ARTINIAN RINGS WHICH ARE NOT GENERALIZED RICKART

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Abstract. In this note, we show that there exist non-unital right artinian rings which are not generalized Rickart. In particular, we provide examples to show that, [16, Corollary 2.31] is not true for non-unital artinian rings.

Throughout this note R denotes an associative ring without unity and $*$ is used to indicate an involution on a ring. An idempotent element $x \in R$ is called a *projection* if $x^* = x$. Rickart [13] in 1946 studied C*-algebras which satisfy the condition that the right annihilator of every single element is generated by a projection. Rickart also showed that all von Neumann algebras satisfy this property. These algebras were later named *Rickart C*-algebras* by Kaplansky.

Kaplansky in 1950 showed that von Neumann algebras satisfy a stronger annihilator condition, namely, that these are rings with identity in which the right annihilator of any nonempty subset is generated by an idempotent. He termed a ring with this property a Baer ring to honor R. Baer who had studied this condition in 1940. Also, a $*$-ring R is called a Baer $*$-ring if the right annihilator of every nonempty subset is generated by a projection as a right ideal. Kaplansky recognized that the notions of a Baer ring and a Baer $*$-ring provide a framework to study the algebraic properties of operator algebras and each is interesting in its own right. The theory of Baer rings, Baer $*$-rings, and AW*-algebras (C*-algebras which are Baer $*$-rings) is studied in [2] and [5].

Maeda in 1960 defined a Rickart ring. He called a ring *right (left) Rickart* if the right (left) annihilator of any single element is generated by an idempotent. It is clear that every Baer ring is right and left Rickart. The same year, Hattori introduced the notion of a *right p.p.*-ring, namely a ring in which every principal right ideal is projective. It was later discovered that a right Rickart ring is precisely the same as a right p.p.-ring. Also, Berberian in [2] defined a Rickart $*$-ring, if the right annihilator of any single element is generated by a projection.

Recall from [8], that a ring R is *generalized right principally projective* (*generalized right p.p.* for short) if for any $x \in R$, the right ideal $x^n R$ is projective for some positive integer n, depending on x, or equivalently, if for any element $x \in R$, the right annihilator $r_R(x^n)$ is generated by an idempotent for some positive integer n. Left cases may be defined analogously. Generalized p.p.-rings (which are also called generalized Rickart rings) have been studied in [8],[9], [10] and [11].

In [16, Theorem 2.30], Ungor et al. show that every finitely generated module over a right artinian ring is π-*Rickart*. They deduce that every right artinian ring is generalized right p.p., see [16, Corollary 2.31].

Now we show that there exist non-unital right artinian rings, which are not generalized Rickart. In particular, this shows that, [16, Corollary 2.31], is not true for non-unital artinian rings.

We do not know any example of a unital artinian ring which is not generalized p.p.

Key words and phrases. artinian ring, group ring, generalized Rickart $*$-ring, generalized p.p.-ring.

Theorem 1. *Let R be a non-unital right artinian ring with the following properties:*

(*i*) *For $a \in R$ and each positive integer n, $2^n a = 0$ implies that $a = 0$;*
(*ii*) *The equation $2x^2 - x = 0$ has only the trivial solution in R;*

and let $S = RG$ be the group ring of the group $G = \{e, g\}$ of order 2 over the ring R. Then S is a right artinian ring which is not generalized right p.p.

Proof. By [7, p.217, Exercise 2], S is right artinian. But, we show that S is not a generalized right p.p.-ring and hence it is not a generalized Rickart $*$-ring. Note that for each positive integer n, $(e+g)^n = 2^{n-1}e + 2^{n-1}g$, $(e-g) \in r_S(e+g)^n$, so $r_S(e+g)^n \neq 0$. If $ae+bg \in S$ is a nontrivial idempotent such that $ae+bg \in r_S(e+g)^n$, then we must have $2^{n-1}a + 2^{n-1}b = 0$, $ab+ba = b$ and $a^2+b^2 = a$. So $b = -a$, then $2a^2 = a$ which contradicts the assumption that the equation $2x^2 - x = 0$ has no nontrivial solutions in R. Thus $r_S(e+g)^n$ can not be generated by an idempotent of S. □

Proposition 2. *If a ring R satisfies the assumptions of Theorem 1, then the triangular matrix ring $T_n(R)$ also satisfies these properties.*

Proof. It is clear that $T_n(R)$ satisfies the assumption (i) of Theorem 1. Let $M = (a_{ij}) \in T_n(R)$, where $a_{ij} = 0$ for $i > j$. Suppose $2M^2 = M$. Then for $1 \leq i \leq n$, $2a_{ii}^2 - a_{ii} = 0$, so $a_{ii} = 0$. Now $2M^2 = M$ implies that $a_{12} = a_{23} = \cdots = a_{n-1,n} = 0$. By continuing this process, we conclude that $a_{ij} = 0$, for all $1 \leq i \leq j \leq n$, so $M = 0$ and $T_n(R)$ satisfies the assumption (ii) of Theorem 1. □

We now provide some examples which satisfy the assumptions of Theorem 1, see [3] for more details.

Example 3. Let R be one of the following finite rings of order p^2:

(1) $A = \langle a \mid p^2 a = 0, a^2 = pa \rangle$;
(2) $B = \langle a \mid p^2 a = 0, a^2 = 0 \rangle$;
(3) $C = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0 \rangle$;
(4) $D = \langle a, b \mid pa = pb = 0, a^2 = b^2 = 0 \rangle$.

where p is a prime number $\neq 2$. Now, let $S = RG$ be the group ring of the group $G = \{e, g\}$ of order 2 over the ring R. Since S is a finite ring, S is artinian. Also, since characteristic of R is p^2 and $(p^2, 2^n) = 1$, for each n, R satisfies Condition (i) of Theorem 1.

If $R = A$ and $x \in R$, then

$$2x^2 = \begin{cases} 0 & \text{if } x = kp, \text{ where } k = 1, \cdots, p-1 \\ 2px & \text{otherwise.} \end{cases}$$

If $R = B$ and $x \in R$, then $2x^2 = 0$.
If $R = C$ and $x \in R$, then

$$2x^2 = \begin{cases} 0 & \text{if } x = kb, k = 1, \cdots, p-1 \\ mb \text{ (for some } m) & \text{otherwise.} \end{cases}$$

If $R = D$ and $x \in R$, then $2x^2 = 0$.
Thus R satisfies Condition (ii) of Theorem 1. So S is an artinian ring which is not generalized right p.p.

Since by Proposition 2, $T_n(R)$ satisfies the assumptions of Theorem 1, the group ring $T_n(R)G$ of the group $G = \{e, g\}$ of order 2 over the triangular matrix ring $T_n(R)$, for each $n \geq 2$, is also an artinian ring which is not generalized right p.p.

Let R be a ring. Consider the subring $T(R, n)$ of the triangular matrix ring $T_n(R)$, with $n \geq 2$, consisting all n by n triangular matrices with constant diagonals. We can denote elements of $T(R, n)$ by $(a_1, a_2, \ldots, a_n)$, then $T(R, n)$ is a ring with addition pointwise and multiplication given by

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1b_1, a_1b_2 + a_2b_1, \ldots, a_1b_n + a_2b_{n-1} + \cdots + a_nb_1),$$

for each $a_i, b_j \in R$.

On the other hand, there is a ring isomorphism $\varphi : R[x]/(x^n) \to T(R, n)$, given by

$$\varphi(a_1 + a_2x + \cdots + a_nx^{n-1} + (x^n)) = (a_1, a_2, \ldots, a_n),$$

with $a_i \in R$ and $1 \leq i \leq n$. So $T(R, n) \cong R[x]/(x^n)$, where $R[x]$ is the ring of polynomials in an indeterminant x and (x^n) is the ideal generated by x^n.

Proposition 4. *Let R be an abelian ring. Then R is generalized right p.p. if and only if $T(R, n)$ is generalized right p.p.*

Proof. The proof is similar to that of [4, Proposition 3]. □

Proposition 5. *A ring R is right (left) artinian if and only if the ring $T(R, n)$ is right (left) artinian.*

Proof. The proof is similar to that of [12, Corollary 4.3]. □

Example 6. Let S be the group ring considered in Example 3. Then by Example 3, Propositions 4 and 5, the ring $\mathfrak{S} := T(S, n)$ is an artinian ring which is not generalized right p.p.

Proposition 7 ([1], Theorem 3.14)**.** *Let R be a ring and G be a group. If the group ring RG is generalized right p.p., then so is R.*

Repeatedly applying Proposition 7 to existing examples, such as Example 3 or Example 6, one can construct new examples from old.

Example 8. Let S be the group ring considered in Example 3 or the ring $\mathfrak{S}$ in the Example 6 and H be any finite group. Then by Proposition 7, Examples 3, 6 and [7, p. 217, Exercise 2], the new group ring SH (respectively $\mathfrak{S}H$) is an artinian ring that is also not generalized right p.p.

Let R be a ring and G be a group. If R has an involution $*$ itself then we have a natural involution $\circledast$ on the group ring RG, induced by the inversion in the group G, given by

$$(\sum a_g g)^{\circledast} = \sum a_g^* g^{-1}.$$

Recall from [1], that a ring R with an involution $*$ is called a *generalized Rickart $*$-ring* if for each $x \in R$, the right annihilator $r_R(x^n)$ is generated by a projection for some positive integer n, depending on x. These rings are generalization of Rickart $*$-rings. There are large classes of both finite and infinite dimensional Banach $*$-algebras which are generalized Rickart $*$-rings, but they are not Rickart $*$.

Theorem 9. *Let R be a non-unital right artinian ring with the following properties:*

(i) *For $a \in R$ and each positive integer n, $3^n a = 0$ implies that $a = 0$;*
(ii) *The equation $3x^2 + x = 0$ has only the trivial solution in R;*

and let $S = RG$ be the group ring of the group $G = \{e, g, g^2\}$ of order 3 over the ring R. Then the group ring S is a right artinian ring which is not generalized Rickart $*$.

Proof. By [7, p.217, Exercise 2], S is right artinian. But, we show that S is not a generalized Rickart $*$-ring. Note that for each positive integer n, $(e + g + g^2)^n = 3^{n-1}e + 3^{n-1}g + 3^{n-1}g^2$, $(e - g) \in r_S(e+g)^n$, so $r_S(e+g+g^2)^n \neq 0$. If $ae+bg+cg^2 \in S$ is a nontrivial projection such that $ae + bg + cg^2 \in r_S(e + g + g^2)^n$, then we must have

$$(3^{n-1}e + 3^{n-1}g + 3^{n-1}g^2)(ae + bg + cg^2) = 0;$$
$$(ae + bg + cg^2)(ae + bg + cg^2) = (ae + bg + cg^2);$$
$$(ae + bg + cg^2) = (ae + bg + cg^2)^* = (ae + cg + bg^2).$$

So $3^{n-1}a + 3^{n-1}b + 3^{n-1}c = 0$, $a^2 + bc + cb = a$, $c^2 + ab + ba = b$, $b^2 + ac + ca = c$ and $b = c$. Then $a = -2b$ and $b^2 + ab + ba = b$. Hence $3b^2 + b = 0$, which contradicts the assumption that the equation $3x^2 + x = 0$ has no nontrivial solutions in R. Thus $r_S(e + g + g^2)^n$ can not be generated by a projection of S. □

Proposition 10. *If a ring R satisfies the assumptions of Theorem 9, then for each n, the ring $T_n(R)$ also satisfies these properties.*

Proof. The proof is similar to that of Proposition 2. □

The following examples also satisfy the assumptions of Theorem 9.

Example 11. Let R be one of the following finite rings of order p^2:

(1) $A = \langle a \mid p^2a = 0, a^2 = pa\rangle$;
(2) $B = \langle a \mid p^2a = 0, a^2 = 0\rangle$;
(3) $C = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0\rangle$;
(4) $D = \langle a, b \mid pa = pb = 0, a^2 = b^2 = 0\rangle$.

where $p \neq 3$ is a prime number. Now, let $S = T_n(R)G$ be the group ring of the group $G = \{e, g, g^2\}$ of order 3 over the ring $T_n(R)$. Since S is a finite ring, S is artinian. Also, since characteristic of R is p^2 and $(p^2, 3^n) = 1$, for each n, R satisfies Condition (i) of Theorem 9.

If $R = A$ and $x \in R$, then

$$3x^2 = \begin{cases} 0 & \text{if } x = kp, \text{ where } k = 1, \cdots, p-1 \\ mb \text{ (for some } m) & \text{otherwise.} \end{cases}$$

If $R = B$ and $x \in R$, then $3x^2 = 0$.
If $R = C$ and $x \in R$, then

$$3x^2 = \begin{cases} 0 & \text{if } x = kb, k = 1, \cdots, p-1 \\ mb \text{ (for some } m) & \text{otherwise.} \end{cases}$$

If $R = D$ and $x \in R$, then $3x^2 = 0$.
Thus R satisfies Condition (ii) of Theorem 9. So S is an artinian ring which is not generalized Rickart $*$.

We define an involution on $T_n(R)$ given by $A^* = (a^*_{\ell k})$, where $\ell = n - j + 1$ and $k = n - i + 1$, for $A = (a_{ij}) \in T_n(R)$ (see [1]). Since by Proposition 10, $T_n(R)$ satisfies the assumptions of Theorem 9, the group ring $T_n(R)G$ of the group $G = \{e, g, g^2\}$ of order 3 over the triangular matrix ring $T_n(R)$, for each $n \geq 2$, is also an artinian ring which is not generalized Rickart-$*$.

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