RIGIDITY OF SPIN FILL-INS WITH NON-NEGATIVE SCALAR CURVATURE

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Abstract. We establish new mean curvature rigidity theorems of spin fill-ins with non-negative scalar curvature using two different spinorial techniques. Our results address two questions by Miao and Gromov, respectively. The first technique is based on extending boundary spinors satisfying a generalized eigenvalue equation via the Fredholm alternative for an APS boundary value problem, while the second is a comparison result in the spirit of Llarull and Lott using index theory. We also show that the latter implies a new Witten-type integral inequality for the mass of an asymptotically Schwarzschild manifold which holds even when the scalar curvature is not assumed to be non-negative.

1. Introduction

Fill-ins and extensions with non-negative scalar curvature play a major role in mathematical relativity and geometry.

On the one hand, they provide an important tool in many proofs. In [9; 13; 16; 23] manifolds with boundary are filled in by disks, in [33] a collar fill-in is used to connect the boundary to a minimal surface, and in [36; 41] asymptotically flat extensions of compact manifolds with boundary are constructed. The common thread in all of these constructions is that the fill-ins and extensions allow the applications of theorems which hold for manifolds without boundary or with minimal boundary such as the positive mass theorem or the Riemannian Penrose inequality. The above methods can even be combined as in [16; 32]. Moreover, fill-ins can be used to simplify the topology of the underlying manifold, see for instance [1; 10].

On the other hand, fill-ins and extensions are used directly in several important definitions such as Bray's inner mass [9] and the Bartnik mass [8] which is in the subject of much current research.

In view of this plethora of applications, it is of great significance to better understand when fill-ins with non-negative scalar curvature exist and what their properties are. In this article, we address two questions asked by Miao [34, Question 2] and Gromov [19, page 3] concerning such fill-ins.

Let us start by recalling the notion of a fill-in with non-negative scalar curvature. For a Riemannian manifold M with boundary ∂M, we denote by $\mathrm{H}_{\partial M}$ the mean curvature of ∂M. We adopt the convention that the boundary of the unit disk $\mathrm{D}^n \subset \mathbb{R}^n$ has mean curvature $n-1$.

Definition 1.1. Let Σ be an $(n-1)$-dimensional connected closed manifold. Given a metric g_Σ on Σ and a smooth function $h\colon \Sigma \to \mathbb{R}$, we say that an n-dimensional connected compact Riemannian manifold with boundary (M,g) is a non-negative scalar curvature (NNSC) fill-in for (Σ, g_Σ, h) if $(\partial M, g|_{\partial M}) = (\Sigma, g_\Sigma)$ and $\mathrm{H}_{\partial M} = h$.

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We say that h is the mean curvature of the fill-in (M, g, h). When Σ is spin, we say that (M, g) is a spin NNSC fill-in if M is spin and induces the spin structure on $\Sigma = \partial M$.

Shi–Wang–Wei's extension theorem [37, Theorem 1] establishes that, if M is a compact manifold with boundary Σ and g_Σ is a Riemannian metric on Σ, then g_Σ extends to a metric of positive scalar curvature on M. This shows that every closed null-bordant Riemannian manifold admits a NNSC fill-in. However, their construction produces a fill-in with negative mean curvature. Miao [34, Question 2] asked whether every closed null-bordant Riemannian manifold (Σ, g_Σ) admits a NNSC fill-in with positive mean curvature. Our first result answers Miao's question in the negative in the spin setting.

Theorem 1.2. *Let Σ be a closed spin manifold of dimension $n \geq 3$ with $n \equiv 0, 1, 3$ or $7 \mod 8$. Then there exists a metric g_Σ on Σ such that every spin NNSC fill-in of (Σ, g_Σ) with non-negative mean curvature is Ricci flat with minimal boundary.Every Berger sphere (S^3, g) with scalar curvature $\mathrm{scal} \leq -24$ has this property.*

Theorem 1.2 relies on a general extension principle for spinors on the boundary, which we exhibit in Theorem 3.1, combined with classical existence results of harmonic spinors on these Berger spheres due to Hitchin [25] and, in the more general cases due to further results of both Hitchin [25] as well as Bär [3]. We remark that harmonic spinors can often be produced independently of the metric using the Atiyah–Singer index theorem, e.g. if $\widehat{\mathrm{A}}(\Sigma) \neq 0$, but then Σ would not be null-bordant (see [27, Chapter III, §11, (11.21)]), so it would not make sense to talk about fill-ins in these situations. The significance of Theorem 3.1 is that it does not rely on the index theorem and in Theorem 1.2 we can thus treat special metrics on null-bordant manifolds such as S^{4k+3}.

The same extension result also shows that mean-convex spin domains admitting a parallel spinor are extremal with respect to spin fill-ins in the sense of the next theorem.

Theorem 1.3. *Let (N, g_N) be a spin manifold which admits a parallel spinor and has a compact mean-convex boundary $\Sigma = \partial N$, that is, $\mathrm{H}_{\partial N} \geq 0$. Let g_Σ be the induced metric on Σ and let $h \in \mathrm{C}^\infty(\Sigma, \mathbb{R})$ with $h \geq \mathrm{H}_{\partial N}$. Then every spin NNSC fill-in of (Σ, g_Σ, h) admits a parallel spinor and in this case $h = \mathrm{H}_{\partial N}$.*

Note that, in general, it is important that a spin NNSC fill-in requires an extension of the given spin structure on Σ to the filled-in manifold M, see also the discussion in Remark 3.2 below. However, in those cases where Σ only admits one spin structure—such as the main example S^3 of Theorem 1.2—the theorem thus already precludes the existence of a fill-in with positive mean curvature as long as it would admit any spin structure.

Next, we investigate a positive upper bound of a spin NNSC fill-in of a closed Riemannian spin manifold Σ in terms of the metric properties of Σ. Let us start with recalling the following notion due to Gromov [19, Section 1], see also [20, Section 4.3].

Definition 1.4. Let (Y, g_Y) be a k-dimensional closed orientable Riemannian manifold. The *hyperspherical radius* of (Y, g_Y), denoted by $\mathrm{Rad}_{\mathrm{S}^k}(Y, g_Y)$, is the supremum of the numbers $R > 0$ such that there exists a smooth $\frac{1}{R}$-Lipschitz map $f\colon (Y, g_Y) \to (\mathrm{S}^k, g_{\mathrm{S}^k})$ of non-zero degree.

Gromov [19, Section 2] proved that if Σ^{n-1} is a closed Riemannian spin manifold, h is a smooth function on Σ, and M is a spin fill-in of Σ, then $h \leq (n-1)/R$, where R is the hyperspherical radius of Σ. Moreover, he conjectured [19, Page 3] that disks are rigid, that is, the equality is achieved if and only if M is a disk in Euclidean space. Our second main result establishes this conjecture.

Theorem 1.5. *For $n \geq 2$, let (Σ, g_Σ) be an $(n-1)$-dimensional closed connected Riemannian spin manifold. Let $h\colon \Sigma \to \mathbb{R}$ be a smooth function, and let (M, g) be a spin NNSC fill-in of (Σ, g_Σ, h). Then*

$$\min_{p\in\Sigma} h(p) \leq \frac{n-1}{\mathrm{Rad}_{\mathrm{S}^{n-1}}(\Sigma, g_\Sigma)}. \tag{1.1}$$

Furthermore, equality is achieved if and only if (M, g) is the round disk in Euclidean space of radius $R = \mathrm{Rad}_{\mathrm{S}^{n-1}}(\Sigma, g_\Sigma)$ and $h = \frac{n-1}{R}$.

One wishes to apply a scalar- and mean curvature rigidity result in the spirit of Goette–Semmelmann [17] and Lott [31], see specifically Wang–Xie [40, Theorem 3.2], to a map $\Sigma \to \mathrm{S}^{n-1}$ realizing the hyperspherical radius. However, due to its definition in terms of a supremum, *a priori* we do not have a *smooth* map precisely realizing the hyperspherical radius. We solve this issue by establishing the following almost rigidity statement for maps to a fixed bounded strictly convex smooth domain in Euclidean space from a fixed spin NNSC manifold with boundary, which is also of independent interest.

Theorem 1.6. *For $n \geq 3$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded, strictly convex domain with smooth boundary. Let M be a connected compact Riemannian spin manifold with connected boundary such that $\mathrm{scal}_M \geq 0$. Fix $p \in [1, \infty)$. Then for every $\varepsilon > 0$ there exists a $\delta = \delta(M, \Omega, p, \varepsilon) > 0$ such that the following holds:*

For every smooth map $f\colon \partial M \to \partial\Omega$ satisfying

- $\mathrm{Lip}(f) \leq 1 + \delta$,
- $\mathrm{H}_{\partial M} \geq \mathrm{H}_{\partial\Omega} \circ f - \delta$,
- $\deg(f) \neq 0$,

there exists an isometry $\phi\colon \partial M \to \partial\Omega$ with $\mathrm{II}_{\partial M} = \phi^* \mathrm{II}_{\partial\Omega}$ such that f is ε-close to ϕ in $\mathrm{W}^{1,p}(\partial M, \mathbb{R}^n)$. Moreover, in this case M is flat and isometric to Ω.

The proof of this theorem is based on carefully analyzing the fundamental spinorial integral inequality established in Proposition 4.4, which can be interpreted as a version of a result of Lott [31, Theorem 1.1] "with coefficients" in the spirit of Listing [29], a special case of which is also used by Brendle in [11]. The existence of spinors used in this theorem is provided by index theory which in this case works regardless of the dimension's parity.

Another possible approach to Theorem 1.5 would be to first extract a convergent subsequence from a sequence of maps almost realizing the hyperspherical radius and then prove rigidity directly for the limiting Lipschitz map. Then the potential difficulty would lie in the fact that this map is *a priori* not necessarily smooth. In the situation of Llarull's theorem, a related low regularity Lipschitz rigidity result was recently established by Cecchini–Hanke–Schick [14], where also singular metrics of $\mathrm{W}^{1,p}$-regularity ($p > n$) are considered. In our present situation, Theorem 1.6 immediately implies a rigidity result for low-regularity Lipschitz maps with smooth metrics as stated in the next corollary. However, Theorem 1.6 provides stronger control than merely a Lipschitz rigidity result because using an a priori compactness argument only yields $\mathrm{C}^{0,\alpha}$-subconvergence to a Lipschitz map rather than in $\mathrm{W}^{1,p}$.

Corollary 1.7. *For $n \geq 3$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded, strictly convex domain with smooth boundary. Let M be a connected compact Riemannian spin manifold with boundary such that $\mathrm{scal}_M \geq 0$. If $f\colon \partial M \to \partial\Omega$ is a (not necessarily smooth) 1-Lipschitz map of non-zero degree such that $\mathrm{H}_{\partial M} \geq \mathrm{H}_{\partial\Omega} \circ f$, then f is an isometry and M is flat and isometric to Ω.*

In the case when the map f in Corollary 1.7 is *smooth*, its statement already follows from results of Wang–Xie [40, Theorem 3.2], see also [39, Theorem 1.1].

Finally, we show in Section 6 that the same techniques—that is, an index-theoretic existence result of a spinor and the integral inequality from Proposition 4.4—also imply a Witten-type integral formula [42] for the ADM-mass of an asymptotically Schwarzschild manifold. Interestingly, unlike Witten's approach, this formula does not use non-negative scalar curvature, and it is still valid when the underlying manifold is only asymptotically Schwarzschild up to first order instead of second order. In particular this gives yet another proof of the Riemannian positive mass theorem and thereby addresses another question of Gromov [19, p. 3] on the relation between the theorems of Llarull, Goette–Semmelmann and Lott [17; 30; 31] to the positive mass theorem.

The paper is organized as follows. In Section 2, we recall the basic notation and convention on spinor bundles and Dirac operators. In Section 3, we prove the extension result for spinors (Theorem 3.1) and use it to deduce Theorems 1.2 and 1.3. Section 4 is dedicated to proving the fundamental integral inequality (Proposition 4.4) needed for the remaining results. In Section 5, we establish Theorem 1.6, from which we derive Theorem 1.5 and Corollary 1.7. Finally, in Section 6, we discuss the application to asymptotically flat manifolds.

2. Notation and conventions

2.1. **The boundary Dirac operator and Weitzenböck formula.** Let M be a Riemannian manifold and $S \to M$ a Dirac bundle in the sense of Gromov–Lawson [18, §1; 27, Chapter II §5] (for instance the spinor bundle if M is a spin manifold). We denote with c the Clifford multiplication and with $\mathcal{D} = \sum_{i=1}^{n} \mathrm{c}(e^i)\nabla_{e_i}$ the Dirac operator on S. We will study boundary value problems in this context, for the general theory of which we refer to Bär–Ballmann [5; 6]. To this end, we denote the *interior* unit normal field of ∂M by ν_M and denote by $\mathrm{H}_{\partial M}$ the mean curvature with respect to ν_M. Next, we fix the *boundary Dirac operator*

$$\mathcal{A} \coloneqq \sum_{i=1} \mathrm{c}^{\partial}(e_i)\nabla^{\partial}_{e_i} = \frac{1}{2}\mathrm{H}_{\partial M} - \mathrm{c}(\nu_M)\mathcal{D} - \nabla_{\nu_M} \colon \mathrm{C}^{\infty}(\partial M, S^{\partial}) \to \mathrm{C}^{\infty}(\partial M, S^{\partial}), \tag{2.1}$$

where $S^{\partial} = S|_{\partial M}$, $\mathrm{c}^{\partial}(\omega) = \mathrm{c}(\omega)\,\mathrm{c}(\nu)$, $\nabla^{\partial}_{\xi} = \nabla_{\xi} + \frac{1}{2}\mathrm{c}^{\partial}(\nabla_{\xi}\nu)$. This operator $\mathcal{A}$ can be used as a canonical adapted operator on the boundary, see [6, Appendix 1] to study boundary value problems for $\mathcal{D}$. Note that $\mathcal{A}\,\mathrm{c}(\nu_M) = -\,\mathrm{c}(\nu_M)\mathcal{A}$.

We then have the integrated Bochner–Lichnerowicz–Weitzenböck formula for $\Psi \in \mathrm{C}^{\infty}(M, S)$,

$$\begin{aligned} \|\mathcal{D}\Psi\|^2_{\mathrm{L}^2(M)} = \|\nabla\Psi\|^2_{\mathrm{L}^2(M)} + \left(\mathcal{R}^S\Psi, \Psi\right)_{\mathrm{L}^2(M)} \\ + \int_{\partial M} \frac{1}{2}\mathrm{H}_{\partial M}\,|\Psi|^2 - \langle \mathcal{A}\Psi, \Psi\rangle\,\mathrm{dS} \end{aligned} \tag{2.2}$$

see [6, Appendix 1, equation (27)]. Here $\mathcal{R}^S$ is given by

$$\mathcal{R}^S = \sum_{i<j} \mathrm{c}(e^i)\,\mathrm{c}(e^j)\mathrm{R}^S_{e_i,e_j}.$$

In the case that M is spin and S its spinor bundle, we have $\mathcal{R}^S = \frac{\mathrm{scal}}{4}$, in which case the formula (2.2) is called the integrated *Schrödinger–Lichnerowicz formula*.

2.2. **Generalized APS boundary conditions.** We need to impose boundary conditions to work with the Dirac operator on manifolds with boundary. Given any self-adjoint first-order operator A on the boundary adapted to $\mathcal{D}$ in the sense of [6,

§3.2], which might not necessarily be the canonical adapted operator described in Section 2.1 above, we can define the corresponding (generalized) Atiyah–Patodi–Singer [2] (APS) boundary condition by imposing on sections $\Psi \in \mathrm{H}^1(M,S)$ that

$$\chi_{[0,\infty)}(A)\,(\Psi|_{\partial M}) = 0,$$

where $\chi_{[0,\infty)}(A)$ is the L^2-orthogonal projection onto the non-negative part of the spectrum of A. This is always an elliptic boundary condition [6, Example 4.21].

In Section 3 we will apply this to the case $A = \mathcal{A} + h$, where $h \in \mathrm{C}^\infty(\partial M, \mathbb{R})$ is some smooth function on the boundary and $\mathcal{A}$ is the canonical adapted operator. The adjoint condition of $\chi_{[0,\infty)}(\mathcal{A}+h)\,(\Psi|_{\partial M}) = 0$ is given by

$$\chi_{(0,\infty)}(\mathcal{A}-h)\,(\Psi|_{\partial M}) = 0,$$

compare [6, §4.3], where $\chi_{(0,\infty)}$ denotes the L^2-orthogonal projection onto the positive part of the spectrum. Note the subtle but important difference between non-negative and positive parts of the spectrum as well as the change in sign in front of the function h.

2.3. **Spin maps and local boundary conditions.** In this subsection, we set up another boundary value problem, namely the one we need for the proof of Theorem 1.5. This is merely a conceptual elaboration of the approach used by Lott [31] that has also been used by several other authors in recent times [11; 12; 40].

Let $f\colon (M,\partial M) \to (N,\partial N)$ be a smooth spin map between Riemannian manifolds with boundary of the same dimension n, where for simplicity we assume that both M and N are oriented. Being a spin map in this case means that $w_2(\mathrm{T}M) = f^* w_2(\mathrm{T}N)$, or equivalently that the bundle $\mathrm{T}M \oplus f^*\mathrm{T}N$ endowed with the pseudo-Riemannian bundle metric $g_M \oplus (-g_N)$ admits a spin structure.[1]

Recall that the Clifford algebra $\mathrm{Cl}_{n,n} = \mathrm{Cl}(\mathbb{R}^n \oplus \mathbb{R}^n, (\delta, -\delta))$ has a canonical irreducible representation $\mathbf{c}\colon \mathrm{Cl}_{n,n} \xrightarrow{\cong} \mathrm{End}(\bigwedge \mathbb{R}^n)$ which is generated by the standard Clifford actions on the exterior algebra

$$\begin{aligned}\mathbf{c}(v,0)\alpha &\coloneqq \mathrm{c}(v)\alpha \coloneqq v \wedge \alpha - \mathrm{i}_v\,\alpha,\\ \mathbf{c}(0,v)\alpha &\coloneqq \bar{\mathrm{c}}(v)\alpha \coloneqq v \wedge \alpha + \mathrm{i}_v\,\alpha.\end{aligned} \tag{2.3}$$

Let S be the complexified spinor bundle associated to a chosen spin structure on $\mathrm{T}M \oplus f^*\mathrm{T}N$ and the canonical representation $\mathbf{c}$, that is,

$$S = \left(\mathrm{P}_{\mathrm{Spin}_n \times_{\mathbb{Z}/2} \mathrm{Spin}_{-n}}(\mathrm{T}M \oplus f^*\mathrm{T}N) \times_{\mathrm{Spin}_n \times_{\mathbb{Z}/2} \mathrm{Spin}_{-n}, \mathbf{c}} \bigwedge \mathbb{R}^n\right) \otimes \mathbb{C}.$$

Note that S admits two different Clifford actions, one for vector fields $\xi \in \mathfrak{X}(M)$ on M which we denote by $\mathrm{c}(\xi) = \mathbf{c}(\xi, 0) \in \mathrm{End}(S)$ and the other for vector fields $\eta \in \mathfrak{X}(N)$ on N which we denote by $\bar{\mathrm{c}}(\eta) = \mathbf{c}(0, f^*\eta) \in \mathrm{End}(S)$. By construction, the usual Clifford algebra relations hold which we spell out in the following for convenience of the reader.

$$\begin{gathered}\mathrm{c}(\xi)^2 = -|\xi|^2, \quad \bar{\mathrm{c}}(\eta)^2 = |\eta|^2, \quad \mathrm{c}(\xi)\,\bar{\mathrm{c}}(\eta) = -\,\bar{\mathrm{c}}(\eta)\,\mathrm{c}(\xi),\\ \mathrm{c}(\xi)^* = -\,\mathrm{c}(\xi), \quad \bar{\mathrm{c}}(\eta)^* = \bar{\mathrm{c}}(\eta).\end{gathered}$$

With respect to the Clifford action of M, the bundle S is a Dirac bundle in the sense of Gromov–Lawson and we have the Dirac operator $\mathcal{D} = \sum_{i=1}^n \mathrm{c}(e_i)\nabla_{e_i}$. In

[1] Since both M and N are oriented, this is by coincidence equivalent to $\mathrm{T}M \oplus f^*\mathrm{T}N$ having a spin structure also with respect to a positive definite bundle metric. But in general the pseudo-Riemannian perspective is the topologically appropriate point of view, compare also Tony [38, §1.2 and §4], and it avoids having to introduce factors of $\sqrt{-1}$ at some places in our proofs.

this case, the interior curvature term $\mathcal{R}^S$ in the Bochner–Lichnerowicz–Weitzenböck formula (2.2) is explicitly given by $\mathcal{R}^S = \frac{\mathrm{scal}}{4} + \mathcal{R}^N$, where

$$\mathcal{R}^N = -\frac{1}{2}\sum_{i<j} \mathrm{c}(e^i \wedge e^j)\,\bar{\mathrm{c}}\left(\mathrm{R}^{\mathrm{T}N}(\mathrm{d}f(e_i)\wedge \mathrm{d}f(e_j))\right). \tag{2.4}$$

Again we need to impose suitable boundary conditions. However, in this case there are specific local boundary conditions well-adapted to the situation at hand. To describe these, let ν_M and ν_N denote the interior unit normals of the boundaries of M and N, respectively, and let $s\colon \partial M \to \{\pm 1\}$ be a choice of sign for each connected component of ∂M. Then we may impose the local boundary condition

$$\mathrm{c}(\nu_M)\psi = s\,\bar{\mathrm{c}}(\nu_N)\psi \quad \text{on } \partial M. \tag{2.5}$$

Rewriting this as $\chi(\psi) = s\psi$ with $\chi = \bar{\mathrm{c}}(\nu_N)\,\mathrm{c}(\nu_M)$ (and using $\chi\,\mathrm{c}(\nu_M) = -\,\mathrm{c}(\nu_M)\chi$ as well as Lemma 4.1 below), one verifies that this is a self-adjoint elliptic boundary condition. Note that for $s = 1$, this corresponds to the boundary condition used by Lott [31]. Moreover, the bundle S inherits a $\mathbb{Z}/2$-grading $S = S^+ \oplus S^-$from the standard even/odd grading on $\bigwedge \mathbb{R}^n$ with respect to which $\mathcal{D}$ is odd. As usual, we can thus define the index

$$\mathrm{ind}(\mathcal{D}, s) = \dim\ker(\mathcal{D}^+, s) - \dim\ker(\mathcal{D}^-, s),$$

where we take the kernel on smooth (or equivalently H^1-) sections satisfying the boundary condition (2.5). A similar setup in a related context was considered in [15, §2–3], compare also [7, Appendix B].

We consider two standard examples. The first is that this reduces to differential forms in case $M = N$.

Example 2.1. Let $M = N$ be the same Riemannian manifold and $f = \mathrm{id}$. Then $\mathrm{T}M \oplus f^*(\mathrm{T}M) = \mathrm{T}M \oplus \mathrm{T}M$ and this bundle always admits a spin structure with respect to the bundle metric $g_M \oplus (-g_M)$. The exterior algebra $\bigwedge \mathrm{T}M$ is itself a fiberwise irreducible bundle of $\mathrm{Cl}(\mathrm{T}M \oplus \mathrm{T}M, g_M \oplus (-g_M))$-modules, so we may just take $S = \bigwedge \mathrm{T}M \otimes \mathbb{C}$ and, in light of (2.3), $\mathcal{D} = d + d^*$ is the de Rham operator on complex-valued differential forms. By the same token, for $s = 1$ the boundary condition (2.5) then becomes the $\mathrm{i}_{\nu_M}\alpha = 0$—in other words, absolute boundary conditions. In particular, $\ker(\mathcal{D}, 1)$ is isomorphic to the de Rham cohomology of M and so $\mathrm{ind}(\mathcal{D}, 1) = \chi(M)$. Similarly, for $s = -1$ it means $\nu_M \wedge \alpha = 0$—that is, relative boundary conditions. In general, it would mean using absolute boundary conditions on some boundary components and relative on the others.

Convention 2.2. In the remaining part of this paper, we will assume that every domain in Euclidean space is bounded and has smooth boundary.

The second example is where $N = \Omega \subset \mathbb{R}^n$ is a convex domain, which is the case we will use in the proof of Theorem 1.5.

Example 2.3. Let $N = \Omega \subset \mathbb{R}^n$ be a convex domain. Then $\mathrm{T}\Omega = \Omega \times \mathbb{R}^n$ is the trivial bundle and hence $\mathrm{T}M \oplus f^*\mathrm{T}\Omega = \mathrm{T}M \oplus \mathbb{R}^n$. In this case, the condition of f being spin is equivalent to M admitting a spin structure. Moreover, given a spin structure on M with its principal Spin_n-bundle, we then obtain

$$S = \left(\mathrm{P}_{\mathrm{Spin}_n}(\mathrm{T}M) \times_{\mathrm{Spin}_n,\mathrm{c}} \bigwedge \mathbb{R}^n\right) \otimes \mathbb{C}.$$

Note that as a $\mathrm{Cl}_{n,0}$-module the exterior algebra $\bigwedge \mathbb{R}^n$ is not irreducible and it splits into multiple irreducible submodules. For instance, if n is even, we can identify S with $\not{S}_M \otimes \not{S}_{\mathbb{R}^n}$, where $\not{S}$ refers to the usual complex spinor bundles. The latter being trivial, we can further observe that $S \cong \bigoplus_{i=1}^m \not{S}_M$, where $m = \dim \not{S}_{\mathbb{R}^n} = 2^{\frac{n}{2}}$, so a section of S can be identified with a system of spinors on M—this is essentially the point of view taken by Brendle [11]. However, we will not make use of this observation and instead work with the abstract construction of the bundle, which works regardless of the dimension's parity.

We also note that the bundle S and its Dirac operator $\mathcal{D}$ do not depend on the map f or the domain Ω at all, but the boundary condition (2.5) does. We then have

$$\operatorname{ind}(\mathcal{D}, 1) = \deg(f).$$

In the even-dimensional case, this is a consequence of Lott [31], but again there is no reason to restrict to the even-dimensional case and we provide a direct argument of this index formula in Appendix A independently of the dimension's parity, see Theorem A.3.

3. NNSC fill-ins with non-negative mean curvature

In this section, we answer Miao's question [34, Question 2] restricted to spin fill-ins and establish fill-in extremality of spin domains admitting a parallel spinor. Both results rely on the following general extension theorem.

Theorem 3.1. *Let (Σ^{n-1}, g_Σ) be a closed spin manifold, and let $h\colon \Sigma \to [0,\infty)$ be a smooth non-negative function such that there exists a non-trivial spinor $\psi \in \mathrm{C}^\infty(\Sigma, \not{S}_\Sigma)$ satisfying $\not{D}_\Sigma \psi = \frac{1}{2} h\psi$. Then any NNSC spin fill-in (M, g) of (Σ, g_Σ) with $\mathrm{H}_{\partial M} \geq h$ admits a parallel spinor and thus is Ricci flat. Moreover, in this case $\mathrm{H}_{\partial M} = h$.*

Proof. Let $S \to M$ denote the spinor bundle of M with Dirac operator $\mathcal{D}$. Then its restriction to the boundary S^∂ can be identified with one or two copies, depending on dimension parity, of the spinor bundle $\not{S}_\Sigma$ such that the canonical boundary operator $\mathcal{A}$ identifies with $\not{D}_\Sigma$ on each copy. Thus by assumption there exists a section $\psi_0 \in \mathrm{C}^\infty(\Sigma, S^\partial)$ such that $\mathcal{A}\psi_0 = \frac{1}{2} h\psi_0$.

We consider the Dirac operator $\mathcal{D}\colon \mathrm{H}^1(M, S; B) \to \mathrm{L}^2(M, S)$ subject to the boundary condition B given by the spectral projection $\chi_{[0,\infty)}(\mathcal{A} - \frac{1}{2}h)$ as described in Section 2.2. Then the adjoint boundary condition B^{ad} is determined by the projection $\chi_{(0,\infty)}(\mathcal{A} + \frac{1}{2}h)$. We now apply the Fredholm alternative.

First consider the case that the operator $\mathcal{D}\colon \mathrm{H}^1(M, S; B) \to \mathrm{L}^2(M, S)$ is surjective. Then choose a section $\Psi_0 \in \mathrm{C}^\infty(M, S)$ such that $\Psi_0|_\Sigma = \psi_0$. By surjectivity, we can furthermore choose Ψ_1 such that $\mathcal{D}\Psi_1 = -\mathcal{D}\Psi_0$ and $\chi_{[0,\infty)}(A - \frac{1}{2}h)(\psi_1) = 0$ with $\psi_1 := \Psi_1|_\Sigma$. We then set $\Psi = \Psi_0 + \Psi_1$. Then $\mathcal{D}\Psi = 0$ and hence by (2.2) we have

$$\begin{aligned}
0 &\geq \|\nabla\Psi\|^2_{\mathrm{L}^2} + \int_\Sigma \langle (\tfrac{1}{2}\,\mathrm{H}_{\partial M} - \mathcal{A})\Psi, \Psi\rangle \,\mathrm{dS} \\
&\geq \|\nabla\Psi\|^2_{\mathrm{L}^2} + \int_\Sigma \langle (\tfrac{1}{2}h - \mathcal{A})\Psi, \Psi\rangle \,\mathrm{dS} \\
&= \|\nabla\Psi\|^2_{\mathrm{L}^2} + \int_\Sigma \langle (\tfrac{1}{2}h - \mathcal{A})\psi_1, \psi_1\rangle \,\mathrm{dS} + \int_\Sigma \langle (\tfrac{1}{2}h - \mathcal{A})\psi_0, \psi_0\rangle \,\mathrm{dS} \\
&= \|\nabla\Psi\|^2_{\mathrm{L}^2} + \int_\Sigma \langle (\tfrac{1}{2}h - \mathcal{A})\psi_1, \psi_1\rangle \,\mathrm{dS} \geq \|\nabla\Psi\|_{\mathrm{L}^2} + \lambda_0 \int_\Sigma |\psi_1|^2 \,\mathrm{dS} \geq 0,
\end{aligned}$$

where $\lambda_0 > 0$ is the smallest positive eigenvalue of the operator $\frac{1}{2}h - \mathcal{A}$. This means that $\nabla\Psi = 0$, $\psi_1 = 0$ and hence $\Psi|_\Sigma = \psi_0$. So we have found the desired parallel spinor Ψ. Moreover, it then follows from (2.1) that $\mathcal{A}\psi_0 = \frac{1}{2}\,\mathrm{H}_{\partial M}\,\psi_0$. But since by assumption we have $\mathcal{A}\psi_0 = \frac{1}{2}h\psi_0$, this means that $h = \mathrm{H}_{\partial M}$.

On the other hand, if the operator $\mathcal{D}\colon \mathrm{H}^1(M, S; B) \to \mathrm{L}^2(M, S)$ is not surjective, then the operator $\mathcal{D}$, subject to the adjoint condition B^{ad}, must have a non-trivial element in the kernel Ψ, that is, $\mathcal{D}\Psi = 0$ and $\chi_{(0,\infty)}(\mathcal{A} + \frac{1}{2}h)(\Psi|_\Sigma) = 0$. By (2.2),

$$0 = \|\mathcal{D}\Psi\|^2_{\mathrm{L}^2(M)} \geq \|\nabla\Psi\|^2_{\mathrm{L}^2} + \int_\Sigma \langle (\tfrac{1}{2}\,\mathrm{H}_{\partial M} - \mathcal{A})\Psi, \Psi\rangle\,\mathrm{dS}$$
$$\geq \int_\Sigma \tfrac{1}{2}\,\mathrm{H}_{\partial M}\,|\Psi|^2\,\mathrm{dS} + \int_\Sigma \langle (-\tfrac{1}{2}h - \mathcal{A})\Psi, \Psi\rangle\,\mathrm{dS} \geq 0,$$

where we crucially use non-negativity of h and $\mathrm{H}_{\partial M}$. We conclude that $\nabla\Psi = 0$, which provides the desired parallel spinor for this case, as well as $0 = \mathrm{H}_{\partial M} \geq h \geq 0$ and so also $\mathrm{H}_{\partial M} = h = 0$. □

We are now ready to prove the first two of our main theorems.

Proof of Theorem 1.2. On the Berger sphere with constant scalar curvature $\mathrm{scal} \leq -24$ there is a solution to the Dirac equation according to [25, page 37]. Similarly, Bär showed in [3] that on every closed spin manifold of dimension $n \equiv 3 \mod 4$, there exists a metric admitting a harmonic spinor, and Hitchin did so in [25, Theorem 4.5] for dimension $n \equiv 0, 1, 7 \mod 8$. Hence, the result follows from Theorem 3.1 with $h = 0$. □

Proof of Theorem 1.3. Given a parallel spinor on N, its restriction to Σ provides a spinor ψ satisfying $\mathcal{A}\psi = \frac{1}{2}\,\mathrm{H}_{\partial N}\,\psi$ because of (2.1). Then we are in the situation of Theorem 3.1 and any spin NNSC fill-in (M, g) of (Σ, g_Σ, h) with $h \geq \mathrm{H}_{\partial N}$ admits a parallel spinor and satisfies $h = \mathrm{H}_{\partial N}$. □

Remark 3.2. The concept of a NNSC spin fill-in in the statement of Theorem 3.1 also includes the spin structure in a subtle way. Indeed, we only obtain a result on fill-ins M that extend the particular spin structure of Σ on whose spinor bundle the given spinor ψ lives. That this is a necessary restriction can already be seen in the case of S^1 which admits two spin structures, one of which extends to the disk but has no harmonic spinors, whereas the other admits a harmonic spinor but does not extend to the disc. In particular, the presence of the latter spin structure does not contradict the fact that the disk is an NNSC fill-in of S^1 with positive mean curvature. Indeed, following Section 2.1, the spinor bundle of the disc restricted to the circle S^1 can be identified with the trivial bundle $\mathrm{S}^1 \times \mathbb{C}^2$ with canonical boundary operator

$$\mathcal{A} = \begin{pmatrix} -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta} + \frac{1}{2} & 0 \\ 0 & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta} + \frac{1}{2} \end{pmatrix}$$

which visibly does not have a kernel. A more intrinsic description is that sections of the spinor bundle on S^1 associated to the spin structure restricted from the disc can be identified with 2π-anti-periodic functions on $\mathbb{R}$. The other spin structure on S^1—the one that does *not* extend to the disc—corresponds to the trivial principal spin bundle and hence is just the trivial bundle $\mathrm{S}^1 \times \mathbb{C}$ on which the Dirac operator $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}$ acts. It clearly has a kernel given by the constant functions. See for instance [4] for more details on these and other examples of this nature.

However, note that this restriction is less severe in the case of the main example in Theorem 1.2, the Berger sphere, since S^3 has only one spin structure.

Remark 3.3. The proof of Theorem 3.1 has the curious feature that both cases of the Fredholm alternative lead to existence of a spinor with the desired property. One might be tempted to ask more generally whether in the setting of the theorem any spinor on the boundary $\psi_0 \in \mathrm{C}^\infty(\Sigma, S)$ satisfying $\mathcal{A}\psi_0 = \frac{1}{2}h\psi_0$ can be extended to a parallel spinor on a spin NNSC fill-in with $\mathrm{H} \geq h$.

The proof already shows that this is the case whenever the first part of the Fredholm alternative applies. Moreover, the second case can only occur if $h = 0$. So if $h \neq 0$, any such spinor indeed extends to a parallel spinor on the fill-in.

However, if $h = 0$ and we have a non-trivial spinor ψ_0 at the boundary satisfying $\mathcal{A}\psi_0 = 0$, then $\psi_1 = \mathrm{c}(\nu_M)\psi_0$ also satisfies $\mathcal{A}\psi_1 = 0$. But it is not possible to simultaneously extend both ψ_0 and ψ_1 to harmonic spinors on the spin fill-in M, because if Ψ_i was a harmonic spinor on M extending ψ_i for $i = 0, 1$, then we would have

$$0 = (\mathcal{D}\Psi_0, \Psi_1)_{\mathrm{L}^2(M)} - (\Psi_0, \mathcal{D}\Psi_1)_{\mathrm{L}^2(M)} = \int_\Sigma \langle \psi_0, \mathrm{c}(\nu_M)\psi_1 \rangle \,\mathrm{dS} = -\int_\Sigma |\psi_0|^2 \,\mathrm{dS} \neq 0.$$

Relatedly, non-negativity of h cannot be dropped from the hypotheses of Theorem 3.1 because if $\mathcal{A}\psi_0 = \frac{1}{2}h\psi_0$, then we have $\mathcal{A}\psi_1 = \frac{1}{2}(-h)\psi_1$ for $\psi_1 = \mathrm{c}(\nu_M)\psi_0$. For instance, the sphere S^{n-1} for $n \geq 3$ admits a spinor ψ_1 with $\not{D}_{\mathrm{S}^{n-1}}\psi_1 = -\frac{n-1}{2}\psi_1$ even though the hemisphere in S^n is a spin PSC fill-in of S^{n-1} with $\mathrm{H} = 0 \geq -\frac{n-1}{2}$. More drastically, the boundary Dirac operator always has arbitrarily negative eigenvalues.

Remark 3.4. A closely related result was previously claimed by Hijazi–Montiel–Zhang [21, Theorem 6] formulated in terms of the smallest non-negative eigenvalue $\lambda_1 \geq 0$ of the boundary Dirac operator $\mathcal{A}$, which can viewed as a special case of Theorem 3.1 for constant $h = 2\lambda_1$. However, the statement and proof of [21, Theorem 6] has a gap because it is claimed that if $\frac{\mathrm{H}_{\partial M}}{2} \geq \lambda_1$, any spinor satisfying $\mathcal{A}\psi = \lambda_1\psi$ on the boundary extends to a parallel spinor, which is not true in the case $\lambda_1 = 0$ as we have discussed in Remark 3.3. This is also precisely the case needed to establish our Theorem 1.2. The root of the problem already occurs in [21, Theorem 2] which does not account for the possibility of having harmonic spinors on the boundary in which case the APS boundary value problem is not self-adjoint.

Nevertheless, the first part of the statement in [21, Theorem 6], namely that for any NNSC fill-in (M, g) we have $\lambda_1 \geq \min_\Sigma \frac{\mathrm{H}_{\partial M}}{2}$, remains true even if $\lambda_1 = 0$, as it is a consequence of our Theorem 3.1.

We also mention that a recent paper due to Raulot [35] treats a similar eigenvalue estimate in the setting of initial data sets and fill-ins satisfying the dominant energy condition.

4. The main integral formula

In this section, we estimate the terms involving $\mathcal{R}^N$ and $\mathcal{A}$ in the Schrödinger–Licnerowicz formula (2.2) to obtain the main integral inequality Proposition 4.4. We start with the following commutation relation.

Lemma 4.1.

$$\mathcal{A}\,\bar{\mathrm{c}}(\nu_N) = \bar{\mathrm{c}}(\nu_N)\mathcal{A} + \mathrm{c}(\nu_M)\sum_{i=1}^{n-1}\mathrm{c}(e^i)\,\bar{\mathrm{c}}(-\nabla_{\mathrm{d}f(e_i)}\nu_N)$$

Proof. Let ψ be a smooth section of S along ∂M. First observe that $\nabla^\partial_\xi \bar{\mathrm{c}}(\nu_N)\psi = \bar{\mathrm{c}}(\nabla_{\mathrm{d}f(\xi)}\nu_N)\psi + \bar{\mathrm{c}}(\nu_N)\nabla^\partial_\xi\psi$. We thus have

$$\begin{aligned}
\mathcal{A}\,\bar{\mathrm{c}}(\nu_N)\psi &= \sum_{i=1}^{n-1}\mathrm{c}(e^i)\,\mathrm{c}(\nu_M)\nabla^\partial_{e_i}\bar{\mathrm{c}}(\nu_N)\psi \\
&= \sum_{i=1}^{n-1}\left(\mathrm{c}(e^i)\,\mathrm{c}(\nu_M)\,\bar{\mathrm{c}}(\nabla_{\mathrm{d}f(e_i)}\nu_N)\psi + \mathrm{c}(e^i)\,\mathrm{c}(\nu_M)\,\bar{\mathrm{c}}(\nu_N)\nabla^\partial_{e_i}\psi\right) \\
&= \left(\mathrm{c}(\nu_M)\sum_{i=1}^{n-1}\mathrm{c}(e^i)\,\bar{\mathrm{c}}(-\nabla_{\mathrm{d}f(e_i)}\nu_N)\psi\right) + \bar{\mathrm{c}}(\nu_N)\mathcal{A}\psi. \qquad \square
\end{aligned}$$

To estimate the curvature terms further, we will use various norms of a linear map $T\colon W \to V$ between Euclidean vector spaces W, V, in particular the trace norm $|T|_{\mathrm{tr}}$, the operator norm $|T|_{\mathrm{op}}$, and the singular values $\sigma_i(T)$, see Appendix B for details.

In the following, we will use the notation $\mathrm{W}_N = -\nabla \nu_N \colon \mathrm{T}\partial N \to \mathrm{T}\partial N$ to denote the Weingarten map of ∂N.

Lemma 4.2. *Suppose that ψ satisfies the boundary condition (2.5), that is, $\mathrm{c}(\nu_M)\psi = s\,\bar{\mathrm{c}}(\nu_N)\psi$. Then pointwise on ∂M we have*

$$\langle \mathcal{A}\psi, \psi\rangle \leq \frac{1}{2}|\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}}|\psi|^2 - \frac{\sigma_{\min}(\mathrm{W}_N \circ \mathrm{d}f)}{4}\ |\bar{\mathrm{c}}(U-)\psi - s\,\mathrm{c}(-)\psi|^2,$$

where $U_x \colon \mathrm{T}_x\partial M \to \mathrm{T}_{f(x)}\partial N$ is an isometry coming from a polar decomposition of $\mathrm{W}_N \circ \mathrm{d}_x f \colon \mathrm{T}_x\partial M \to \mathrm{T}_{f(x)}\partial N$ and $\sigma_{\min} = \sigma_{\min}(\mathrm{W}_N \circ \mathrm{d}f) \geq 0$ denotes the smallest singular value of $\mathrm{W}_N \circ \mathrm{d}f$.

Proof. It follows from the fact that $\mathcal{A}$ anticommutes with $\mathrm{c}(\nu_M)$ and $\bar{\mathrm{c}}(\nu_N)\,\mathrm{c}(\nu_M)\psi = s\psi$ that

$$\begin{aligned}
\langle \mathcal{A}\psi, \psi\rangle =& s\langle \mathcal{A}\psi, \bar{\mathrm{c}}(\nu_N)\,\mathrm{c}(\nu_M)\psi\rangle = s\langle \bar{\mathrm{c}}(\nu_N)\mathcal{A}\psi, \mathrm{c}(\nu_M)\psi\rangle \\
=& s\langle (\bar{\mathrm{c}}(\nu_N)\mathcal{A} - \mathcal{A}\,\bar{\mathrm{c}}(\nu_N))\psi, \mathrm{c}(\nu_M)\psi\rangle - s\langle \mathcal{A}\,\bar{\mathrm{c}}(\nu_N)\,\mathrm{c}(\nu_M)\psi, \psi\rangle \\
=& s\langle (\bar{\mathrm{c}}(\nu_N)\mathcal{A} - \mathcal{A}\,\bar{\mathrm{c}}(\nu_N))\psi, \mathrm{c}(\nu_M)\psi\rangle - \langle \mathcal{A}\psi, \psi\rangle.
\end{aligned}$$

Hence, Lemma 4.1 implies

$$\begin{aligned}
\langle \mathcal{A}\psi, \psi\rangle &= -\frac{s}{2}\sum_{i=1}^{n-1} \langle \mathrm{c}(\nu_M)\,\mathrm{c}(e_i)\,\bar{\mathrm{c}}((\mathrm{W}_N \circ \mathrm{d}f)(e_i))\psi, \mathrm{c}(\nu_M)\psi\rangle \\
&= -\frac{s}{2}\sum_{i=1}^{n-1} \langle \mathrm{c}(e_i)\,\bar{\mathrm{c}}((\mathrm{W}_N \circ \mathrm{d}f)(e_i))\psi, \psi\rangle\,.
\end{aligned}$$

Now choose an orthonormal basis (e_i) of $\mathrm{T}_x\partial M$ such that $(\mathrm{W}_N \circ \mathrm{d}f)(e_i) = \sigma_i \bar{e}_i$, where $\bar{e}_i = Ue_i$ is an orthonormal basis of $\mathrm{T}_{f(x)}\partial N$ and $\sigma_i = \sigma_i(\mathrm{W}_N \circ \mathrm{d}_x f) \geq 0$ are the singular values of $\mathrm{W}_N \circ \mathrm{d}_x f$. Then $|\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}} = \sum_{i=1}^{n-1}\sigma_i$. We thus obtain

$$\begin{aligned}
\langle \mathcal{A}\psi, \psi\rangle &= -\frac{s}{2}\sum_{i=1}^{n-1} \langle \mathrm{c}(e_i)\,\bar{\mathrm{c}}((\mathrm{W}_N \circ \mathrm{d}f)(e_i))\psi, \psi\rangle \\
&= -\frac{s}{2}\sum_{i=1}^{n-1} \sigma_i \underbrace{\langle \mathrm{c}(e_i)\,\bar{\mathrm{c}}(\bar{e}_i)\psi, \psi\rangle}_{-|\psi|^2 \leq \cdots \leq |\psi|^2} \\
&= \frac{1}{2}|\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}}|\psi|^2 - \frac{1}{2}\sum_{i=1}^{n-1}\sigma_i \underbrace{\left(|\psi|^2 + s\,\langle \mathrm{c}(e_i)\,\bar{\mathrm{c}}(\bar{e}_i)\psi, \psi\rangle\right)}_{\geq 0} \\
&\leq \frac{1}{2}|\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}}|\psi|^2 - \frac{\sigma_{\min}}{2}\sum_i \left(|\psi|^2 + s\,\langle \mathrm{c}(e_i)\,\bar{\mathrm{c}}(\bar{e}_i)\psi, \psi\rangle\right) \\
&= \frac{1}{2}|\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}}|\psi|^2 - \frac{\sigma_{\min}}{2}\sum_i \left(|\psi|^2 - s\,\langle \bar{\mathrm{c}}(\bar{e}_i)\psi, \mathrm{c}(e_i)\psi\rangle\right) \\
&= \frac{1}{2}|\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}}|\psi|^2 - \frac{\sigma_{\min}}{4}\sum_i |\bar{\mathrm{c}}(Ue_i)\psi - s\,\mathrm{c}(e_i)\psi|^2 \\
&= \frac{1}{2}|\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}}|\psi|^2 - \frac{\sigma_{\min}}{4}\ |\bar{\mathrm{c}}(U-)\psi - s\,\mathrm{c}(-)\psi|^2.
\end{aligned}$$

∎

The same argument also shows:

Lemma 4.3. *For a section $\Psi \in \mathrm{C}^\infty(M, S)$, pointwise on M we have*

$$\langle \mathcal{R}^N \Psi, \Psi \rangle \geq -\frac{1}{2} |\mathrm{R}^{\mathrm{T}N} \circ (\mathrm{d}f \wedge \mathrm{d}f)|_{\mathrm{tr}} |\Psi|^2.$$

Proof. Choose an orthonormal basis (ω_α) of $\Lambda^2 \mathrm{T}_x M$ such that $\mathrm{R}^{\mathrm{T}N} \circ (\mathrm{d}f \wedge \mathrm{d}f)(\omega_\alpha) = \lambda_\alpha \bar{\omega}_\alpha$, where $(\bar{\omega}_\alpha)$ is an orthonormal basis of $\Lambda^2 \mathrm{T}_{f(x)} N$ and $\lambda_\alpha \geq 0$. Then $|\mathrm{R}^{\mathrm{T}N} \circ (\mathrm{d}f \wedge \mathrm{d}f)|_{\mathrm{tr}} = \sum_\alpha \lambda_\alpha$. It follows from (2.4) that

$$\langle \mathcal{R}^N \Psi, \Psi \rangle = -\frac{1}{2} \sum_\alpha \lambda_\alpha \langle \mathrm{c}(\omega_\alpha)\, \bar{\mathrm{c}}(\bar{\omega}_\alpha) \Psi, \Psi \rangle$$

and thus $\langle \mathcal{R}^N \Psi, \Psi \rangle \geq -\frac{1}{2} |\mathrm{R}^{\mathrm{T}N} \circ (\mathrm{d}f \wedge \mathrm{d}f)|_{\mathrm{tr}} |\Psi|^2$. □

Combining Lemmas 4.2 and 4.3 we thus obtain the main integral inequalities:

Proposition 4.4. *For any section $\Psi \in \mathrm{C}^\infty(M, S)$ which satisfies the boundary condition (2.5), we have the folllowing inequality.*

$$\begin{aligned} \|\mathcal{D}\Psi\|^2_{\mathrm{L}^2(M)} \geq \|\nabla \Psi\|^2_{\mathrm{L}^2(M)} &+ \frac{1}{4} (\mathrm{scal}\, \Psi, \Psi)_{\mathrm{L}^2(M)} - \frac{1}{2} \int_M |\mathrm{R}^{\mathrm{T}N} \circ \mathrm{d}f \wedge \mathrm{d}f|_{\mathrm{tr}} |\Psi|^2 \, \mathrm{dV} \\ &+ \frac{1}{2} \int_{\partial M} (\mathrm{H}_{\partial M} - |\mathrm{W}_N \circ \mathrm{d}f|_{\mathrm{tr}}) \, |\Psi|^2 \, \mathrm{dS} \\ &+ \frac{1}{4} \int_{\partial M} \sigma_{\min}(\mathrm{W}_N \circ \mathrm{d}f) |\bar{\mathrm{c}}(U-)\psi - s\, \mathrm{c}(-)\psi|^2 \, \mathrm{dS}, \end{aligned}$$

where $U_x \colon \mathrm{T}_x \partial M \to \mathrm{T}_{f(x)} \partial N$ is a measurable section of bundle isometries coming from a polar decomposition of $\mathrm{W}_N \circ \mathrm{d}_x f \colon \mathrm{T}_x \partial M \to \mathrm{T}_{f(x)} \partial N$.

Proposition 4.4 together with the matrix Hölder inequality (B.1) yields the estimate of Goette–Semmelmann [17], Listing [29], and Lott [31] because if the curvature operator of N is non-negative, then $|\mathrm{R}^{\mathrm{T}N}|_{\mathrm{tr}} = \frac{\mathrm{scal}_N}{2}$, and if the boundary is convex, then $|\mathrm{W}_N|_{\mathrm{tr}} = \mathrm{H}_{\partial N}$. In the following corollary, we focus attention on the curvature term at the boundary and combine it with a quantitative version of the Hölder inequality given in Lemma B.1.

Corollary 4.5. *In particuar, if the boundary ∂N is convex and $\lambda \colon \partial M \to (0, \infty)$ is a function with $\lambda \geq |\mathrm{d}f|_{\mathrm{op}}$, then*

$$\begin{aligned} \|\mathcal{D}\Psi\|^2_{\mathrm{L}^2(M)} \geq \|\nabla \Psi\|^2_{\mathrm{L}^2(M)} &+ \frac{1}{4} (\mathrm{scal}_M\, \Psi, \Psi)_{\mathrm{L}^2(M)} - \frac{1}{2} \int_M |\mathrm{R}^{\mathrm{T}N} \circ \mathrm{d}f \wedge \mathrm{d}f|_{\mathrm{tr}} |\Psi|^2 \, \mathrm{dV} \\ &+ \frac{1}{2} \int_{\partial M} (\mathrm{H}_{\partial M} - (\mathrm{H}_{\partial N} \circ f) \cdot \lambda) \, |\Psi|^2 \, \mathrm{dS} \\ &+ \frac{1}{4} \int_{\partial M} \frac{\sigma_{\min}(\mathrm{W}_N)}{\lambda} |\mathrm{d}f - \lambda U|_2^2 |\Psi|^2 \, \mathrm{dS} \\ &+ \frac{1}{4} \int_{\partial M} \sigma_{\min}(\mathrm{W}_N \circ \mathrm{d}f) |\bar{\mathrm{c}}(U-)\Psi - s\, \mathrm{c}(-)\Psi|^2 \, \mathrm{dS}, \end{aligned}$$

where $U_x \colon \mathrm{T}_x \partial M \to \mathrm{T}_{f(x)} \partial N$ is a measurable section of bundle isometries coming from a polar decomposition of $\mathrm{W}_N \circ \mathrm{d}_x f \colon \mathrm{T}_x \partial M \to \mathrm{T}_{f(x)} \partial N$.

In particular, if in this situation $\mathcal{D}\Psi = 0$, $\mathrm{scal}_M \geq |\mathrm{R}^{\mathrm{T}N} \circ \mathrm{d}f \wedge \mathrm{d}f|_{\mathrm{tr}}$ and $\mathrm{H}_{\partial M} \geq (\mathrm{H}_{\partial N} \circ f) \cdot \lambda$, then $\nabla \Psi = 0$. If furthermore ∂N is strictly convex, then $\lambda^{-1} \mathrm{d}f = U \colon \mathrm{T}\partial M \to \mathrm{T}\partial N$ is a bundle isometry and we have $\bar{\mathrm{c}}(\mathrm{d}f(\xi))\Psi = s\lambda\, \mathrm{c}(\xi)\Psi$ on ∂M for any smooth vector field ξ tangent to ∂M.

In our main application, the curvature of N vanishes, so the interior curvature term $|\mathrm{R}^{\mathrm{T}N} \circ \mathrm{d}f \wedge \mathrm{d}f|_{\mathrm{tr}}$ drops out completely. This is why we stop here to keep the notation reasonably light, but an analogous analysis can be applied to this interior term, including a more refined version of Lemma 4.3 that keeps some of the dropped terms in the same way as Lemma 4.2 does for the boundary term.

5. Almost rigidity for maps to Euclidean domains

In this section, we prove our main almost rigidity theorem for maps to Euclidean domains and deduce rigidity for Gromov's hyperspherical radius estimate.

Proposition 5.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded strictly convex domain with smooth boundary $\partial\Omega$. Let M be a connected compact Riemannian spin manifold with connected boundary such that $\mathrm{scal}_M \geq 0$. Let $f_i \colon \partial M \to \partial\Omega$ be a sequence of smooth maps such that, for each $i \in \mathbb{N}$, we have that*

- $\mathrm{Lip}(f_i) \leq 1 + \frac{1}{i}$,
- $\mathrm{H}_{\partial M} \geq \mathrm{H}_{\partial\Omega} \circ f_i - \frac{1}{i}$,
- $\deg(f_i) \neq 0$.

Then there exists a smooth isometry $f \colon \partial M \to \partial\Omega$ such that $\mathrm{II}_{\partial M} = f^* \mathrm{II}_{\partial\Omega}$ and a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ such that $f_{i_k} \to f$ in $\mathrm{W}^{1,p}(\partial M, \mathbb{R}^n)$ for every $p < \infty$.

Proof. Extend the maps f_i to obtain maps $F_i \colon (M, \partial M) \to (\Omega, \partial\Omega)$ of the same degrees and such that $\mathrm{d}F_i(\nu_M) = \nu_\Omega$. Since by Example 2.3 we have $\mathrm{ind}(\mathcal{D}, 1) = \deg(F_i) = \deg(f_i) \neq 0$, we find non-trivial harmonic spinors Ψ_i with $\|\Psi_i\|^2_{\mathrm{L}^2(M)} = 1$ satisfying the boundary condition $\mathrm{c}(\nu_M)\Psi_i = \bar{\mathrm{c}}_i(\nu_\Omega)\Psi_i$. Now the estimate given by Corollary 4.5 (applied with $\lambda = 1 + \frac{1}{i}$ and $s = 1$) shows that there exists some fixed constant $\kappa > 0$ (not depending on i) such that for all i we have

$$\begin{aligned}(5.1) \qquad \frac{\kappa}{i} \int_{\partial M} |\Psi_i|^2 \,\mathrm{dS} \geq \|\nabla\Psi_i\|^2_{\mathrm{L}^2(M)} \\ + \| |\mathrm{d}f_i - (1 + \tfrac{1}{i}) U_i|_2 |\Psi_i| \|^2_{\mathrm{L}^2(\partial M)} \\ + \| \sqrt{\sigma_{\min}(\mathrm{W}_\Omega \circ \mathrm{d}f_i)}\, (\bar{\mathrm{c}}(U_i -)\Psi_i - \mathrm{c}(-)\Psi_i) \|^2_{\mathrm{L}^2(\partial M)}.\end{aligned}$$

Here U_i denotes a measurable bundle isometry $\mathrm{T}\partial M \to f_i^* \mathrm{T}\partial\Omega$ such that $\mathrm{W}_\Omega \circ \mathrm{d}f_i = P_i \circ U_i$ is a polar decomposition of $\mathrm{W}_\Omega \circ \mathrm{d}f_i$. Note that while U_i is uniquely determined and smooth on any subset where $\mathrm{d}f_i$ is invertible because we have assumed W_Ω to be strictly positive, we may have non-uniqueness and jumps in the presence of non-regular points of f_i, which is why a priori we can only assume U_i to be a measurable section.

By the trace inequality applied to the function $|\Psi_i|$, we also obtain a constant $C_\mathrm{T} \geq 0$ not depending on i such that

$$(5.2) \qquad \int_{\partial M} |\Psi_i|^2 \,\mathrm{dS} \leq C_\mathrm{T} \|\Psi_i\|^2_{\mathrm{W}^{1,2}(M)} = C_\mathrm{T} \left(1 + \|\nabla\Psi_i\|^2_{\mathrm{L}^2(M)}\right)$$

for all i. Plugging (5.2) into (5.1) shows that the right-hand side of (5.1) tends to zero as $i \to \infty$, that is,

$$(5.3) \qquad \|\nabla\Psi_i\|_{\mathrm{L}^2(M)} \to 0,$$

$$(5.4) \qquad \| |\mathrm{d}f_i - (1 + \tfrac{1}{i}) U_i|_2 |\Psi_i| \|_{\mathrm{L}^2(\partial M)} \to 0,$$

$$(5.5) \qquad \| \sqrt{\sigma_{\min}(\mathrm{W}_\Omega \circ \mathrm{d}f_i)}\, (\bar{\mathrm{c}}(U_i -)\Psi_i - \mathrm{c}(-)\Psi_i) \|_{\mathrm{L}^2(\partial M)} \to 0.$$

In particular, the sequence (Ψ_i) is uniformly bounded in $\mathrm{W}^{1,2}(M)$, so after passing to a subsequence we may assume that $\Psi_i \to \Psi$ in $\mathrm{L}^2(M)$ for some spinor Ψ with $\|\Psi\|_{\mathrm{L}^2(M)} = 1$ by the Rellich theorem. But then (5.3) further implies that this convergence already takes place in $\mathrm{W}^{1,2}(M)$ and $\nabla\Psi = 0$, in particular Ψ is smooth on all of M. Moreover, by the trace theorem,

$$(5.6) \qquad \Psi_i|_{\partial M} \to \Psi|_{\partial M} \text{ in } \mathrm{L}^2(\partial M).$$

Since $\mathrm{d}f_i$ and U_i are uniformly bounded in $\mathrm{L}^\infty(\partial M)$, the convergence (5.6) together with (5.4) and (5.5) means that as $i \to \infty$,

$$\|\mathrm{d}f_i - U_i\|_{\mathrm{L}^2(\partial M)} \to 0,$$
$$\|\sqrt{\sigma_{\min}(\mathrm{W}_\Omega \circ \mathrm{d}f_i)}\,(\bar{\mathrm{c}}(U_i -)\Psi - \mathrm{c}(-)\Psi)\,\|_{\mathrm{L}^2(\partial M)} \to 0,$$

respectively. After passing to a further subsequence, we can then ensure that pointwise almost everywhere on ∂M we have

$$|\mathrm{d}f_i - U_i|^2 \to 0, \qquad (5.7)$$
$$\sigma_{\min}(\mathrm{W}_\Omega \circ \mathrm{d}f_i)\;|\,\bar{\mathrm{c}}(U_i -)\Psi - \mathrm{c}(-)\Psi|^2 \to 0.$$

At a point $x \in \partial M$ where $|\mathrm{d}_x f_i - (U_i)_x| \to 0$, we have $\liminf_{x\to\infty} \sigma_{\min}(\mathrm{W}_\Omega \circ \mathrm{d}f_i) > 0$ and so we deduce that

$$|\,\bar{\mathrm{c}}(U_i -)\Psi - \mathrm{c}(-)\Psi| \to 0 \quad \text{pointwise almost everywhere on } \partial M. \qquad (5.8)$$

Next (5.8) implies that there exists a measurable section U of $\mathrm{Hom}(\mathrm{T}\partial M, \mathbb{R}^n)$ which is a fiberwise isometric embedding $\mathrm{T}_x\partial M \hookrightarrow \mathbb{R}^n$ such that that $U_i \to U$ pointwise almost everywhere. This is because we have an injective bundle map

$$\mathrm{Hom}(\mathrm{T}_x\partial M, \mathbb{R}^n) \hookrightarrow \mathrm{Hom}(\mathrm{T}_x\partial M, S_x), \quad T \mapsto \bar{\mathrm{c}}(T -)\Psi. \qquad (5.9)$$

Together with (5.7) this implies $\mathrm{d}f_i \to U$ pointwise almost everywhere on ∂M. Again using that $\mathrm{d}f_i$ and U are uniformly bounded $\mathrm{L}^\infty(\partial M)$, this implies that $\mathrm{d}f_i \to U$ in $\mathrm{L}^p(\partial M, \mathrm{Hom}(\mathrm{T}\partial M, \mathbb{R}^n))$ for every $p < \infty$ by dominated convergence.

Finally, we use Arzelà–Ascoli to pass to another subsequence such that $f_i \to f$ in C^0, where $f\colon \partial M \to \partial\Omega$ is a Lipschitz map. Since we already know that $\mathrm{d}f_i$ converges in L^p, it follows that (f_i) is a Cauchy sequence also in $\mathrm{W}^{1,p}(\partial M, \mathbb{R}^n)$ and hence the convergence $f_i \to f$ takes place in $\mathrm{W}^{1,p}(\partial M, \mathbb{R}^n)$ for every $p < \infty$. Furthermore, it follows that $\mathrm{d}f = U$ almost everywhere and so $\mathrm{d}f$ is an isometry almost everywhere.

Finally, we deduce from (5.8) that

$$\bar{\mathrm{c}}(\mathrm{d}f(\xi))\Psi = \mathrm{c}(\xi)\Psi \qquad (5.10)$$

almost everywhere for any smooth vector field ξ tangent to ∂M. Now since the parallel spinor Ψ is smooth, this means already that $\mathrm{d}f\colon \mathrm{T}\partial M \to \mathbb{R}^n$ is smooth, again because of the smooth bundle embedding (5.9). Thus f is smooth on ∂M. But then $f\colon \partial M \to \partial\Omega$ is a smooth local isometry and hence an isometry because $\partial\Omega$ is diffeomorphic to the sphere.

To obtain the result on the second fundamental forms, first observe that Ψ also satisfies the boundary condition $\bar{\mathrm{c}}(\nu_\Omega)\Psi = \mathrm{c}(\nu_M)\Psi$, where $\bar{\mathrm{c}}(\nu_\Omega)$ is defined in terms of the limiting map f. Differentiating this equality in the direction of a vector field ξ tangential to ∂M, we obtain

$$\mathrm{c}(\nabla_\xi \nu_M)\Psi = \bar{\mathrm{c}}(\nabla_{\mathrm{d}f(\xi)}\nu_\Omega)\Psi \underset{(5.10)}{=} \mathrm{c}((\mathrm{d}f)^{-1}\nabla_{\mathrm{d}f(\xi)}\nu_\Omega)\Psi,$$

and thus $\nabla_\xi \nu_M = (\mathrm{d}f)^{-1}\nabla_{\mathrm{d}f(\xi)}\nu_\Omega$ which proves the desired statement $\mathrm{II}_{\partial M} = f^*\,\mathrm{II}_{\partial\Omega}$. □

The next proposition already follows from [40, Theorem 3.2]. We include a self-contained proof, which does not make use of the analysis on manifolds with corners for the odd-dimensional case.

Proposition 5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded strictly convex Euclidean domain. Let M be a connected compact Riemannian spin manifold with boundary such that $\mathrm{scal}_M \geq 0$. Let $f\colon \partial M \to \partial\Omega$ be an isometry satisfying $\mathrm{II}_{\partial M} = f^*\,\mathrm{II}_{\partial\Omega}$. Then M is isometric to Ω.*

Proof. We work in the same setup as the proof of Proposition 5.1 above. That is, extend f to a smooth map $F\colon (M, \partial M) \to (\Omega, \partial\Omega)$ of the same degree such that $\mathrm{d}F(\nu_M) = \nu_\Omega$. Since by Example 2.3 $\operatorname{ind}(\mathcal{D}, 1) = \deg(F) = \deg(f) \neq 0$, we find a non-trivial harmonic spinor Ψ satisfying the boundary condition $\mathrm{c}(\nu_M)\Psi = \bar{\mathrm{c}}(\nu_\Omega)\Psi$. By Corollary 4.5, Ψ is parallel and it satisfies

$$\mathrm{c}(\xi)\Psi = \bar{\mathrm{c}}(\mathrm{d}F(\xi))\Psi \quad \text{on } \partial M \text{ for any } \xi \in \mathrm{C}^\infty(\partial M, \mathrm{T}M), \tag{5.11}$$

compare (5.10) above and which for the normal component $\xi = \nu_M$ is just the boundary condition.

Let $\bar{e}_1, \dots, \bar{e}_n$ denote a constant orthonormal basis of $\mathbb{R}^n$. For each $i = 1, \dots, n$ define a vector field $\xi_i \in \mathfrak{X}(M)$ by $\langle \xi_i, -\rangle = \langle \bar{\mathrm{c}}(\bar{e}_i)\Psi, \mathrm{c}(-)\Psi\rangle$. Since Ψ and $\bar{e}_i$ are parallel, the vector field ξ_i is also parallel. Using (5.11) on ∂M we have

$$\begin{aligned}\langle \xi_i, \xi\rangle = \langle \bar{\mathrm{c}}(\bar{e}_i)\Psi, \bar{\mathrm{c}}(\mathrm{d}F(\xi))\Psi\rangle &= -\langle \bar{\mathrm{c}}(\bar{e}_i)\,\bar{\mathrm{c}}(\mathrm{d}F(\xi))\Psi, \Psi\rangle + 2\langle \bar{e}_i, \mathrm{d}F(\xi)\rangle|\Psi|^2 \\ &= -\langle \bar{\mathrm{c}}(\bar{e}_i)\,\mathrm{c}(\xi)\Psi, \Psi\rangle + 2\langle \bar{e}_i, \mathrm{d}F(\xi)\rangle|\Psi|^2 \\ &= -\langle \xi_i, \xi\rangle + 2\langle \bar{e}_i, \mathrm{d}F(\xi)\rangle|\Psi|^2,\end{aligned}$$

where we have used the Clifford relation $\bar{\mathrm{c}}(v)\,\bar{\mathrm{c}}(w) + \bar{\mathrm{c}}(w)\,\bar{\mathrm{c}}(v) = 2\langle v, w\rangle$. Thus

$$\langle \xi_i, \xi\rangle = \langle \bar{e}_i, \mathrm{d}F(\xi)\rangle|\Psi|^2 \quad \text{on } \partial M \text{ for any } \xi \in \mathrm{C}^\infty(\partial M, \mathrm{T}M). \tag{5.12}$$

In particular, assuming the parallel spinor Ψ is normalized such that $|\Psi| = 1$, then (5.12) implies that $\mathrm{d}F(\xi_i) = \bar{e}_i$ on ∂M. Since $\mathrm{d}F$ is an isometry at the boundary, this means that $\xi_1, \dots, \xi_n$ is an orthonormal frame of $\mathrm{T}M$ along ∂M. But the vector fields ξ_i are parallel, so they form a parallel orthonormal frame of $\mathrm{T}M$ on all of M. This means that the metric on M is flat.

Using that $\mathrm{II}_{\partial M} = f^* \mathrm{II}_\Omega$, we glue $\mathbb{R}^n \setminus \Omega$ to M, obtaining an open manifold equipped with a complete flat C^2-metric (see e.g. [36, Lemma 4.1]). By the classification theorem for flat manifolds, we deduce that M is a domain in $\mathbb{R}^n$. Using again that $\mathrm{II}_{\partial M} = f^* \mathrm{II}_{\partial\Omega}$, we deduce that M and Ω are related via a Euclidean motion by the fundamental theorem of hypersurfaces. □

Proof of Theorem 1.6. The first part of the theorem is just a restatement of Proposition 5.1. Indeed, assume by contradiction that for some $p \in [1, \infty)$ and $\varepsilon_0 > 0$, we have that for every $\delta > 0$ there exists a map $f\colon \partial M \to \partial\Omega$ as in the statement of the theorem but which is at least ε_0-far away in $\mathrm{W}^{1,p}(\partial M, \mathbb{R}^n)$ from any isometry $\phi\colon \partial M \to \partial\Omega$ satisfying $\phi^* \mathrm{II}_{\partial\Omega} = \mathrm{II}_{\partial M}$. Letting $\delta = \frac{1}{i}$, this would lead to a sequence $f_i\colon \partial M \to \partial\Omega$ contradicting Proposition 5.1. Hence the first part of the theorem is established.

But if we have an isometry $\phi\colon \partial M \to \partial\Omega$ on the boundary satisfying $\phi^* \mathrm{II}_{\partial\Omega} = \mathrm{II}_{\partial M}$, then M and Ω are also isometric by Proposition 5.2. □

Proof of Theorem 1.5. Equivalently, we show that, if $h \geq (n-1)/R$, then M is isometric to the disc D^n_R of radius R. Since Σ has hyperspherical radius R, for every $\delta > 0$ there exists a smooth map $f\colon \partial M \to \mathrm{S}^n_R$ such that $\operatorname{Lip}(f) \leq 1 + \delta$ and $\deg(f) \neq 0$. Since M is a fill-in, $\mathrm{H}_{\partial M} \geq (n-1)/R = \mathrm{H}_{\partial \mathrm{D}^n_R} \circ f$. By Theorem 1.6, M and D^n_R are isometric in case $n \geq 3$. The case $n = 2$ follows from Gauß–Bonnet. □

Proof of Corollary 1.7. For any $\delta > 0$, we can C^0-approximate f by a smooth map $f_\delta\colon \partial M \to \partial\Omega$ such that $\operatorname{Lip}(f_\delta) \leq 1 + \delta$.[2] By uniform continuity of $\mathrm{H}_{\partial\Omega}$, we can further assume that $\mathrm{H}_{\partial M} \geq \mathrm{H}_{\partial\Omega} \circ f_\delta - \delta$. Fix $p > n$. Then Theorem 1.6 proves that M is isometric to Ω and that f_δ can be assumed to be arbitrarily $\mathrm{W}^{1,p}$-close to an

[2]Using mollification in small coordinate patches on ∂M we can obtain such an approximation as a map $\tilde{f}_\delta\colon \partial M \to \mathbb{R}^n$. Projecting back to $\partial\Omega$ along a small tubular neighborhood yields the desired approximation $f_\delta\colon \partial M \to \partial\Omega$.

isometry. Hence f_δ is C^0-close to an isometry. In sum, we can C^0-approximate the original 1-Lipschitz map $f\colon \partial M \to \partial\Omega$ arbitrarily well by isometries $\partial M \to \partial\Omega$ and so f itself must be an isometry because metric isometries are closed in C^0. □

6. Applications to asymptotically flat manifolds

Witten [42] showed that, on an asymptotically flat spin manifold M^n of non-negative scalar curvature, the following identity holds:

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_M (4|\nabla\psi|^2 + \mathrm{scal}\,|\psi|^2)\,\mathrm{dV}. \tag{6.1}$$

Here, ψ is a harmonic spinor asymptotic to a constant spinor with norm 1 at ∞. For the definitions of the mass m and asymptotic flatness, we refer to [28]. One drawback of this formula is that it needs non-negative scalar curvature since otherwise the existence of ψ is not guaranteed. Having such a mass formula without the $\mathrm{scal} \geq 0$ requirement has been for instance exploited in [24, Theorem 1.5]. Interestingly, Proposition 4.4 implies such a Witten-type integral formula without imposing any conditions on the scalar curvature.

Given $k \geq 1$ and $s > n-2$, we say that (M^n, g) is C^k_{-s}-asymptotically Schwarzschild of mass m if in the asymptotically flat coordinate system

$$\left(g - \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 - r^2 g_{S^{n-1}}\right) \in \mathrm{C}^k_{-s}\,. \tag{6.2}$$

Moreover, we denote with M_r the portion of M where in the asymptotically flat coordinate system $|x| \leq r$.

Theorem 6.1. *Let $\delta > 0$ and let (M^n, g) be a smooth spin manifold which is $\mathrm{C}^1_{-(n-2)-\delta}$-asymptotically Schwarzschild of mass m. Then for every radius $r \gg 1$ there exists a harmonic spinor ψ_r on M_r normalized to $\|\psi_r\|^2_{L^2(\partial M_r)} = |\partial M_r|$ such that*

$$m \geq \frac{1}{2(n-1)\omega_{n-1}} \int_{M_r} (4|\nabla\psi_r|^2 + \mathrm{scal}\,|\psi_r|^2)\,\mathrm{dV} + \mathcal{O}(r^{-\delta}). \tag{6.3}$$

Since manifolds which are asymptotically Schwarzschild are dense within arbitrary asymptotically flat manifolds, the above theorem implies the Riemannian positive mass theorem. We also remark that the spinors in our proof live in a different bundle compared to the ones in Witten's argument.

Proof. Let $r \gg 1$ and consider the coordinate sphere ∂M_r in the asymptotically flat end. Since (M^n, g) is asymptotically Schwarzschild, we obtain

$$\mathrm{H}_r = \frac{n-1}{r}\left(1 - \frac{m}{r^{n-2}} + \mathcal{O}(r^{-(n-2)-\delta})\right) \tag{6.4}$$

for the mean curvature H_r of ∂M_r. Moreover, the induced metric g_r on ∂M_r satisfies

$$g_r = g_{S^{n-1}_r} + \mathcal{O}(r^{-(n-2)-\delta}), \tag{6.5}$$

where $g_{S^{n-1}_r} = r^2 g_{S^{n-1}}$ is the round metric on the sphere of radius r. The latter condition implies that the map $f_r = \mathrm{id} : \partial M_r \to S_r$ is Lipschitz with Lipschitz constant L_r satisfying

$$L_r = 1 + \mathcal{O}(r^{-(n-2)-\delta}). \tag{6.6}$$

Now as described in Section 5, we can extend f_r to a map F_r from M_r to the ball B_r of radius r, and solve on the corresponding twisted spin bundle S the Dirac equation

$\mathcal{D}\psi_r = 0$ with boundary condition $\mathrm{c}(\nu_{M_r})\psi_r = \bar{\mathrm{c}}(\nu_{B_r})\psi_r$. Combining equations (6.4), (6.5) and (6.6) with Proposition 4.4, which in this situation states

$$\int_{M_r} (4|\nabla\psi_r|^2 + \mathrm{scal}\,|\psi_r|^2)\,\mathrm{dV} \leq 2\int_{\partial M_r} \left(L_r \frac{n-1}{r} - \mathrm{H}_r\right) |\psi_r|^2\,\mathrm{dS},$$

the result follows. □

We expect that this approach to also work in the initial data set setting, and we refer [12] for the corresponding Schrödinger–Licnerowicz formula. Also, compare this to [10] and [22], where integral formulas for the mass are obtained without assuming non-negative scalar curvature or the dominant energy condition.

Appendix A. The index theorem

In this appendix, we exhibit a quick cut-and-paste argument to derive the index formula for the boundary value problem studied in Section 2.3 for the case of maps to Euclidean domains in any dimension parity. Suppose from now on that we are in the setting of Section 2.3. Given a vector field $\eta \in \mathfrak{X}(N)$, we consider the following perturbed Dirac operator

$$\mathcal{D}_\eta = \mathcal{D} + \bar{\mathrm{c}}(\eta).$$

Expanding the term $|\mathcal{D}_\eta\psi|^2$ and integrating by parts the term $\langle\bar{\mathrm{c}}(\eta)\psi, \mathcal{D}\psi\rangle$, we obtain the following formula

$$\begin{aligned} \|\mathcal{D}_\eta\psi\|^2_{\mathrm{L}^2(M)} = \|\mathcal{D}\psi\|^2_{\mathrm{L}^2(M)} &+ \int_M \langle(\bar{\mathrm{c}}(\eta)\mathcal{D} + \mathcal{D}\,\bar{\mathrm{c}}(\eta))\psi, \psi\rangle + |\eta|^2|\psi|^2\,\mathrm{dV} \\ &+ \int_{\partial M} \langle \mathrm{c}(\nu_M)\,\bar{\mathrm{c}}(\eta)\psi, \psi\rangle\,\mathrm{dS}. \end{aligned} \tag{A.1}$$

Note that $\bar{\mathrm{c}}(\eta)\mathcal{D} + \mathcal{D}\,\bar{\mathrm{c}}(\eta) = \sum_{i=1}^n \mathrm{c}(e_i)\,\bar{\mathrm{c}}(\nabla_{\mathrm{d}f(e_i)}\eta)$ by a similar argument as in the proof of Lemma 4.1. Moreover, if ψ satisfies the boundary condition $\mathrm{c}(\nu_M)\psi = s\,\bar{\mathrm{c}}(\nu_N)\psi$, then

$$\begin{aligned} \langle \mathrm{c}(\nu_M)\,\bar{\mathrm{c}}(\eta)\psi, \psi\rangle &= -\,\langle\bar{\mathrm{c}}(\eta)\,\mathrm{c}(\nu_M)\psi, \psi\rangle \\ &= -\,s\langle\bar{\mathrm{c}}(\eta)\,\bar{\mathrm{c}}(\nu_N)\psi, \psi\rangle \\ &= s\langle\bar{\mathrm{c}}(\nu_N)\,\bar{\mathrm{c}}(\eta)\psi, \psi\rangle - 2s\langle\eta, \nu_N\rangle|\psi|^2 \\ &= -\,\langle \mathrm{c}(\nu_M)\,\bar{\mathrm{c}}(\eta)\psi, \psi\rangle + 2s\langle\eta, -\nu_N\rangle|\psi|^2, \end{aligned} \tag{A.2}$$

where we have used the Clifford relation $\bar{\mathrm{c}}(v)\,\bar{\mathrm{c}}(w) + \bar{\mathrm{c}}(w)\,\bar{\mathrm{c}}(v) = 2\langle v, w\rangle$. Thus we conclude that $\langle \mathrm{c}(\nu_M)\,\bar{\mathrm{c}}(\eta)\psi, \psi\rangle = s\langle\eta, -\nu_N\rangle|\psi|^2$.

We obtain the following lemma:

Lemma A.1. *Let $s\colon \partial N \to \{\pm 1\}$ and suppose there exists a nowhere-vanishing vector field $\eta \in \mathfrak{X}(N)$ such that $s\langle\eta, -\nu_N\rangle \geq 0$ on ∂N. Then the operator $\mathcal{D}$ subject to the boundary condition $\mathrm{c}(\nu_M)\psi = s\,\bar{\mathrm{c}}(\nu_N)\psi$ has vanishing index.*

Proof. We have $\mathrm{ind}(\mathcal{D}, s) = \mathrm{ind}(\mathcal{D}_{\lambda\eta}, s)$ for any $\lambda \geq 0$ by homotopy invariance of the index. But (A.1) implies that under the boundary condition given by s the operator $\mathcal{D}_{\lambda\eta}$ is invertible for λ sufficiently large and hence has vanishing index. □

Lemma A.2. *Suppose that we have a decomposition $N = N_0 \cup_\Upsilon N_1$ and $M = M_0 \cup_\Sigma M_1$ respecting all structures, where M_i and N_i are codimension zero submanifolds with boundary and the gluing happens along the additional boundary components not present in M and N, denoted by $\Upsilon \subseteq \partial N_i$ and $\Sigma \subseteq \partial M_i$. We also assume that the map $f\colon M \to N$ restricts to $f_i\colon (M_i, \partial M_i) \to (N_i, \partial N_i)$. Let $s\colon \partial N \to \{\pm 1\}$ and extend this to $s_i\colon \partial N_i \to \{\pm 1\}$ such that $s_0|_\Upsilon = -s_1|_\Upsilon$. Then*

$$\mathrm{ind}(\mathcal{D}, s) = \mathrm{ind}(\mathcal{D}_0, s_0) + \mathrm{ind}(\mathcal{D}_1, s_1).$$

Proof. This is a consequence of the splitting theorem of Bär–Ballmann [6, Theorem 6.5], compare also [15, Theorem 3.6]. ☐

Theorem A.3. *Let $N = \Omega$ be a convex domain in $\mathbb{R}^n$ and $s = 1$. Then*

$$\operatorname{ind}(\mathcal{D}, 1) = \deg(f)$$

Proof. We proceed in several steps of successive generality.

(1) The case where $M = N$ is a convex domain in $\mathbb{R}^n$ with identical metrics and $f = \mathrm{id}$ reduces to Example 2.1 because $\chi(N) = 1$.

(2) Now assume that $f\colon M \to N$ is a diffeomorphism, where N is a convex domain in $\mathbb{R}^n$. Then by homotopy invariance of the index we may assume that f is an isometry. Thus we obtain a bundle isomorphism $\mathrm{T}M \oplus f^*\mathrm{T}N \cong \mathrm{T}M \oplus \mathrm{T}M$ which preserves the pseudo-Riemannian metric and connection. If f is also orientation preserving, this is an orientation preserving bundle isomorphism and hence lifts to the associated spinor bundles, in which case we are immediately reduced to (1) because then $\operatorname{ind}(\mathcal{D}, 1) = 1 = \deg f$. Otherwise, if f is orientation reversing, then the induced isomorphism $\mathrm{T}M \oplus f^*\mathrm{T}N \cong \mathrm{T}M \oplus \mathrm{T}M$ is also orientation reversing and the effect on the spinor bundle is a reversal of the $\mathbb{Z}/2$-grading. Hence $\operatorname{ind}(\mathcal{D}, 1) = -1 = \deg(f)$.

(3) Next we consider a map $f\colon M \to N$ which on each connected component of M restricts to a diffeomorphism onto the convex domain N. Then the desired index formula follows directly from (2) by additivity over connected components.

(4) Now we are ready to treat the general case. To this end, let $y_0 \in N$ be an interior point of N which is a regular value of f. Choose a smooth vector field η on N which satisfies $\eta|_{\partial N} = -\nu_N$ and which is equal to the vector field $y \mapsto y - y_0$ in a disk neighborhood $N_0 \subset N^\circ$ around y_0 and such that η has no zeroes other than y_0. By possibly shrinking the disk N_0 further, we can also assume that $f\colon f^{-1}(N_0) \to N_0$ is a diffeomorphism on each connected component of $M_0 := f^{-1}(N_0)$ and M_0 has a smooth boundary $\Sigma := \partial M_0$. Let N_1 denote the closure of the complement of N_0 and $M_1 := f^{-1}(N_1)$ as well as $\Upsilon := \partial N_0$. Let $s_i\colon \partial N_i \to \{\pm 1\}$ be defined by setting $s_0(\Upsilon) = 1$, $s_1(\Upsilon) = -1$ and $s_1(\partial N) = 1$. Then by construction η has no zeroes on N_1 and we have $s_1\langle \eta, -\nu_{N_1}\rangle > 0$ on $\partial N_1 = \Upsilon \sqcup \partial N$, where ν_{N_1} denotes the unit normal of ∂N_1 pointing inside N_1. Thus Lemma A.1 implies that $\operatorname{ind}(\mathcal{D}_{1,s_1}) = 0$. Moreover, Lemma A.2 implies that $\operatorname{ind}(\mathcal{D}_s) = \operatorname{ind}(\mathcal{D}_{0,s_0}) + \operatorname{ind}(\mathcal{D}_{1,s_1}) = \operatorname{ind}(\mathcal{D}_{0,s_0})$. So it suffices to show that $\operatorname{ind}(\mathcal{D}_{0,s_0}) = \deg(f)$, but we have treated this in (3). ☐

Remark A.4. By elaborating on this argument further and using the Poincaré–Hopf index formula [26] for vector fields, it is possible to show the general formula $\operatorname{ind}(\mathcal{D}, 1) = \deg(f) \cdot \chi(N)$ in this way, where N is not necessarily a convex domain in $\mathbb{R}^n$. See Tony [38] where this idea is carried out in a different setting.

Appendix B. A quantitative matrix Hölder inequality

Let W, V be real Euclidean vector spaces and assume for simplicity that $\dim W = \dim V = n < \infty$. Let $T\colon W \to V$ be a linear map. We can always find a polar decomposition $T = P \circ U$, where $U\colon W \to V$ is an isometry and $P\colon V \to V$ is a non-negative self-adjoint operator. The operator P is always uniquely determined by T and given by the formula $P = \sqrt{T \circ T^*}$, whereas the isometry U is only unique if T is invertible. The eigenvalues $\sigma_1, \ldots, \sigma_n \geq 0$ of P are called the *singular values* of the operator T. The *trace norm* of T is defined as $|T|_{\mathrm{tr}} = \operatorname{tr}(P)$, or in other words as the sum of the singular values. The *operator norm* of T, denoted by $|T|_{\mathrm{op}}$, is the maximum of its singular values. More generally, we may also define the *Schatten norms* by $|T|_p = \operatorname{tr}(P^p)^{1/p}$ for any $1 \leq p < \infty$. For $p = 2$, this is known as the *Hilbert–Schmidt norm*. Given linear maps between $T\colon W \to V$ and $S\colon V \to V$, we have the following "Hölder inequality"

$$|S \circ T|_1 \leq |S|_p \; |T|_q, \tag{B.1}$$

whenever $1/p + 1/q = 1$.

We will only use the case $p = 1, q = \infty$ and when S is a non-negative endomorphism. In this case we have the following quantitative estimate which we will need in our (almost) rigidity argument:

Lemma B.1. *Let V, W be finite dimensional real Euclidean vector spaces of the same dimension, $T\colon W \to V$ a linear map and $S\colon V \to V$ a self-adjoint non-negative endomorphism. Let $\sigma_{\min} = \sigma_{\min}(S) \geq 0$ denote the smallest eigenvalue of S. Furthermore, let $P \circ U$ be a polar decomposition of $S \circ T$, that is, $P \circ U = S \circ T$ with $U\colon W \to V$ an isometry and $P\colon V \to V$ a self-adjoint non-negative endomorphism. Then for any $\lambda \geq |T|_{\mathrm{op}}$, we have*

$$\tfrac{\sigma_{\min}}{2} |T - \lambda U|_2^2 \leq |S|_{\mathrm{tr}} \; \lambda^2 - |S \circ T|_{\mathrm{tr}} \lambda.$$

In particular, if $T \neq 0$,

$$\tfrac{\sigma_{\min}}{2|T|_{\mathrm{op}}} |T - |T|_{\mathrm{op}} U|_2^2 \leq |S|_{\mathrm{tr}} \; |T|_{\mathrm{op}} - |S \circ T|_{\mathrm{tr}}.$$

Proof. Let (e_i) be an orthonormal basis of W. Then

$$\begin{aligned}
\tfrac{\sigma_{\min}}{2} |T - \lambda U|_2^2 &= \tfrac{\sigma_{\min}}{2} \sum_i |T(e_i) - \lambda U(e_i)|^2 \\
&= \tfrac{\sigma_{\min}}{2} \sum_i |T(e_i)|^2 + \lambda^2 |U(e_i)|^2 - 2\lambda \langle T(e_i), U(e_i) \rangle \\
&\leq \sigma_{\min} \sum_i \left(\lambda^2 - \lambda \langle U^* T(e_i), e_i \rangle \right) \\
&= \sigma_{\min} \operatorname{tr} \left(\lambda^2 \operatorname{id}_W - \lambda U^* \circ T \right) \\
&= \sigma_{\min} \operatorname{tr} \left(\lambda^2 \operatorname{id}_V - \lambda T \circ U^* \right) \\
&\leq \operatorname{tr} \left(\lambda^2 S - \lambda S \circ T \circ U^* \right) \\
&= \operatorname{tr} \left(\lambda^2 S - \lambda P \right) = |S|_{\mathrm{tr}} \; \lambda^2 - |S \circ T|_{\mathrm{tr}} \lambda.
\end{aligned}$$

□

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