \end{aligned}$$

$$\int_0^T \langle \partial_t r_{\varepsilon}(t), v(t) \rangle dt + \int_Q \nu_r \nabla r_{\varepsilon} \cdot \nabla v dx dt - \int_Q \phi i_{\varepsilon} v dx dt = 0,$$

for all $v \in L^2(0, T; V)$. We note that $\frac{\beta i_{\varepsilon}}{s_{\varepsilon} + i_{\varepsilon} + r_{\varepsilon} + \varepsilon}$ converges a.e. and weak* in $L^{\infty}(Q)$ to a function g which is bounded, staying between 0 and β_M , and is equal to $\frac{\beta i}{s+i+r}$ when $s+i+r > 0$. On the other hand, since $s_{\varepsilon} \rightarrow s$ a.e., it turns out that the limit product gs should satisfy (3.49): indeed, $s = 0$ when $s+i+r = 0$. The same conclusion holds also for the similar term in the second variational equality.

For the integral on the right-hand side of the second equation, we have that

$$\int_Q \chi(i_{\varepsilon}) \nabla s_{\varepsilon} \cdot \nabla v dx dt \rightarrow \int_Q \chi(i) \nabla s \cdot \nabla v dx dt \text{ as } \varepsilon \rightarrow 0. \quad (3.52)$$

Indeed, it holds that

$$\chi(i_\varepsilon)\nabla v \rightarrow \chi(i)\nabla v \text{ strongly in } (L^2(Q))^d,$$

by virtue of (3.4)-(3.5) and the Lebesgue dominated convergence theorem; then, on account of the weak convergence of ∇s_ε to ∇s in $(L^2(Q))^d$, due to (3.51), (3.52) follows.

As for the other integrals in the variational equalities there is no trouble for passing to the limit, we can conclude that (s, i, r) is a weak solution to (2.15)-(2.19).

In order to prove (3.50), we sum up the equations (2.15), (2.16), (2.17), then test the sum by 1. Using the boundary conditions and integrating from 0 to t , it is a standard matter to arrive at (3.50). This ends the proof. \square

Now, under the additional hypothesis that s_0 is bounded from below by a positive constant, i.e.,

$$\frac{1}{s_0} \in L^\infty(\Omega) \quad (3.53)$$

we prove the following result.

Proposition 3.3 *Let the assumptions of Theorem 3.2 hold. In addition, we assume (3.53). Then there exists a constant $s_m > 0$ such that*

$$s_m \leq s \leq s_M \text{ a.e. in } Q. \quad (3.54)$$

Proof. We would like to test equation (2.15) by the function $v = -ps^{-p-1}$, for $p \geq 1$, then deduce a uniform estimate for $\left\|\frac{1}{s}\right\|_p$ and then pass to the limit as $p \rightarrow \infty$. The chosen test function is not in V , so that it should be replaced by a smooth approximation, as we rigorously did before in the calculation of the estimate for the function $\hat{j}(i)$ in the proof of Theorem 3.1. However, here we shall skip the rigorous approach and perform only the formal computation. Hence, we test (2.15) by the function $v = -ps^{-p-1}$ and obtain

$$\frac{d}{dt} \int_\Omega \frac{1}{s^p} dx + \int_\Omega \nu_s p(p+1) s^{-p-2} |\nabla s|^2 dx = \int_\Omega \frac{\beta i}{s+i+r} p s^{-p} dx. \quad (3.55)$$

We observe that

$$\left| \frac{|\nabla s|}{s^{1+p/2}} \right|^2 = \left| \frac{2}{p} \nabla s^{-p/2} \right|^2$$

and integrate (3.55) from 0 to t . Using (3.1) and (3.2) we obtain

$$\begin{aligned} & \int_\Omega \left| \frac{1}{s(t)} \right|^p dx + 4\nu_m \frac{p+1}{p} \int_0^t \int_\Omega |\nabla s^{-p/2}|^2 dx d\tau \\ & \leq \int_\Omega \left| \frac{1}{s_0} \right|^p dx + \beta_M p \int_0^t \int_\Omega \left| \frac{1}{s(\tau)} \right|^p dx d\tau. \end{aligned}$$

By applying the Gronwall lemma we deduce that

$$\begin{aligned} & \int_\Omega \left| \frac{1}{s(t)} \right|^p dx + 4\nu_m \frac{p+1}{p} \int_0^t \int_\Omega |\nabla s^{-p/2}|^2 dx d\tau \\ & \leq e^{\beta_M p T} \int_\Omega \left| \frac{1}{s_0} \right|^p dx, \text{ for all } t \in [0, T], \end{aligned}$$

whence we have

$$\left\| \frac{1}{s(t)} \right\|_{L^p(\Omega)} \leq e^{\beta_M T} \left\| \frac{1}{s_0} \right\|_{L^p(\Omega)}, \quad \text{for all } t \in [0, T].$$

Then we pass to the limit as $p \rightarrow \infty$ and obtain that

$$\|1/s\|_{L^\infty(0,T;L^\infty(\Omega))} \leq e^{\beta_M T} \|1/s_0\|_{L^\infty(\Omega)},$$

whence

$$s \geq \frac{e^{-\beta_M T}}{\|1/s_0\|_{L^\infty(\Omega)}} =: s_m \quad \text{a.e. in } Q.$$

We point out that another consequence of this argument is

$$\|s^{-p/2}\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C, \quad \text{for all } p \in [1, +\infty), \quad (3.56)$$

with C independent of p . This concludes the proof. \square

The previous result implies that

$$0 < s_m \leq n = s + i + r \quad \text{a.e. in } Q. \quad (3.57)$$

Next, we are going to prove some additional regularity properties for s .

Proposition 3.4 *Under the same assumptions as in Theorem 3.2, we assume in addition (3.53) and*

$$\nu_s \text{ is constant, } \beta \in L^2(0, T; V), \quad s_0 \in W. \quad (3.58)$$

Then, the component s of the solution (s, i, r) found by Theorem 3.2 is also a strong solution of (2.15) and satisfies

$$\|s\|_{H^1(0,T;V) \cap L^\infty(0,T;W)} \leq C. \quad (3.59)$$

Proof. In preparation of the following calculations we set the notation

$$\psi(s, i, r) = \frac{si}{s + i + r} \quad (3.60)$$

and, on account of (3.48), claim that $0 \leq \psi(s, i, r) \leq s_M$ a.e. in Q .

We first test (2.15) by $\partial_t s$ and integrate with respect to time. Then, by the Young inequality we obtain

$$\begin{aligned} & \int_0^t \int_\Omega |\partial_t s|^2 dx d\tau + \frac{\nu_s}{2} \int_\Omega |\nabla s(t)|^2 dx \\ &= \frac{\nu_s}{2} \int_\Omega |\nabla s_0|^2 dx - \int_0^t \int_\Omega \beta \psi(s, i, r) (\partial_t s) dx d\tau \\ &\leq \frac{\nu_s}{2} \|s_0\|_V^2 + \frac{1}{2} \int_0^t \int_\Omega |\partial_t s|^2 dx d\tau + \frac{1}{2} \beta_M^2 s_M^2 T \text{meas}(\Omega), \end{aligned}$$

which obviously implies that $\|s\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C$. By a subsequent comparison of terms in (2.15) we infer that $\|\Delta s\|_{L^2(0,T;H)} \leq C$, whence by elliptic regularity we find out that

$$\|s\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C. \quad (3.61)$$

Now we are going to prove (3.59). We proceed formally by testing (2.15) by $-\Delta \partial_t s$ and integrating from 0 to t . We have

$$\begin{aligned} & \int_0^t \int_\Omega |\nabla \partial_t s|^2 dx d\tau + \frac{\nu_s}{2} \int_\Omega |\Delta s(t)|^2 dx \\ & \leq \frac{\nu_s}{2} \int_\Omega |\Delta s_0|^2 dx - \int_0^t \int_\Omega \psi(s, i, r) \nabla \beta \cdot \nabla (\partial_t s) dx d\tau \\ & \quad - \int_0^t \int_\Omega \beta \nabla \psi(s, i, r) \cdot \nabla (\partial_t s) dx d\tau. \end{aligned} \quad (3.62)$$

In the last two terms on the right-hand side we apply the Young inequality to absorb the resulting contribution $\frac{1}{2} \int_0^t \int_\Omega |\nabla \partial_t s|^2$ with the respective term on the left-hand side. Besides, we observe that

$$\int_0^t \int_\Omega |\psi(s, i, r)|^2 |\nabla \beta|^2 dx d\tau \leq C \quad (3.63)$$

since $|\psi(s, i, r)|^2 \leq s_M^2$ and $\beta \in L^2(0, T; V)$. Now, we calculate

$$\nabla \psi(s, i, r) = \frac{i(i+r)}{(s+i+r)^2} \nabla s + \frac{s(s+r)}{(s+i+r)^2} \nabla i - \frac{si}{(s+i+r)^2} \nabla r \quad (3.64)$$

and note that the absolute value of every fraction is smaller than 1, so that $\nabla \psi(s, i, r)$ is bounded in $(L^2(Q))^d$ from (3.43). In conclusion, by (3.62), (3.58) and (3.61) it is straightforward to deduce (3.59). \square

Thanks to the previous results we are now able to prove the uniqueness of the solution.

Theorem 3.5 *Let (3.1)-(3.6), (3.53), (3.58) hold and assume in addition that there exists a constant L_M such that*

$$|\chi(x, t, v_1) - \chi(x, t, v_2)| \leq L_M |v_1 - v_2| \quad \text{for all } v_1, v_2 \geq 0, \text{ a.e. } (x, t) \in Q. \quad (3.65)$$

Then, the problem (2.15)-(2.19) has a unique weak solution (s, i, r) .

Proof. Let (s_j, i_j, r_j) , $j = 1, 2$, be two weak solutions to (2.15)-(2.19) and, within this proof, set $s := s_1 - s_2$, $i := i_1 - i_2$, $r := r_1 - r_2$. We consider the difference of the equations (2.15), add the term $\nu_s s$ to both sides, then test by $\partial_t s$. At the same time, we take the difference of the equations (2.16), testing by i , and the difference of the

equations (2.17), testing by r . Then we add all the resultants and obtain

$$\begin{aligned}
& \int_{Q_t} |\partial_t s|^2 dx d\tau + \frac{\nu_s}{2} \|s(t)\|_V^2 + \frac{1}{2} \|i(t)\|_H^2 + \nu_m \int_{Q_t} |\nabla i|^2 dx d\tau \\
& + \int_{Q_t} \phi i^2 dx d\tau + \frac{1}{2} \|r(t)\|_H^2 + \nu_m \int_{Q_t} |\nabla r|^2 dx d\tau \\
\leq & \int_{Q_t} \nu_s s \partial_t s dx d\tau + \int_{Q_t} \beta(\psi(s_1, i_1, r_1) - \psi(s_2, i_2, r_2))(i - \partial_t s) dx d\tau \\
& + \int_{Q_t} (\chi(i_1) \nabla s_1 - \chi(i_2) \nabla s_2) \cdot \nabla i dx d\tau + \int_{Q_t} \phi i r dx d\tau, \tag{3.66}
\end{aligned}$$

where the notation ψ from the previous proof has been used. Now we treat some terms. By a direct verification we see that there is a positive constant C_ψ such that

$$|\psi(s_1, i_1, r_1) - \psi(s_2, i_2, r_2)| \leq C_\psi(|s| + |i| + |r|)$$

and by repeated use of the Young inequality we deduce that

$$\begin{aligned}
& \left| \int_{Q_t} \nu_s s (\partial_t s) dx d\tau + \int_{Q_t} \phi i r dx d\tau \right. \\
& \quad \left. + \int_{Q_t} \beta(\psi(s_1, i_1, r_1) - \psi(s_2, i_2, r_2))(i - \partial_t s) dx d\tau \right| \\
\leq & \frac{1}{2} \int_{Q_t} |\partial_t s|^2 dx d\tau + C \int_{Q_t} (s^2 + i^2 + r^2) dx d\tau.
\end{aligned}$$

Then, by (3.65) we infer that

$$\begin{aligned}
& \int_{Q_t} (\chi(i_1) \nabla s_1 - \chi(i_2) \nabla s_2) \cdot \nabla i dx d\tau \\
= & \int_{Q_t} (\chi(i_1) - \chi(i_2)) \nabla s_1 \cdot \nabla i dx d\tau + \int_{Q_t} \chi(i_2) \nabla s \cdot \nabla i dx d\tau \\
\leq & L_M \int_0^t \|i(\tau)\|_{L^4(\Omega)} \|\nabla s_1(\tau)\|_{L^4(\Omega)} \|\nabla i(\tau)\|_H d\tau \\
& + \chi_M \int_0^t \|\nabla s(\tau)\|_H \|\nabla i(\tau)\|_H d\tau.
\end{aligned}$$

In the first term we use the bound in (3.59) (note that $W \subset W^{1,4}(\Omega)$ continuously) and the compactness of the embedding $V \subset L^4(\Omega)$, which, along with the Lions lemma (see, e.g., [27, p. 59], give

$$\begin{aligned}
& L_M \int_0^t \|i(\tau)\|_{L^4(\Omega)} \|\nabla s_1(\tau)\|_{L^4(\Omega)} \|\nabla i(\tau)\|_H d\tau \\
\leq & \frac{\nu_m}{2} \int_{Q_t} |\nabla i|^2 dx d\tau + C \int_0^t \|s_1\|_{L^\infty(0,T;W)}^2 \|i(\tau)\|_{L^4(\Omega)}^2 d\tau \\
\leq & \frac{\nu_m}{2} \int_{Q_t} |\nabla i|^2 dx d\tau + \delta \int_0^t \|i(\tau)\|_V^2 d\tau + C_\delta \int_0^t \|i(\tau)\|_H^2 d\tau
\end{aligned}$$

for all $\delta > 0$ with a related constant C_δ . Letting $\delta < \nu_m/2$, we can collect everything and apply the Gronwall lemma to show that the left-hand side of (3.66) is equal to zero for all $t \in [0, T]$, whence $s = i = r = 0$ and the theorem is completely proved. \square

4 Analysis of a reduced system

In practice, let us point out that it makes sense to assume ν_s and ν_r small, and in particular $\nu_s, \nu_r \ll \nu_i$, for a few reasons. First, the s and r populations do not spread disease. Second, introducing diffusion in these compartments introduces an irreversible process in which the populations move from high-to-low concentration areas, tending to equilibrium over time. This is not realistic as, generally speaking, people's mobility is transient, and they will return to an initial starting location. Consequently, over the time-scales relevant in the current work (days/weeks), we do not expect the spatial distribution of the susceptible or recovered populations to vary significantly. Accordingly, it is often appropriate to directly set $\nu_s = \nu_r = 0$, see [32]. In view of these considerations, the asymptotic behavior of our system (2.15)-(2.19), as the two coefficients ν_s and ν_r tend to 0, turns out to be worth of investigation.

From now on, ν_s, ν_r are assumed to be constant. Our intention is showing that as $\nu_s, \nu_r \rightarrow 0$ the solution to (2.15)-(2.19) converges in some topology to a solution of the following problem

$$\partial_t s + \frac{\beta i}{s + i + r} s = 0, \text{ in } Q, \quad (4.1)$$

$$\partial_t i - \nabla \cdot (\nu_i \nabla i - \chi(i) \nabla s) + \phi i - \frac{\beta s}{s + i + r} i = 0, \text{ in } Q, \quad (4.2)$$

$$\partial_t r - \phi i = 0, \text{ in } Q, \quad (4.3)$$

$$(\nu_i \nabla i - \chi(i) \nabla s) \cdot \mathbf{n} = 0, \text{ on } \Sigma, \quad (4.4)$$

$$s(0) = s_0, i(0) = i_0, r(0) = r_0, \text{ in } \Omega. \quad (4.5)$$

Namely, we are going to prove the following convergence result.

Theorem 4.1 *Let (3.1)-(3.6), (3.53), and*

$$\beta \in L^2(0, T; V), \quad s_0 \in V, \quad r_0 \in V \quad (4.6)$$

hold, where ν_s and ν_r in (3.1) are now replaced by two arbitrary sequences $\nu_{s,n}$ and $\nu_{r,n}$ of positive numbers monotonically decreasing and converging to 0 as $n \rightarrow \infty$. Denote by (s_n, i_n, r_n) a weak solution in $L^2(0, T; V')$ to

$$\partial_t s_n - \nabla \cdot (\nu_{s,n} \nabla s_n) + \frac{\beta i_n}{s_n + i_n + r_n} s_n = 0, \text{ in } Q, \quad (4.7)$$

$$\partial_t i_n - \nabla \cdot (\nu_i \nabla i_n - \chi(i_n) \nabla s_n) + \phi i_n - \frac{\beta s_n}{s_n + i_n + r_n} i_n = 0, \text{ in } Q, \quad (4.8)$$

$$\partial_t r_n - \nabla \cdot (\nu_{r,n} \nabla r_n) - \phi i_n = 0, \text{ in } Q, \quad (4.9)$$

$$\nabla s_n \cdot \mathbf{n} = (\nu_i \nabla i_n - \chi(i_n) \nabla s_n) \cdot \mathbf{n} = \nabla r_n \cdot \mathbf{n} = 0, \text{ on } \Sigma, \quad (4.10)$$

$$s_n(0) = s_0, i_n(0) = i_0, r_n(0) = r_0, \text{ in } \Omega, \quad (4.11)$$

whose existence and the regularity properties

$$s_m \leq s_n \leq s_M, i_n \geq 0, r_n \geq 0 \text{ a.e. in } Q, \quad (4.12)$$

are guaranteed by Theorem 3.2 and Proposition 3.3. Then, the estimate

$$\begin{aligned} & \|s_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|i_n\|_{H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)} \\ & + \|r_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq C \end{aligned} \quad (4.13)$$

holds for some constant C independent of $\nu_{s,n}, \nu_{r,n}$. Moreover, there exist a triplet (s, i, r) and a subsequence, still denoted by n , such that

$$\begin{aligned} & s_n \rightarrow s, r_n \rightarrow r \\ & \text{weakly in } H^1(0, T; H), \text{ weak}^* \text{ in } L^\infty(0, T; V), \\ & \text{strongly in } C([0, T]; H) \text{ and a.e. in } Q, \end{aligned} \quad (4.14)$$

$$\begin{aligned} & \nu_{s,n} s_n \rightarrow 0, \nu_{r,n} r_n \rightarrow 0 \\ & \text{strongly in } L^2(0, T; W) \text{ and a.e. in } Q, \end{aligned} \quad (4.15)$$

$$\begin{aligned} & i_n \rightarrow i \\ & \text{weakly in } H^1(0, T; V') \cap L^2(0, T; V), \text{ weak}^* \text{ in } L^\infty(0, T; H), \\ & \text{strongly in } L^2(0, T; H) \text{ and a.e. in } Q, \end{aligned} \quad (4.16)$$

as $n \rightarrow \infty$, with the limit triplet (s, i, r) being a weak solution to (4.1)-(4.5), with (4.2) to be understood in $L^2(0, T; V')$.

Proof. From Theorem 3.2 it is clear that

$$(s_n, i_n, r_n) \in (H^1(0, T; V') \cap C([0, T]; H) \cap L^2(0, T; V))^3$$

for all $n \in \mathbb{N}$. In addition, as the initial data s_0 and r_0 are in V (see (4.6)) and the terms in the equations (4.7) and (4.9),

$$\frac{\beta i_n}{s_n + i_n + r_n} s_n \text{ and } -\phi i_n \text{ are (at least) in } L^2(0, T; H),$$

from the parabolic regularity theory (cf., e.g., [26]) it turns out that both

$$s_n, r_n \text{ lie in } H^1(0, T; H) \cap C([0, T]; V) \cap L^2(0, T; W)$$

and strongly solve (4.7), (4.9). Then, we are allowed to test (4.7) by $s_n - \Delta s_n$ and (4.9) by $r - \Delta r_n$, integrate by parts using the boundary conditions (4.10) and integrate from

0 to t . Next, we add the resultants obtaining

$$\begin{aligned}
& \frac{1}{2} \|s_n(t)\|_V^2 + \nu_{s,n} \int_{Q_t} (|\nabla s_n|^2 + |\Delta s_n|^2) + \int_{Q_t} \frac{\beta i_n}{s_n + i_n + r_n} |s_n|^2 \\
& + \frac{1}{2} \|r_n(t)\|_V^2 + \nu_{r,n} \int_{Q_t} (|\nabla r_n|^2 + |\Delta r_n|^2) \\
\leq & \frac{1}{2} \|s_0\|_V^2 + \frac{1}{2} \|r_0\|_V^2 - \int_{Q_t} \psi(s_n, i_n, r_n) \nabla \beta \cdot \nabla s_n dx d\tau \\
& - \int_{Q_t} \beta \nabla \psi(s_n, i_n, r_n) \cdot \nabla s_n dx d\tau + \int_{Q_t} \phi(i_n r_n + \nabla i_n \cdot \nabla r_n) dx d\tau, \quad (4.17)
\end{aligned}$$

for all $t \in [0, T]$, where we have used the notation ψ from (3.60). Note that all terms on the left-hand side are nonnegative and, in view of (4.6) and (3.63), (3.64) that can be repeated here, we easily infer that

$$\begin{aligned}
& - \int_{Q_t} \psi(s_n, i_n, r_n) \nabla \beta \cdot \nabla s_n dx d\tau - \int_{Q_t} \beta \nabla \psi(s_n, i_n, r_n) \cdot \nabla s_n dx d\tau \\
\leq & \|\nabla \beta\|_{L^2(0,T;H)}^2 + C \int_0^t \|\nabla s_n(\tau)\|_H^2 d\tau \\
& + \beta_M \int_0^t (\|\nabla i_n(\tau)\|_H + \|\nabla r_n(\tau)\|_H) \|\nabla s_n(\tau)\|_H d\tau.
\end{aligned}$$

On the other hand, we have that

$$\int_{Q_t} \phi(i_n r_n + \nabla i_n \cdot \nabla r_n) dx d\tau \leq 2\phi \int_0^t \|i_n(\tau)\|_V \|r_n(\tau)\|_V d\tau.$$

At this point, we control the right-hand side of (4.17) by the last inequalities and consider the estimate (3.32), which can be repeated here for i_n without any modification. We add the two resulting inequalities and obtain

$$\begin{aligned}
& \frac{1}{2} \|s_n(t)\|_V^2 + \nu_{s,n} \int_{Q_t} (|\nabla s_n|^2 + |\Delta s_n|^2) + \frac{1}{2} \|r_n(t)\|_V^2 \\
& + \nu_{r,n} \int_{Q_t} (|\nabla r_n|^2 + |\Delta r_n|^2) + \frac{1}{2} \|i_n(t)\|_H^2 + \nu_m \int_0^t \|\nabla i_n(\tau)\|_H^2 d\tau \\
\leq & C + C \int_0^t \|\nabla s_n(\tau)\|_H^2 d\tau + \beta_M \int_0^t (\|\nabla i_n(\tau)\|_H + \|\nabla r_n(\tau)\|_H) \|\nabla s_n(\tau)\|_H d\tau \\
& + 2\phi \int_0^t \|i_n(\tau)\|_V \|r_n(\tau)\|_V d\tau + \beta_M \int_0^t \|i_n(\tau)\|_H^2 d\tau \\
& + \chi_M \int_0^t \|\nabla s_n(\tau)\|_H \|\nabla i_n(\tau)\|_H d\tau, \quad (4.18)
\end{aligned}$$

where we warn that now the positive constant ν_m (see (3.1)) is a bound from below only for the diffusion coefficient ν_i . Next, we use the Young inequality for the terms on

the right-hand side of (4.18), taking care in particular of

$$\begin{aligned}
& \beta_M \int_0^t \|\nabla i_n(\tau)\|_H \|\nabla s_n(\tau)\|_H d\tau + 2\phi \int_0^t \|i_n(\tau)\|_V \|r_n(\tau)\|_V d\tau \\
& + \chi_M \int_0^t \|\nabla s_n(\tau)\|_H \|\nabla i_n(\tau)\|_H d\tau \\
& \leq \frac{\nu_m}{2} \int_0^t \|\nabla i_n(\tau)\|_H^2 d\tau + C \int_0^t (\|i_n(\tau)\|_H^2 + \|s_n(\tau)\|_V^2 + \|r_n(t)\|_V^2) d\tau.
\end{aligned}$$

After that, we apply the Gronwall lemma and, due to the elliptic regularity theory as well, we find out that

$$\begin{aligned}
& \|s_n\|_{L^\infty(0,T;V)} + \sqrt{\nu_{s,n}} \|s_n\|_{L^2(0,T;W)} + \|i_n\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \\
& + \|r_n\|_{L^\infty(0,T;V)} + \sqrt{\nu_{r,n}} \|r_n\|_{L^2(0,T;W)} \leq C,
\end{aligned} \tag{4.19}$$

for some uniform constant C . Now, on account of (4.19) and by comparison of terms in equations (4.7), (4.8), (4.9) we deduce that

$$\|\partial_t s_n\|_{L^2(0,T;H)} + \|\partial_t i_n\|_{L^2(0,T;V')} + \|\partial_t r_n\|_{L^2(0,T;H)} \leq C,$$

which completes the proof of (4.13). Having shown these estimates, all the properties previously derived in the proofs of Theorem 3.2 and Proposition 3.3 continue to hold, in particular for every $n \in \mathbb{N}$ the function i_n is non negative and s_n satisfies (3.25) with s_m , s_M fixed and independent of $\nu_{s,n}$, $\nu_{r,n}$.

Now, thanks to (4.13) and (4.19) we can conclude that there are a subsequence of n , still denoted by n , and a triplet (s, i, r) such that, by weak and weak* compactness, along with the Ascoli theorem, the convergences (4.14)-(4.16) hold. Moreover, by passing to the limit in (4.7)-(4.11), as

$$\nabla \cdot (\nu_{s,n} \nabla s_n) \rightarrow 0, \quad \nabla \cdot (\nu_{r,n} \nabla r_n) \rightarrow 0 \quad \text{strongly in } L^2(Q) \text{ as } n \rightarrow \infty,$$

and applying the same arguments as in the previous limit procedures we easily show that (s, i, r) is a weak solution to (4.1)-(4.5) and conclude the proof. \square

5 Numerical simulations

In this section, we perform a suite of ten numerical simulations of the reduced model (4.1)-(4.5) in the Italian region of Lombardy. Across the ten cases, we vary the values of the chemotaxis parameter μ_i , diffusion coefficient ν_i , and capacity term C_0 as detailed in Table 2. We are primarily interested in the following basic questions:

1. How do changes in μ_i , ν_i and C_0 affect the model behavior, both quantitatively and qualitatively?
2. Does incorporating a chemotaxis term allow for a more realistic description of airborne infectious disease transmission in human populations? Specifically, is the model more able to account for

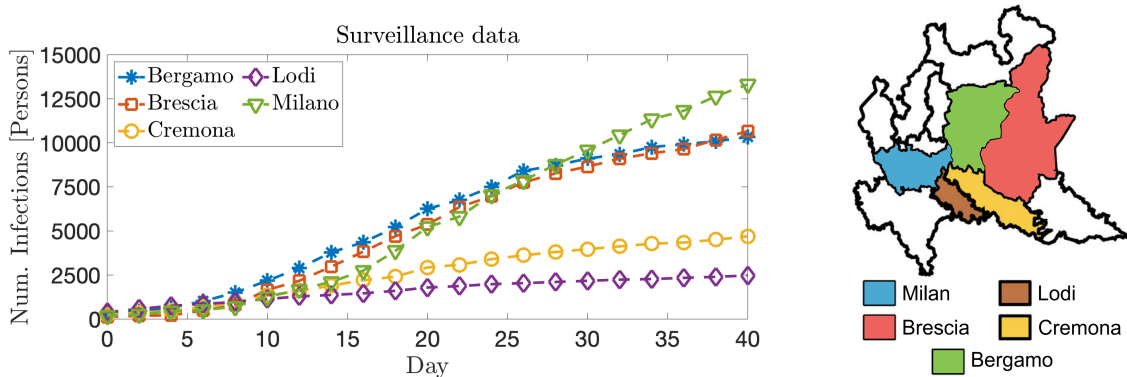


Figure 1: Left: observed cumulative infections across five relevant provinces in Lombardy (data and figure retrieved from [22]). Right: geographic locations of each province.

- (a) Transmission over large geographic distances?
- (b) The effects of population density and density patterns on transmission?
- (c) The relatively rapid appearance of the disease and subsequent sustained transmission in major urban centers, even in the absence of an infected population at initialization and/or time space-varying parameters?

Note that existing PDE models of infectious disease, including those studied in [7, 19, 20, 34], have difficulty in reproducing some or all of these behaviors.

To evaluate the model, we compare the cumulative case counts in the early stages of COVID-19 pandemic with surveillance data. Despite the use of COVID-19 for the example, the introduced model should not be regarded as a model of COVID-19, and the subsequent simulations should not be interpreted as attempts to simulate COVID-19 in a rigorous manner. The choice of this example is based primarily on practical considerations, as the severity, scope, and recency of the COVID-19 pandemic have led to a large amount of publicly available, reasonably high-quality spatiotemporal data. However, due to well-known difficulties in interpreting such data quantitatively ([3, 4]), we restrict our comparisons to a qualitative analysis of cumulative incidence in the provinces of Lodi, Cremona, Bergamo, Brescia, and Milano. A geographic depiction of the relevant locations and the corresponding surveillance data (courtesy of [22]) is provided in Figure 1.

The outbreak initially begins in the provinces of Lodi and Cremona (particularly concentrated in the town of Codogno, province of Lodi). However, within the first several weeks, the outbreak becomes far more concentrated in the more populous northern provinces of Bergamo, Brescia and Milano. In our numerical simulations, we hope to observe a similar spatiotemporal dynamics consistent with reality; that is, the initial outbreak in Lodi and Cremona is quickly overshadowed by greater contagion in Bergamo, Brescia, and Milano. Further, ideally we observe these dynamics without changes to the problem parameterization. While updating parameters in order to more precisely account for e.g. lockdowns and other restrictions may be helpful for a detailed model of COVID-19, in the present work we are more interested in simply observing if

Sim.	μ_i	ν_i	C_0	β	ϕ
1	$0.0 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$1.0 \frac{\text{km}^2}{\text{Days}}$	0 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
2	$0.0 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$2.5 \frac{\text{km}^2}{\text{Days}}$	0 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
3	$0.01 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$1.0 \frac{\text{km}^2}{\text{Days}}$	50 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
4	$0.01 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$1.0 \frac{\text{km}^2}{\text{Days}}$	200 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
5	$0.01 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$2.5 \frac{\text{km}^2}{\text{Days}}$	50 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
6	$0.01 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$2.5 \frac{\text{km}^2}{\text{Days}}$	200 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
7	$0.02 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$1.0 \frac{\text{km}^2}{\text{Days}}$	50 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
8	$0.02 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$1.0 \frac{\text{km}^2}{\text{Days}}$	200 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
9	$0.02 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$2.5 \frac{\text{km}^2}{\text{Days}}$	50 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$
10	$0.02 \frac{\text{km}^2 \cdot \text{Persons}}{\text{Days}}$	$2.5 \frac{\text{km}^2}{\text{Days}}$	200 Persons	0.175 Days^{-1}	$1/18 \text{ Days}^{-1}$

Table 2: Parameter values for each of the ten numerical simulations for Lombardy.

the chemotaxis model can capture the more general spatio-temporal features observed in reality. Hence, we believe that these core behaviors 2a-2c listed above should be observed, even for a simple, time-constant parameterization.

We reconstructed the geometry and defined the initial susceptible and infected compartments following [20]. Regarding the numerical solution, we employed a triangular mesh consisting of 53,506 triangles, giving an average spatial resolution of .44 kilometers. The solution was advanced in time using a BDF2 scheme, using Heun’s method at the first time-step to ensure second-order temporal accuracy throughout the simulated time period. We do not discuss the numerical solution of the problem further, however, we note that several relevant considerations regarding the numerical solution for problems of this type can be found in [20,39], and apply equally to the current problem.

The results of the simulations are presented in Fig. 2 (for $\nu_i=1.0 \text{ km}^2 \cdot \text{Days}^{-1}$) and Fig. 3 (for $\nu_i=2.5 \text{ km}^2 \cdot \text{Days}^{-1}$). For both values of ν_i , we see many of the same general trends. In the absence of chemotaxis ($\mu_i = 0$), the numerical simulations are not qualitatively consistent with the surveillance data. In particular, the outbreak remains most severe in the province of Lodi, and growth is similar in the Milan, Bergamo, and Cremona regions. Once chemotaxis is introduced, the simulations begin to show a better qualitative agreement with the surveillance data. Around day 15, the rate of growth in the Bergamo, Brescia, and Milan regions begins to significantly outpace Cremona and Lodi. Doubling μ_i from .01 to .02 further exacerbates these effects, and results in even faster growth Bergamo, Brescia, and particularly Milan. While increasing μ_i

does appear to lead to increased transmission in Cremona, the extent of this increase is substantially smaller than in Bergamo, Brescia and Milan. Interestingly, increasing μ_i appears to *reduce* transmission in Lodi.

In comparison to the effect of chemotaxis, the effect of the diffusion parameter ν_i is relatively small. Nonetheless, we do observe several important differences between the two simulated values. In the absence of chemotaxis, increased diffusion appears to reduce transmission in each province, with the exception of Milan, where transmission increases slightly. This makes sense, as Milan is significantly further away from the initially infected areas; hence, increasing the diffusion leads to the infection reaching Milan more quickly. However, in the other regions, diffusion primarily serves to reduce the local infection concentration, and hence overall transmission. However, when chemotaxis is present, the effect of diffusion becomes more complex. Notably, increased diffusion leads to *increased* transmission in Bergamo, Brescia, and Milan; however, it still results in decreased transmission in Lodi and Cremona.

Finally, the capacity parameter C_0 did not appear to greatly affect transmission for any value of μ_i or ν_i . This may simply be due to the small range of values considered in the present work, however. A more thorough investigation of this parameter, over a larger range of values, may be necessary to make more definitive conclusions about its importance.

Qualitatively, we also depict the infected compartment over the entire region for days 1, 5, 10, and 15 for $\nu_i = 1.0$ and $\mu_i = 0.00, 0.01, 0.02$ in Figs. 4-6, respectively. When looking at the purely-diffusive model (Fig. 4), we see that the infection concentration remains primarily in the initially affected regions and diffuses in a regular, local, pattern, independent of the local population density. In contrast, the chemotaxis model (Figs. 5, 6) shows a significantly more complex spatiotemporal evolution. In particular, by day 5, we already see the appearance of nonlocalized dynamics, with the appearance of isolated hotspots in Milan. These continue to grow, and by day 15 have become a significant source of new transmission. Furthermore, additional pockets of transmission, appear northwest of Bergamo and Brescia (occurring in significant population centers), by day 15. In contrast, the purely diffusive model is unable to account for such dynamics, whose existence is well-supported by the available surveillance data as well as other sources (see e.g. [36]).

Overall, our considered simulations suggest that a chemotaxis model may provide a superior description of airborne infectious disease transmission in human populations, as compared to a purely diffusive model. Our numerical simulations demonstrate that the chemotaxis model provides results more consistent with the spatiotemporal surveillance data observed in airborne epidemics. In particular, the chemotaxis model is able to better recreate the propagation of an airborne disease over large geographic distances to reach major population centers. Furthermore, these behaviors all occur the models' behavior, and will occur even when model parameters are not adjusted in time and/or space.

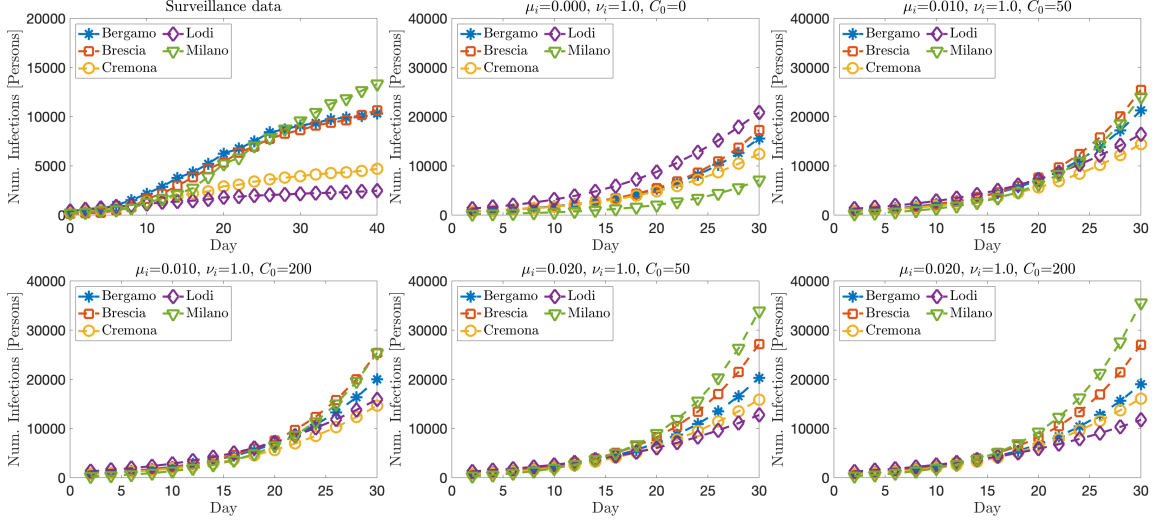


Figure 2: Cumulative simulated incidence in the provinces of Bergamo, Brescia, Lodi, Milano, and Cremona over 30 days, for $\nu_i = 1.0 \text{ km}^2 \cdot \text{Days}^{-1}$ and varying values of μ_i and C_0 .

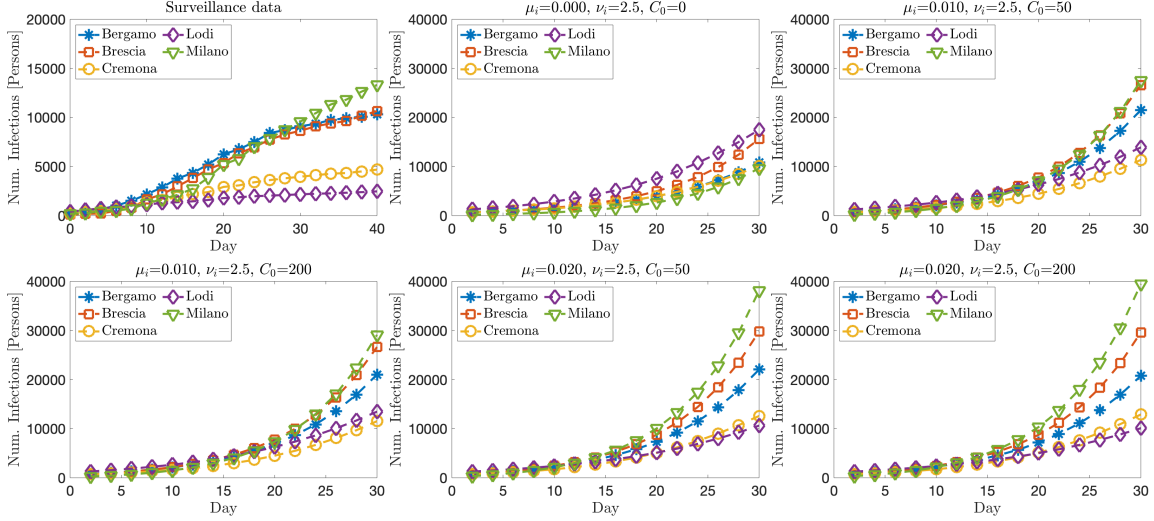


Figure 3: Cumulative simulated incidence in the provinces of Bergamo, Brescia, Lodi, Milano, and Cremona over 30 days, for $\nu_i = 2.5 \text{ km}^2 \cdot \text{Days}^{-1}$ and varying values of μ_i and C_0 .

6 Conclusions

In the present work, we have introduced a novel PDE model for the airborne transmission of infectious disease in human populations. The novelty of the model is in the addition of a nonlinear, chemotaxis-like term to a standard reaction-diffusion model. While the standard reaction-diffusion models state that the infection should propagate from areas of high-to-low concentration of infection, the chemotaxis term further postulates that additional transmission should occur along the inverse gradient of the

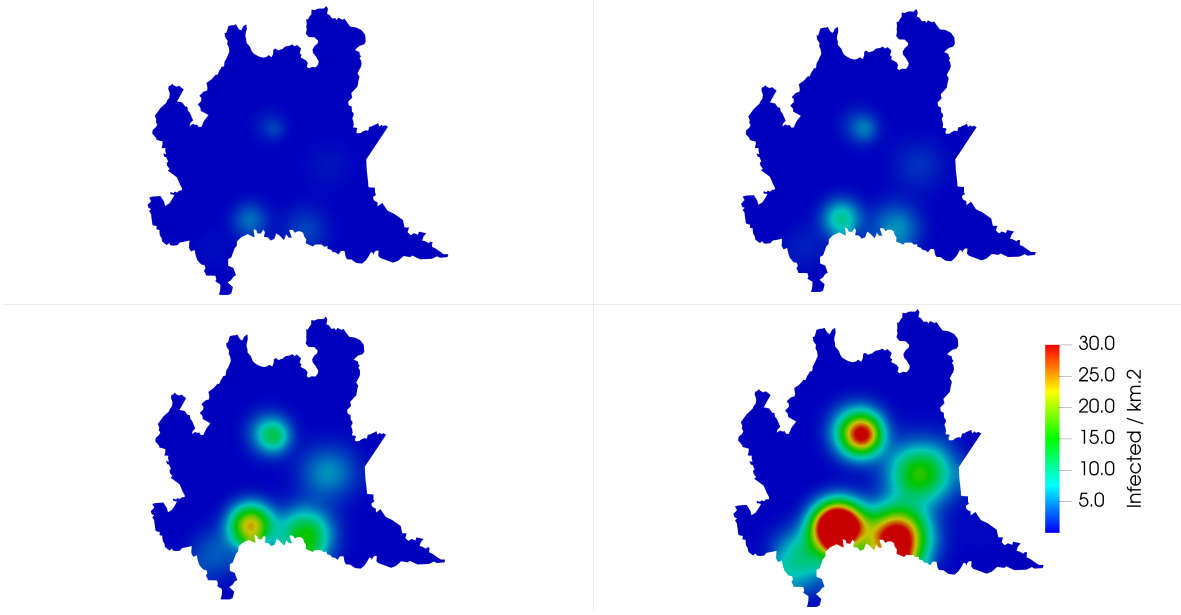


Figure 4: Left-to-right, top-to-bottom: The infected compartment on days 1, 5, 10, and 15 for $\nu_i = 1.0$, $\mu_i = 0$

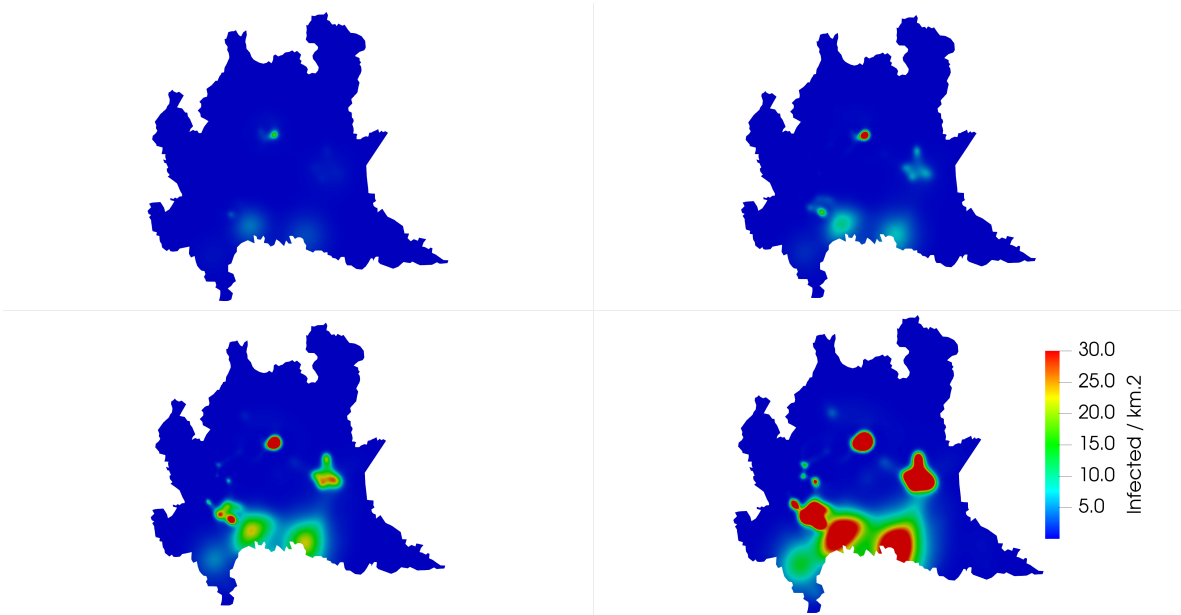


Figure 5: Left-to-right, top-to-bottom: The infected compartment on days 1, 5, 10, and 15 for $\nu_i = 1.0$, $\mu_i = 0.01$

susceptible population; that is, from areas of low-to-high concentration of susceptible individuals. We provided an intuitive explanation motivating the reasoning behind model and literature supporting this idea.

We then proceeded demonstrating the mathematical well-posedness of the model, and show that the model provides unique solutions under reasonable assumptions, for

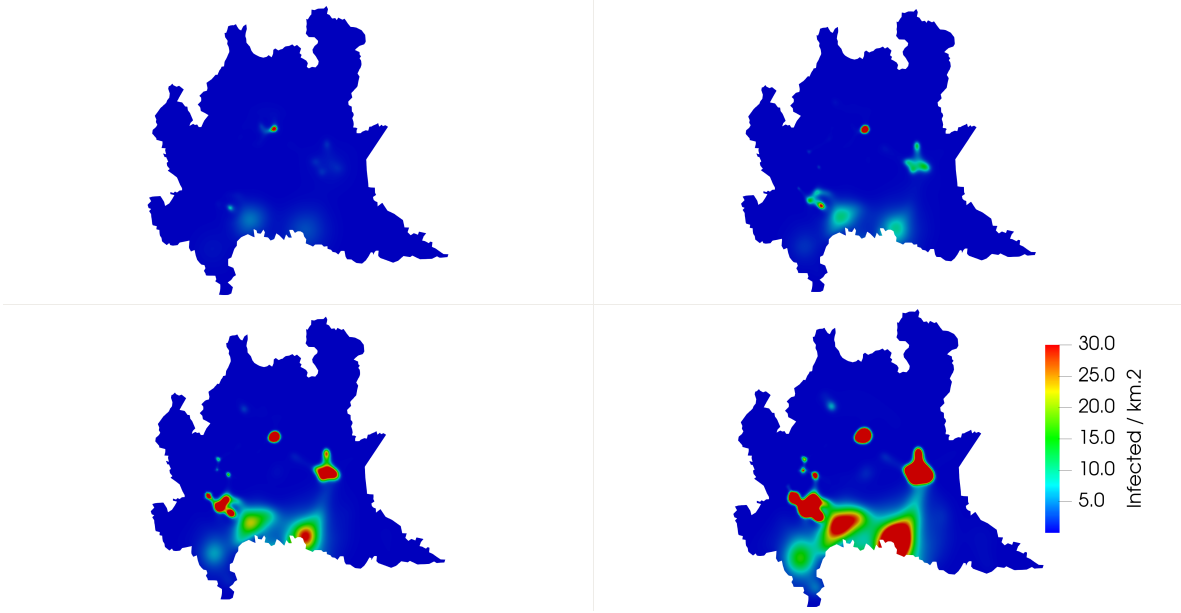


Figure 6: Left-to-right, top-to-bottom: The infected compartment on days 1, 5, 10, and 15 for $\nu_i = 1.0$, $\mu_i = 0.025$

sufficiently smooth initial conditions. Furthermore, we also proved that solutions still exist for epidemiologically relevant parameters, including the scenario in which diffusion and chemotaxis are only considered in the infected population, and the susceptible and removed compartments are purely reactive.

This particular scenario was then used as the basis for a numerical simulation study, in which we simulated the propagation of an infectious disease in the Italian region of Lombardy for varying parameter values. Qualitatively, we found that the chemotaxis model was able to account for important dynamics observed in the surveillance data; however, the purely-diffusive model was unable to account for these dynamics. The results of these simulations suggest that the chemotaxis model may be more well-suited for modeling airborne infectious disease in human populations, as compared to a diffusion model.

This work can be extended in several ways. In order to simulate a particular infectious disease at a quantitative level, rather than just evaluating basic qualitative agreement, likely requires a more sophisticated compartmental structure, able to account for hospitalizations, asymptomatic individuals, population demographic characteristics, and other important considerations. Nonetheless, we do not expect that these extensions will affect the mathematical results shown herein, which should easily generalize to more complex compartmental structures. Furthermore, applying the chemotaxis model to a wider range of diseases and geographic areas is necessary to provide a further examination of the models' behavior. From the numerical point of view, a more detailed analysis of numerical methods, including splitting and stabilization schemes, may be of importance when extending the model to larger-scale problems. Finally a mathematical analysis based on some epidemiological considerations, including the derivation of a basic reproduction number, is also a direction for future work.

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