# Reduction of the effective population size in a branching particle system in the moderate mutation-selection regime 

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#### Abstract

We consider a system of particles performing a one-dimensional dyadic branching Brownian motion with positive drift $\beta \in(0,1)$, branching rate $1 / 2$, killed at $L(\beta)>0$, and reflected at 0 . The killing boundary $L(\beta)$ is chosen so that the total population size is approximately constant, proportional to $N \in \mathbb{N}$. This branching system is interpreted as a population accumulating deleterious mutations.

We prove that, when the typical width of the cloud of particles is of order $c \log (N), c \in(0,1)$, the demographic fluctuations of the system converge to a Feller diffusion on the time scale $N^{1-c}$. In addition, we show that the limiting genealogy of the system comprises only binary mergers and that these mergers are concentrated in the vicinity of the reflective boundary. This model is a version of the branching Brownian motion with absorption studied by Berestycki, Berestycki and Schweinsberg to describe the effect of natural selection on the genealogy of a population accumulating beneficial mutations. In the latter case, the genealogical structure of the system is described by a Bolthausen-Sznitman coalescent on a logarithmic time scale. In this work, we show that, when the population size in the fittest class is mesoscopic, namely of order $N^{1-c}$, the genealogy of the system is given by a Kingman coalescent on a polynomial time scale.

Keywords. Branching Brownian Motion, spinal decomposition, selection, mutation, method of moments.


## Contents

1 Introduction ..... 2
2 Outline of the proof ..... 8
3 Heat kernel estimates ..... 13
4 Many-to-few formula and convergence of moments ..... 19
5 Survival probability (0-th moment) ..... 28
6 Convergence of genealogies ..... 33
7 Convergence of the demographic fluctuations ..... 34

## 1 Introduction

The evolutionary dynamics of an asexual population accumulating beneficial mutations is relatively well understood [25, 2, 33, 31. The genealogy is given by a multiple mergers coalescent, describing the rapid spread and fixation of beneficial mutations due to natural selection, which results in a reduction of genetic diversity.

However, when mutations are deleterious, i.e. when they decrease individuals' fitness, the population undergoes two opposite forces, natural selection and mutational pressure. The resulting evolutionary dynamic turns out to be complex and a complete picture of the underlying mechanisms is still lacking.

### 1.1 The Wright-Fisher model with (deleterious) mutations and selection in the weak-selection regime

The Wright-Fisher model with mutations and selection is a classical model for the evolutionary dynamics of an asexual population undergoing selection. In this model, the population size is constant, equal to $N \in \mathbb{N}$ and the system evolves in discrete time. Its dynamics depend on two parameters: the mutation rate $\mu_{N}>0$ and the strength of the selection $s_{N} \in \mathbb{R}$. At each step, the population is renewed. First, each of the $N$ new individuals independently chooses a parent from the previous generation. The probability that a parent carrying $k$ mutations is selected is proportional to $\left(1+s_{N}\right)^{k}$. Each newborn inherits the mutations of its parents. Second, the $i$-th individual acquires $X_{i}$ new mutations, where $\left(X_{i}\right)_{1 \leq i \leq N}$ is a sequence of i.i.d. Poisson $\left(\mu_{N}\right)$ random variables.

The mutations are said to be deleterious (resp. beneficial) when $s_{N}<0$ (resp. $s_{N}>0$ ). When the mutations are deleterious, the type with the smallest number of mutations is referred to as the best class. Because the mutations are unidirectional, the current best class eventually disappears: we say that the ratchet clicks [24]. This phenomenon is known as Muller's ratchet. A number of authors studied versions of this model to estimate the rate of the ratchet and the underlying mechanisms of the evolution 14, 21, 8, 27.

Under the assumption $1 \ll N \mu_{N} \ll N$, Etheridge, Pfaffelhuber and Wakolbinger 14 observe a phase transition in the Wright-Fisher dynamics. Their analysis, which is based on a diffusion approximation of the size of the best class, allows them to pinpoint the key quantity

$$
\gamma:=\frac{\mu_{N}}{s_{N} \log \left(N \mu_{N}\right)},
$$

and to state the following rule of thumb. For $\gamma>1 / 2$, the rate of the ratchet is of the order $\left(N \mu_{N}\right)^{\gamma} / N$, whereas for $\gamma<1 / 2$, it is exponentially slow in $\left(N \mu_{N}\right)^{1-\gamma}$. When the mutation rate $\mu_{N}$ is of the form $N^{-b}$, for some $b \in(0,1)$, the quantity $\gamma$ is constant if and only if there exists a constant $c>0$ such that

$$
\begin{equation*}
\frac{\mu_{N}}{s_{N}}=c \log (N) \tag{1}
\end{equation*}
$$

The assumption (1) is referred to as the moderate mutation-selection regime 21].
On the other hand, it is known [18 that between two clicks the distribution of types is well approximated by a Poisson distribution with parameter $\mu_{N} / s_{N}$. Under the assumption (1), we observe that the bulk of this distribution is concentrated around $c \log (N)$ and that the best class contains $N e^{-\frac{\mu_{N}}{s_{N}}}=N^{1-c}$ individuals.

In the present work, we propose a branching particle system as a model for a population accumulating deleterious mutations. The model will be set up in such a way that the typical width of the type distribution is given by $c \log (N)$ for some $c \in(0,1)$. We conjecture that this model captures the underlying evolutionary mechanisms leading to Muller's ratchet.

### 1.2 The model

Let $\beta \in(0,1)$ and define

$$
\begin{equation*}
L_{\beta}:=\frac{1}{\sqrt{1-\beta^{2}}}\left(\arctan \left(-\frac{\sqrt{1-\beta^{2}}}{\beta}\right)+\pi\right) . \tag{2}
\end{equation*}
$$

In this paper, we consider a dyadic branching diffusion on $\left[0, L_{\beta}\right]$. We assume that;
(i) The generator $\mathcal{G}$ of a single particle is given by the differential operator

$$
\begin{aligned}
& \mathcal{G} f(x)=\frac{1}{2} \partial_{x x} f(x)+\beta \partial_{x} f(x) \\
& f^{\prime}(0+)=0, \quad f^{\prime}\left(L_{\beta}-\right)=0 .
\end{aligned}
$$

In words, each particle performs a one-dimensional Brownian motion with drift $\beta$, is killed upon reaching $L_{\beta}$ and reflected at 0 . The particles are assumed to move independently.
(ii) Each particle branches into two particles at rate $1 / 2$.

Let $\mathcal{N}_{t}^{\beta}$ denote the set of particles in the system at time $t$ and for all $v \in \mathcal{N}_{t}$, let $X_{v}^{\beta}=X_{v}^{\beta}(t) \in\left[0, L_{\beta}\right]$ denote the position of particle $v$. Let $Z^{\beta}(t):=\left|\mathcal{N}_{t}^{\beta}\right|$ be the number of particles in the system at time $t$. The branching Brownian motion (BBM) is denoted by $\left(\mathbf{X}_{t}^{\beta}\right)=\left(X_{v}^{\beta}(t), v \in \mathcal{N}_{t}\right)$. We write $\mathbb{P}_{x}$ for the law of the BBM started from a single particle at $x \in\left[0, L_{\beta}\right]$ and $\mathbb{E}_{x}$ for the corresponding expectation. The natural filtration generated by the BBM is denoted by $\left(\mathcal{F}_{t}^{\beta}, t \geq 0\right)$.

This system can be interpreted as a population undergoing selection. In this context, the position of a particle measures its fitness. We refer to Section 1.4 for further details on the biological interpretation of the model.

Critical regime The killing boundary $L_{\beta}$ is set in such a way that the number of particles in the BBM stays roughly constant.

The expected number of particles in the $\mathrm{BBM} \mathbf{X}^{\beta}$ is governed by the PDE

$$
\begin{cases}\partial_{t} u(t, y)=\frac{1}{2} \partial_{y y} u(t, y)-\beta \partial_{y} u(t, y)+\frac{1}{2} u(t, y) &  \tag{A}\\ \beta u(t, 0)-\left.\frac{1}{2} \partial_{y} u(t, y)\right|_{y=0}=0 & \text { (flux at 0), } \\ u\left(t, L_{\beta}\right)=0 & \text { (killing at } \left.L_{\beta}\right) .\end{cases}
$$

Let $p_{t}^{\beta}(x, y)$ be the fundamental solution of A]. It is well-known (see e.g. [23] p.188]) that $p_{t}^{\beta}$ is the density of particles in the BBM:
Lemma 1.1 (many-to-one lemma). For every measurable function $f: \mathbb{R} \mapsto \mathbb{R}$, for all $x \in\left[0, L_{\beta}\right]$ and $t \geq 0$, we have

$$
\begin{equation*}
\mathbb{E}_{x}\left[\sum_{v \in \mathcal{N}_{t}^{\beta}} f\left(x_{v}\right)\right]=\int_{0}^{L} f(y) p_{t}^{\beta}(x, y) d y . \tag{3}
\end{equation*}
$$

In particular, note that, if the system starts with a single particle at $x$ at time 0 , the expected number of particles in any Borel set $B$ at time $t$ is given by $\int_{B} p_{t}^{\beta}(x, y) d y$. Define

$$
\begin{equation*}
g_{t}^{\beta}(x, y):=e^{\beta(x-y)} e^{\frac{\beta^{2}-1}{2} t} p_{t}(x, y) . \tag{4}
\end{equation*}
$$

A straightforward computation shows that $g_{t}^{\beta}$ is the fundamental solution of the self-adjoint PDE

$$
\begin{cases}\partial_{t} u(t, y)=\frac{1}{2} \partial_{y y} u(t, y) &  \tag{B}\\ \beta u(t, 0)-\partial_{y} u(t, y)_{\mid y=0}=0 & \text { (flux at 0) } \\ u\left(t, L_{\beta}\right)=0 & \text { (killing at } \left.L_{\beta}\right) .\end{cases}
$$

For $\beta \in(0,1)$ and $L>0$, consider the Sturm-Liouville problem

$$
\begin{equation*}
\frac{1}{2} v^{\prime \prime}(x)=\lambda v(x), \quad x \in(0, L), \quad v^{\prime}(0)=\beta v(0), \quad v(L)=0 . \tag{SLP}
\end{equation*}
$$

It is known [35, Chapter 4] that the eigenvalues $\lambda_{i} \equiv \lambda_{i}^{\beta, L}$ of (SLP are simple and that the associated eigenvectors $\left(v_{i}\right)_{i \geq 1} \equiv\left(v_{i}^{\beta_{L}}\right)_{i \geq 1}$ can be normalised to form an orthonormal basis of $\mathrm{L}^{2}([0, L])$. A direct calculation shows that the $i$-th eigenvalue of SLP ) is the unique solution of the algebraic equation

$$
\begin{equation*}
\tan \left(\sqrt{-2 \lambda_{i}} L\right)=-\frac{\sqrt{-2 \lambda_{i}}}{\beta}, \quad \sqrt{-2 \lambda_{i}} L \in\left[\left(i-\frac{1}{2}\right) \pi, i \pi\right], \quad \forall i \in \mathbb{N} \tag{5}
\end{equation*}
$$

Thus, for sufficiently large $t$, we expect the density of particles in the dyadic BBM with drift $\beta$, branching at rate $1 / 2$, reflected at 0 and killed at $L$, to be well-approximated by the first term of its spectral decomposition, that is

$$
\begin{equation*}
e^{\beta(y-x)} e^{\frac{1-\beta^{2}}{2} t} e^{\lambda_{1} t} \frac{v_{1}(x) v_{1}(y)}{\left\|v_{1}\right\|^{2}} . \tag{6}
\end{equation*}
$$

One can now choose $L$ in such a way that the expected number of particles neither increases nor decreases exponentially: for this choice of $L$, we say that the BBM is critical. This motivates our definition for $L_{\beta}$. Indeed, for $L=L_{\beta}$, one can show (see (5) that $\lambda_{1}=-\frac{1-\beta^{2}}{2}$ so that $\mathbf{X}^{\beta}$ is critical.

Moreover, one can check that

$$
\begin{equation*}
v_{1}^{\beta}(x) \equiv v_{1}^{\beta, L_{\beta}}(x)=\sin \left(\gamma\left(L_{\beta}-x\right)\right), \quad \text { with } \quad \gamma \equiv \gamma_{\beta}:=\sqrt{1-\beta^{2}} \tag{7}
\end{equation*}
$$

is a positive solution of SLP for $\lambda=-\frac{1-\beta^{2}}{2}$.
Perron-Frobenius eigenvectors Define

$$
\begin{equation*}
\tilde{h}^{\beta}(x)=\tilde{c}^{\beta} e^{\beta x} v_{1}^{\beta}(x), \quad \text { and } \quad h^{\beta}(x)=\frac{1}{\tilde{c}^{\beta}} \frac{2}{L_{\beta}+\beta} e^{-\beta x} v_{1}^{\beta}(x), \quad x \in \Omega_{\beta}:=\left[0, L_{\beta}\right], \tag{8}
\end{equation*}
$$

with

$$
\tilde{c}^{\beta}:=\left(\int_{0}^{L_{\beta}} \tilde{h}^{\beta}(x) d x\right)^{-1}=\frac{1}{\gamma} e^{-\beta L_{\beta}} .
$$

The function $h^{\beta}$ (resp. $\tilde{h}^{\beta}$ ) is a right (resp. left) eigenvector of the differential operator $\mathcal{G}+\frac{1}{2}$ on $\Omega_{\beta}$ and $\tilde{c}^{\beta}$ can be thought as a Perron-Frobenius renormalisation constant. Indeed, we have

$$
\begin{equation*}
\int_{0}^{L_{\beta}} \tilde{h}^{\beta}(x) d x=1, \quad \text { and } \quad \int_{0}^{L_{\beta}} h^{\beta}(x) \tilde{h}^{\beta}(x) d x=1 . \tag{9}
\end{equation*}
$$

Moreover, with this notation, (6) shows that the density $p_{t}^{\beta}(x, y)$ is approximately given by $h^{\beta}(x) \tilde{h}^{\beta}(y)$ so that $\tilde{h}^{\beta}(y) d y$ can be interpreted as the stable configuration of the system and $h^{\beta}(x)$ as the reproductive value of a particle located at $x$.
Remark 1. The function $[0,1) \rightarrow[\pi / 2, \infty), \beta \mapsto L_{\beta}$ is increasing, $L_{0}=\pi / 2$ and $L_{\beta} \rightarrow \infty$ as $\beta \rightarrow 1$. Moreover,

$$
\begin{equation*}
\gamma_{\beta} L_{\beta}=\pi-\gamma_{\beta}+o\left(\gamma_{\beta}\right), \quad \beta \rightarrow 1 . \tag{10}
\end{equation*}
$$

### 1.3 Main results

We are interested in the limiting behaviour of a sequence of critical branching Brownian motions. More precisely, we set $c \in(0,1)$ and consider a sequence of drifts $\left(\beta_{N}\right)_{N \geq N_{0}}$ such that

$$
\begin{equation*}
\forall N \geq N_{0}, \quad L_{\beta_{N}}=c \log (N)+6 \log \log (N) \tag{11}
\end{equation*}
$$

where $N_{0}:=\min \{n \geq 2: c \log (n)+\log \log (n)>\pi / 2\}$. For all $N \geq N_{0}$, we write $\left(\mathbf{X}_{t}^{\beta_{N}}\right) \equiv\left(\mathbf{X}_{t}^{N}\right)$ for the corresponding critical BBM with drift $\beta_{N}$. To ease the notation, we will drop the $\beta$ 's and write

$$
\begin{equation*}
L_{\beta_{N}} \equiv L_{N}, \quad \gamma_{\beta}=\gamma_{N} \quad \Omega_{\beta_{N}}=\Omega_{N}, \quad h^{\beta_{N}} \equiv h^{N}, \quad \tilde{h}^{\beta_{N}} \equiv \tilde{h}^{N}, \quad Z^{\beta_{N}}(t)=Z^{N}(t) \tag{12}
\end{equation*}
$$

and use similar conventions for all the quantities related to the BBM $\mathbf{X}^{N}$. The definition in motivated by the following observation.
Proposition 1.2. Let $\left(\beta_{N}\right)$ be as in 11. Define

$$
\begin{equation*}
\Sigma_{N}(z):=\left(\int_{0}^{z}\left(h^{N}(x)\right)^{2} \tilde{h}^{N}(x) d x\right)^{1 / 2}, \quad z \in \Omega_{N} \tag{13}
\end{equation*}
$$

Then there exists a constant $\sigma>0$ that only depends on $c$, such that

$$
\begin{equation*}
\frac{1}{N^{c}} \Sigma_{N}\left(L_{N}\right)^{2} \rightarrow \sigma^{2}, \quad \text { as } \quad N \rightarrow \infty \tag{14}
\end{equation*}
$$



Figure 1: Stable configuration, reproductive value and reproductive variance for $\beta=0.995$. In both pictures, the red solid line corresponds to the graph of $\tilde{h}^{\beta}$, the blue dotted line to the graph of $h^{\beta}$ and the orange dashed line to the graph of $\Sigma^{2} / \Sigma^{2}\left(L_{\beta}\right)$. Left panel: the green horizontal line corresponds to the position of the rightmost position such that $\Sigma^{2}(z) \leq 0.999 \cdot \Sigma^{2}\left(L_{\beta}\right)$. Right panel: Close-up on the region delineated by the $y$-axis and the green line.

Let

$$
\begin{equation*}
A_{N}=6 \log \log N-\log \log \log (N) \tag{15}
\end{equation*}
$$

Then, as $N \rightarrow \infty$,

$$
\begin{equation*}
\frac{1}{N^{c}} \Sigma_{N}\left(A_{N}\right)^{2} \rightarrow \sigma^{2} \quad \text { and } \quad \frac{1}{N^{1-c}} \int_{0}^{A_{N}} N \tilde{h}^{N}(x) d x \rightarrow a \tag{16}
\end{equation*}
$$

for some constant $a>0$ that only depends on $c$.
The proof of this proposition is the object of Section 3.2 . We now give a brief interpretation of these quantities. As we shall see, the function $\tilde{h}^{N}$ can be interpreted as the stable configuration of the BBM $\mathbf{X}^{N}$ and $h^{N}$ as the reproductive values of the particles. Hence, $\Sigma^{2}$ can be thought of as the reproductive variance of the BBM and allows us to measure the effective population size of the system. Indeed, (16) shows that the mass of the integral $\Sigma^{2}$ is concentrated in the interval $\left[0, A_{N}\right]$. In this region, the expected number of individuals is of order $N^{1-c}$ (see (16)). This region will be referred to as the best class in the BBM (and should correspond to the best class in the WF model, see Section 1.1. Moreover, one can check that the bulk of the distribution $\tilde{h}^{N}$ is concentrated at a distance of order 1 of $L_{N}$. Recall that the drift $\beta_{N}$ has been chosen so that $L_{N}$ is of the order $c \log (N)$, in accordance with (1). This is illustrated in Figure 1

In this paper, we prove that the limiting behaviour of the sequence of branching Brownian motions $\mathbf{X}^{N}$ is governed by the particles located between 0 and $A_{N}$.

The process started from a single particle. In this paragraph, we describe this limiting behaviour when the sequence of $\mathrm{BBM} \mathbf{X}^{N}$ is started from a single particle at a fixed location $x>0$. In particular, note that for $N$ large enough, $x \in\left[0, L_{N}\right]$. As a first step, we derive precise asymptotics for the survival probability of BBMs.

Theorem 1 (Kolmogorov estimate). Let $t>0$. Uniformly in $x \in\left[0, L_{N}\right]$,

$$
\left|N \mathbb{P}_{x}\left(Z^{N}\left(t N^{1-c}\right)>0\right)-\frac{2}{\sigma^{2} t} h^{N}(x)\right| \rightarrow 0, \quad \text { as } N \rightarrow \infty
$$

where $\sigma^{2}$ is as in Proposition 1.2.
This theorem suggests that the typical time scale of evolution of $\left(Z^{N}(t)\right)$ is of order $N^{1-c}$. The following Yaglom law describes the scaling limit of $Z^{N}$ on that time scale.

Theorem 2 (Yaglom law). Let $t>0$ and $x>0$. Assume that $\mathbf{X}_{0}^{N}$ consists of a single particle at $x$. Conditional on $\left\{Z^{N}\left(t N^{1-c}\right)>0\right\}$, we have

$$
\frac{1}{N} Z^{N}\left(t N^{1-c}\right) \rightarrow \frac{\sigma^{2} t}{2} \mathcal{E}, \quad \text { as } N \rightarrow \infty
$$

in distribution, where $\mathcal{E}$ is an exponential random variable of mean 1.
We now describe the genealogy of the system. For two particles $v_{1}, v_{2} \in \mathcal{N}_{t}^{N}$, we denote by $d_{t}^{N}\left(v_{1}, v_{2}\right)$ the time to the most recent common ancestor (MRCA) of $v_{1}$ and $v_{2}$. Intuitively, our next result shows that the rescaled distance matrix of $k$ individuals sampled in the BBM converges to the distance matrix of $k$ individuals sampled from a critical Galton-Watson process [29. This limiting object is defined as follows. Let $t>0$ and $U$ be a random variable on $[0, t]$. Define $U^{\theta}$ such that

$$
\begin{equation*}
\forall s \leq t, \quad \mathbb{P}\left(U^{\theta} \leq s\right):=\frac{(1+\theta) \mathbb{P}(U \leq s)}{1+\theta \mathbb{P}(U \leq s)}=\frac{(1+\theta)(s / t)}{1+\theta(s / t)} \tag{17}
\end{equation*}
$$

Let $\left(U_{i}^{\theta} ; i \in\{1, \ldots, k\}\right)$ be $k$ i.i.d. copies of $U^{\theta}$ and set

$$
\forall 1 \leq i<j \leq k, \quad U_{i, j}^{\theta}=U_{j, i}^{\theta}:=\max \left\{U_{l}^{\theta}: l \in\{i, \cdots, j-1\}\right\}
$$

Define the random distance matrix $\left(H_{i, j}\right):=\left(H_{i, j} ; i \neq j \in[k]\right)$ such that for every bounded and continuous function $\phi: \mathbb{R}^{k^{2}} \rightarrow \mathbb{R}$,

$$
\begin{equation*}
\mathbb{E}\left[\phi\left(\left(H_{i, j}\right)\right)\right]=k \int_{0}^{\infty} \frac{1}{(1+\theta)^{2}}\left(\frac{\theta}{1+\theta}\right)^{k-1} \mathbb{E}\left[\phi\left(\left(U_{i, j}^{\theta}\right)\right)\right] d \theta \tag{18}
\end{equation*}
$$

Theorem 3 (Genealogy). Let $t>0$ and $x \in \mathbb{R}$. Assume that the $B B M \mathbf{X}^{N}$ starts with a single particle at $x$ and condition on the event $\left\{Z^{N}\left(t N^{1-c}\right)>0\right\}$. Sample $k$ particles $\left(v_{1}, \ldots, v_{k}\right)$ in $\mathbf{X}^{N}$ at time $t N^{1-c}$. The rescaled distance matrix $\left(\frac{1}{N^{1-c}} d_{t N^{1-c}}^{N}\left(v_{i}, v_{j}\right)_{i, j}\right)$ converges to $\left(H_{\sigma_{i}, \sigma_{j}}\right)$ in distribution, where $H$ is as in 18) and $\sigma$ is an independent permutation of $\{1, \ldots, k\}$.

Demographic fluctuations in the fitness wave Alternatively, one can describe this scaling limit when the process starts with approximately $N$ particles in a wave-like configuration. In this setting, the system should exhibit a "fitness wave" behaviour: the population size should stay of order $N$ and quickly settle in a stable configuration. We expect the demographic fluctuations in the system to be given by a Feller diffusion, a 2-stable continuous-state branching process.

A continuous-state branching process (CSBP), see e.g. [4, 3, is a $[0, \infty]$-valued Markov process $(\Xi(t), t \geq 0)$ whose transition function satisfies the branching property $p_{t}(x+y, \cdot)=p_{t}(x, \cdot) * p_{t}(y, \cdot)$. In words, this means that the sum of two independent copies of the process starting from $x$ and $y$ has the same finite-dimensional distribution as the process starting from $x+y$. It is well-known that continuous-state branching processes can be characterised by their branching mechanism, which is a function $\Psi:[0, \infty) \rightarrow \mathbb{R}$. If $(\Xi(t), t \geq 0)$ is a continuous-state branching process with branching mechanism $\Psi$, then for all $\lambda \geq 0$,

$$
\begin{equation*}
\mathbb{E}\left[e^{-\lambda \Xi(t)} \mid \Xi_{0}=x\right]=e^{-x u_{t}(\lambda)} \tag{19}
\end{equation*}
$$

where $u_{t}(\lambda)$ can be obtained as the solution to the differential equation

$$
\begin{equation*}
\frac{\partial}{\partial t} u_{t}(\lambda)=-\Psi\left(u_{t}(\lambda)\right), \quad \quad u_{0}(\lambda)=\lambda \tag{20}
\end{equation*}
$$

The Feller diffusion is a 2-stable CSBP. As a result, its branching mechanism is quadratic, of the form

$$
\begin{equation*}
\Psi(q)=\frac{a^{2}}{2}-b q, \quad a^{2} \geq 0, b \in \mathbb{R} \tag{21}
\end{equation*}
$$

For $x \geq 0$, the Feller diffusion started from $x$ can also be constructed as the solution of the stochastic differential equation

$$
d \Xi_{t}=a \sqrt{\Xi_{t}} d B_{t}+b \Xi_{t} d t, \quad \Xi_{0}=0
$$

where $B$ is a standard Brownian motion.


Figure 2: Right panel: Genealogical structure spanned by 4 individuals sampled at time $t_{N}=t N^{1-c}$. Left panel: spatial trajectories of the ancestral lineages.

Theorem 4. Let $\sigma^{2}$ be as in Proposition 1.2. Assume that, as $N \rightarrow \infty$,

$$
\begin{equation*}
\frac{1}{N} \sum_{v \in \mathcal{N}_{0}^{N}} h^{N}\left(X_{v}^{N}(0)\right) \rightarrow z_{0}, \quad \text { in probability } \tag{22}
\end{equation*}
$$

for some $z_{0}>0$. Then, the finite-dimensional distributions of the processes $\left(\frac{1}{N} Z^{N}\left(t N^{1-c}\right)\right)$ converge to that of a Feller diffusion $\left(\Xi_{t}, t \geq 0\right)$ with branching mechanism $\Psi(q)=\frac{\sigma^{2}}{4} q$, started from $z_{0}$.

Biological interpretation Our results suggest that the evolutionary dynamics in the fitness wave are governed by a set of particles of size $N^{1-c}$, behaving like a neutral population, that is a population with no selection. Indeed, it is known [4] that the genealogy of a population whose demographic fluctuations are described by Feller diffusion is given by a time-changed Kingman coalescent. In the present work, we show that the genealogical structure spanned by $k$ individuals sampled at a fixed time horizon only comprises binary mergers (see Theorem 3). As we shall see (Section 4), this approach also allows us to show that these mergers are all located in the interval $\left[0, A_{N}\right]$, where the expected population size is of order $N^{1-c}$. This fact is illustrated in Figure 2

### 1.4 Related models

Branching Brownian motion with absorption Our model is a variation of the BBM with absorption studied by Berestycki, Berestycki and Schweinsberg [2] to investigate the effect of natural selection on the genealogy of a population. In [2], the authors consider a BBM with drift -1 and branching rate $1 / 2$, in which the particles are killed when they reach 0 . In this framework, each individual in the population is represented by a point $x \geq 0$, measuring its fitness. The selective pressure is seen as a moving barrier with constant speed 1, killing all the individuals whose fitness is too low. The fitness of each individual evolves according to a Brownian motion, describing the accumulation of mutations. Roughly speaking, a particle located far to the right corresponds to an individual that accumulated many beneficial mutations, faster than the rest of the population. As a result, this individual produces a large number of offspring before getting absorbed at $x=0$. In this case, it was proved in [2] that the genealogy of the system is given by a Bolthausen-Sznitman coalescent [6] and that the fluctuations in the population size are described by Neveu's continuous state branching process (a CSBP with jumps) 26.

In the present work, we model individuals accumulating deleterious mutations: the fitness of a particle is now given by $L(\beta)-x \in[0, L(\beta)]$. While the particles can accumulate an arbitrarily large number of beneficial mutations in [2], in our setting, the fittest individuals are the ones with the smallest number of mutations. In the BBM, this lower bound on the minimal number of mutations is modeled thanks to a reflexion at 0 . This
reflective boundary at 0 has the effect of a cut-off, preventing the formation of multiple mergers in the genealogy, or equivalently, the emergence of jumps in the limiting fluctuations of the system.

Simplifications of the Wright-Fisher model with deleterious mutations The analysis of the WF dynamics presented in Section 1.1 turns out to be complex. The constant population size induces strong dependencies between the particles. Continuous diffusion approximations have been proposed to tackle this problem 14, 27. Alternatively, simplifications of the selection mechanisms have been suggested to make the discrete model more tractable [21, 8].

In this article, we consider a different type of approximation. As in [21, 8, we consider an individual-based model, in which we replace natural selection by "truncation selection". Instead of picking parents at random accordingly to their fitness, we kill particles whose fitness is too low. This approach relies on an assumption that is widely accepted in the biological literature (see [19] for an extensive bibliography), i.e. the evolutionary dynamics are driven by a small number of individuals, belonging to the fittest classes. Since the effect of the regulation mechanism on these individuals is negligible, the whole dynamics does not depend on its particular form.

In addition, we assume that individuals accumulate many mutations, each of them having a small impact on their fitness (see e.g. 11] for some motivation from biology). This assumption is crucial to approximate the evolution of fitness by a Brownian motion [1].

Method of moments and reproductive variance in travelling waves In this work, we use a method of moment for marked metric measure spaces derived in [16] to prove the joint convergence of the population size and genealogy of the population. This method has already been used in 32, 16, to investigate the genealogy of a BBM with inhomogeneous branching rate, negative drift and killing at 0 . In this framework, the BBM is seen as a toy model for what happens at the tip of an invasion front in a cooperating population. For a certain range of parameters, the reproductive variance of the system scales like $N^{c}$, for some $c \in(0,1)$, as in Proposition 1.2 In both cases, the time scale for the scaling limit of the demographic fluctuations (and thus the genealogy) is given by $N / \Sigma^{2}=N^{1-c}$. However, the invasion mechanisms in [16] are very different from the evolutionary dynamics described in the previous section. The main difference lies in the fact that the reproductive variance is concentrated in a region in which the expected number of individuals is microscopic. In [16], this results in jumps in the scaling limit of the demographic fluctuations (a $(2-c)$-stable CSBP) and in a multiple merger limiting genealogy (precisely, a Beta $(c, 2-c)$-coalescent) of the invasion front. This is illustrated in Figure 3

## 2 Outline of the proof

Our approach relies on a method of moments for random metric measure spaces introduced in [15]. As in [32, 16], we will enrich the structure of the BBM to prove the joint convergence of the genealogy and the size of the system started from a single particle. The convergence of the demographic fluctuations to Feller's diffusion will be deduced from our Yaglom law and the uniform converge of the moments.

In Section 2.1, we first define a topology on the set of metric measure spaces ( mm spaces) following 17. In section 2.2. we introduce our limiting random mm space, the Coalescent Point Process (CPP). This limiting object is known to be the scaling limit of critical Galton-Watson processes with finite variance 29. In Section 2.3 , we encode the genealogy of the system at time $t N^{1-c}$ as a random mm space. Next, in Section 2.4 we explain how to compute the moments of mm space associated to the BBM thanks to a spinal decomposition of the process $\mathbf{X}^{N}$. The remainder of this section is dedicated to the sketch of the proof for the convergence of the moments of our BBM.

### 2.1 Random Metric Measure Spaces (mm spaces)

A metric measure space $(X, d, \mu)$ is a complete separable metric space $(X, d)$ equipped with a finite measure $\nu$. Let $|X|:=\nu(X)$ and note that we do not require $\mu$ to be a probability measure. We say that $(X, d, \mu)$ and $\left(X^{\prime}, d^{\prime}, \mu^{\prime}\right)$ are equivalent if there exists an isometry $\varphi$ between the supports of $\mu$ and $\mu^{\prime}$ such that $\mu^{\prime}$ is the pushed forward of $\mu$ by $\varphi$. We denote by $\mathbb{M}$ the set of equivalence classes of mm spaces.


Figure 3: Fitness wave versus spatial wave. (a) Fitness wave: stable configuration $\tilde{h}^{\beta}$ is the BBM $\mathbf{X}^{\beta}$. The reproductive variance is concentrated in the interval $[0, A]$ where the expected number of particles is of order $N^{1-c}$ (see Proposition 1.2. (b) Spatial wave: stable configuration in the BBM defined in 34 to describe the genealogy at the tip of spatial fronts. In the semi pushed regime, the reproductive variance if or order $N^{2-\alpha}$ for some $\alpha \in(1,2)$. In this case, the variance is concentrated in an interval of the form $[L, L-A]$ for some constant $A$. For this interval, the expected number of particles is of order $N^{1-\alpha} \ll 1$.

Definition 1 (Polynomials [17, Definition 2.3]). A functional $\Phi: \mathbb{M} \rightarrow \mathbb{R}$ is a polynomial of degree $k \in \mathbb{N}$ if there exists a bounded continuous function $\phi:[0, \infty)^{\binom{k}{2}} \rightarrow \mathbb{R}$ such that

$$
\left.\Phi((X, d, \mu))=\int_{X^{k}} \phi\left(d\left(v_{i}, v_{j}\right)_{1 \leq i<j \leq k}\right)\right) \prod_{i=1}^{k} \mu\left(d v_{i}\right)
$$

We write $\Pi$ for the set of all polynomials.
Definition 2 (Gromov-weak topology [17, Definition 2.8]). The Gromov-weak topology is defined as the topology induced by the polynomials, that is, the smallest topology making all polynomials continuous. A sequence $\left(X^{n}, d^{n}, \mu^{n}\right)$ is said to converge to $(X, d, \mu)$ in $\mathbb{M}$ with respect to the Gromov-weak topology if and only if $\Phi\left(X^{n}, d^{n}, \mu^{n}\right)$ converges to $\Phi(X, d, \mu)$ for all $\Phi \in \Pi$.
Definition 3. A random $m m$ space is a random variable with values in $\mathbb{M}$, endowed with the Gromov-weak topology and the associated Borel $\sigma$-field.

Many properties of the marked Gromov-weak topology are derived in 10 under the additional assumption that $\mu$ is a probability measure. The next result shows that $\boldsymbol{\Pi}$ forms a convergence determining class only when the limit satisfies a moment condition, which is a well-known criterion for a real variable to be identified by its moments, see for instance [12, Theorem 3.3.25].
Proposition 2.1 (Convergence criterion [9, Lemma 2.7]). Suppose that ( $X, d, \mu$ ) is a random mm space satisfying

$$
\begin{equation*}
\limsup _{p \rightarrow \infty} \frac{\mathbb{E}\left[|X|^{p}\right]^{1 / p}}{p}<\infty \tag{23}
\end{equation*}
$$

Let $\left(\left(X^{n}, d^{n}, \mu^{n}\right)\right)$ be a sequence of random $m m$ spaces. If

$$
\lim _{n \rightarrow \infty} \mathbb{E}\left[\Phi\left(X^{n}, d^{n}, \mu^{n}\right)\right]=\mathbb{E}[\Phi(X, d, \nu)]
$$

for all $\Phi \in \Pi$, then, the sequence $\left.\left(\left(X^{n}, d^{n}, \mu^{n}\right]\right)\right)$ converges in distribution to $(X, d, \nu)$ for the marked Gromovweak topology.

For $\Phi \in \Pi$ and a random mm space $(X, d, \mu)$, the quantity $\mathbb{E}[(\Phi(X, d, \mu))]$ is called the moment of $(X, d, \mu)$ associated to $\Phi$.

Remark 2. Consider a random mm space $(X, d, \mu)$. Assume that $\mathbb{E}\left[|X|^{k}\right]<\infty$. The moments of a random mm can be rewritten as

$$
\mathbb{E}[\Phi(X, d, \nu)]=\mathbb{E}\left[|X|^{k}\right] \times \frac{1}{\mathbb{E}\left[|X|^{k}\right]} \mathbb{E}\left[|X|^{k} \phi\left(d\left(v_{i}, v_{j}\right), i \neq j \in[k]\right)\right]
$$

where $\left(v_{i}\right)$ are $k$ points sampled uniformly at random and $|X|=\mu(X)$ is thought of as the total population size. As a consequence, the moments of a random mmm are obtained by biasing the population size by its $k^{\text {th }}$ moment and then picking $k$ individuals uniformly at random.

Let $(X, d, \mu)$ be a random marked metric measure space, we denote by $\mathbb{E}(\Phi(X, d, \nu))$ the moment of $[X, d, \mu]$ associated to $\Phi$.

### 2.2 The Brownian Coalescent Point Processes

In this section, we recall the construction of the Brownian Coalescent Point Process. This mm space is known to be the limiting genealogy of critical Galton-Watson processes with finite variance [29. As we shall see, the sequence of mm spaces associated to the sequence of BBMs $\mathbf{X}^{N}$ also converges to a Brownian CPP.

Let $T>0$. Consider a Poisson Point Process $\mathcal{P}$ on $\mathbb{R}_{+} \times \mathbb{R}_{+}$with intensity $d t \otimes x^{-2} d x$. Define

$$
Y_{T}=\inf \{y \geq 0:(t, y) \in \mathcal{P}, t>T\}
$$

and

$$
d_{P}(x, y)=\sup \{t:(z, t) \in \mathcal{P} \text { and } x \leq z \leq y\}, \quad 0<x<y<Y_{T}
$$

The Brownian Coalescent Point Process with height $T$ is defined as the random metric measure space

$$
M_{C P P_{T}}:=\left(\left[0, Y_{T}\right], d_{P}, \mathrm{Leb}\right)
$$

Proposition 2.2 (Moments of the Brownian CPP). Let $k \in \mathbb{N}$. Let $\left(U_{1}, \ldots, U_{k-1}\right)$ be i.i.d. uniform random variables on $[0, T]$. Define

$$
U_{i, j}=\max \left\{U_{i}, \ldots, U_{j-1}\right\}
$$

Then for any bounded continuous function $\phi:[0, \infty)\binom{k}{2} \rightarrow \mathbb{R}$ with associated polynomial $\Phi$, we have

$$
\mathbb{E}\left[\Phi\left(M_{C P P_{T}}\right)\right]=k!T^{k} \mathbb{E}\left[\phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right]
$$

where $\sigma$ is an independent uniform permutation of $[k]$.
Proof. The proof can be found in [15, Proposition 4].
Remark 3. The random variable $Y_{T}$ is an exponential random variable with parameter $T$. It is interpreted as the size of the population.
Proposition 2.3 (Sampling from the CPP). Let $k \in \mathbb{N}$ and sample $k$ points, denoted by $\left(v_{1}, \ldots, v_{k}\right)$, uniformly at random from the CPP. Then $\left(\left(d_{T}\left(v_{i}, v_{j}\right)\right)_{i, j}\right)$ is identical in law to $\left(\left(H_{\sigma_{i}, \sigma_{j}}\right)\right)$ where $\left(H_{i, j}\right)$ is defined as in (18).

Proof. The proof can be found in [5, Proposition 4.3].

### 2.3 The Metric Space associated to the BBM

Let $t>0$ and recall that $\mathcal{N}_{t}^{N}$ denotes the set of particles alive at time $t$ in the BBM $\mathbf{X}^{N}$. Set

$$
\begin{equation*}
\mu_{t}^{N}:=\sum_{v \in \mathcal{N}_{t}^{N}} \delta_{v}, \quad \text { and } \quad \forall v, v^{\prime} \in \mathcal{N}_{t}^{N}, \quad d_{t}^{N}\left(v, v^{\prime}\right)=t-\left|v \wedge v^{\prime}\right| \tag{24}
\end{equation*}
$$

where $\left|v \wedge v^{\prime}\right|$ denotes the most recent time when $v$ and $v^{\prime}$ had a common ancestor. Let $M_{t}^{N}:=\left[\mathcal{N}_{t}^{N}, d_{t}^{N}, \mu_{t}^{N}\right]$ be the resulting random mm . Finally, set

$$
\bar{\mu}_{t}^{N}:=\frac{1}{N} \sum_{v \in \mathcal{N}_{t N}^{N}} \delta_{v}, \quad \text { and } \quad \forall v, v^{\prime} \in \mathcal{N}_{t N}^{N}, \quad \bar{d}_{t}^{N}\left(v, v^{\prime}\right)=\left(t-\frac{1}{N}\left|v \wedge v^{\prime}\right|\right)
$$

and define the rescaled mm space $\bar{M}_{t}^{N}:=\left[\mathcal{N}_{t N}^{N}, \bar{d}_{t}^{N}, \bar{\mu}_{t}^{N}\right]$. In order to prove Theorem 3 we will establish the following convergence result.
Theorem 5. Let $t>0$. Let $M_{C P P_{t}}$ be a Brownian CPP with height $\frac{\sigma^{2}}{2} t$. Conditional on the event $\left\{Z_{t N^{1-c}}^{N}>0\right\}$, the sequence $\left(\bar{M}_{t} ; N \in \mathbb{N}\right)$ converges weakly to $M_{C P P_{t}}$ with respect to the Gromov-weak topology.

### 2.4 The $k$-spine tree and the many-to-few formula

In this section, we introduce the $k$-spine tree associated to the BBM and state our many-to-few formula. This formula will be used to compute the moments of $\bar{M}_{t}$.
Definition 4. The spine process $\left(\zeta_{t}^{N}\right)_{t \geq 0}$ is the diffusion with generator

$$
\begin{equation*}
\frac{1}{2} f^{\prime \prime}(x)+\frac{v_{1}^{\prime}(x)}{v_{1}(x)} f^{\prime}(x), \quad f^{\prime}(0)=0, \quad f\left(L_{N}\right)=0 \tag{25}
\end{equation*}
$$

In the following $q_{t}^{N}(x, y)$ denotes the probability kernel of the spine process. The generator (25) is the Doob $\left(h^{N}\right)$-transform of $\mathcal{G}+\frac{1}{2}$. In particular,

$$
q_{t}^{N}(x, y)=\frac{h^{N}(y)}{h^{N}(x)} p_{t}^{N}(x, y)
$$

The next result is standard (see e.g. [13]).
Proposition 2.4. The spine process has a unique invariant measure given by

$$
\begin{equation*}
\Pi^{N}(d x)=h(x)^{N} \tilde{h}^{N}(x) d x \tag{26}
\end{equation*}
$$

We now move to the definition of the $k$-spine tree. Let $\left(U_{1}, \ldots, U_{k-1}\right)$ be independent random variables uniformly distributed in $[0, t]$. Define

$$
\begin{equation*}
\forall 1 \leq i<j \leq k-1, \quad U_{i, j}=U_{j, i}=\max \left\{U_{i}, \ldots, U_{j-1}\right\} . \tag{27}
\end{equation*}
$$

Let $\mathbb{T}$ be the unique tree of depth $t$ with $k$ leaves such that the tree distance between the $i$-th and the $j$-th leaves is given $U_{i, j}$. This tree is ultrametric and planar in the sense that

$$
\forall i, j, l \in[k], \quad U_{i, j} \leq U_{i, l} \vee U_{l, j}
$$

(ultrametric) and the inequality becomes an equality if $i<l<j$ (planar). The depth of the first branching point in the $k$-spine tree is thus given by

$$
\tau=\max _{i \in[k-1]} U_{i} .
$$

Marks are then assigned as follows. On each branch of the tree, the mark evolves according to the spine process $\zeta^{N}$ and it branches into two independent diffusions at each branching point of $\mathbb{T}$. The resulting planar marked ultrametric tree will be denoted by $\mathcal{T}$ and referred to as the $k$-spine tree. We will denote by $Q_{x}^{N, k, t}$ the distribution of the $k$-spine associated to $\mathbf{X}^{N}$ of height $t$ rooted at $x$.

In the following, $\mathcal{B}$ will denote the set of $k-1$ branching points of the $k$-spine and $\mathcal{L}$ will denote the set of $k$ leaves. We will denote by $\zeta_{v}^{N}$ the mark (or the position) of the node $v \in \mathcal{B} \cup \mathcal{L}$. Finally, $\left(V_{i} ; i \in[k]\right)$ is the enumeration of the leaves from left to right in the $k$-spine (i.e., $V_{i}$ is the leaf with label $i$ ).
Theorem 6 (Many-to-few). Recall the definition of $M_{t}^{N}$ from Section 2.3. Let $t>0$ and $x>0$. Let $k \in \mathbb{N}$, let $\phi:[0, \infty)\binom{k}{2} \rightarrow \mathbb{R}$ bounded continuous function and denote by $\Phi$ the associated polynomial. Then

$$
\mathbb{E}_{x}\left[\Phi\left(M_{t}^{N}\right)\right]=k!h(x) t^{k-1} Q_{x}^{N, k, t}\left(\Delta^{N} \phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right), \quad \text { with } \quad \Delta^{N}:=\left(\frac{1}{2}\right)^{k-1} \prod_{v \in \mathcal{B}} h^{N}\left(\zeta_{v}^{N}\right) \prod_{i=1}^{k} \frac{1}{h^{N}\left(\zeta_{V_{i}}^{N}\right)}
$$

where $\left(U_{i, j}\right)$ is as in 27) and $\sigma$ is an independent random permutation of $[k]$.
The proof of this result can be found in Section 4


Figure 4: $k$-spine with $k=3$. Left panel: planar tree $\mathbb{T}$ generated from 2 i.i.d. uniform random variables $\left(U_{1}, U_{2}\right)$. Right panel: branching 1 -spines running along the branches of the tree $\mathbb{T}$.

Definition 5 (Accelerated $k$-spine). Consider the spine process accelerated by $N^{1-c}$, i.e. the transition kernel of the spine process is now given by $q_{t N^{1-c}}^{N}(x, y)$. We denote this kernel by $\bar{q}_{t}^{N}(x, y)$. Consider the same planar structure as before, i.e., the depth is $t$ and the distance between points at time $t$ is given by (27). We denote by $\bar{Q}_{x}^{N, k, t}$ the distribution of the $k$-spine tree obtained by running accelerated spines along the branches. For any vertex $v$ in the accelerated $k$-spine tree, $\bar{\zeta}_{v}$ will denote the mark of the vertex $v$.
Proposition 2.5 (Rescaled many-to-few). Recall the definition of $\bar{M}_{t}^{N}$ from Section 2.3. Let $t>0$ and $x \in \Omega_{N}$. Let $k \in \mathbb{N}$, let $\phi:[0, \infty)^{\binom{k}{2}} \rightarrow \mathbb{R}$ bounded continuous function and denote by $\Phi$ the associated polynomial. Then

$$
\mathbb{E}_{x}\left[\Phi\left(\bar{M}_{t}^{N}\right)\right]=\frac{1}{N^{(k-1) c+1}} k!h(x) t^{k-1} \bar{Q}_{x}^{N, k, t}\left(\Delta^{N} \phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right), \quad \text { with } \quad \bar{\Delta}^{N}:=\left(\frac{1}{2}\right)^{k-1} \prod_{v \in \mathcal{B}} h^{N}\left(\bar{\zeta}_{v}^{N}\right) \prod_{i=1}^{k} \frac{1}{h^{N}\left(\bar{\zeta}_{V_{i}}^{N}\right)}
$$

where $\left(U_{i, j}\right)$ is as in 27) and $\sigma$ is an independent random permutation of $\{1, \ldots, k\}$.
Proof. This is a direct consequence Theorem 6 after rescaling the measure $\mu^{N}$ by $N$ and time by $N^{1-c}$.
The proof of Theorem 5 relies on the following convergence result.
Proposition 2.6. Let $t>0$. Let $k \in \mathbb{N}$, let $\phi:[0, \infty)^{\binom{k}{2}} \rightarrow \mathbb{R}$ bounded continuous function and denote by $\Phi$ the associated polynomial. Then, uniformly in $x \Omega_{N}$,

$$
\frac{1}{N^{(k-1) c}} \bar{Q}_{x}^{N, k, t}\left(\bar{\Delta}^{N} \phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right) \rightarrow\left(\frac{\sigma^{2}}{2}\right)^{k-1} \mathbb{E}\left(\phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right) .
$$

### 2.5 Heuristics and organisation of the paper

Theorem 2 and Theorem 3 can be deduced from Theorem 5 The idea behind the proof of Theorem 5 consists in identifying the limiting moments of the mm space $\bar{M}_{t}^{N}$ to that of a Brownian CPP of height $t$ (see Proposition 2.2 using a spinal decomposition (see Proposition 2.5 and Proposition 2.6). The convergence criterion for mm spaces given in Proposition 2.1 then yields the result.

We now give a brief heuristics to explain why Proposition 2.6 should hold. By definition of $\bar{\Delta}^{N}$,

$$
\bar{Q}_{x}^{N, k, t}\left(\bar{\Delta}^{N} \cdot \phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right)=\bar{Q}_{x}^{N, k, t}\left(\left(\prod_{v \in \mathcal{B}} h^{N}\left(\bar{\zeta}_{v}^{N}\right) \prod_{i=1}^{k} \frac{1}{h^{N}\left(\bar{\zeta}_{V_{\sigma_{i}}}^{N}\right)}\right) \cdot \phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right) .
$$

By definition of the accelerated $k$-spine tree, its structure is binary a.s. and the marks along the branches are given by spine processes accelerated by $N^{1-c}$. On the other hand, we will prove in Section 3 that the mixing time
of the spine process is of order $\log (N)^{2} \ll N^{1-c}$. As a consequence, the $\left(\bar{\zeta}_{u}^{N}\right)_{u \in \mathcal{B} \cup \mathcal{L}}$ should be well-approximated by $2 k-1$ i.i.d. random variables with law $\Pi^{N}$ and the RHS of the above should be approximately given by

$$
\left(\frac{1}{2} \int_{0}^{L_{N}}\left(h^{N}(y)\right)^{2} \tilde{h}^{N}(y) d y\right)^{k-1}\left(\int_{0}^{L_{N}} \tilde{h}^{N}(y) d y\right)^{N} \mathbb{E}\left[\phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right]
$$

where we used that $\Pi^{N}=h^{N} \tilde{h}^{N}$ (see (26)). Our Perron-Frobenius renormalisation (9) and Proposition 1.2 then yield the content of Proposition 2.6 .

Once this result is proved, Theorem 2.5 then shows that, for all polynomials $\Phi$, as $N \rightarrow \infty$,

$$
\mathbb{E}_{x}\left[\Phi\left(\bar{M}_{t}^{N}\right)\right] \approx \frac{1}{N} k!\left(\frac{\sigma^{2}}{2} t\right)^{k-1} h(x) \mathbb{E}\left[\phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right]
$$

Conditioning this on the survival of the process and using Theorem 1 we get that, as $N \rightarrow \infty$,

$$
\mathbb{E}_{x}\left[\Phi\left(\bar{M}_{t}^{N}\right) \mid Z_{t N^{1-c}}^{N}>0\right] \rightarrow k!\left(\frac{\sigma^{2}}{2} t\right)^{k} \mathbb{E}\left[\phi\left(U_{\sigma_{i}, \sigma_{j}}\right)\right]
$$

Proposition 2.1 shows that $\bar{M}_{t}^{N}$ converges to a Brownian CPP of depth $\frac{\sigma^{2}}{2} t$.
The convergence of the demographic fluctuations for the BBM started from $N$ particles follows from the Kolmogorov estimate Theorem 1 and the convergence of the moment associated to $\Phi \equiv 1$, provided that these convergence results are uniform in the starting point $x$ of the BBM. The above calculations (applied to $\Phi \equiv 1$ ) indicate that, for the BBM started from a single particle at $x>0$, the process $\bar{Z}_{t}^{N}=\frac{1}{N} Z_{t N^{1-c}}^{N}$ is wellapproximated by a random variable with law

$$
\left(1-\frac{2}{\sigma^{2} t} \frac{h^{N}(x)}{N}\right) \delta_{0}(d x)+\frac{2}{\sigma^{2} t} \frac{h^{N}(x)}{N} \exp \left(-\frac{2 x}{\sigma^{2} t}\right) \frac{2 d x}{\sigma^{2} t} .
$$

Hence, for $q \geq 0$

$$
\mathbb{E}_{x}\left[e^{-q \bar{Z}_{t}^{N}}\right] \approx\left(1-\frac{2}{\sigma^{2} t} \frac{h^{N}(x)}{N}\right)+\frac{2}{\sigma^{2} t} \frac{h^{N}(x)}{N} \frac{\sigma^{2} t}{\sigma^{2} t+2 q}=1-\frac{2}{\sigma^{2} t} \frac{h^{N}(x)}{N} \frac{\frac{\sigma^{2} t}{2} q}{1+\frac{\sigma^{2} t}{2} q} \approx \exp \left(-\frac{h^{N}(x)}{N} \frac{q}{1+\frac{\sigma^{2} t}{2}}\right)
$$

for $N$ large. If we now assume that the initial configurations $\mathcal{N}_{0}^{N}$ are such that $\frac{1}{N} \sum_{v \in \mathcal{N}_{0}^{N}} h\left(X_{v}^{N}(0)\right)$ converges to some $z_{0}>0$ in probability, the branching property shows that

$$
\mathbb{E}\left[e^{-q \bar{Z}_{t}^{N}}\right] \approx \mathbb{E}_{x}\left[\prod_{v \in \mathcal{N}_{0}^{N}} \exp \left(-\frac{h^{N}\left(X_{v}^{N}(0)\right)}{N} \frac{q}{1+\frac{\sigma^{2} t}{2}}\right)\right] \approx \exp \left(-z_{0} \frac{q}{1+\frac{\sigma^{2} t}{2}}\right),
$$

which coincides with the Laplace transform of a Feller diffusion with branching mechanisms $\Psi(q)=\frac{\sigma^{2}}{4} q$

## 3 Heat kernel estimates

In this section, we derive precise estimates on the density of particles $p_{t}$ (see Section 1.2 ) and on the transition kernel of the spine $q_{t}$. Related results have already been proved for Dirichlet boundary conditions. We refer the reader to [30, Lemma 2.1] for general diffusions in bounded domains of $\mathbb{R}^{d}$ and [2, Lemma 5] for the case where 0 and $L$ are both absorbing.

### 3.1 Spectral theory and mixing time

First, we recall classical results from Sturm-Liouville theory following [35, Section 4.6]. Consider the SturmLiouville problem

$$
\begin{equation*}
\frac{1}{2} v^{\prime \prime}(x)=\lambda v(x), \quad x \in(0, L), \quad v^{\prime}(0)=\beta v(0), v(L)=0 . \tag{SLP}
\end{equation*}
$$

(1) A solution of SLP is defined as a function $v:[0, L] \rightarrow \mathbb{R}$ such that $v$ and $v^{\prime}$ are absolutely continuous on $[0, L]$ and satisfies (SLP) a.e. on ( $0, L$ ). In particular, any solution $v$ is continuously differentiable on $[0, L]$. In our particular case, the solutions are also twice differentiable on $[0, L]$ and satisfy $(\mathrm{SLP}$ for all $x \in(0, L)$.
(2) A complex number $\lambda$ is an eigenvalue of the Sturm-Liouville problem (SLP) if Equation $(\overline{S L P})$ has a solution $v$ which is not identically zero on $[0, L]$. This set of eigenvalues will be referred to as the spectrum.
(3) It is known that the set of eigenvalues is infinite, countable and has no finite accumulation point. Besides, it is upper bounded and all the eigenvalues are simple and real so that they can be enumerated

$$
\lambda_{1}>\lambda_{2}>\ldots>\lambda_{n}>\ldots
$$

where

$$
\lambda_{n} \rightarrow-\infty \quad \text { as } \quad n \rightarrow+\infty .
$$

(4) As a consequence, the eigenvector $v_{i}$ associated to $\lambda_{i}$ is unique up to constant multiplies. Furthermore, the sequence of eigenfunctions can be normalised to form an orthonormal sequence of $L^{2}([0, L])$. This orthonormal sequence is complete in $\mathrm{L}^{2}([0, L])$ so that the fundamental solution of PDE $\sqrt{B}$ can be written as

$$
\begin{equation*}
g_{t}(x, y)=\sum_{k=1}^{\infty} e^{\lambda_{k} t} \frac{v_{k}(x) v_{k}(y)}{\left\|v_{k}\right\|^{2}} \tag{28}
\end{equation*}
$$

(5) The function $v_{1}$ does not change sign in $(0, L)$. For $k \geq 2$, the eigenfunction $v_{k}$ has exactly $k-1$ zeros on $(0, L)$.
Finally, we recall from (4) that the density of the $\mathrm{BBM} \mathbf{X}^{\beta}$ is related to $g_{t}$ via the relation

$$
\begin{equation*}
p_{t}^{\beta}(x, y)=e^{\frac{\gamma^{2}}{2} t} e^{\beta(y-x)} g_{t}^{\beta}(x, y), \quad \gamma^{2}=1-\beta^{2} \tag{29}
\end{equation*}
$$

Lemma 3.1. Suppose that $L=L(\beta)$ is as in Definition 41. Then,

$$
p_{t}^{\beta}(x, y)=e^{\frac{1-\beta^{2}}{2} t} e^{\beta(y-x)} \sum_{k \geq 1} e^{-\frac{\gamma_{k}^{2}}{2}} t \frac{v_{k}(x) v_{k}(y)}{\left\|v_{k}\right\|^{2}}, \quad t>0, x, y \in \Omega_{\beta},
$$

where $\gamma_{1}=\gamma=\sqrt{1-\beta^{2}}$ and, for all $k \in \mathbb{N}, \gamma_{k}$ is the unique solution of

$$
\tan \left(\gamma_{k} L_{\beta}\right)=-\frac{\gamma_{k}}{\beta}, \quad \text { such that } \quad \gamma_{k} L \in\left[\left(k-\frac{1}{2}\right) \pi, k \pi\right]
$$

and

$$
v_{k}(x)=\sin \left(\gamma_{k}\left(L_{\beta}-x\right)\right), x \in\left[0, L_{\beta}(\beta)\right], \quad\left\|v_{k}\right\|^{2}=\frac{1}{2}\left(L_{\beta}+\frac{1}{\beta} \cos \left(\gamma_{i} L\right)^{2}\right) .
$$

Moreover, we have

$$
\begin{equation*}
\sin \left(\gamma L_{\beta}\right)=\gamma, \quad \cos \left(\gamma L_{\beta}\right)=-\beta, \quad\left\|v_{1}\right\|^{2}=\frac{L_{\beta}+\beta}{2} \tag{30}
\end{equation*}
$$

Proof. The formula for $p_{t}$ follows directly from points (1) to (6) and from a straightforward calculation. We leave this calculation to the reader. The last part of the result follows from the observation that

$$
\tan \left(\gamma L_{\beta}\right)=-\frac{\gamma}{\beta}, \quad \text { and } \quad \beta^{2}+\gamma^{2}=1, \quad \beta \in(0,1), \gamma \in(0,1)
$$

Lemma 3.2. There exists $L_{0}>0$ such that, for all $L_{\beta}>L_{0}$, we have

$$
\left\{\begin{array}{l}
\gamma\left(\frac{1}{2 \pi} x+1\right) \leq v_{1}(x) \leq \gamma(\beta x+1), \quad 0 \leq x \leq L_{\beta}-\frac{\pi}{2 \gamma} \\
\frac{2 \gamma}{\pi}\left(L_{\beta}-x\right) \leq v_{1}(x) \leq \gamma\left(L_{\beta}-x\right), \quad L_{\beta}-\frac{\pi}{2 \gamma} \leq x \leq L_{\beta}
\end{array}\right.
$$

and for $k \geq 1$, we have

$$
\left|v_{k}(x)\right| \leq \begin{cases}\gamma_{k}(x+2) & 0 \leq x \leq L_{\beta}-\frac{\pi}{2 \gamma} \\ \gamma_{k}\left(L_{\beta}-x\right) & L_{\beta}-\frac{\pi}{2 \gamma} \leq x \leq L_{\beta}\end{cases}
$$

Proof. Using a convexity argument combined with (30), one can show that

$$
\begin{cases}\gamma\left(\frac{2(1-\gamma)}{2 \gamma L_{\beta}-\pi} x+1\right) \leq v_{1}(x) \leq \gamma(\beta x+1), & 0 \leq x \leq L_{\beta}-\frac{\pi}{2 \gamma} \\ \frac{2 \gamma}{\pi}\left(L_{\beta}-x\right) \leq v_{1}(x) \leq \gamma\left(L_{\beta}-x\right), & L_{\beta}-\frac{\pi}{2 \gamma} \leq x \leq L_{\beta} .\end{cases}
$$

Then, Remark 1 shows that, for $L_{\beta}$ large enough, $2 \gamma L_{\beta}-\pi \leq 2 \pi$ and $2(1-\gamma) \geq 1$. Similarly, for $k \geq 1$, we see that

$$
\left|v_{k}(x)\right| \leq \begin{cases}\beta\left|\sin \left(\gamma_{k} L\right)\right| x+\sin \left(\gamma_{k} L_{\beta}\right) & 0 \leq x \leq L_{\beta}-\frac{\pi}{2 \gamma} \\ \gamma_{k}\left(L_{\beta}-x\right) & L_{\beta}-\frac{\pi}{2 \gamma} \leq x \leq L\end{cases}
$$

We then recall from Lemma 3.1 that

$$
\begin{equation*}
\left|\sin \left(\gamma_{k} L\right)\right|=\frac{\gamma_{k}}{\beta}\left|\cos \left(\gamma_{k} L\right)\right| \leq \frac{\gamma_{k}}{\beta} \tag{31}
\end{equation*}
$$

and use that for $L$ large enough, $\beta>1 / 2$.
Corollary 3.3. The same convexity argument shows that

$$
\forall x \in \Omega_{\beta}, \quad \gamma\left(\frac{2(1-\gamma)}{2 \gamma L_{\beta}-\pi} x+1\right) \wedge \frac{2 \gamma}{\pi}\left(L_{\beta}-x\right) \leq v_{1}(x) \leq \gamma(\beta x+1) \wedge \gamma\left(L_{\beta}-x\right)
$$

Lemma 3.4. (i) We have

$$
\forall \beta \in(0,1), \forall t>0, \forall x, y \in \Omega_{\beta}, \quad\left|p_{t}(x, y)-h(x) \tilde{h}(y)\right| \leq 4 e^{-\frac{\pi^{2}}{L^{2}} t}\left(\sum_{k \geq 2} k^{2} e^{-\frac{\left(k^{2}-4\right) \pi^{2}}{L^{2}} t}\right) h(x) \tilde{h}(y) .
$$

(ii) Let $\varepsilon>0$. There exists a constant $C_{\varepsilon}>0$ such that, for all $L_{\beta}$ large enough,

$$
\forall t>c_{\varepsilon}\left(L_{\beta}\right)^{2}, \forall x, y \in \Omega_{\beta}, \quad\left|p_{t}^{\beta}(x, y)-h(x) \tilde{h}(y)\right| \leq \varepsilon h^{\beta}(x) \tilde{h}^{\beta}(y)
$$

(iii) There exists $\alpha>0$ such that, for $L_{\beta}$ large enough,

$$
\forall t>\left(L_{\beta}\right)^{3}, \forall x, y \in \Omega_{\beta}, \quad\left|p_{t}^{\beta}(x, y)-h(x) \tilde{h}(y)\right| \leq e^{-\alpha L} h^{\beta}(x) \tilde{h}^{\beta}(y)
$$

Proof. First, note that $p_{t}$ can be written as

$$
p_{t}(x, y)=h(x) \tilde{h}(y)+e^{\beta(y-x)} \sum_{k \geq 2} e^{\frac{1}{2}\left(\gamma_{1}^{2}-\gamma_{k}^{2}\right) t} \frac{v_{k}(x) v_{k}(y)}{\left\|v_{k}\right\|^{2}} .
$$

Then, remark that for all $k \geq 2$,

$$
\frac{\left\|v_{1}\right\|^{2}}{\left\|v_{k}\right\|^{2}} \leq \frac{1}{1+\frac{1}{\beta L_{\beta}}} \leq 1, \quad \text { and } \quad \gamma^{2}-\gamma_{k}^{2} \leq-\frac{k^{2} \pi^{2}}{2 L_{\beta}^{2}}
$$

For $x \in\left[L_{\beta}-\frac{\pi}{2 \gamma}, L_{\beta}\right]$, we see from Lemma 3.1 and Lemma 3.2 that

$$
\left|\frac{v_{k}(x)}{v_{1}(x)}\right| \leq \frac{\pi}{2} \frac{\gamma_{k}}{\gamma} \leq k \pi
$$

Similarly, for $x \in\left[0, L_{\beta}-\frac{\pi}{2 \gamma}\right]$, we get that

$$
\left|\frac{v_{k}(x)}{v_{1}(x)}\right| \leq\left(\frac{\gamma}{\gamma_{k}}\right) \frac{\frac{1}{2 \pi} x+1}{x+2} \leq 2 k .
$$

Putting all of this together, we finally get that

$$
e^{\beta(y-x)} \sum_{k \geq 2} e^{\frac{1}{2}\left(\gamma_{1}^{2}-\gamma_{k}^{2}\right) t} \frac{\left|v_{k}(x) v_{k}(y)\right|}{\left\|v_{k}\right\|^{2}} \leq\left(4 \sum_{k \geq 2} k^{2} e^{-\frac{k^{2} \pi^{2}}{4 L^{2}} t}\right) h(x) \tilde{h}(y)
$$

which concludes the proof of the lemma.

### 3.2 Proof of Proposition 1.2

For the sake of clarity, we drop the $N$ superscripts/indices in the following calculations and write $L_{N} \equiv L_{\beta}$, $\beta_{N} \equiv \beta, \gamma_{N} \equiv \gamma$ and $v_{1}^{N} \equiv v_{1}$.

Recall from (8) that

$$
\begin{equation*}
\Sigma(z)^{2}=\frac{4 \gamma e^{\beta L}}{(L+\beta)^{2}} \int_{0}^{z} e^{-\beta x} v_{1}(x)^{3} d x=: \frac{4 \gamma e^{\beta L}}{(L+\beta)^{2}} I_{z} . \tag{32}
\end{equation*}
$$

First, we remark that for all $\alpha>0$ and $0 \leq x_{1} \leq x_{2} \leq L$, we have

$$
\begin{equation*}
\left(\beta^{2}+\alpha^{2}\right) \int_{x_{1}}^{x_{2}} e^{\beta x} \sin (\alpha x) d x=\beta\left[e^{\beta x_{2}} \sin \left(\alpha x_{2}\right)-e^{\beta x_{1}} \sin \left(\alpha x_{1}\right)\right]-\alpha\left[e^{\beta x_{2}} \cos \left(\alpha x_{2}\right)-e^{\beta x_{1}} \cos \left(\alpha x_{1}\right)\right] . \tag{33}
\end{equation*}
$$

This equality can be obtained by consecutive integrations by parts. We will also make use of the following identities:

$$
\begin{align*}
& \forall x \in \mathbb{R}, \quad \sin ^{3}(x)=\frac{1}{4}(3 \sin (x)-\sin (3 x)), \quad \cos ^{3}(x)=\frac{1}{4}(3 \cos (x)+\cos (3 x)) \\
& \sin (\gamma L)=\gamma, \quad \cos \gamma L=-\beta, \quad \sin (3 \gamma L)=3 \gamma-4 \gamma^{3}, \quad \cos (3 \gamma L)=3 \beta-4 \beta^{3}, \quad \gamma^{2}+\beta^{2}=1 \tag{34}
\end{align*}
$$

where the second and the third inequalities are established in Lemma 3.1 Thus,

$$
I_{z}=\frac{3}{4} \int_{0}^{z} e^{-\beta x} \sin (\gamma(L-x)) d x-\frac{1}{4} \int_{0}^{z} e^{-\beta x} \sin (3 \gamma(L-x)) d x .
$$

It follows from a change a variable, (33) and (34) that

$$
\begin{aligned}
I_{z, 1} & :=\int_{0}^{z} e^{-\beta x} \sin (\gamma(L-x)) d x=e^{-\beta L} \int_{L-z}^{L} e^{\beta y} \sin (\gamma y) d y \\
& =\beta\left(\sin (\gamma L)-e^{-\beta z} \sin (\gamma(L-z))\right)-\gamma\left(\cos (\gamma L)-e^{-\beta z} \cos (\gamma(L-z))\right) \\
& =2 \beta \gamma+e^{-\beta z}(\gamma \cos (\gamma(L-z))-\beta \sin (\gamma(L-z)))
\end{aligned}
$$

Similarly, we get that

$$
\begin{align*}
I_{z, 3} & :=\int_{0}^{z} e^{-\beta z} \sin (3 \gamma(L-x)) d x=e^{-\beta L} \int_{L-z}^{L} e^{\beta y} \sin (3 \gamma y) d y  \tag{35}\\
& =\left(1+8 \gamma^{2}\right)^{-1}\left[\beta\left(\sin (3 \gamma L)-e^{-\beta z} \sin (3 \gamma(L-z))\right)-3 \gamma\left(\cos (3 \gamma L)-e^{-\beta z} \cos (3 \gamma(L-z))\right)\right] \\
& =\left(1+8 \gamma^{2}\right)^{-1}\left[\beta \sin (3 \gamma L)-3 \gamma \cos (3 \gamma L)+e^{-\beta z}(3 \gamma \cos (3 \gamma(L-z))-\beta \sin (3 \gamma(L-z)))\right] \\
& =\left(1+8 \gamma^{2}\right)^{-1}\left[6 \beta \gamma-16 \beta \gamma^{3}+e^{-\beta z}(3 \gamma \cos (3 \gamma(L-z))-\beta \sin (3 \gamma(L-z)))\right] \\
& =\left(1+8 \gamma^{2}\right)^{-1} \tilde{I}_{z, 3} .
\end{align*}
$$

In particular, for $z=L$, we have

$$
\begin{equation*}
I_{L, 1}=2 \beta \gamma+\gamma e^{-\beta L}, \quad \text { and } \quad I_{L, 3}=\left(1+8 \gamma^{2}\right)^{-1}\left(6 \beta \gamma-16 \beta \gamma^{3}+3 \gamma e^{-\beta L}\right), \tag{36}
\end{equation*}
$$

so that,

$$
I_{L}=\frac{3}{4} I_{L, 1}-\frac{1}{4} I_{L, 3}=\frac{1}{4}\left(64 \gamma^{3}+o\left(\gamma^{3}\right)\right),
$$

which, together with (32, 10) and 11) yields (with $\sigma^{2}=64 \pi^{2} / c^{6}$ ).
Note that

$$
\begin{aligned}
3 I_{z, 1}-\tilde{I}_{z, 3} & =16 \beta \gamma^{3}+e^{-\beta z}(3 \gamma(\cos (\gamma(L-z)-\cos (3 \gamma(L-z))))+\beta(\sin (3 \gamma(L-z))-3 \beta \sin (\gamma(L-z)))) \\
& =16 \beta \gamma^{3}+e^{-\beta z}\left(12 \gamma \sin (\gamma(L-z))^{2} \cos (\gamma(L-z))-4 \beta \sin (\gamma(L-z))^{3}\right) \\
& =16 \beta \gamma^{3}-4 e^{-\beta z} \sin (\gamma(L-z))^{2}(\beta \sin (\gamma(L-z))-3 \gamma \cos (\gamma(L-z)))
\end{aligned}
$$

where we used (34) and the identity $\cos (x)-\cos (y)=-2 \sin ((x-y) / 2) \sin ((x+y) / 2)$ to get the second line. Let us now assume that $1 \ll z \ll L$. Then, $\gamma \ll \gamma z \ll 1$ and a Taylor expansion then shows that

$$
\tilde{I}_{z, 3}=6 \beta \gamma-16 \beta \gamma^{3}+e^{-\beta z}(3 \gamma z+o(\gamma z)),
$$

and that

$$
3 I_{z, 1}-\tilde{I}_{z, 3}=16 \beta \gamma^{3}-4 e^{-\beta z}\left((\gamma z)^{3}+o\left((\gamma z)^{3}\right)\right) .
$$

For $z=A_{N} \equiv A$ (see 15 ), one can check that $z^{3} e^{-\beta z}$ tends to 0 as $N$ goes to $\infty$. Combining this with (35) yields the first part of 16 .

We now compute the quantity

$$
J_{z}:=\int_{0}^{z} \tilde{h}(x) d x=\frac{1}{\gamma} \int_{0}^{z} e^{\beta(x-L)} \sin (\gamma(L-x)) d x=\frac{1}{\gamma} \int_{L-z}^{L} e^{-\beta x} \sin (\gamma x) d x
$$

An integration by part shows that, for all $0 \leq x_{1} \leq x_{2} \leq L$, we have

$$
\left(\beta^{2}+\alpha^{2}\right) \int_{x_{1}}^{x_{2}} e^{-\beta x} \sin (\alpha x) d x=-\beta\left[e^{-\beta x_{2}} \sin \left(\alpha x_{2}\right)-e^{-\beta x_{1}} \sin \left(\alpha x_{1}\right)\right]-\alpha\left[e^{-\beta x_{2}} \cos \left(\alpha x_{2}\right)-e^{-\beta x_{1}} \cos \left(\alpha x_{1}\right)\right]
$$

Putting this together with (34) shows that

$$
J_{z}=\frac{1}{\gamma} e^{-\beta(L-z)}(\beta \sin (\gamma(L-z))+\gamma \cos (\gamma(L-z)))
$$

In particular, we see that $J_{L}=1$ (as mentioned in 9). For $1 \ll z \ll L$, a Taylor expansion yields that

$$
J_{z}=\frac{1}{\gamma} e^{-\beta(L-z)}(\gamma z+o(\gamma z))
$$

Plugging $z=A$ in the above equation shows that $N^{c} J_{A} \rightarrow 6$ as $N$ goes to $\infty$. This gives the second part of (16) (with $a=6$ ).
Remark 4. The above calculations show that for all $1 \ll A \ll L$, as $N \rightarrow \infty$,

$$
\frac{1}{N^{c}} \Sigma(A)^{2} \rightarrow \sigma^{2}, \quad \text { and } \quad N \int_{0}^{A} \tilde{h}(x) d x \sim \frac{N^{1-c}}{c^{6} \log (N)^{6}}\left(A e^{\beta A}\right) .
$$

### 3.3 The Green's function

In this section, we derive the Green's function of the system $\mathbf{X}^{\beta}$ to control its small time behaviour (on the time scale of the Feller diffusion). Recall from Section 1.2 that $g_{t}^{\beta}(x, y)$ from (4) refers to the fundamental solution of the PDE

$$
\left\{\begin{array}{l}
\partial_{t} u(t, y)=\frac{1}{2} \partial_{y y} u(t, y), \quad y \in \Omega_{\beta}  \tag{B}\\
\beta u(t, 0)-\partial_{y} u(t, y)_{\mid y=0}=0, \quad u\left(t, L_{\beta}\right)=0
\end{array}\right.
$$

Let $B_{t}$ be a Brownian motion started from $x \in \Omega_{\beta}$ with generator

$$
\begin{gathered}
\mathcal{A} f(y)=\frac{1}{2} \partial_{y y} f(y), \quad y \in \Omega_{\beta} \\
\beta u(0)-u^{\prime}(0)=0, \quad u\left(L_{\beta}\right)=0
\end{gathered}
$$

and $\left.\tau:=\inf \left\{t>0: B_{t} \notin\left[0, L_{\beta}\right)\right]\right\}$. The Green's function $G$ associated to this differential operator is the unique function such that, for every bounded measurable functions $g$, we have

$$
\mathbb{E}\left[\int_{0}^{\tau} g\left(B_{t}\right) d t\right]=\int_{0}^{\infty} G^{\beta}(x, y) g(y) d y
$$

In particular, we have

$$
\int_{0}^{\infty} g_{s}^{\beta}(x, y) d s=G^{\beta}(x, y)
$$

We know from Section 3.1 that $v_{1} \geq 0$ and $\mathcal{A} v_{1} \leq 0$ on $\Omega_{\beta}$. This implies (see [28, Proposition 4.2.3]) that the Green's function $G$ is finite (i.e. the operator $\mathcal{A}$ is subcritical in the sense of 28, Section 4.3]). The next result gives an explicit formula for the Green function $G$.

Lemma 3.5 ([7] p.19]). We have

$$
G^{\beta}(x, y)= \begin{cases}\gamma(\beta x+1) \cdot \gamma\left(L_{\beta}-y\right) /\left[\gamma^{2}\left(\beta L_{\beta}+1\right)\right] & 0 \leq x \leq y \leq L_{\beta} \\ \gamma(\beta y+1) \cdot \gamma\left(L_{\beta}-x\right) /\left[\gamma^{2}\left(\beta L_{\beta}+1\right)\right] & 0 \leq y \leq x \leq L_{\beta}\end{cases}
$$

A Taylor expansion with Lagrange remainder shows that

$$
\forall x \in \Omega_{\beta}, \quad\left|v_{1}(x)-\gamma(\beta x+1)\right| \leqslant \gamma^{2} x^{2}, \quad \text { and } \quad\left|v_{1}(x)-\gamma\left(L_{\beta}-x\right)\right| \leqslant \gamma^{2}\left(L_{\beta}-x\right)^{2} .
$$

The following lemma is a key tool to estimate the variance in the number of particles produced in a time interval of length $C \log (N)^{2}$.
Lemma 3.6. Let $\varepsilon>0$ and let $c_{\varepsilon}$ be as in Lemma 3.4 (ii). Set $t_{0}=c_{\varepsilon}\left(L_{\beta}\right)^{2}$. There exists a constant $D_{\varepsilon}, B_{\varepsilon}>0$ such that

$$
\forall x \in \Omega_{\beta}, \quad \int_{0}^{t_{0}}\left(\int_{0}^{L} h(y) q_{s}(x, y) d y\right) d s \leq D_{\varepsilon}\left(v_{1}(x)\right)^{-1} \int_{0}^{L} h(y) v_{1}(y) G^{\beta}(x, y) d y . \leq B_{\varepsilon} \gamma^{3} e^{\beta L_{\beta}}
$$

Proof. We know from (4) and Definition 4 that

$$
\begin{equation*}
q_{t}^{\beta}(x, y)=\frac{h^{\beta}(y)}{h^{\beta}(x)} p_{t}^{\beta}(x, y)=e^{\frac{1-\beta^{2}}{2} t} \frac{v_{1}(y)}{v_{1}(x)} g_{t}^{\beta}(x, y) \tag{37}
\end{equation*}
$$

Besides, note that, by definition of $t_{0}$, we have $e^{\frac{1-\beta^{2}}{2} t} \leq C$ for all $t \leq t_{0}$. Thus, we have

$$
\int_{0}^{t_{0}} q_{s}^{\beta}(x, y) d s \leq C \frac{v_{1}(y)}{v_{1}(x)} \int_{0}^{\infty} g_{s}^{\beta}(x, y) d s \leq C \frac{v_{1}(y)}{v_{1}(x)} G^{\beta}(x, y)
$$

Hence, by Fubini's theorem,

$$
\int_{0}^{t_{0}}\left(\int_{0}^{L} h(y) q_{s}(x, y) d y\right) d s=\left(v_{1}(x)\right)^{-1} \int_{0}^{L} h(y) v_{1}(y) G^{\beta}(x, y) d y=:\left(v_{1}(x)\right)^{-1} I_{x}
$$

It then follows from Lemma 3.5 and (8),

$$
I_{x}:=\frac{2}{L+\beta} \frac{\gamma\left(L_{\beta}-x\right)}{\gamma\left(\beta L_{\beta}+1\right)} I_{x, 1}+\frac{2}{L+\beta} \frac{\gamma(\beta x+1)}{\gamma\left(\beta L_{\beta}+1\right)} I_{x, 2},
$$

with

$$
I_{x, 1}:=\int_{0}^{x} e^{\beta(L-y)} v_{1}(y)^{2} \gamma(\beta y+1) d y, \quad I_{x, 2}:=\int_{x}^{L} e^{\beta(L-y)} v_{1}(y)^{2} \gamma\left(L_{\beta}-y\right) d y
$$

The second part of Lemma 3.5 implies that

$$
I_{x, 1}=e^{\beta L}\left(\int_{0}^{x} e^{-\beta y} v_{1}(y)^{3} d y+\gamma^{2} \int_{0}^{x} e^{-\beta y} v_{1}(y)^{2} y^{2} d y\right)
$$

One can then check using a convexity argument that, for $L$ large enough,

$$
I_{x, 1} \leq C e^{\beta L} \gamma^{3}(1 \wedge x)
$$

and that the same holds for $I_{x, 2}$ with $L-x$ instead of $x$. As a result, we get that

$$
\frac{2}{L+\beta} \frac{\gamma\left(L_{\beta}-x\right)}{\gamma\left(\beta L_{\beta}+1\right)} I_{x, 1} \leq C e^{\beta L_{\beta}} \gamma^{4} \frac{\gamma\left(L_{\beta}-x\right)(1 \wedge x)}{\gamma\left(\beta L_{\beta}+1\right)} \leq C e^{\beta L} \gamma^{3} e^{\beta L_{\beta}},
$$

where the second inequality is deduced from Lemma 3.2 . The second part of the integral can be bounded with the exact same arguments.

## 4 Many-to-few formula and convergence of moments

The proof of Theorem 1 and Theorem 4 requires to estimate moments of quantities that depend on the position of the particles in the BBM at time $t N^{1-c}$. These calculations can be performed thanks to a many-to-few formula, similar to that given in Theorem 6, provided that we enrich the structure of the mm space of the BBM to keep track of the positions of the particles. This approach is detailed in Section 4.1 Section 4.4 is dedicated to the proof of Proposition 2.6

### 4.1 Marked metric measure space

Let $\left(E, d_{E}\right)$ be a fixed complete separable metric space, referred to as the marked space. In our application $E=\Omega_{\beta}$ is endowed with the usual distance on the real line and the marks will be seen as positions in $\Omega_{\beta}$.
Definition 6 ( 10 ). (i) A marked metric measure space ( mmm space) is a triple ( $X, d, \nu$ ) such that $(X, d)$ is a complete and separable metric space and $\nu$ is a finite measure on $X \times E$. We denote by $\mathbb{M}^{E}$ the set of equivalence classes of mmm spaces.
(ii) Let $(X, d, \nu)$ be a mmm space. Consider the application

$$
R_{k}:=\left\{\begin{array}{l}
(X \times E)^{k} \rightarrow \mathbb{R}_{+}^{k^{2}} \times E^{k} \\
\left(\left(v_{i}, x_{i}\right) ; i \leq k\right) \mapsto\left(d\left(v_{i}, v_{j}\right), x_{i} ; i, j \leq k\right)
\end{array}\right.
$$

that maps $k$ points in $X \times E$ to the matrix of pairwise distances and marks. The marked distance matrix distribution $\nu_{k, X}$ of $(X, d, \nu)$ is defined as the pushforward of $\nu^{\otimes k}$ by the map $R_{k}$, that is $\nu_{k, X}=\nu^{\otimes k} \circ R_{k}^{-1}$.
(iii) A functional $\Psi: \mathbb{M}^{E} \rightarrow \mathbb{R}$ is a polynomial if, there exists $k \in \mathbb{N}$ and a bounded continuous function $\psi:[0, \infty)\left(\begin{array}{c}\binom{k}{2}\end{array} \times E^{k} \rightarrow \mathbb{R}\right.$ such that

$$
\Psi(X, d, \nu)=\left\langle\nu_{k, X}, \psi\right\rangle=\int_{(X \times E)^{k}} \psi\left(d\left(v_{i}, v_{j}\right)_{1 \leq i<j \leq k},\left(x_{i}\right)_{1 \leq i \leq k}\right) \prod_{i=1}^{k} \nu\left(d v_{i} \times d x_{i}\right)
$$

Definition 7. The marked Gromov-weak topology is the topology on $\mathbb{M}^{E}$ induced by the set of polynomials. A random $m m m$ space is a random variable with values in $\mathbb{M}^{E}$, endowed with the marked Gromov-weak topology and the associated Borel $\sigma$-field.

The mmm space associated to the BBM Fix $t>0$ and recall that $\mathcal{N}_{t}^{N}$ refers to the set of particles alive at time $t$ in the BBM $\mathbf{X}^{N}$. Let

$$
\mathbf{M}_{t}^{N}:=\left(\mathcal{N}_{t}^{N}, d_{t}^{N}, \nu_{t}^{N}\right), \quad \text { with } \quad \nu_{t}^{N}:=\sum_{v \in \mathcal{N}_{t}^{N}} \delta_{v, X_{t}^{N}(v)},
$$

and $d_{t}^{N}$ as defined in 24). As for the mm space associated to the BBM (see Section 2.3), we are interested in the rescaled version of this object,

$$
\overline{\mathbf{M}}_{t}^{N}:=\left(\mathcal{N}_{t N^{1-c}}^{N}, \bar{d}_{t}^{N}, \bar{\nu}_{t}^{N}\right), \quad \text { with } \quad \bar{\nu}_{t}^{N}:=\frac{1}{N} \sum_{v \in \mathcal{N}_{t N^{1-c}}^{N}} \delta_{v, X_{t}^{N}(v)}
$$

and $d_{t}$ is rescaled in the same way as in Section 2.3
Theorem 7. Let $t>0, x \in \Omega_{N}$, and $k \in \mathbb{N}, \phi:[0, \infty)^{\binom{k}{2}} \rightarrow \mathbb{R}$ and $\left(\varphi_{i}, i \in[k]\right)$ be bounded continuous functions. Let

$$
\psi\left(d\left(v_{i}, v_{j}\right)_{1 \leq i<j \leq k},\left(x_{i}\right)_{1 \leq i \leq k}\right)=\phi\left(d\left(v_{i}, v_{j}\right)_{1 \leq i<j \leq k}\right) \prod_{i=1}^{k} \varphi_{i}\left(x_{i}\right)
$$

and denote by $\Psi$ the associated polynomial. Recall the definition of the $k$-spine tree distribution $Q^{k, t}$ from Section 2.4. We have

$$
\mathbb{E}_{x}\left[\Psi\left(\mathbf{M}_{t}^{N}\right)\right]=k!h^{N}(x) t^{k-1} Q_{x}^{k, t}\left(\Delta^{N} \phi\left(\left(U_{\sigma(i), \sigma(j)}\right)\right) \prod_{i=1}^{k} \varphi_{i}\left(\zeta_{V_{\sigma(i)}}\right)\right)
$$

where $\Delta^{N}$ is as in Theorem 6 and $\sigma$ is an independent permutation of $[k]$.

Remark 5. If we set $\varphi_{i} \equiv 1$ in the previous result, we obtain the many-to-few lemma for the unmarked space stated in Theorem 6. On the other hand, setting $\phi \equiv 1$ and $\varphi_{i} \equiv h^{N}$ boils down to computing the factorial moments of the random variable

$$
\begin{equation*}
Y(t)^{N}=\sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}(t)\right) \tag{38}
\end{equation*}
$$

### 4.2 Marked planar ultrametric distance matrices

The proof of Theorem 7 relies on a planarisation argument, similar to that developed in 32, Section 3]. We repeat this argument here for the sake of completeness. The main idea of the proof is to use the branching property to derive a recursion formula satisfied both by the moments of the planarised BBM and the spine measure. This formula will also be used in the next sections to compute the moments of the BBM by induction.

The genealogy of our planarised BBM will be encoded by marked binary ultrametric matrices that we now define. We say that a matrix $\left(U_{i, j}\right)_{1 \leq i, j \leq k}$ is planar ultrametric if

$$
\forall i<j<l, \quad U_{i, l}=U_{i, j} \vee U_{j, l} .
$$

Moreover, the matrix $\left(U_{i, j}\right)$ is said to be binary if

$$
U_{i, j} \neq U_{k, l} \quad \Leftrightarrow \quad(i, j) \neq(k, l) \quad \text { and } \quad(i, j) \neq(l, k) .
$$

We denote by $\mathbf{U}_{k}$ the set of binary planar ultrametric matrices of size $k$. Let $\mathbf{U}_{k}^{*}=\mathbf{U}_{k} \times E^{k}$ be the set of marked binary planar distance matrices.

The BBM is planarised by giving an Ulam-Harris label $p_{v} \in \cup_{n \in \mathbb{N}\{0,1\}^{n}}$ to each individual. These labels are assigned recursively:
(i) We label the root with $\emptyset$.
(ii) The label does not vary between two branching points.
(iii) At each branching point $v$, we distribute the labels $\left(p_{v}, 0\right)$ and ( $p_{v}, 1$ ) uniformly among the two children. The child $\left(p_{v}, 0\right)$ (resp. $\left.\left(p_{v}, 1\right)\right)$ is said to be the left (resp. the right) child of the individual $v$.
We denote by $\mathcal{N}_{t}^{N, p l}$ the set of particles at time $t$ in the planarised version of the BBM $\mathbf{X}^{N}$. The Ulam-Harris labelling $\left(p_{v}\right)$ induces an order on $\mathcal{N}_{t}^{N, p l}$ (the lexicographic order). In particular, for every $v_{1}<\ldots<v_{k} \in \mathcal{N}_{t}^{N, p l}$, the marked distance matrix of the sample $\vec{v}=\left(v_{1}, \ldots, v_{k}\right)$, defined as

$$
U^{N}(\vec{v})=\left(d_{t}^{N}\left(v_{i}, v_{j}\right)_{1 \leq i, j \leq k}\right)
$$

is an element of $\mathbf{U}_{k}$.
The recursion formula will be obtained by decomposing each sample $v_{1}<\ldots<v_{k}$ into two subfamilies. This will be achieved by partitioning $[k]$ as follows. For $\left(\left(U_{i, j}\right),\left(x_{i}\right)\right) \in \mathbf{U}_{k}^{*}$, define $\tau(U)=\max _{i \neq j} U_{i, j}$. In words, $\tau(U)$ is the time to the most recent common ancestor (MRCA) of the sample whose distance matrix is given by $U$. We say that the integers $i$ and $j$ are in the same block iff $U_{i, j}<\tau(U)$. Since $U$ is a binary planar ultrametric matrix, there exists an integer $n \leq k-1$ such that this partition can be written as $\{\{1, \ldots, n\},\{n+1, \ldots, k\}\}$. We write $T_{0}(U)$ and $T_{1}(U)$ for the corresponding sub-matrices obtained from this partition and $\left|T_{0}(U)\right|,\left|T_{1}(U)\right|$ for the sizes of the two blocks. Let

$$
\begin{equation*}
d_{e}: U \mapsto\left(\tau(U), T_{0}(U), T_{1}(U)\right) \tag{39}
\end{equation*}
$$

be the application that maps marked distance matrices to its decomposition. This application is continuous on $\mathbf{U}_{k}^{*}$. In particular, this map is continuous on $\mathbf{U}_{k}$.

Note that for a sample $v_{1}<\ldots<v_{k} \in \mathcal{N}_{t}^{N, p l}$ and $U=U(\vec{v}), T_{0}(U)$ (resp. $T_{1}(U)$ ) are the marked distance matrices of the descendants of the left (resp. right) child of the MRCA.
Proposition 4.1. Let $k \in \mathbb{N}, t>0$ and $x \in \Omega_{N}$. Let $R_{x}^{N, k, t}$ be the measure on $\mathbf{U}_{k}^{*}$ such that for every bounded measurable functional $F: \mathbf{U}_{k}^{*} \rightarrow \mathbb{R}$,

$$
R_{x}^{N, k, t}(F)=\mathbb{E}_{x}\left[\sum_{v_{1}<\ldots<v_{k} \in \mathcal{N}_{t}^{N, p l}} F(U(\vec{v}))\right] .
$$

Then, we have

$$
R_{x}^{N, k, t}(F)=h^{N}(x) t^{k-1} Q_{x}^{N, k, t}\left(\Delta^{N} F\left(\left(U_{i, j}\right),\left(\zeta_{V_{i}}\right)\right)\right),
$$

where $\left(U_{i, j}\right)$ is as in 27, $\Delta^{N}$ is as in Theorem 6 and $Q_{x}^{N, k, t}$ as is Section 2.4.
A (sketch of) proof will be given in Section 4.3 for this proposition. We refer to 32 for more details.

### 4.3 Recursion formula for the $k$-spine tree

The proof of Proposition 4.1 relies on a recursion formula for functionals $F: \mathbf{U}_{k}^{*} \rightarrow \mathbb{R}$ of the product form

$$
\begin{equation*}
F(U)=\mathbf{1}_{\left\{\left|T_{0}(U)\right|=n,\left|T_{1}\right|=k-n\right\}} f(\tau(U)) F_{0}\left(T_{0}(U)\right) F_{1}\left(T_{1}(U)\right), \tag{40}
\end{equation*}
$$

where $f: \mathbb{R}^{+} \rightarrow \mathbb{R}, F_{0}: \mathbf{U}_{n}^{*} \rightarrow \mathbb{R}$ and $F_{1}: \mathbf{U}_{k-n}^{*} \rightarrow \mathbb{R}$ are bounded continuous functions and $1 \leq n \leq k-1$. We will show that for such functionals, the family of biased spine measures $\mathbf{L}$, defined by

$$
\frac{d \mathbf{L}_{x}^{N, k, t}}{d Q_{x}^{N, k, t}}=t^{k-1} \Delta^{N}
$$

satisfies a recursion formula. By a slight abuse of notation, for functionals $F: \mathbf{U}_{k}^{*} \rightarrow \mathbb{R}$, we write

$$
\begin{equation*}
\mathbf{L}_{x}^{N, k, t}(F)=t^{k-1} Q_{x}^{N, k, t}\left(\Delta^{N} F\left(\left(U_{i, j}\right),\left(\zeta_{V_{i}}\right)\right)\right) \tag{41}
\end{equation*}
$$

Thanks to this projection, $\mathbf{L}_{x}^{N, k, t}$ can be seen as a measure on $\mathbf{U}_{k}^{*}$ and Proposition 4.1 can be written as

$$
\begin{equation*}
R_{x}^{N, k, t}(F)=h^{N}(x) \mathbf{L}_{x}^{N, k, t}(F) \tag{42}
\end{equation*}
$$

This identity relies on the following observation.
Proposition 4.2. Let $F$ be a functional of the product form as in 40. Then

$$
\begin{equation*}
\mathbf{L}_{x}^{N, k, t}(F)=\frac{1}{2} \int_{0}^{t} f(s) \mathbb{E}_{x}\left[h^{N}\left(\zeta_{t-s}^{N}\right) \mathbf{L}_{\zeta_{t-s}^{N}}^{N, n, s}\left(F_{0}\right) \mathbf{L}_{\zeta_{t-s}^{N}}^{N, k-n, s}\left(F_{1}\right)\right] d s \tag{43}
\end{equation*}
$$

Proof. This is a direct consequence of the construction of the $k$-spine. A detailed proof can be found in 32, Section 3].

As explained in Section 2.5 the appropriate rescaling of the system is to accelerate time by $N^{1-c}$ and to rescale space by $N$. On the other hand, we expect the mass of the rescaled $k$-spine trees to be of order $N^{(k-1) c}$ (see Proposition 2.6. We will thus work on the family of rescaled biased spine measure $\overline{\mathbf{L}}$,

$$
\begin{equation*}
\overline{\mathbf{L}}_{x}^{N, k, t}(F)=\frac{1}{N^{(k-1) c}} \bar{Q}_{x}^{N, k, t}\left(\bar{\Delta}^{N} F\left(\left(\left(U_{i, j}\right),\left(\bar{\zeta}_{V_{i}}\right)\right)\right)\right. \tag{44}
\end{equation*}
$$

and prove that these measures converge to the desired limits.
Corollary 4.3. Let $F$ be a functional of the product form as in 40. Then,

$$
\overline{\mathbf{L}}_{x}^{N, k, t}(F)=\frac{1}{2 N^{c}} \int_{0}^{t} f(s) \mathbb{E}_{x}\left[h^{N}\left(\bar{\zeta}_{t-s}^{N}\right) \overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, n, s}\left(F_{0}\right) \overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, k-n, s}\left(F_{1}\right)\right] d s
$$

Proof of Proposition 4.1. The proof goes along the same lines as 32, Section 3] and we prove the result by induction. The case $k=1$ is the well-known 'many-to-one' lemma (Lemma 1.1. It shows that, for a bounded measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$
\begin{equation*}
R_{x}^{N, 1, t}(f)=\int_{\Omega_{\beta}} f(y) p_{t}^{N}(x, y) d y=h^{N}(x) \int_{\Omega_{\beta}} f(y) \frac{1}{h^{N}(y)} q_{t}(x, y) d y \tag{45}
\end{equation*}
$$

where the second inequality is a consequence of Definition 4 For $k \geq 2$ and $F$ of the product form 40), we use the Markov property and the many-to-one lemma to show that

$$
\begin{aligned}
R_{x}^{N, k, t}(F) & =\mathbb{E}_{x}\left(\frac{1}{2} \int_{0}^{t} f(s) \sum_{v \in \mathcal{N}_{t-s}^{p l}} \mathbb{E}_{X_{v}^{N}}\left(\sum_{\substack{v_{1}<\cdots<v_{n} \\
\left|v_{i}\right|=s}} F_{0}(U(\vec{v}))\right) \mathbb{E}_{X_{v}^{N}}\left(\sum_{\substack{w_{1}<\cdots<w_{k-n} \\
\left|w_{i}\right|=s}} F_{1}(U(\vec{w}))\right) d s\right) \\
& =h(x) \int_{0}^{t} \frac{1}{2} f(s) \int_{y \in \Omega_{N}} \frac{q_{t-s}(x, y)}{h(y)} \mathbb{E}_{y}\left(\sum_{\substack{ \\
v_{1}<\cdots<v_{n} \\
\left|v_{i}\right|=s}} F_{0}(U(\vec{v}))\right) \mathbb{E}_{y}\left(\sum_{\substack{w_{1}<\cdots<w_{k-n} \\
\left|w_{i}\right|=s}} F_{1}(U(\vec{w}))\right) d y d s \\
& =h(x) \int_{0}^{t} \frac{1}{2} f(s) \int_{y \in \Omega_{N}} \frac{q_{t-s}^{N}(x, y)}{h^{N}(y)} R_{y}^{N, n, s}\left(F_{0}\right) R_{y}^{N, k-n, s}\left(F_{1}\right) d y d s .
\end{aligned}
$$

The result then follows by induction using Proposition 4.2 We refer to [32, Section 3] for further details.

### 4.4 Convergence of the planar moments of the mm space

Lemma 4.4 (Rough bounds). Let $k \geq 1$ and $T>0$. There exists a constant 4.4 $\equiv 44.4(k, T)$ such that

$$
\forall 0<T \leq T, \forall x \in \Omega_{N}, \forall N, \quad \overline{\mathbf{L}}_{x}^{N, k, t}(1) \leq 4.4
$$

Proof. Let $t_{0}:=\frac{\log (N)^{2}}{N^{1-c}}$. For $k=1$, the existence of $4.4(1, T)$ follows from Lemma 3.4 and (9) for $t \geq t_{0}$ and from a coupling with a BBM with no reflective boundary for $t \in\left[0, t_{0}\right]$. In the latter case, a Girsanov transform shows that the expected number of particles in the system is bounded by $\exp \left(\frac{1-\beta^{2}}{2} t_{0} N^{1-c}\right)$, which is bounded (see Remark 1).

For $k=2$, we see from Corollary 4.3 that

$$
\overline{\mathbf{L}}_{x}^{N, 2, t}(1)=\frac{1}{2 N^{c}} \int_{0}^{t} \mathbb{E}_{x}\left[h^{N}\left(\bar{\zeta}_{t-s}^{N}\right) \overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, 1, s}(1)^{2}\right] d s \leq \mathbb{4 . 4}(1, T)^{2} \frac{1}{2 N^{c}} \int_{0}^{t} \int_{\Omega_{N}} h^{N}(y) \bar{q}_{s}^{N}(x, y) d y d s
$$

We see from Lemma 3.4 that, for $N$ large enough,

$$
\forall t \geq t_{0}, \quad \bar{q}_{s}^{N}(x, y) \leq c_{1} \Pi^{N}(y)
$$

so that, by Proposition 1.2

$$
\frac{1}{2 N^{c}} \int_{t_{0}}^{t} \int_{\Omega_{N}} h^{N}(y) \bar{q}_{s}^{N}(x, y) d y d s \leq c_{1} \sigma^{2} t
$$

We now estimate the second part of the integral thanks to the Green's function. First, recall from Definition 4 and 29 that

$$
\begin{equation*}
q_{t}^{N}(x, y)=\frac{h^{N}(y)}{h^{N}(x)} p_{t}^{N}(x, y)=\frac{v_{1}^{N}(y)}{v_{1}^{N}(x)} e^{\frac{1-\left(\beta_{N}\right)^{2}}{2} t} g_{t}^{N}(x, y) \tag{46}
\end{equation*}
$$

Remark 1 shows that the exponential term in the RHS of the above is bounded by a constant for all $t \leq t_{0} N^{1-c}$, so that

$$
\forall t \leq t_{0}, \quad \int_{0}^{t_{0}} \bar{q}_{s}^{N}(x, y) d s=\frac{1}{N^{1-c}} \int_{0}^{t_{0}} q_{u}^{N}(x, y) d u \leq \frac{C}{N^{1-c}} \frac{v_{1}^{N}(y)}{v_{1}^{N}(x)} G^{N}(x, y)
$$

Using Lemma 3.6 we see that

$$
\overline{\mathbf{L}}_{x}^{N, 2, t_{0}}(1) \leq \frac{1}{N}\left(\gamma_{N}\right)^{3} e^{\beta_{N} L_{N}} \leq C \frac{\left(L_{N}\right)^{3}}{N^{c-1}} \xrightarrow{N \rightarrow \infty} 0
$$

This concludes the proof of the result for $k=2$.
Let us now assume that the result holds for $k-1 \geq 2$. It follows from Corollary 4.3 that

$$
\overline{\mathbf{L}}_{x}^{N, k, t}(1)=\frac{1}{2 N^{c}} \int_{0}^{t} \mathbb{E}_{x}\left[h^{N}\left(\bar{\zeta}_{s}^{N}\right) \sum_{n=1}^{k} \overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, n, t-s}(1) \overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, k-n, t-s}(1)\right] d s
$$

By induction, we get that

$$
\forall t \leq T, \quad \overline{\mathbf{L}}_{x}^{N, k, t}(1) \leq\left(\frac{1}{2 N^{c}} \int_{0}^{t} \mathbb{E}_{x}\left[h^{N}\left(\bar{\zeta}_{t-s}^{N}\right)\right] d s\right)\left(\sum_{n=1}^{k} \underset{44.4}{ }(n) \underset{4.4}{ }(k-n)\right) .
$$

Note that the first factor on the RHS of the above equation is $\overline{\mathbf{L}}_{x}^{N, 2, t}(1)$. We can then conclude by using the first part of the proof.

The second result of this section shows that the convergence of moments is uniform in the starting point $x$. Lemma 4.5. Let $t>0$ and $k \in \mathbb{N}$. Let $F$ be a functional of the product form 40. Then, as $N \rightarrow \infty$,

$$
\sup _{x \in \Omega_{N}}\left|\overline{\mathbf{L}}_{x}^{N, k, t}(F)-\overline{\mathbf{L}}_{\Pi N^{N}}^{N, k, t}(F)\right| \rightarrow 0
$$

Proof. Let $\varepsilon>0$ and $c_{\varepsilon}$ be as in Lemma 3.4 Define $t_{1}=c_{\varepsilon}\left(L_{N}\right)^{2} / N^{1-c}$ and let $D_{t}^{N}(x, y)=q_{t}^{N}(x, y)-\Pi^{N}(y)$. Our recursion formula (see Corollary 4.3) shows that

$$
\begin{aligned}
\overline{\mathbf{L}}_{x}^{N, k, t}(F)-\overline{\mathbf{L}}_{\Pi^{N}}^{N, k, t}(F) & =\frac{1}{2 N^{c}} \int_{0}^{t-t_{1}} f(s) \int_{\Omega_{N}} h^{N}(y) D_{(t-s) N^{1-c}}^{N}(x, y) \overline{\mathbf{L}}_{y}^{N, n, s}\left(F_{0}\right) \overline{\mathbf{L}}^{N, k-n, s}\left(F_{1}\right) d y d s \\
& +\frac{1}{2 N^{c}} \int_{t_{1}}^{t-t_{1}} f(s) \int_{\Omega_{N}} h^{N}(y) q_{(t-s) N^{1-c}}^{N}(x, y) \overline{\mathbf{L}}_{y}^{N, n, s}\left(F_{0}\right) \overline{\mathbf{L}}^{N, k-n, s}\left(F_{1}\right) d y d s \\
& +\frac{1}{2 N^{c}} \int_{t_{1}}^{t-t_{1}} f(s) \int_{\Omega_{N}} h^{N}(y) \Pi_{(t-s) N^{1-c}}^{N}(y) \overline{\mathbf{L}}_{y}^{N, n, s}\left(F_{0}\right) \overline{\mathbf{L}}^{N, k-n, s}\left(F_{1}\right) d y d s \\
& =: A_{1}+A_{2}+A_{3} .
\end{aligned}
$$

Let us first bound $A_{1}$. We deduce from Lemma 3.4 that for $N$ large enough,

$$
\begin{equation*}
\forall u \geq t_{1}, \forall x, y \in \Omega_{N}, \quad\left|D_{u N}^{N}(x, y)\right| \leq \varepsilon \Pi^{N}(y) \tag{47}
\end{equation*}
$$

We then deduce from Lemma 4.4 and Proposition 1.2 that, for $N$ large enough,

$$
\left|A_{1}\right| \leq \frac{1}{N^{c}} C \varepsilon \int_{\Omega_{N}} h^{N}(y) \Pi^{N}(y) d y \leq C \varepsilon \frac{1}{N^{c}}\left(\Sigma^{N}\left(L_{N}\right)\right)^{2} \leq C \varepsilon
$$

The quantity $A_{3}$ can be bounded in the exact same way, using that $t_{1} \rightarrow 0$ as $N \rightarrow \infty$.
We prove that $A_{2}$ is also small using the Green's function of the BBM. First, we use Lemma 4.4 and the triangle inequality to see that

$$
\left|A_{2}\right| \leq \frac{1}{N^{c}} C \int_{0}^{t_{1}} \int_{\Omega_{N}} h^{N}(y) \bar{q}_{s}(x, y) d y d s \leq \frac{1}{N} C \int_{0}^{t_{1} N^{1-c}} h(y) q_{s}(x, y) d y d s
$$

It then follows from Lemma 3.6 that

$$
\left|A_{2}\right| \leq C\left(L_{N}\right)^{3} N^{c-1} \leq \varepsilon,
$$

for $N$ large enough. This concludes the proof of the result.
Lemma 4.6. Let $t>0$ and $k \in \mathbb{N}$. Let $F$ be a functional of the product form 40. Uniformly in $x \in \Omega_{N}$,

$$
\overline{\mathbf{L}}_{x}^{N, k, t}(F) \xrightarrow{N \rightarrow \infty} \mathcal{L}^{k, t}(F),
$$

where $\mathcal{L}$ is the unique family of measures on $\cup_{k \in \mathbb{N}} \mathbf{U}^{k}$ characterised by

$$
\forall t \geq 0, \quad \mathcal{L}^{1, t}(1)=1
$$

and for all $k \geq 2$ and all functionals $F$ of the product form (40),

$$
\mathcal{L}^{k, t}(F)=\frac{\sigma^{2}}{2} \int_{0}^{t} f(s) \mathcal{L}^{n, s}\left(F_{0}\right) \mathcal{L}^{k-n, s}\left(F_{1}\right) d s
$$

Proof. According to Lemma 4.5 it is enough to prove the result for the biased spine measure rooted at a random location, distributed according to $\Pi^{N}$.

We prove the result by induction. For $k=1$,

$$
\overline{\mathbf{L}}_{\Pi^{N}}^{N, 1, t}(1)=\mathbb{E}_{\Pi^{N}}\left[1 / h^{N}\left(\zeta_{t}^{N}\right)\right]=\int_{\Omega_{N}} \tilde{h}^{N}(y) d y=1
$$

where we use $\sqrt{9}$ to get the last equality. For $k \geq 2$, we recall from Corollary 4.3 that

$$
\overline{\mathbf{L}}_{\Pi^{N}}^{N, k, t}(F)=\frac{1}{2 N^{c}} \int_{0}^{t} f(s) \int_{\Omega_{N}} \Pi^{N}(y) h^{N}(y) \overline{\mathbf{L}}_{y}^{N, n, s}\left(F_{0}\right) \overline{\mathbf{L}}_{y}^{N, k-n, s}\left(F_{1}\right) d y d s
$$

It then follows from Proposition 1.2 the dominated convergence theorem that

$$
\frac{1}{2 N^{c}} \int_{0}^{t} f(s) \int_{\Omega_{N}} \Pi^{N}(y) h^{N}(y) \overline{\mathbf{L}}_{y}^{N, n, s}\left(F_{0}\right) \overline{\mathbf{L}}_{y}^{N, k-n, s}\left(F_{1}\right) d y d s \rightarrow \frac{\sigma^{2}}{2} \int_{0}^{t} f(s) \mathcal{L}^{n, s}\left(F_{0}\right) \mathcal{L}^{k-n, s}\left(F_{1}\right) d s
$$

### 4.5 Planar moments of the CPP

Recall the definition of the Brownian CPP $M_{C P P_{T}}$ of height $T$ as introduced in Section 2.2, hence

$$
M_{C P P_{T}}=\left(\left[0, Y_{T}\right], d_{P}, \mathrm{Leb}\right),
$$

where $Y_{T}$ is exponentially distributed with parameter $1 / T$. We define a planar moment of order $k$ of $M_{C P P_{T}}$ with respect to $\Phi$ as

$$
\begin{equation*}
\mathcal{R}^{k, T}(\Phi)=\mathcal{R}^{k, T}\left(\Phi\left(M_{C P P_{T}}\right)\right)=T^{-1} \mathbb{E}\left[\int_{\substack{v_{1}, \ldots, v_{k} \in\left[0, Y_{T}\right] \\ v_{1}<\ldots<v_{k}}} \phi(U(\vec{v})) \prod_{i=1}^{k} d v_{i}\right], \tag{48}
\end{equation*}
$$

where we recall that $U(\vec{v})$ is the distance matrix spanned by $\vec{v}$, i.e. we have

$$
\begin{equation*}
U_{i, j}=d_{P}\left(v_{i}, v_{j}\right), \quad i, j \leq k . \tag{49}
\end{equation*}
$$

In particular, the functionals considered in this definition can be restricted to planar matrices.
Proposition 4.7 (planar moments of the CPP). Let $T>0$ and $k \in \mathbb{N}$. Let $F$ be a functional of the product form (40) and $R^{k, T}$ be as in 48). Then we have the following recursive formula for $R^{k, T}$

$$
\mathcal{R}^{k, T}(F)=\int_{0}^{T} f(s) \mathcal{R}^{n, s}\left(F_{0}\right) \mathcal{R}^{n, s}\left(F_{1}\right) d s
$$

Proof. We have for any polynomial $\Phi$

$$
\begin{aligned}
\mathcal{R}^{k, T}(\Phi) & =\frac{\mathbb{E}\left[Y_{T}^{k}\right]}{T} \mathbb{E}\left[\frac{Y_{T}^{k}}{\mathbb{E}\left[Y_{T}^{k}\right]} \frac{1}{Y_{T}^{k}} \int_{\left(v_{1}, \ldots, v_{k}\right) \in\left[0, Y_{T}\right]^{k}} \phi\left(\left(d_{P}\left(v_{i}, v_{j}\right)_{i, j \leq k}\right)\right) \prod_{i=1}^{k} d v_{i}\right] \\
& =\frac{\mathbb{E}\left[Y_{T}^{k}\right]}{k!T} \mathbb{E}\left[\frac{Y_{T}^{k}}{\mathbb{E}\left[Y_{T}^{k}\right]} \mathbb{E}\left[\phi\left(d_{P}\left(V_{i}^{*} Y_{T}, V_{j}^{*} Y_{T}\right)_{i, j \leq k}\right)\right]\right],
\end{aligned}
$$

where $V_{1}, \ldots, V_{k}$ are i.i.d. uniform random variables on $[0,1]$ and $V_{1}^{*}, \ldots, V_{k}^{*}$ are the respective order statistics. Recalling that $Y_{T}$ is an exponential random variable with parameter $\theta=1 / T$, we get that $\mathbb{E}\left[Y_{T}^{k}\right]=k!\theta^{-k}$ and

$$
\begin{aligned}
\mathbb{E}\left[\frac{Y_{T}^{k}}{\mathbb{E}\left[Y_{T}^{k}\right]} \phi\left(d_{P}\left(V_{i}^{*} Y_{T}, V_{j}^{*} Y_{T}\right)_{i, j \leq k}\right)\right] & =\int_{0}^{\infty} \frac{\theta^{k+1} y^{k}}{k!} e^{-\theta y} \mathbb{E}\left[\phi\left(d_{P}\left(V_{i}^{*} y, V_{j}^{*} y\right)_{i, j \leq k}\right)\right] d y \\
& =\mathbb{E}\left[\phi\left(d_{P}\left(V_{i}^{*} W, V_{j}^{*} W\right)_{i, j \leq k}\right)\right],
\end{aligned}
$$

where $W$ is $\Gamma$-distributed with parameters $k+1$ and $\theta$. Altogether, we arrive at

$$
\begin{equation*}
\mathcal{R}^{k, T}(\Phi)=\theta^{-k+1} \mathbb{E}\left[\phi\left(d_{P}\left(V_{i}^{*} W, V_{j}^{*} W\right)_{i, j \leq k}\right)\right] \tag{50}
\end{equation*}
$$

Now, making use of the definition of $F$ in and the formula derived above, we get

$$
\begin{equation*}
\mathcal{R}^{k, T}(F)=\theta^{-k+1} \mathbb{E}\left[f\left(\tau\left(U_{k}\right)\right) F_{0}\left(T_{0}\left(U_{k}\right)\right) F_{1}\left(T_{1}\left(U_{k}\right)\right) \mathbf{1}_{\left|T_{0}\left(U_{k}\right)\right|=n} \mathbf{1}_{\left|T_{1}\left(U_{k}\right)\right|=k-n}\right], \tag{51}
\end{equation*}
$$

where $U_{k}$ is the random binary ultrametric matrix generated by $U_{i, j}=d_{P}\left(V_{i}^{*} W, V_{j}^{*} W\right)$. As a first step we aim for a more precise understanding of that distance matrix. Let us define

$$
H_{i}^{*}=d_{P}\left(V_{i}^{*} W, V_{i+1}^{*} W\right), \quad i \leq k-1,
$$

to be the genealogical distance between individual $i$ and $i+1$. Recall, that the metric $d_{P}$ is induced by the Poisson Point Process $\mathcal{P}$ on $\mathbb{R}_{+}^{2}$ with intensity $d t \otimes x^{-2} d x$ by setting

$$
\begin{equation*}
d_{P}(x, y)=\sup \{t:(z, t) \in \mathcal{P}, \text { and } x \leq z \leq y\}, 0<x<y<Y_{T} . \tag{52}
\end{equation*}
$$

By standard properties of the Gamma distribution one sees that the length of the intervals $\left[V_{i}^{*} W, V_{i+1}^{*} W\right.$ ] are independent exponentially distributed with parameter $\theta=1 / T$. Hence,

$$
\mathbb{P}\left(H_{i}^{*} \leq s\right)=\mathbb{P}\left(\left|\mathcal{P}\left(\left(V_{i}^{*} W, V_{i+1}^{*} W\right) \times(s, T)\right)\right|=0\right)=\int_{0}^{\infty} \frac{1}{T} e^{-\frac{y}{T}} e^{-y\left(\frac{1}{s}-\frac{1}{T}\right)} d y=\frac{s}{T},
$$

i.e. the distances $H_{i}^{*}$ are independent and uniform distributed on $[0, T]$. Returning to 51 we have seen that $U_{k}$ can be obtained as

$$
U_{i, j}=U_{j, i}=\max \left\{H_{i}^{*}, \ldots, H_{j-1}^{*}\right\}, \quad 1 \leq i \leq j \leq k
$$

with the convention $\max \{\emptyset\}=0$. For the event in the expectation of equation 51 the deepest branching point needs to be $H_{n}^{*}$ which happens with probability $1 /(k-1)$. The height of this branching point is then distributed as the maximum of $k-1$ independent uniform random variables, hence with density $(k-1) s^{k-2} / T^{k-1} \mathbf{1}_{[0, T]}(s)$. From now on we condition on the event $\left\{s=H_{n}^{*}=\max \left\{H_{i}^{*}, i \leq k-1\right\}\right\}$ and examine the distribution of $\mathcal{P}$ under that condition, as well as the resulting consequence for the distance matrices $T_{0}\left(U_{k}\right)$ and $T_{1}\left(U_{k}\right)$.

Note that the position $P\left(H_{n}^{*}\right)$ in $[0, W]$ where $H_{n}^{*}$ is attained in $[0, W]$ is uniformly distributed, since the Poisson Point process is homogenous. This results in another biasing of $W$, since we are now having $k+1$ points which are sampled uniformly, resulting in a length $\widehat{W}$ which is $\Gamma_{k+2, \theta}$-distributed. In addition, since $H_{n}^{*}=s$, $P\left(H_{n}^{*}\right)$ is distributed as a random variable with $\Gamma_{n+1, \frac{1}{s}}$-distribution. Since there is no point $(x, t)$ such that $t \geq s$ with $x \in\left[P\left(H_{n}^{*}\right), Y_{T}\right]$, the length of this interval is by the same reasoning has a $\Gamma_{k-n+1, \frac{1}{s}}$-distributed.

In conclusion this splits the random interval into two parts having respective distributions $\Gamma_{n+1, \frac{1}{s}}$ and $\Gamma_{k-n+1, \frac{1}{s}}$ and by properties of the $\Gamma$-distribution, these are independent. Therefore, we get (still under the condition $\left.H_{n}^{*}=s \wedge H_{n}^{*}=\max \left\{H_{i}^{*}, i \leq k-1\right\}\right)$

$$
\begin{equation*}
\left\{V_{i}^{*} \widehat{W}, i \leq n\right\} \stackrel{d}{=}\left\{V_{i}^{0, *} W_{0}, i \leq n\right\} \tag{53}
\end{equation*}
$$

where $\left\{V_{i}^{0, *}, i \leq n\right\}$ are distributed as the order statistics of $n$ standard independent uniform random variables and $W_{0}$ has $\Gamma_{n+1, \frac{1}{s}}$-distribution. The same holds true for

$$
\begin{equation*}
\left\{V_{i}^{*} \widehat{W}, i>n\right\} \stackrel{d}{=}\left\{V_{i}^{1, *} W_{1}, i \leq k-n\right\}, \tag{54}
\end{equation*}
$$

where $W_{1}$ is distributed according to $\Gamma_{k-n+1, \frac{1}{s}}$. Moreover, the random variables in (53) and 54) are independent. The induced metric by $\mathcal{P}$ on $\left[0, W_{1}\right]$ as in 52 ) is in law equal to the metric induced by a Brownian CPP of height $s$, and the same holds true for the other interval and both are independent due to the fact that the points were generated by a PPP.

Therefore, continuing with 51) and by integrating against the height of the deepest branching point, we get

$$
\begin{aligned}
& \theta^{-k+1} \mathbb{E}\left[f\left(\tau\left(U_{k}\right)\right) F_{0}\left(T_{0}\left(U_{k}\right)\right) F_{1}\left(T_{1}\left(U_{k}\right)\right) \mathbf{1}_{\left|T_{0}\left(U_{k}\right)\right|=n} \mathbf{1}_{\left|T_{1}\left(U_{k}\right)\right|=k-n}\right] \\
& =\theta^{-k+1} \int_{0}^{T}(k-1) \frac{s^{k-2}}{T^{k-1}} f(s) \frac{1}{k-1} \mathbb{E}\left[F_{0}\left(T_{0}\left(U^{*}\right)\right) F_{1}\left(T_{1}\left(U^{*}\right)\right) \mid H_{n}^{*}=s, H_{n}^{*}=\max \left\{H_{i}^{*}, i \leq k-1\right\}\right] d s \\
& =\int_{0}^{T} s^{k-2} f(s) \mathbb{E}\left[F_{0}\left(d_{P}\left(V_{i}^{*} W, V_{j}^{*} W\right)_{i, j \leq n}\right) F_{1}\left(d_{P}\left(V_{i}^{*} W, V_{j}^{*} W\right)_{i, j>n}\right) \mid H_{n}^{*}=s, H_{n}^{*}=\max \left\{H_{i}^{*}, i \leq k-1\right\}\right] d s . \\
& =\int_{0}^{T} s^{k-2} f(s) \mathbb{E}\left[F_{0}\left(d_{P}\left(V_{i}^{0, *} W_{0}, V_{j}^{0, *} W_{0}\right)_{i, j \leq n}\right)\right] \mathbb{E}\left[F_{1}\left(d_{P}\left(V_{i}^{1, *} W_{1}, V_{j}^{1, *} W_{1}\right)_{i, j \leq k-n}\right)\right] d s .
\end{aligned}
$$

As a last step we use 50. for $R^{n, s}$ (resp. $R^{k-n, s}$ ) to arrive at

$$
\mathcal{R}^{k, T}(F)=\int_{0}^{T} f(s) \mathcal{R}^{n, s}\left(F_{0}\right) \mathcal{R}^{k-n, s}\left(F_{1}\right) d s
$$

To make the Brownian CPP of height $T$ comparable to our branching Brownian motion until time $t$ we need to introduce a rescaling, mimicking the reproductive variance. To this end we set $T=t \frac{\sigma^{2}}{2}$ and define

$$
\widetilde{\mathcal{R}}^{k, t}(\cdot)=\left(\frac{\sigma^{2}}{2}\right)^{k-1} \mathcal{R}^{k, t}(\cdot) .
$$

The following result shows how a linear time change affects the planar moments of the Brownian CPP.
Corollary 4.8. Let $F$ be a functional of the product form as in 40 and $\widetilde{\mathcal{R}}^{k, t}$ as above. Let $T=t \frac{\sigma^{2}}{2}$, then it holds

$$
\widetilde{\mathcal{R}}^{k, t}(F)=\frac{\sigma^{2}}{2} \int_{0}^{t} f(s) \widetilde{\mathcal{R}}^{n, s}\left(F_{0}\right) \widetilde{\mathcal{R}}^{k-n, s}\left(F_{1}\right) d s
$$

In addition, it holds

$$
\frac{d \mathcal{R}^{k, T}}{d \widetilde{\mathcal{R}}^{k, t}}=1
$$

Proof. The first part follows immediately from Proposition 4.7. For the second part notice that by Proposition 2.2 for any polynomial $\Phi$

$$
\begin{aligned}
\mathcal{R}^{k, T}(\Phi) & =\frac{1}{T} \mathbb{E}\left[\int_{\substack{v_{1}<\ldots<v_{k} \\
v_{1}, \ldots, v_{k} \in\left[0, Y_{T}\right]}} \phi(U(\vec{v})) \prod_{i=1}^{k} d v_{i}\right]=T^{k-1} k!\mathbb{E}\left[\phi\left(\left(U_{\sigma_{i}, \sigma_{j}}\right)_{i, j \leq k} 1_{\{U \text { is planar }\}}\right)\right] \\
& =T^{k-1} \mathbb{E}\left[\phi\left(\left(U_{i, j}\right)_{i, j \leq k}\right)\right]=\left(\frac{t \sigma^{2}}{2}\right)^{k-1} \mathbb{E}\left[\phi\left(\left(U_{i, j}\right)_{i, j \leq k}\right)\right],
\end{aligned}
$$

where $U_{i, j}, i, j \leq k$ are as in 17 ) and $\sigma$ is an independent permutation of $[k]$. By the exact same line of arguments one obtains

$$
\widetilde{\mathcal{R}}^{k, t}(\Psi)=\left(\frac{t \sigma^{2}}{2}\right)^{k-1} \mathbb{E}\left[\phi\left(\left(U_{i, j}\right)_{i, j \leq k}\right)\right],
$$

proving that the measures agree on all polynomials. Since the set of polynomials is separating, the claim follows.

### 4.6 Moments of the additive martingale

Our first lemma is a rough bound on the variance of the additive martingale for the unscaled process. This bound will be needed to estimate the probability of survival of the system.
Lemma 4.9 (Rough bound on the variance of the unscaled martingale). Let $F: \mathbf{U}_{2}^{*} \rightarrow \mathbf{R}$ be of the form

$$
\begin{equation*}
F\left(\left(U,\left(x_{i}\right)\right)\right)=\prod_{i=1}^{2} h^{N}\left(x_{i}\right) \tag{55}
\end{equation*}
$$

There exists a constant 4.9 such that, for all $t>0$,

$$
\mathbf{L}^{N, 2, t}(F) \leq 4.9\left(t \vee \log (N)^{3}\right) N^{c}
$$

Proof. This is a direct consequence of Lemma 3.6 Indeed,

$$
\mathbf{L}^{N, 2, t}(F)=\int_{0}^{t}\left(\int_{0}^{L} q_{s}^{N}(x, y) h^{N}(y) d y\right) d s
$$

Let us now set $c_{1}$ as in Lemma 3.4 (applied to $\varepsilon=1$ ). Let $t_{1}=c_{1}\left(L_{N}\right)^{2}$. It follows from Lemma 3.6 that

$$
\int_{0}^{t_{1}}\left(\int_{0}^{L} q_{s}^{N}(x, y) h^{N}(y) d y\right) d s \leq C \log (N)^{3} N^{c}
$$

The remaining part of the integral is bounded by the integral of the stationary distribution : it follows from Lemma 3.4 that

$$
\int_{t_{1}}^{t}\left(\int_{0}^{L} q_{s}^{N}(x, y) h^{N}(y) d y\right) d s \leq 2\left(t-t_{1}\right)\left(\int_{0}^{L} h^{N}(y)^{2} \tilde{h}^{N}(y) d y\right) d s \leq C t N^{c}
$$

where we used Proposition 1.2 to obtain the last inequality.
For the next step we use recursion formula to obtain the planar moments of the additive martingale, for this reason we consider a functional $F^{N}: U_{k}^{*} \rightarrow \mathbb{R}$ of the following form. Let $U=\left(\left(U_{i, j}\right),\left(x_{i}\right)\right) \in U_{k}^{*}$, set

$$
\begin{equation*}
F^{N}(U)=\prod_{i=1}^{k} h^{N}\left(x_{i}\right)=\sum_{n=1}^{k-1} \mathbf{1}_{\left\{\left|T_{0}(U)=n\right|\right\}} F_{0}^{N}\left(T_{0}(T)\right) F_{1}^{N}\left(T_{1}(T)\right), \tag{56}
\end{equation*}
$$

where $F_{0}^{N}=\prod_{i=1}^{k} h^{N}\left(x_{i}\right) \mathbf{1}_{\left\{x_{i} \in T_{0}\right\}}$ and $F_{1}^{N}=\prod_{i=1}^{k} h^{N}\left(x_{i}\right) \mathbf{1}_{\left\{x_{i} \in T_{1}\right\}}$. Due to Corollary 4.3 we get for all $k \geq 2$

$$
\begin{equation*}
\overline{\mathbf{L}}_{x}^{N, k, t}\left(F^{N}\right)=\frac{1}{k-1} \sum_{n=1}^{k-1} \frac{1}{2 N^{c}} \int_{0}^{t} \mathbb{E}_{x}\left[h^{N}\left(\bar{\zeta}_{t-s}^{N}\right) \bar{L}_{\bar{\zeta}_{t-s}^{N}}^{N, n, s}\left(F_{0}^{N}\right) \bar{L}_{\bar{\zeta}_{t-s}^{N}}^{N, k-n, s}\left(F_{1}^{N}\right)\right] d s \tag{57}
\end{equation*}
$$

Proposition 4.10. Let $F^{N}$ be a functional of the form in (56), then it holds for all $k \geq 1, t>0$

$$
\overline{\mathbf{L}}_{x}^{N, k, t}\left(F^{N}\right) \rightarrow\left(\frac{\sigma^{2} t}{2}\right)^{k-1}, \quad \text { as } N \rightarrow \infty, \text { uniformly in } x \in \Omega_{N}
$$

Proof. We prove the statement by induction, for $k=1$, we have by 44

$$
\begin{aligned}
\overline{\mathbf{L}}_{x}^{N, 1, t}\left(F^{N}\right) & =\bar{Q}_{x}^{N, 1, t}\left(\bar{\Delta}^{N} h^{N}\left(\bar{\zeta}_{V_{i}}\right)\right) \\
& =\bar{Q}_{x}^{N, 1, t}(1)=1,
\end{aligned}
$$

hence the statement holds for $k=1$. Now assume $k=2$, then by 57,

$$
\begin{aligned}
\lim _{N \rightarrow \infty} \overline{\mathbf{L}}_{x}^{N, 2, t}\left(F^{N}\right) & =\lim _{N \rightarrow \infty} \frac{1}{2 N^{c}} \int_{0}^{t} \mathbb{E}_{x}\left[\overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N, 1, s}}^{N}\left(F_{0}^{N}\right) \overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N, 1, s}}^{N}\left(F_{1}^{N}\right) h^{N}\left(\bar{\zeta}_{t-s}^{N}\right)\right] \\
& =\lim _{N \rightarrow \infty} \frac{1}{2 N^{c}} \int_{0}^{t} \mathbb{E}_{x}\left[h^{N}\left(\bar{\zeta}_{t-s}^{N}\right)\right] d s
\end{aligned}
$$

Making use of Definition 4 the bounds in Lemma 3.4 and the same reasoning as in the proof of Lemma 4.5, we can approximate the above by

$$
\lim _{N \rightarrow \infty} \frac{1}{2 N^{c}} \int_{t_{1}}^{t} \int_{\Omega_{N}}\left(h^{N}(y)\right)^{2} \tilde{h}^{N}(y)(1+o(1)) d y
$$

with the same $t_{1}=t_{1}(N) \rightarrow 0$ as in the proof of Lemma 4.5 By Proposition 1.2 and the dominated convergence theorem, one gets

$$
\lim _{N \rightarrow \infty} \frac{1}{2 N^{c}} \int_{t_{1}}^{t} \int_{\Omega_{N}}\left(h^{N}(y)\right)^{2} \tilde{h}^{N}(y)(1+o(1)) d y=\frac{\sigma^{2} t}{2} .
$$

Assume that the statement holds for $k-1 \geq 2$, then for the induction step again relying on 57, we obtain

$$
\overline{\mathbf{L}}_{x}^{N, k, t}\left(F^{N}\right)=\sum_{n=1}^{k-1} \frac{1}{(k-1) 2 N^{c}} \int_{0}^{t} \mathbb{E}_{x}\left[\overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, n, s}\left(F_{0}^{N}\right) \overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, k-n, s}\left(F_{1}^{N}\right) h^{N}\left(\bar{\zeta}_{t-s}^{N}\right)\right] d s
$$

By the induction argument, $\overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, n, s}\left(F_{0}^{N}\right) \rightarrow\left(\sigma^{2} t / 2\right)^{n-1}$ and $\overline{\mathbf{L}}_{\bar{\zeta}_{t-s}^{N}}^{N, k-n, s}\left(F_{1}^{N}\right) \rightarrow\left(\sigma^{2} t / 2\right)^{k-n-1}$ uniformly in the position. As previously, by Proposition 1.2 and the dominated convergence Theorem, the claim follows.

Let $\bar{Y}^{N}(t)$ be the rescaled additive martingale, i.e.

$$
\begin{equation*}
\bar{Y}^{N}(t)=\frac{1}{N} \sum_{v \in \mathcal{N}_{t N^{1-c}}^{N}} h^{N}\left(X_{v}\left(t N^{1-c}\right)\right), \quad t \geq 0 . \tag{58}
\end{equation*}
$$

Corollary 4.11. Let $k \geq 1$, then it holds as $N \rightarrow \infty$,
$\frac{N}{h^{N}(x)} \mathbb{E}_{x}\left[\left(\bar{Y}(t)^{N}\right)^{k}\right]=\frac{1}{N^{k-1} h^{N}(x)} \mathbb{E}_{x}\left[\sum_{v_{1}, \ldots, v_{k} \in \mathcal{N}_{t N^{1-c}}^{N}} \prod_{i=1}^{k} h^{N}\left(X_{v_{i}}\left(t N^{1-c}\right)\right)\right] \rightarrow k!\left(\frac{t \sigma^{2}}{2}\right)^{k-1}, \quad$ uniformly in $x \in \Omega_{N}$,
Proof. The proof is a direct consequence of Proposition 4.10 and noting that by Theorem 7 and 44

$$
\begin{align*}
\frac{N}{h^{N}(x)} \mathbb{E}_{x}\left[\left(\bar{Y}(t)^{N}\right)^{k}\right] & =k!\frac{\left(t N^{1-c}\right)^{k-1}}{N^{k-1}} Q_{x}^{k, t N^{1-c}}\left(F^{N}\right)  \tag{59}\\
& =k!\overline{\mathbf{L}}_{x}^{N, k, t}\left(F^{N}\right) \tag{60}
\end{align*}
$$

## 5 Survival probability (0-th moment)

The section is devoted to the proof of Theorem 1. To this end, we prove the following convergence result on the probability of survival of the unrescaled process.
Proposition 5.1. For all $\varepsilon>0$, there exists $N_{*}=N_{*}(\varepsilon)$ such that

$$
\forall N>N_{*}, \forall t \geq \log (N)^{5}, \forall x \in \Omega_{N} \quad(1-\varepsilon) \frac{h^{N}(x)}{t} \frac{2}{\Sigma^{2}} \leq \mathbb{P}_{x}\left(Z^{N}(t)>0\right) \leq(1+\varepsilon) \frac{h^{N}(x)}{t} \frac{2}{\Sigma^{2}}
$$

The content of Theorem 1 then follows from this proposition by setting $t=s N^{1-c}$ and recalling from Proposition 1.2 that $\frac{1}{N^{c}} \Sigma^{2} \rightarrow \sigma^{2}$ converges to $\sigma^{2}$ as $N$ goes to $\infty$. In this section, we denote

$$
f(t, x) \equiv f_{N}(t, x)=\mathbb{P}_{x}\left(Z^{N}(t)>0\right), \quad t \geq 0, x \in\left[0, L_{N}\right]
$$

and

$$
a(t) \equiv a_{N}(t)=\int_{0}^{L_{N}} f_{N}(t, x) \tilde{h}^{N}(x) d x, \quad t \geq 0
$$

For the sake of clarity, we will often omit the $N$ superscript but one must remember that these quantities have an implicit dependence on $N$. Essentially, we will show that for $N$ and $t$ large enough

$$
\begin{equation*}
f(t, x) \approx a(t) h(x), \tag{61}
\end{equation*}
$$

and that $t \rightarrow a(t)$ has the desired asymptotics. This strategy is standard [20, 30, 32], but one needs to check carefully that all the estimates are uniform in time. The proof of the result relies on the following observation.
Lemma 5.2. For all $N \geq N_{0}$,

$$
\forall t>0, \quad \frac{d}{d t} a_{N}(t)=-\frac{1}{2} \int_{0}^{L_{N}} f_{N}(t, x)^{2} \tilde{h}^{N}(x) d x
$$

Proof. By definition of $a_{N} \equiv a$, we see that

$$
\frac{d}{d t} a(t)=\int_{0}^{L_{N}} \partial_{t} f(t, x) \tilde{h}(x) d x
$$

Yet, $f_{N}$ is solution to the FKPP equation

$$
\left\{\begin{array}{l}
\partial_{t} f(t, x)=\frac{1}{2} \partial_{x x} f(t, x)+\beta \partial_{x} f(t, x)+\frac{1}{2}\left(f(t, x)-f(t, x)^{2}\right) \\
f\left(t, L_{\beta}\right)=0,\left.\quad \partial_{x} f(t, x)\right|_{x=0}=0
\end{array}\right.
$$

An integration by part then yields

$$
\begin{aligned}
\dot{a}(t)= & {\left[\frac{1}{2} \partial_{x} f(t, x) \tilde{h}(x)\right]_{0}^{L_{\beta}}-\frac{1}{2} \int_{0}^{L_{\beta}} \partial_{x} f(t, x) \tilde{h}^{\prime}(x) d x+[\beta f(t, x) \tilde{h}(x)]_{0}^{L_{\beta}} } \\
& -\beta \int_{0}^{L_{\beta}} f(t, x) \tilde{h}^{\prime}(x) d x+\frac{1}{2} \int_{0}^{L_{\beta}} f(t, x) \tilde{h}(x) d x-\frac{1}{2} \int_{0}^{L_{\beta}} f(t, x)^{2} \tilde{h}(x) d x \\
= & {\left[-\frac{1}{2} f(t, x) \tilde{h}^{\prime}(x)\right]_{0}^{L_{\beta}}-\beta f(t, 0) \tilde{h}^{\prime}(0)+\int_{0}^{L_{\beta}} f(t, x)\left(\frac{1}{2} \tilde{h}^{\prime \prime}(x)-\beta \tilde{h}^{\prime}(x)-\frac{1}{2} \tilde{h}(x)\right) d x-\frac{1}{2} \int_{0}^{L_{\beta}} f(t, x)^{2} \tilde{h}(x) d x } \\
= & -\frac{1}{2} \int_{0}^{L_{\beta}} f(t, x)^{2} \tilde{h}(x) d x,
\end{aligned}
$$

where we use that $\frac{1}{2} \tilde{h}^{\prime \prime}-\beta \tilde{h}^{\prime}-\frac{1}{2} \tilde{h}$ on $\left(0, L_{\beta}\right)$, that $f\left(t, L_{\beta}\right)=0$ and that $\frac{1}{2} \tilde{h}^{\prime}(0)-\beta \tilde{h}(0)=0$ to get the last line.

### 5.1 Rough bounds

Lemma 5.3. Let $N \geq N_{0}$. Then,

$$
\forall t>0, \quad a_{N}(t) \leq \frac{2}{t}
$$

Proof. This is a direct consequence of Lemma 5.2 and Jensen's inequality. Indeed, we have

$$
\forall t>0, \quad \frac{d}{d t} a_{N}(t)=-\frac{1}{2} \int_{0}^{L_{N}} f_{N}(t, x)^{2} \tilde{h}^{N}(x) d x \leq-\frac{1}{2}\left(\int_{0}^{L_{N}} f_{N}(t, x) \tilde{h}^{N}(x) d x\right)^{2}=-\frac{1}{2} a_{N}(t)^{2} .
$$

Integrating this inequality then yields that

$$
\forall t>0, \quad a_{N}(t) \leq \frac{1}{\frac{1}{a_{N}(0)}+\frac{1}{2} t} \leq \frac{2}{t}
$$

Lemma 5.4 (Lower bound on the probability of survival). There exists $c_{1}>0$ such that, for $N$ large enough,

$$
\begin{equation*}
\forall t \geq \log (N)^{3}, \quad f_{N}(t, x) \geq \frac{c_{1}}{t N^{c}} h^{N}(x) \tag{62}
\end{equation*}
$$

Proof. The proof is adapted from [20, Lemmas 7.2] and relies on a change of measure combined with Jensen's inequality. Let $\tilde{\mathbb{P}}_{x}^{N, t}$ be the probability measure absolutely continuous w.r.t. to $\mathbb{P}_{x}$ whose Radon-Nikodym derivative is given by

$$
\begin{equation*}
\frac{d \tilde{\mathbb{P}}_{x}^{N, t}}{d \mathbb{P}_{x}}=\frac{1}{h^{N}\left(X_{v}(t)\right)} \sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}(t)\right) . \tag{63}
\end{equation*}
$$

This change of measure combined with Jensen's inequality yields

$$
\frac{\mathbb{P}_{x}\left(Z^{N}(t)>0\right)}{h^{N}(x)}=\tilde{\mathbb{P}}_{x}^{N, t}\left[\frac{1}{\sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}(t)\right)}\right] \geq \frac{1}{\tilde{\mathbb{P}}_{x}^{N, t}\left[\sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}(t)\right)\right]}
$$

Yet, we see from 63) that

$$
\begin{equation*}
\tilde{\mathbb{P}}_{x}^{N, t}\left[\sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}(t)\right)\right]=\frac{\mathbb{E}_{x}\left[\left(\sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}(t)\right)\right)^{2}\right]}{h^{N}(x)} . \tag{64}
\end{equation*}
$$

We then note that

$$
\mathbb{E}_{x}\left[\left(\sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}\right)\right)^{2}\right]=\mathbb{E}_{x}\left[\sum_{v \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}\right)^{2}+\sum_{v \neq w \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}\right) h^{N}\left(X_{w}\right)\right] \leq \mathbb{E}_{x}\left[\sum_{v \neq w \in \mathcal{N}_{t}^{N}} h^{N}\left(X_{v}\right) h^{N}\left(X_{w}\right)\right] .
$$

We know that,

$$
\mathbb{E}_{x}\left[\sum_{v \neq w \in \mathcal{N}_{t}} h\left(X_{v}\right) h\left(X_{w}\right)\right]=2 h(x) \mathbf{L}_{x}^{N, 2, t}(F)
$$

with $F\left(U,\left(x_{i}\right)\right)=\prod_{i=1}^{2} h\left(x_{i}\right)$. We conclude the proof by recalling from Lemma 4.9 that, for $N$ large,

$$
\begin{equation*}
\forall t>\log (N)^{3}, \quad \mathbf{L}_{x}^{N, 2, t}(F) \leq C t N^{c} . \tag{65}
\end{equation*}
$$

Corollary 5.5. There exists a constant $c_{2}>0$ such that, for $N$ large enough,

$$
\forall t \geq \log (N)^{3}, \quad \frac{d}{d t} a_{N}(t) \leq-\frac{c_{2}}{t^{2}} \frac{1}{N^{c}} \quad \text { and } \quad a(t) \geq \frac{1}{N^{c}} \frac{c_{2}}{t} .
$$

Proof. It follows from Lemma 5.2, Lemma 5.4 and Proposition 1.2 that

$$
\begin{equation*}
\forall N, \forall t>1, \quad \dot{a}_{N}(t) \leq-\left(\frac{c_{1}}{t N^{c}}\right)^{2} \int_{\Omega_{N}}\left(h^{N}(x)\right)^{2} \tilde{h}^{N}(x) d x \leq-\frac{c_{2}}{t^{2}} \frac{1}{N^{c}} \tag{66}
\end{equation*}
$$

The second part follows from an integration: for $0<t<s$,

$$
a_{N}(s)-a_{N}(t) \leq \frac{c_{2}}{N^{c}}\left(\frac{1}{s}-\frac{1}{t}\right) .
$$

Yet we know from Lemma 5.3 that $a_{N}(s) \rightarrow 0$ as $s \rightarrow \infty$ (for fixed $N$ ). Letting $s \rightarrow \infty$ provides the desired lower bound.

Lemma 5.6. Let $\alpha^{\prime}>0$ such that, for $N$ large enough,

$$
\forall t \geq \log (N)^{4}, \quad f_{N}(t, x) \leq\left(1+O\left(\log (N)^{-1}\right)\right) a_{N}(t) h^{N}(x)
$$

Proof. Let $t_{0}=\log (N)^{3}$. We know from the branching property that, for all $t>t_{0}$ and $x \in \Omega_{N}$,

$$
\begin{equation*}
f_{N}(t, x)=\mathbb{P}_{x}\left(\bigcup_{v \in \mathcal{N}_{t_{0}}^{N}}\left\{Z_{t-t_{0}}^{N,(v)}>0\right\}\right) \tag{67}
\end{equation*}
$$

where $Z_{t-t_{0}}^{N,(v)}$ denotes the number of descendants at time $t$ of the particle $v$ living at time $t_{0}$. A union bound combined with Lemma 1.1 then shows that

$$
f_{N}(t, x) \leq \mathbb{E}_{x}\left[\sum_{v \in \mathcal{N}_{t_{0}}} \mathbb{P}_{X_{v}}\left(Z_{t-t_{0}}^{N,(v)}>0\right)\right]=\int_{\Omega_{N}} p_{t_{0}}(x, y) f_{N}\left(t-t_{0}, y\right) d y
$$

For $t \geq 2 t_{0}$, it follows from Lemma 3.4 (iii) that

$$
f_{N}(t, x) \leq\left(1+O\left(N^{-\tilde{\alpha}}\right) a_{N}\left(t-t_{0}\right),\right.
$$

for some $\tilde{\alpha}>0$. The mean value theorem combined with Corollary 5.5 then shows that

$$
\forall t>2 t_{0}, \quad\left|a_{N}\left(t-t_{0}\right)-a_{N}(t)\right| \leqslant \frac{1}{N^{c}} \frac{c_{2} t_{0}}{\left(t-t_{0}\right)^{2}}
$$

We conclude the by remarking that Corollary 5.5 also implies that, for $N$ large enough,

$$
\forall t \geq \log (N)^{4}, \quad \frac{1}{N^{c}} \frac{t_{0}}{\left(t-t_{0}\right)^{2}}=O\left(\log (N)^{-1}\right) a(t)
$$

Lemma 5.7. Let $\varepsilon>0$. For $N$ large enough,

$$
\begin{equation*}
\forall t \geq \log (N)^{4}, \quad N^{c}\left(\log (N)^{3}\right) a_{N}(t) \leq \varepsilon \tag{68}
\end{equation*}
$$

Proof. Let $1 \ll A \ll L$ and write $a_{N}$ as

$$
\begin{equation*}
a_{N}(t)=\int_{0}^{A} f_{N}(t, x) \tilde{h}^{N}(x) d x+\int_{A}^{L_{N}} f_{N}(t, x) \tilde{h}^{N}(x) d x \tag{69}
\end{equation*}
$$

We know from Remark 4 that, for $N$ large enough

$$
\begin{equation*}
\int_{0}^{A} f_{N}(t, x) \tilde{h}^{N}(x) d x \leq \int_{0}^{A} \tilde{h}^{N}(x) d x \leq \frac{2 A e^{\beta A}}{c^{6} N^{c} \log (N)^{6}} . \tag{70}
\end{equation*}
$$

On the other hand, Lemma 5.6 yields that

$$
\forall t \geq \log (N)^{4}, \quad \int_{A}^{L} f_{N}(t, x) \tilde{h}^{N}(x) d x \leq\left(1+O\left(\log (N)^{-1}\right) a_{N}(t) \int_{A}^{L} \tilde{h}^{N}(x) h^{N}(x) .\right.
$$

An explicit calculation then shows that

$$
\int_{A}^{L} \tilde{h}^{N}(x) h^{N}(x) d x \leq 1-\frac{A}{L} .
$$

Let us choose $A=\log \log (N)$. The previous estimates imply that

$$
\forall t \geq \log (N)^{4}, \quad \int_{A}^{L} f_{N}(t, x) \tilde{h}^{N}(x) d x \leq\left(1-\frac{C \log \log (N)}{\log (N)}\right) a_{N}(t)
$$

for some constant $C>0$. Combining this with 69 and 70 shows that, for $N$ large enough,

$$
\forall t \geq \log (N)^{4}, \quad \frac{C \log \log (N)}{\log (N)} a_{N}(t) \leq \frac{2 \log \log (N)}{c^{6} N^{c} \log (N)^{5}}
$$

This concludes the proof of the lemma.

### 5.2 Precise asymptotic for the probability of survival

Lemma 5.8. Let $\varepsilon>0$. There exists $N_{1}=N_{1}(\varepsilon)$, such that, for all $N \geq N_{1}$ and $t \geq \log (N)^{4}$,

$$
\forall x \in \Omega_{N}, \quad(1-\varepsilon) a_{N}(t) h(x) \leq f_{N}(t, x) \leq(1+\varepsilon) a_{N}(t) h(x) .
$$

Proof. Step 1. Upper bound. This is a direct consequence of Lemma 5.6
Step 2. Lower bound. Let $t_{0}=\log (N)^{3}$ Combining the second Bonferroni inequality and 67), we get that

$$
\begin{align*}
f_{N}(t, x) & \geq \mathbb{E}_{x}\left[\sum_{v \in \mathcal{N}_{t_{0}}} \mathbb{P}_{X_{v}}\left(Z_{t-t_{0}}^{N,(v)}>0\right)\right]-\frac{1}{2} \mathbb{E}_{x}\left[\sum_{v \neq w \in \mathcal{N}_{t_{0}}} \mathbb{P}_{X_{v}}\left(Z_{t-t_{0}}^{N,(v)}>0\right) \mathbb{P}_{X_{w}}\left(Z_{t-t_{0}}^{(w)}>0\right)\right] \\
& \geq \mathbb{E}_{x}\left[\sum_{v \in \mathcal{N}_{t_{0}}} \mathbb{P}_{X_{v}}\left(Z_{t}^{N,(v)}>0\right)\right]-\frac{1}{2} \mathbb{E}_{x}\left[\sum_{v \neq w \in \mathcal{N}_{t_{0}}} \mathbb{P}_{X_{v}}\left(Z_{t-t_{0}}^{N,(v)}>0\right) \mathbb{P}_{X_{w}}\left(Z_{t-t_{0}}^{N,(w)}>0\right)\right] . \tag{71}
\end{align*}
$$

As for the upper bound, one can prove from 67 that for $N$ large enough,

$$
\begin{equation*}
\forall t \geq \log (N)^{4}, \forall x \in \Omega_{N}, \quad \mathbb{E}_{x}\left[\sum_{v \in \mathcal{N}_{t_{0}}} \mathbb{P}_{X_{v}}\left(Z_{t}^{(v)}>0\right)\right]=\int_{0}^{L} p_{t_{0}}(x, y) f(t, y) d y \geq(1-\varepsilon) a(t) h(x) . \tag{72}
\end{equation*}
$$

The second term on the RHS of $\sqrt[71]{ }$ is a moment of order 2 and can be expressed thanks to the many-to-two lemma

$$
\begin{equation*}
\mathbb{E}_{x}\left[\sum_{v \neq w \in \mathcal{N}_{t_{0}}^{N}} \mathbb{P}_{X_{v}}\left(Z_{t-t_{0}}^{N,(v)}>0\right) \mathbb{P}_{X_{w}}\left(Z_{t-t_{0}}^{N,(w)}>0\right)\right]=2 h(x) \mathbf{L}_{x}^{N, 2, t_{0}}(G), \tag{73}
\end{equation*}
$$

with $G\left(U,\left(x_{i}\right)\right)=\prod_{i=1}^{2} f_{N}\left(t-t_{0}, x_{i}\right)$. Using Lemma 5.6 again, we get that, for $N$ large enough,

$$
\forall t \geq \log (N)^{4}, \quad G\left(U,\left(x_{i}\right)\right) \leq(1+\varepsilon) a(t)^{2} \prod_{i=1}^{2} h\left(x_{i}\right)
$$

so that, by Lemma 4.9

$$
\mathbf{L}_{x}^{N, 2, t_{0}}(G) \leq C\left(L_{N}\right)^{3} N^{c} .
$$

This upper bound combined with Lemma 5.7 shows that, for $N$ large enough,

$$
\forall t \geq \log (N)^{4}, \quad f(t, x) \geq(1-\varepsilon) a(t) h(x)-C\left(L_{N}\right)^{3} N^{c} h(x) a(t)^{2} \geq(1-2 \varepsilon) h(x) a(t),
$$

which concludes the proof of the lemma.
Proof of Proposition 5.1. Let $t_{1}:=\log (N)^{4}$ Combining Lemma 5.2 and Lemma 5.8, we see that, for $N$ large enough,

$$
\forall t \geq \log (N)^{5}, \quad \frac{1}{(1+\varepsilon) \frac{\Sigma^{2}}{2}\left(t-t_{1}\right)+\frac{1}{a\left(t_{1}\right)}} \leq a(N) \leq \frac{1}{(1-\varepsilon) \frac{\Sigma^{2}}{2}\left(t-t_{1}\right)+\frac{1}{a\left(t_{1}\right)}} .
$$

Choosing $N$ large enough, we get that

$$
(1-\varepsilon) \frac{2}{\Sigma^{2}} \frac{1}{t N} \leq a(t N) \leq(1+\varepsilon) \frac{2}{\Sigma^{2}} \frac{1}{t N}
$$

The result then follows from Lemma 5.8

## 6 Convergence of genealogies

This section is devoted to the proof of Theorem 3 .
Proof of Theorem 5. The result follows from Proposition 2.1 To use this result, we first need to 'unplanarise' the moments calculated in Section 4.4 Let $\phi:[0, \infty)\binom{k}{2} \rightarrow \mathbb{R}$ be a bounded continuous function and $F: \mathbf{U}_{k} \rightarrow \mathbb{R}$ be defined as

$$
F(U)=\frac{1}{k!} \sum_{\sigma \in S_{k}} \phi\left(\left(U_{\sigma(i), \sigma(j)}\right)\right),
$$

where $S_{k}$ is the set of permutation of $\{1, \ldots, k\}$. Denote by $\Phi$ the polynomial associated to $\phi$. Remark that

$$
\mathbb{E}_{x}\left[\sum_{v_{1} \neq \ldots \neq v_{k} \in \mathcal{N}_{t}^{N}} \phi\left(d_{t}^{N}\left(v_{i}, v_{j}\right)\right)\right]=k!\mathbb{E}_{x}\left[\sum_{v_{1}<\ldots<v_{k} \in \mathcal{N}_{t}^{N, p l}} F(U(\vec{v}))\right]
$$

In addition, we know from Lemma 4.6 and Corollary 4.8 that

$$
\begin{equation*}
\left|\frac{1}{N^{(k-1) c}} \mathbb{E}_{x}\left[\sum_{v_{1}<\ldots<v_{k} \in \mathcal{N}_{t N^{1-c}, ~}^{N, p l}} F(U(\vec{v}))\right]-h^{N}(x)\left(\frac{\sigma^{2}}{2} t\right)^{k-1}\right| \xrightarrow[N \rightarrow \infty]{ } 0 \tag{74}
\end{equation*}
$$

Conditioning on $Z^{N}\left(t N^{1-c}\right)>0$ and using Theorem1. we then obtain

$$
\mathbb{E}_{x}\left[\sum_{v_{1} \neq \ldots \neq v_{k} \in \mathcal{N}_{t N^{1-c}}^{N}} \phi\left(d_{t}^{N}\left(v_{i}, v_{j}\right)\right) \mid Z^{N}\left(t N^{1-c}\right)>0\right] \rightarrow k!\left(\frac{\sigma^{2}}{2} t\right)^{k}
$$

In now remains to show that

$$
\left|\mathbb{E}_{x}\left[\Phi\left(\bar{M}_{t}^{N}\right) \mid Z^{N}\left(t N^{1-c}\right)>0\right]-\mathbb{E}_{x}\left[\sum_{v_{1} \neq \ldots \neq v_{k} \in \mathcal{N}_{t}^{N}} \phi\left(d_{t}^{N}\left(v_{i}, v_{j}\right)\right) \mid Z^{N}\left(t N^{1-c}\right)>0\right]\right| \xrightarrow[N \rightarrow \infty]{\longrightarrow} 0
$$

This boils down to proving that

$$
\lim _{N \rightarrow \infty} \mathbb{E}_{x}\left[\sum_{\left(v_{1}, \ldots, v_{k}\right) \in \mathcal{N}_{t N^{1-c}}^{N}} \mathbf{1}_{\cup_{1 \leq i<j \leq k}\left\{v_{i}=v_{j}\right\}} \mid Z^{N}\left(t N^{1-c}\right)>0\right]=0
$$

which can be easily deduced from a union bound combined with an induction. Putting all of this together, we get that

$$
\lim _{N \rightarrow \infty} \mathbb{E}_{x}\left[\Phi\left(\bar{M}_{t}^{N}\right) \mid Z^{N}\left(t N^{1-c}\right)>0\right]=k!\left(\frac{\sigma^{2}}{2} t\right)^{k}
$$

which concludes the proof of the result by Proposition 2.1 .
Proof of Theorem 3. This result is a consequence of Theorem 5 The proof goes along the exact same lines as Theorem 2 in 5], where the convergence of the population size and of the genealogy is deduced from the convergence of the mmm space to the Brownian CPP. We recall the main steps of the argument for completeness. The maps

$$
[X, d, \nu] \mapsto|X|, \quad[X, d, \nu] \mapsto\left[X, d, \frac{\nu}{|X|}\right]
$$

are continuous w.r.t. the Gromov-weak topology. Conditional on survival, $\bar{M}_{t}^{N}$ to the Brownian CPP with height $\frac{\sigma^{2}}{2} t$ converges in the Gromov-weak topology. Hence, (i) follows from the fact that the limiting CPP has a total mass exponentially distributed with mean $\frac{\sigma^{2} t}{2}$.

We now prove (ii). Let $[X, d, \nu]$ be a general random mmm space. Sample $k$ points ( $v_{1}, \cdots, v_{k}$ ) uniformly at random with replacement. Then

$$
\mathbb{E}\left[\phi\left(\left(d\left(v_{i}, v_{j}\right)\right)\right)\right]
$$

is the moment of order $k$ of $\left[X, d, \frac{\nu}{|X|}\right]$ associated to $\Phi$. Since, conditional on survival, $\bar{M}_{t}^{N}$ converges to a Brownian CPP, (ii) follows from Proposition 2.3

## 7 Convergence of the demographic fluctuations

In this section, we proof that the demographic fluctuations of the system are well approximated by a Feller diffusion when the system starts with $N$ particles in a suitable configuration at time $t=0$. In practice, we will show that this result holds for the additive martingale $Y$ (see 38) and prove that $Y_{t}$ is a suitable approximation for $Z_{t}$ for $t$ large. For the remainder of the section, we will use the following notations:

- We denote by $\bar{Y}^{N}(t)$ the re-scaled additive martingale at time $t \geq 0$,

$$
\bar{Y}^{N}(t):=\frac{1}{N} Y_{t N^{1-c}}^{N}=\frac{1}{N} \sum_{v \in \mathcal{N}_{t N^{1-c}}^{N}} h^{N}\left(X_{v}\left(t N^{1-c}\right)\right) .
$$

- Let $\mathbb{P}_{(x, t)}$ be the law of the BBM started from a single particle at $x$ at time $t$ and by $\mathbb{E}_{(x, t)}$ the corresponding expectation.
- $\varepsilon_{N}=\varepsilon_{N}(x)$ refers to a quantity that tends to 0 as $N \rightarrow \infty$, uniformly in $x \in[0, L]$. We write $O(\cdot)$ for a quantity that is bounded in absolute value by a constant times the quantity inside the parenthesis.
Lemma 7.1. Let $y_{0} \geq 0$. Assume that

$$
\bar{Y}^{N}(0) \rightarrow_{p} y_{0} \quad \text { as } N \rightarrow \infty, \quad \text { and } \quad \sup _{N} \mathbb{E}\left[\bar{Y}^{N}(0)\right]<\infty .
$$

Then, the finite-dimensional distributions of the processes $\left(\bar{Y}^{N}(t)\right)$ converge to the finite-dimensional distributions of a 2-stable $\operatorname{CSBP}\left(\Xi_{t}\right)$ starting from $y_{0}$, as $N \rightarrow \infty$.

Proof. We start by proving the one-dimensional convergence of the process $\bar{Y}^{N}$. Let $t>0$ and $\lambda \in \mathbb{R}$.
Step 1. Power series of the Laplace transform. Using that $\left\{Z^{N}(t)=0\right\} \subset\left\{Y^{N}(t)=0\right\}$, we see that

$$
\mathbb{E}_{x}\left[e^{-\lambda \bar{Y}^{N}(t)}\right]=\mathbb{P}_{x}\left(\bar{Z}^{N}(t)=0\right)+\mathbb{E}_{x}\left[e^{-\lambda \bar{Y}^{N}(t)} \mid \bar{Z}^{N}(t)>0\right] \mathbb{P}_{x}\left(\bar{Z}^{N}(t)>0\right)
$$

Remarking that $\mathbb{P}\left(\bar{Y}^{N}(t)=0 \mid \bar{Z}^{N}(t)>0\right)=0$, we obtain

$$
\mathbb{E}_{x}\left[e^{-\lambda \bar{Y}^{N}(t)} \mid \bar{Z}^{N}(t)>0\right]=\mathbb{E}_{x}\left[\left.\sum_{K \geq 1} \frac{(-\lambda)^{K}}{K!}\left(\bar{Y}^{N}(t)\right)^{K} \right\rvert\, \bar{Z}^{N}(t)>0\right]
$$

On the other hand we have by Corollary 4.11 and Theorem 1

$$
\begin{aligned}
\mathbb{E}_{x}\left[\left(\bar{Y}^{N}(t)\right)^{K} \mid Z^{N}(t)>0\right] & =\frac{h^{N}(x)}{N \mathbb{P}_{x}\left(\bar{Z}^{N}(t)>0\right)} \frac{N}{h^{N}(x)} \mathbb{E}_{x}\left[\left(\bar{Y}^{N}(t)\right)^{K}\right] \\
& \rightarrow K!\left(\frac{\sigma^{2} t}{2}\right)^{K}, \quad \text { uniformly in } x \in \Omega_{N}
\end{aligned}
$$

which is the $K$-th moment of an exponential random variable with rate $\frac{2}{\sigma^{2} t}$. It then follows from standard domination arguments that for some $\Lambda>0$ and all $\lambda \in(-\Lambda, \Lambda)$,

$$
\sum_{K \geq 1} \frac{(-\lambda)^{K}}{K!} \mathbb{E}_{x}\left[\left(\bar{Y}^{N}(t)\right)^{K} \mid \bar{Z}^{N}(t)>0\right] \rightarrow \frac{2}{\sigma^{2} t \lambda+2}, \quad N \rightarrow \infty, \quad \text { uniformly in } x \in \Omega_{N}
$$

Hence, there exists an $N_{0}=N_{0}(\lambda)$ big enough, such that for all $N \geq N_{0}$ it holds

$$
\sum_{K \geq 1} \mathbb{E}_{x}\left[\left.\left|\frac{(-\lambda)^{K}}{K!}\left(\bar{Y}^{N}(t)\right)^{K}\right| \right\rvert\, \bar{Z}^{N}(t)>0\right]<\infty .
$$

Therefore, by Fubini-Tonelli we conclude that for all $\lambda \in(\Lambda, \Lambda)$ $\mathbb{E}_{x}\left[e^{-\lambda \bar{Y}^{N}(t)} \mid \bar{Z}^{N}(t)>0\right]=\sum_{K \geq 1} \mathbb{E}_{x}\left[\left.\frac{(-\lambda)^{K}}{K!}\left(\bar{Y}^{N}(t)\right)^{K} \right\rvert\, \bar{Z}^{N}(t)>0\right] \rightarrow \frac{2}{\sigma^{2} t \lambda+2}, \quad N \rightarrow \infty, \quad$ uniformly in $x \in \Omega_{N}$.
Proposition 5.1 yields that $\mathbb{P}_{x}\left(\bar{Z}^{N}(t)>0\right) \rightarrow 0$. Hence, for all $\lambda \in(-\Lambda, \Lambda)$ and some $\varepsilon_{N} \rightarrow 0$, which precise value might change from line to line, we get

$$
\begin{aligned}
\mathbb{E}_{x}\left[e^{-\lambda \bar{Y}^{N}(t)}\right] & =1-\left(\left(1+\varepsilon_{N}\right) \frac{2}{\sigma^{2} t \lambda+2}-1\right) \mathbb{P}_{x}\left(\bar{Z}^{N}(t)>0\right) \\
& =\exp \left(-\mathbb{P}_{x}\left(\bar{Z}^{N}(t)>0\right) \frac{\sigma^{2} \lambda t+\varepsilon_{N}}{2+\sigma^{2} t \lambda}\right)
\end{aligned}
$$

Finally, conditioning on the initial configuration, we get by independence that for all $\lambda \in(-\Lambda, \Lambda)$,

$$
\begin{align*}
\mathbb{E}\left[e^{-\lambda \bar{Y}^{N}(t)}\right] & =\mathbb{E}\left[\prod_{v \in \mathcal{N}_{0}} \mathbb{E}_{x_{v}}\left[e^{-\lambda \bar{Y}^{N}(t)}\right]\right]=\exp \left(\sum_{v \in \mathcal{N}_{0}}-\mathbb{P}_{x_{v}}\left(\bar{Z}^{N}(t)>0\right) \frac{\sigma^{2} \lambda t+\varepsilon_{N}}{2+\sigma^{2} t \lambda}\right) \\
& =\exp \left(-\frac{1}{N} \sum_{v \in \mathcal{N}_{0}} h^{N}\left(x_{v}\right) \frac{N}{h^{N}\left(x_{v}\right)} \mathbb{P}_{x_{v}}\left(\bar{Z}^{N}(t)>0\right) \frac{\sigma^{2} \lambda t+\varepsilon_{N}}{2+\sigma^{2} t \lambda}\right) \\
& \rightarrow \exp \left(-y_{0} \frac{\lambda}{1+\frac{\sigma^{2} t}{2} \lambda}\right), \quad \text { as } N \rightarrow \infty, \tag{75}
\end{align*}
$$

by assumption and 1 .
Step 2. Tightness. The sequence $\left(\bar{Y}^{N}(t), N \in \mathbb{N}\right)$ is bounded in $L^{1}$. Indeed, it follows from Proposition 2.5 that

$$
\mathbb{E}\left[\bar{Y}^{N}(t)\right]=\mathbb{E}\left[\bar{Y}^{N}(0)\right]<\sup _{N} \mathbb{E}\left[\bar{Y}^{N}(0)\right]<\infty .
$$

Thus, $\bar{Y}^{N}(t)$ is tight and there exists a subsequence of $\left(\bar{Y}^{N}(t), N \in \mathbb{N}\right)$ that converges weakly to some random variable $Y^{\infty}(t)$.

Step 3. Characterisation of the limit. We see from 75 that

$$
\begin{equation*}
\mathbb{E}\left[e^{-\lambda Y^{\infty}(t)}\right]=\exp \left(-\frac{\lambda}{1+\frac{\Sigma^{2} t}{2} \lambda} y_{0}\right), \quad \forall \lambda \in(-\Lambda, \Lambda) \tag{76}
\end{equation*}
$$

Note that the RHS of 76 is precisely the Laplace transform of $\Xi_{t}$. It now remains to show that this implies that $Y^{\infty}(t) \stackrel{(d)}{=} \Xi_{t}$. Since the Laplace transforms of $Y^{\infty}(t)$ and $\Xi_{t}$ are equal on $(-\Lambda, \Lambda)$, they have the same moments. In addition, (76) shows that $Y^{\infty}(t)$ satisfies Cramér's condition (i.e. there exists a constant $c>0$ such that $\mathbb{E}\left[e^{-\lambda Y^{\infty}(t)}\right]<\infty$ for all $\left.\lambda \in(-c, c)\right)$. This implies (see e.g. [22] Corollary 15.33) that the distribution of $Y^{\infty}(t)$ is determined by its moments so that $Y^{\infty}(t) \stackrel{(d)}{=} \Xi_{t}$. Step 2 then yields that

$$
\bar{Y}^{N}(t) \Rightarrow \Xi_{t}, \quad N \rightarrow \infty
$$

The finite-dimensional convergence stems from this result by induction.
It now remains to show that the additive martingale is a good approximation of the process $Z$.
Lemma 7.2. Let $t>0$. Assume that

$$
\sup _{N} \mathbb{E}\left[\bar{Y}^{N}(0)\right]<\infty
$$

Then we have

$$
\left|\bar{Z}^{N}(t)-\bar{Y}^{N}(t)\right| \rightarrow_{p} 0, \quad \text { as } \quad N \rightarrow \infty
$$

Proof. Fix $\eta>0$. Let us prove that for $N$ large enough,

$$
\mathbb{P}\left(\left|\bar{Z}^{N}(t)-\bar{Y}^{N}(t)\right|>2 \eta\right)<\eta .
$$

First note that

$$
\left|\bar{Z}^{N}(t)-\bar{Y}^{N}(t)\right| \leq\left|\bar{Y}^{N}(t)-\bar{Y}^{N}((1-\delta) t)\right|+\left|\bar{Y}^{N}((1-\delta) t)-\bar{Z}^{N}(t)\right|
$$

Recall that $\bar{Y}$ is a martingale so that $\mathbb{E}\left[\bar{Y}^{N}(t) \mid \mathcal{F}_{(1-\delta) t}\right]=Y^{N}((1-\delta) t)$. It then follows from Chebyshev's inequality that

$$
\begin{equation*}
\mathbb{P}\left(\left|\bar{Y}^{N}(t)-\bar{Y}^{N}((1-\delta) t)\right| \geq \eta \mid \mathcal{F}_{(1-\delta) t}\right) \leq \eta^{-2} \operatorname{Var}\left(\bar{Y}^{N}(t) \mid \mathcal{F}_{(1-\delta) t}\right) \tag{77}
\end{equation*}
$$

Note that conditional on $\mathcal{F}_{(1-\delta) t}$, the particles alive at time $(1-\delta) t N^{1-c}$ evolve independently between times $(1-\delta) t N^{1-c}$ and $t N^{1-c}$. Hence, the conditional variance $\operatorname{Var}\left(\bar{Y}^{N}(t) \mid \mathcal{F}_{(1-\delta) t}\right)$ is equal to the sum of the variances of the contributions to $\bar{Y}^{N}(t)$ from the particles alive at time $(1-\delta) t N^{1-c}$. Yet, we have from Corollary 4.11

$$
\frac{N}{h^{N}(x)} \mathbb{E}_{(x,(1-\delta) t)}\left[\bar{Y}^{N}(t)^{2}\right]=\frac{N}{h^{N}(x)} \mathbb{E}_{x}\left[\bar{Y}^{N}(\delta t)^{2}\right] \rightarrow \frac{\sigma^{2} t \delta}{2}, \quad \text { uniformly in } x \in \Omega_{N} .
$$

As a result, for $N$ large enough (that only depends on $t$ and $\delta$ ), we have

$$
\operatorname{Var}\left(\bar{Y}^{N}(t) \mid \mathcal{F}_{(1-\delta) t}\right) \leq \frac{1}{N} \sum_{v \in \mathcal{N}_{(1-\delta) t N}} h^{N}\left(x_{v}\right) \frac{N}{h^{N}\left(x_{v}\right)} \mathbb{E}_{\left(x_{v},(1-\delta) t N\right.}\left[\left(\bar{Y}^{N}(t)\right)^{2}\right] \leq 2 \sigma^{2} \delta t \bar{Y}^{N}((1-\delta) t)
$$

Putting this together with 77 , we get that

$$
\begin{align*}
\mathbb{P}\left(\left|\bar{Y}^{N}(t)-\bar{Y}^{N}((1-\delta) t)\right| \geq \frac{\eta}{2}\right) & =\mathbb{E}\left[\mathbb{P}\left(\left.\left|\bar{Y}^{N}(t)-\bar{Y}^{N}((1-\delta) t)\right| \geq \frac{\eta}{2} \right\rvert\, \mathcal{F}_{(1-\delta) t}\right)\right] \\
& \leq 8 \eta^{-2} \sigma^{2} \delta t \mathbb{E}\left[\bar{Y}^{N}((1-\delta) t)\right]=8 \eta^{-2} \sigma^{2} \delta t \mathbb{E}\left[\bar{Y}^{N}(0)\right] . \tag{78}
\end{align*}
$$

Similarly, we get from the many-to-one lemma (Lemma 1.1, that

$$
\begin{aligned}
\mathbb{E}\left[\bar{Z}^{N}(t) \mid \mathcal{F}_{(1-\delta) t N}\right] & =\sum_{x_{v} \in \mathcal{N}_{(1-\delta) t N^{1-c}}^{N}} h^{N}\left(x_{v}\right) Q_{x_{v}}^{N, 1, \delta t N^{1-c}}\left(1 / h^{N}\left(\bar{\zeta}_{\delta}\right)\right) \\
& =\sum_{x_{v} \in \mathcal{N}_{(1-\delta) t N^{1-c}}^{N}} h^{N}\left(x_{v}\right) \int_{\Omega_{N}} \frac{p_{\delta}^{N}\left(x_{v}, y\right)}{h^{N}(x)} d y \\
& \leq \sum_{x_{v} \in \mathcal{N}_{(1-\delta) t N^{1-c}}^{N}} h^{N}\left(x_{v}\right)\left(1+\varepsilon_{N}\right) \int_{\Omega_{N}} \tilde{h}^{N}(y) d y=\left(1+\varepsilon_{N}\right) \bar{Y}^{N}((1-\delta) t),
\end{aligned}
$$

where we applied Lemma 3.4 (ii) using that $\delta N^{1-c} \gg \log (N)^{3}$.
The conditional variance $\operatorname{Var}\left(\bar{Z}^{N}(t) \mid \mathcal{F}_{(1-\delta) t N^{1-c}}\right)$ can then be bounded similarly, using Proposition 5.1

$$
\frac{N}{h^{N}(x)} \mathbb{E}_{(x,(1-\delta) t)}\left[\left(\bar{Z}^{N}(t)\right)^{2}\right]=\frac{N}{h^{N}(x)} \mathbb{E}_{(x,(1-\delta) t)}\left[\left(\bar{Z}^{N}(t)\right)^{2} \mid\left(\bar{Z}^{N}(t)\right)^{2}>0\right] \mathbb{P}_{(x,(1-\delta) t)}\left(\left(\bar{Z}^{N}(t)\right)^{2}>0\right) \leq 2 \sigma^{2} \delta t
$$

for $N$ large enough (that only depends on $t$ and $\delta$ ). Hence,

$$
\operatorname{Var}\left(\bar{Z}^{N}(t) \mid \mathcal{F}_{(1-\delta) t}\right) \leq \sum_{v \in \mathcal{N}_{(1-\delta) t N^{1-c}}^{N}} \mathbb{E}_{\left(x_{v},(1-\delta) t\right)}\left[\left(\bar{Z}^{N}(t)\right)^{2}\right] \leq 2 \sigma^{2} \delta t \bar{Y}^{N}((1-\delta) t)
$$

Chebyshev's inequality then yields

$$
\begin{equation*}
\mathbb{P}\left(\left|\bar{Z}^{N}(t)-\bar{Y}^{N}((1-\delta) t)\right|>\frac{\eta}{2}\right) \leq 8 \eta^{-2} \sigma^{2} \delta t \mathbb{E}\left[\bar{Y}^{N}(0)\right] \tag{79}
\end{equation*}
$$

Finally, note that $\mathbb{E}[\bar{Y} N(0)]$ is uniformly bounded by assumption, hence for small enough delta it follows from (78) and 79)

$$
\mathbb{P}\left(\left|\bar{Z}^{N}(t)-\bar{Y}^{N}(t)\right|>2 \eta\right)<\eta .
$$

This concludes the proof of the lemma.
Proof of Theorem 4. The result is a consequence of Lemma 7.1 and Lemma 7.2

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