EXTENDED GENUS FIELDS OF ABELIAN EXTENSIONS OF RATIONAL FUNCTION FIELDS

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ABSTRACT. In this paper we obtain the extended genus field of a finite abelian extension of a global rational function field. We first study the case of a cyclic extension of prime power degree. Next, we use that the extended genus fields of a composite of two cyclotomic extensions of a global rational function field is equal to the composite of their respective extended genus fields, to obtain our main result. This result is that the extended genus field of a general finite abelian extension of a global rational function field, is given explicitly in terms of the field and of the extended genus field of its "cyclotomic projection".

1. INTRODUCTION

The concepts of genus field and extended (or narrow) genus field depend on the respective concepts of Hilbert Class Field (HCF) and extended (or narrow) Hilbert Class Field. The theory of the genus goes back to Gauss. The HCF concept is much more recent. The first to translate the theory of Gauss on genus to "modern terms", was Hilbert. Nowadays it may be used to study the "easy" part of the HCF of a finite extension of the field of rational numbers.

The first to give a definition of a genus field for number fields, was H. Hasse, who defined the genus field of a quadratic extension of $\mathbb{Q}$. Since the genus field is related with the HCF, one natural way to study genus fields is by means of class field theory. However, we may study genus fields of abelian extensions of the rational field by means of Dirichlet characters.

For number fields, the definition of Hilbert and extended Hilbert Class Field are canonically given as the maximal abelian unramified and the maximal abelian unramified at the finite primes of the field respectively. The definition of the genus field is not absolute, as is the HCF, but depends on an extension. A. Fröhlich gave a general definition of genus fields for any number field. Fröhlich definition is also canonical.

We are interested in global function fields. In this context, there are several different definitions of HCF of a global field K depending on which aspect we are interested in. In this paper we study the extended genus field of a finite abelian extension K/k where $k = \mathbb{F}_q(T)$ is a global rational function field. Let $\mathfrak{p}_\infty$ be infinite prime of k. Then we define HCF of K as the maximal unramified abelian extension K_H of K such that the infinite primes of K (those above $\mathfrak{p}_\infty$) decompose fully in K_H. B. Anglès and J.-F. Jaulent ([1]) give the same concept by means of the idèle norm subgroup corresponding to K_H. They also define the extended HCF of

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any global field K by means of the norm subgroup of K_{H^+} in the idèle group J_K of K. We use the definitions of Anglès and Jaulent of K_H and K_{H^+} to define the genus $K_{\mathfrak{ge}}$ and the extended genus $K_{\mathfrak{gex}}$ fields of K with respect to the extension K/k and compare our findings with the concepts of extended genus fields given in [8] and in [3].

We first study the case of a cyclic extension of k of prime power degree l^n. We consider four possible type of primes l: (1) $l = p$, the characteristic of k (Artin-Schreier-Witt case); (2) $l \nmid q-1$, $l \neq p$; (3) $l^n | q-1$ (Kummer case), and (4) $l^\rho | q-1$, $1 \leq \rho < n$ and $l^n \nmid q-1$ ("semi-Kummer" case). Our main results are Theorems 5.20 and 6.2.

The main tools used in this paper are the Carlitz theory of cyclotomic function fields and class field theory, particularly the concepts of HCF and genus fields developed by Anglès and Jaulent.

2. Notations and general results

For the general Carlitz–Hayes theory of cyclotomic function fields, we refer to [12, Ch. 12] and [9, Cap. 9]. For the results on genus fields of function fields we refer to [2, 5, 6] and [9, Cap. 14].

We will be using the following notation. Let $k = \mathbb{F}_q(T)$ be a global rational function field, where $\mathbb{F}_q$ is the finite field of q elements. Let $R_T = \mathbb{F}_q[T]$ and let R_T^+ denote the set of the monic irreducible elements of R_T. For $N \in R_T$, $k(\Lambda_N)$ denotes the N–th cyclotomic function field where Λ_N is the N–th torsion of the Carlitz module. For a $D \in R_T$ we define $D^* := (-1)^{\deg D} D$.

We will call a field F a *cyclotomic function field* if there exists $N \in R_T$ such that $F \subseteq k(\Lambda_N)$.

Let $N \in R_T$. The Dirichlet characters $\chi \bmod N$ are the group homomorphisms $\chi\colon (R_T/\langle N\rangle)^* \longrightarrow \mathbb{C}^*$. Given a group X of Dirichlet characters modulo N, the *field associated to* X is the fixed field $F = k(\Lambda_N)^H$ where $H = \bigcap_{\chi\in X} \ker \chi$. We say that F corresponds to the group X and that X corresponds to F. We have that $X \cong \operatorname{Hom}(\operatorname{Gal}(F/k), \mathbb{C}^*)$. When X is a cyclic group generated by χ, we have that the field associated to X is equal to $F = k(\Lambda_N)^{\ker \chi}$ and we say that F corresponds to χ.

Given a cyclotomic function field F with Dirichlet group characters X, we have that the ramification index of $P \in R_T^+$ in F/k equals $|X_P|$ where $X_P = \{\chi_P \mid \chi \in X\}$ and χ_P is the P–th component of χ, see [5]. The maximum cyclotomic extension of F unramified at the finite prime divisors is the field that corresponds to $Y := \prod_{P\in R_T^+} X_P$ This field is denoted as $F_{\mathfrak{gex}}$.

We denote the infinite prime of k by $\mathfrak{p}_\infty$. That is, $\mathfrak{p}_\infty$ is the pole divisor of T and $1/T$ is an uniformizer for $\mathfrak{p}_\infty$.

Given a finite extension K/k and a definition of HCF K_H and of extended HCF K_{H^+} of K, the respective genus and extended genus field of K with respect to k are the extensions KL such that L is the maximal abelian extension of k contained in K_H and in K_{H^+}, respectively.

We will use both notations: $e_P(F|k)$ or $e_{F/k}(P)$ to denote the ramification index of the prime P of k in F. For the place $\mathfrak{p}_\infty$ we use the notation $e_\infty(F|k)$.

When K/k is a finite abelian extension, it follows from the Kronecker–Weber Theorem that there exist $N \in R_T$, $n \in \mathbb{N}\cup\{0\}$ and $m \in \mathbb{N}$ such that $K \subseteq {}_nk(\Lambda_N)_m$, where, for any F, $F_m := F\mathbb{F}_{q^m}(T)$, for any $N \in R_T$, ${}_nk(\Lambda_N) := L_nk(\Lambda_N)$, and

L_n is the maximum subfield of $k(\Lambda_{1/T^n})$ where $\mathfrak{p}_\infty$ is totally and wildly ramified. Then we define

$$E := \mathcal{M}K \cap k(\Lambda_N) \tag{2.1}$$

where $\mathcal{M} = L_n k_m$. Then $K_{\mathfrak{ge}} = E_{\mathfrak{ge}}^{\mathcal{H}} K$ where $\mathcal{H}$ is the decomposition group of the infinite primes in $KE_{\mathfrak{ge}}/K$ (see [2, Theorem 2.2]). The group $\mathcal{H}$ is also the decomposition group of the infinite primes of K in KE/K.

For any global function field L, $\mathbb{P}_L$ denotes the set of all places of L.

For $x \in \mathbb{Z}$, $v_l(x)$ denotes the valuation of x at l. That is, $v_l(x) = \gamma$ if $l^\gamma | x$ and $l^{\gamma+1} \nmid x$. We write $v_l(0) = \infty$.

3. Basic results

One result on ramification of tamely ramified extensions, used frequently, is the following theorem.

Theorem 3.1 (Abhyankar's Lemma). *Let L/K be a separable of global function fields. Assume that $L = K_1K_2$ with $K \subseteq K_i \subseteq L$, $1 \leq i \leq 2$. Let $\mathfrak{p}$ be a prime divisor of K and $\mathfrak{P}$ a prime divisor in L above $\mathfrak{p}$. Let $\mathfrak{P}_i := \mathfrak{P} \cap K_i$, $i = 1, 2$. If at least one of the extensions K_i/K is tamely ramified at $\mathfrak{p}$, then*

$$e_{L/K}(\mathfrak{P}|\mathfrak{p}) = \operatorname{lcm}[e_{K_1/K}(\mathfrak{P}_1|\mathfrak{p}), e_{K_2/K}(\mathfrak{P}_2|\mathfrak{p})],$$

where $e_{L/K}(\mathfrak{P}|\mathfrak{p})$ denotes the ramification index. □

Next, we present some basic facts on finite cyclic groups and we apply them to the case of a finite field.

Let G be a cyclic group of order n, say $G = \langle a \rangle$. Let Λ_m the unique subgroup of G or order m where $m|n$. We have $\Lambda = \langle a^{n/m} \rangle$. Let $t \in \mathbb{N}$ and let $G^t := \{x \in G \mid x = y^t \text{ for some } y \in G\}$. We have $G^t = \operatorname{im} \varphi_t$, where $\varphi_t \colon G \to G$ is given by $\varphi_t(x) = x^t$.

Note that if $t \in \mathbb{N}$ and $d = \gcd(t, n)$, then $G^d = G^t$, namely, if $\alpha, \beta \in \mathbb{Z}$ are such that $\alpha t + \beta n = d$, we have

$$G^d = G^{\alpha t + \beta n} = (G^\alpha)^t (G^n)^\beta \subseteq G^t \cdot 1 = G^t.$$

Conversely, let $t = \kappa d$ with $\kappa \in \mathbb{N}$. Then $G^t = (G^\kappa)^d \subseteq G^d$.

We also have that $\Lambda_d = \Lambda_t$ since if $x \in \Lambda_t$, then $x^t = 1 = (x^t)^\alpha \cdot (x^n)^\beta = x^{\alpha t + \beta n} = x^d$ so that $\Lambda_t \subseteq \Lambda_d$. Conversely, if $t = \kappa d$ and if $x \in \Lambda_d$ then $1 = x^t = (x^d)^\kappa = 1^\kappa = 1$.

Now, if $d|n$, then $G^d = \Lambda_{n/d}$ because we have the exact sequence

$$1 \longrightarrow \Lambda_d \longrightarrow G \xrightarrow{\varphi_d} G^d \longrightarrow 1,$$

obtaining $|G^d| = \frac{|G|}{|\Lambda_d|} = \frac{n}{d} = |\Lambda_{n/d}|$ and, if $x \in G^d$, there exists $y \in G$ such that $x = y^d$ that implies $x^{n/d} = (y^d)^{n/d} = y^n = 1$ so that $G^d \subseteq \Lambda_{n/d}$. Thus $G^d = \Lambda_{n/d}$.

We apply the previous basic results to the multiplicative groups of the finite field $\mathbb{F}_q^*$ that is a cyclic group of $q-1$ elements.

Lemma 3.2. *Let l be a prime number and $n \in \mathbb{N}$, with $l^n | q-1$. Let $F := \mathbb{F}_q\big(\sqrt[l^n]{\beta}\big)$ with $\beta \in \mathbb{F}_q^*$. Then $F = \mathbb{F}_{q^{l^s}}$ for some $0 \leq s \leq n$.*

Proof. Let $\mu = \sqrt[l^n]{\beta}$. Then $\mu^{l^n} = \beta \in \mathbb{F}_q^*$. Set s, $0 \leq s \leq n$ to be the minimal non-negative integer such that $\mu^{l^s} = \theta \in \mathbb{F}_q^*$. If $s = 0$, then $\mu = \theta \in \mathbb{F}_q^*$ and $f(X) = X^{l^s} - \theta = X - \theta$ is irreducible.

For any s, we will see that $f(X) = X^{l^s} - \theta$ is an irreducible polynomial. We have $f(X) = X^{l^s} - \theta = \prod_{j=1}^{l^s}(X - \zeta_{l^s}^j \mu)$, where ζ_m denotes a primitive m-th root of unity. Let $\mathcal{G} := \operatorname{Gal}(\mathbb{F}_q(\mu)/\mathbb{F}_q)$. Let $\sigma \in \mathcal{G}$, $\sigma \neq \operatorname{Id}$ and let $\sigma(\mu) = \zeta_{l^s}^j \mu$, where $j = j_0 l^b$, with $\gcd(j_0, l) = 1$. We choose an element $\sigma \in \mathcal{G}$ such that b is minimal.

We have $\sigma(\mu) = \zeta_{l^s}^{j_0 l^b} \mu = \zeta_{l^{s-b}}^{j_0} \mu$. Let $i_0 \in \mathbb{Z}$ be such that $j_0 i_0 \equiv 1 \bmod l^s$. Then $\sigma^{i_0}(\mu) = \zeta_{l^{s-b}}^{j_0 i_0}(\mu) = \zeta_{l^{s-b}} \mu$. This $\mathcal{G}$ is a cyclic group and of order l^{s-b} and

$$\prod_{\varepsilon=1}^{l^{s-b}} (X - \zeta_{l^{s-b}}^{\varepsilon} \mu) = X^{l^{s-b}} - \mu^{l^{s-b}} \in \mathbb{F}_q[X].$$

Hence $\mu^{l^{s-b}} \in \mathbb{F}_q^*$. Therefore $b = 0$, $|\mathcal{G}| = l^s$, and $X^{l^s} - \theta = \operatorname{Irr}(\mu, X, \mathbb{F}_q)$ is irreducible.

It follows that $[\mathbb{F}_q(\mu) : \mathbb{F}_q] = |\mathcal{G}| = l^s$ and $F = \mathbb{F}_q(\mu) = \mathbb{F}_q(\sqrt[l^n]{\beta}) = \mathbb{F}_q(\sqrt[l^s]{\theta}) = \mathbb{F}_{l^s}$. $\square$

Remark 3.3. We have $\mathbb{F}_q(\sqrt[l^n]{\beta}) = \mathbb{F}_{q^{l^s}}$, where s is the minimal non-negative integer such that $\mu^{l^s} \in \mathbb{F}_q^*$, with $\mu = \sqrt[l^n]{\beta}$.

Corollary 3.4. *With the above notations, if $m \in \mathbb{N}$, we have $[\mathbb{F}_q(\sqrt[l^{n+m}]{\beta}) : \mathbb{F}_q] = l^{s+m}$.*

Proof. Set $\delta = \sqrt[l^{n+m}]{\beta}$, then $\delta^{l^m} = \mu = \sqrt[l^n]{\beta}$, and $\mu^{l^s} = \delta^{l^{m+s}} = \theta \in \mathbb{F}_q^*$. Clearly $m + s$ is minimal. $\square$

We consider $\mu = \sqrt[l^n]{\beta}$, with $\beta \in \mathbb{F}_q^*$ and $[\mathbb{F}_q(\sqrt[l^n]{\beta} : \mathbb{F}_q] = l^s$ for some $0 \leq s \leq n$. Set $\mu^{l^s} = \theta \in \mathbb{F}_q^*$, $\beta = \mu^{l^n} = (\mu^{l^s})^{l^{n-s}} = \theta^{l^{n-s}}$. Hence $\beta \in G^{l^{n-s}}$, where $G = \mathbb{F}_q^*$.

In case that there would exist $\varepsilon \in \mathbb{F}_q^*$ such that $\beta = \varepsilon^{l^{n-s+1}}$ it would imply that $\mu^{l^n} = \varepsilon^{l^{n-s+1}}$ and thus $\mu = (\varepsilon^{l^{n-s+1}})^{1/l^n} = \varepsilon^{l^{n-s+1-n}} = \varepsilon^{l^{-s+1}} = \sqrt[l^{s-1}]{\varepsilon}$. Therefore $X^{l^{s-1}} - \varepsilon$ has μ as a root. It would follow that $[\mathbb{F}_q(\mu) : \mathbb{F}_q] \leq l^{s-1} < l^s$ contrary to our hypothesis, Therefore $\beta \in G^{l^{n-s}} \setminus G^{l^{n-s+1}}$.

Conversely, if $\beta \in G^{l^{n-s}} \setminus G^{l^{n-s+1}}$, then $\beta = \kappa^{l^{n-s}}$ where κ in not an l-power. Therefore

$$\mu^{1/l^n} = (\kappa)^{l^{n-s}})^{1/l^n} = \kappa^{l^{-s}} = \sqrt[l^s]{\kappa},$$

hence $\mathbb{F}_q(\sqrt[l^n]{\mu}) \subseteq \mathbb{F}_{q^{l^s}}$. In case that $\mathbb{F}_q(\sqrt[l^n]{\mu}) \subseteq \mathbb{F}_{q^{l^{s-1}}}$ it would follow that $\mu^{l^{s-1}} \in \mathbb{F}_q^*$ contrary to our hypothesis.

We have proved:

Theorem 3.5. *We have that $[\mathbb{F}_q(\sqrt[l^n]{\beta}) : \mathbb{F}_q] = l^s$ if and only if $\beta \in (\mathbb{F}_q^*)^{l^{n-s}} \setminus (\mathbb{F}_q^*)^{l^{n-s+1}}$.* $\square$

A basic result on cyclic groups of prime power degree that we need is the following.

Proposition 3.6. *Let G be a cyclic group of order l^τ with l a prime number. Given $H_1, H_2 < G$, then $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. In particular $H_1 \cap H_2 = H_j$ with $j = 1$ or $j = 2$ and if $H_1 \neq \{\operatorname{Id}\}$ and $H_2 \neq \{\operatorname{Id}\}$, then $H_1 \cap H_2 \neq \{\operatorname{Id}\}$.*

Proof. By cyclicity, G has a unique subgroup of each of the divisors of $|G| = l^\tau$. These subgroups are $\Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_\tau$ with $|\Lambda_i| = l^i$. The result follows. $\square$

4. Extended genus fields and class field theory

First we establish the definition of *extended genus fields* according to Anglès and Jaulent [1]. For the terminology and notations we refer to [11].

We have that $k_\infty \cong \mathbb{F}_q\big(\big(\frac{1}{T}\big)\big)$ is the completion of k at $\mathfrak{p}_\infty$. Let $x \in k_\infty^*$. Then x is written uniquely as

$$x = \left(\frac{1}{T}\right)^{n_x} \lambda_x \varepsilon_x \quad \text{with} \quad n_x \in \mathbb{Z}, \quad \lambda_x \in \mathbb{F}_q^* \quad \text{and} \quad \varepsilon_x \in U_\infty^{(1)},$$

where $U_\infty^{(1)} = U_{\mathfrak{p}_\infty}^{(1)}$ is the group of the one units of k_∞. We write $\pi_\infty := 1/T$, which is an uniformizer at $\mathfrak{p}_\infty$.

The *sign function* is defined as $\phi_\infty \colon k_\infty^* \longrightarrow \mathbb{F}_q^*$ given by $\phi_\infty(x) = \lambda_x$ for $x \in k_\infty^*$.

We have that ϕ_∞ is an epimorphism and $\ker \phi_\infty = \langle \pi_\infty \rangle \times U_\infty^{(1)}$.

For a finite separable extension L of k_∞, we define the *sign of* L^* by the morphism $\phi_L := \phi_\infty \circ \mathrm{N}_{L/k_\infty} \colon L^* \longrightarrow \mathbb{F}_q^*$. We have $\frac{L^*}{\ker \phi_L} \cong A \subseteq \mathbb{F}_q^*$.

For a global function field L, let $\mathcal{P}$ be the set of places of L dividing $\mathfrak{p}_\infty$. We define the following subgroups of the group of idèles J_L as:

$$U_L := \prod_{v|\infty} L_v^* \times \prod_{v\nmid\infty} U_{L_v} \quad \text{and} \quad U_L^+ := \prod_{v|\infty} \ker \phi_{L_v} \times \prod_{v\nmid\infty} U_{L_v}, \tag{4.1}$$

where we denote $v \nmid \infty$ if $v \notin \mathcal{P}$ and $v|\infty$ if $v \in \mathcal{P}$. The groups $U_L L^*$ and $U_L^+ L^*$ are open subgroups of J_L, the idèle group of L.

Definition 4.1. *Let K/k be a finite abelian extension. Then the* Hilbert Class Field (HCF) *K_H and the* extended HCF *K_{H^+} of K are the fields corresponding to the idèle subgroups $U_K K^*$ and $U_K^+ K^*$ of J_K respectively. By class field theory (*[9, Theorem 17.6.198]*), the respective genus $K_{\mathfrak{ge}}$ and extended genus fields $K_{\mathfrak{gex}}$ with respect to the extension K/k, correspond to the idèle subgroups $(\mathrm{N}_{K/k} U_K)k^*$ and $(\mathrm{N}_{K/k} U_K^+)k^*$ of J_k respectively.*

By class field theory we have

$$\mathrm{Gal}(K_H/K) \cong J_K/U_K K^*.$$

We have that K_{H^+}/K is an unramified extension at the finite prime divisors of K, $K_H \subseteq K_{H^+}$ and

$$\mathrm{Gal}(K_{H^+}/K) \cong J_K/U_K^+ K^*.$$

Let K/k be a finite abelian extension then, with the notation given above, if $E = \mathcal{M}K \cap k(\Lambda_N)$, we have $K_{\mathfrak{gex}} = DK$ for some subfield $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} \subseteq D \subseteq E_{\mathfrak{gex}}$ for some decomposition group $\mathcal{H}$ (see [11]). In most cases we have $\mathcal{H} = \{\mathrm{Id}\}$. In this case, $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} = E_{\mathfrak{gex}}$ and $K_{\mathfrak{gex}} = E_{\mathfrak{gex}} K$.

In this paper, we particularly study the case $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} \neq E_{\mathfrak{gex}}$. A general result is the following.

Proposition 4.2. *Let K/k be a finite abelian extension and let E be given by* (2.1)*. Then, if $P_1, \ldots, P_r$ are the finite primes of k ramified in K and*

$$e_{P_j}(E_{\mathfrak{ge}}^{\mathcal{H}}|k) = e_{P_j}(E|k) = e_{P_j}(E_{\mathfrak{ge}}|k),$$

for all $1 \leq j \leq r$, it follows that $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} = E_{\mathfrak{gex}}$.

Proof. The group of Dirichlet characters associated to $E_{\mathfrak{gex}}$ is

$$Y := \prod_{P \in R_T^+} X_P = \prod_{j=1}^{r} X_{P_j},$$

where X is the group of Dirichlet characters associated to E. Each X_{P_j} is cyclic of order $e_{P_j}(E|k)$. The field associated to X_{P_j} is the subfield of $k(\Lambda_{P_j})$ of degree $e_{P_j}(E|k)$ over k. It follows that

$$[E_{\mathfrak{gex}} : k] = \prod_{j=1}^{r} e_{P_j}(E|k).$$

Let Z be the group of Dirichlet characters associated to $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}}$. The only finite primes of k possibly ramified in $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}}$ are $P_1, \ldots, P_r$. The group of Dirichlet characteres to $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}}$ is $\prod_{j=1}^{r} Z_{P_j}$. By hypothesis, $X_{P_j} = Z_{P_j}$ for all $1 \leq j \leq r$. Therefore $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} = E_{\mathfrak{gex}}$. $\square$

Corollary 4.3. *We have* $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} \neq E_{\mathfrak{gex}} \iff$ *there exists* $1 \leq j_0 \leq r$ *such that* $e_{P_{j_0}}(E_{\mathfrak{gex}}^{\mathcal{H}}|k) < e_{P_{j_0}}(E|k) = e_{P_{j_0}}(E_{\mathfrak{ge}}|k) = e_{P_{j_0}}(E_{\mathfrak{gex}}|k)$.

Proof. It follows from Proposition 4.2 and from the facts that $E_{\mathfrak{gex}}/E_{\mathfrak{ge}}$ is not ramified at any finite prime and that $\mathfrak{p}_\infty$ is fully ramified in $E_{\mathfrak{gex}}/E_{\mathfrak{ge}}$. $\square$

Therefore, if $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} \neq E_{\mathfrak{gex}}$, or, equivalently, there exists a finite prime P_j ramified in $E_{\mathfrak{gex}}/E_{\mathfrak{ge}}^{\mathcal{H}}$. The prime $\mathfrak{p}_\infty$ is fully ramified in $E_{\mathfrak{gex}}/E_{\mathfrak{ge}}^{\mathcal{H}}$.

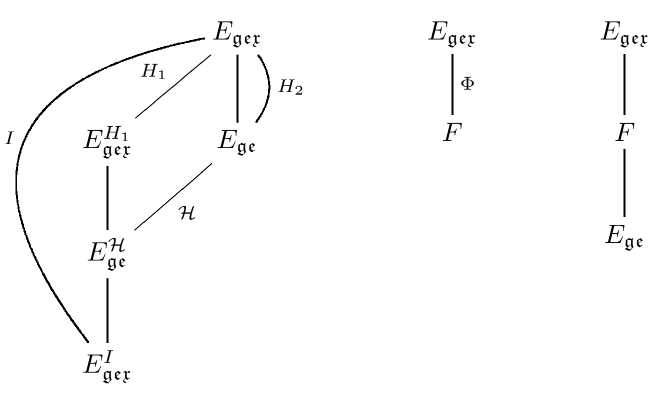
We have that $\mathcal{H} \subseteq I_\infty(E_{\mathfrak{ge}}/k)$, the inertia group of $\mathfrak{p}_\infty$ in the extension $E_{\mathfrak{ge}}/k$.

It follows that $\mathrm{Gal}(E_{\mathfrak{gex}}/E_{\mathfrak{ge}}^{\mathcal{H}}) \subseteq I_\infty(E_{\mathfrak{gex}}/k)$ and we have that $I_\infty(E_{\mathfrak{gex}}/k)$ is a cyclic group of order a power of l. Set $I := I_\infty(E_{\mathfrak{gex}}/k)$, an l–cyclic group. Furthermore $|I| = e_\infty(E_{\mathfrak{gex}}|k)|q-1$.

Using Proposition 3.6, we obtain:

Proposition 4.4. $E_{\mathfrak{gex}} = E_{\mathfrak{ge}}$.

Proof. Let $G := I = I_\infty(E_{\mathfrak{gex}}/k)$, $H_1 := I_{P_i}(E_{\mathfrak{gex}}/E_{\mathfrak{ge}}^{\mathcal{H}})$ and $H_2 := \mathrm{Gal}(E_{\mathfrak{gex}}/E_{\mathfrak{ge}})$. By hypothesis, we have $H_1 \neq \{\mathrm{Id}\}$. Set $\Phi := H_1 \cap H_2$ and $F := E_{\mathfrak{gex}}^{\Phi}$.



Then P_i is fully ramified in $E_{\mathfrak{gex}}/F$ and $E_{\mathfrak{ge}} \subseteq F$. Therefore P_i is fully ramified and non-ramified in $E_{\mathfrak{gex}}/F$. Hence $F = E_{\mathfrak{gex}}$ and $H_1 \cap H_2 = \{\mathrm{Id}\}$. Finally, it follows that $H_2 = \{\mathrm{Id}\}$ and that $E_{\mathfrak{gex}} = E_{\mathfrak{ge}}$. $\square$

In the rest of the section we will focus in the cases $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} \neq E_{\mathfrak{gex}}$.

Definition 4.5. *For a finite abelian extension K/k we define $K^{\mathrm{ext}} := E_{\mathfrak{gex}}K$, where E is given by* (2.1).

In this paper we will proof that always

$$K^{\mathrm{ext}} = K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K,$$

where K/k is any finite abelian extension.

In [8] we define the extended genus field of a finite abelian extension K/k as K^{ext}. In general we have $K_{\mathfrak{gex}} \subseteq K^{\mathrm{ext}}$.

We recall the following theorem from class field theory.

Theorem 4.6. *Let F be a global function field. Let N/F be a finite abelian extension and let $B < C_F$ be the the subgroup of the idèle class group of F corresponding to N. Then, if $\mathbb{F}_q$ is the field of constants of F, $\mathbb{F}_{q^\kappa}$ is the field of constants of N, where*

$$\kappa := \min\{\sigma \in \mathbb{N} \mid \textit{there exists } \tilde{\vec{\alpha}} \in B \textit{ such that } \deg \tilde{\vec{\alpha}} = \sigma\}.$$

Proof. See [9, Teorema 17.6.192]. □

Remark 4.7. Note that if F is any global function field, then the field of the constants of $F_{\mathfrak{gex}}$ and of F_{H^+} is the same. Therefore, in the especial case $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} \neq E_{\mathfrak{gex}}$, if we obtain that if the field of constants of $E_{\mathfrak{gex}}K$ is contained in the one of K_{H^+} then, because $E_{\mathfrak{gex}}K = E_{\mathfrak{ge}}K$ is an extension of constants of $K_{\mathfrak{ge}}$, say $E_{\mathfrak{gex}}K = K_{\mathfrak{ge}}\mathbb{F}_{q^\mu}$ and $K_{\mathfrak{ge}} \subseteq K_H \subseteq K_{H^+}$ and $\mathbb{F}_{q^\mu} \subseteq K_{H^+}$, it follows that $E_{\mathfrak{gex}}K \subseteq K_{H^+}$ and therefore $E_{\mathfrak{gex}}K \subseteq K_{\mathfrak{gex}}$. Since $K_{\mathfrak{gex}} \subseteq E_{\mathfrak{gex}}K$ (see [11]), we obtain $E_{\mathfrak{gex}}K = K_{\mathfrak{gex}}$.

Consider a finite abelian extension K/k. The idèle class subgroup B of the idèle class group C_K, the idèle class group of K, associated to K_{H^+} is

$$B = U_K^+ K^*/K^* = \Big(\prod_{\mathfrak{P}\mid\infty} \ker \phi_{\mathfrak{P}} \times \prod_{\mathfrak{P}\nmid\infty} U_{\mathfrak{P}}\Big) K^*/K^*,$$

where

$$U_{\mathfrak{P}} := U_{K_{\mathfrak{P}}}, \quad \ker \phi_{\mathfrak{P}} := \ker \phi_{K_{\mathfrak{P}}}, \quad \text{and} \quad \phi_{\mathfrak{P}} = \phi_\infty \circ \mathrm{N}_{K_{\mathfrak{P}}|k_\infty}.$$

We denote $\mathrm{N}_{\mathfrak{P}} := \mathrm{N}_{K_{\mathfrak{P}}|k_\infty}$.

On the one hand, if $\vec{\alpha} \in U_K^+$, then $\vec{\alpha} = \big(\alpha_{\mathfrak{P}}\big)_{\mathfrak{P}}$, $\alpha_{\mathfrak{P}} \in U_{\mathfrak{P}}$ for $\mathfrak{P} \nmid \infty$, so that $\deg_{\mathfrak{P}} \alpha_{\mathfrak{P}} = 0$ for $\mathfrak{P} \nmid \infty$. On the other hand, if $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are two infinite primes, $\mathrm{N}_{\mathfrak{P}_1} K_{\mathfrak{P}_1}^* = \mathrm{N}_{\mathfrak{P}_2} K_{\mathfrak{P}_2}^*$. It follows that if we fix an infinite prime $\mathfrak{P}$ of K, then

Lemma 4.8. *We have*

$$\begin{aligned}\kappa :&= \min\{\sigma \in \mathbb{N} \mid \textit{there exists } \vec{\alpha} \in U_K^+ \textit{ such that } \deg \vec{\alpha} = \sigma\}\\ &= \min\{\sigma \in \mathbb{N} \mid \textit{there exists } \tilde{\vec{\alpha}} \in B \textit{ such that } \deg \tilde{\vec{\alpha}} = \sigma\}\\ &= \min\{\sigma \in \mathbb{N} \mid \textit{there exists } x \in K_{\mathfrak{P}}^* \textit{ such that } x \in \ker \phi_{\mathfrak{P}} \textit{ and } \deg_{\mathfrak{P}} x = \sigma\}.\end{aligned}$$ □

5. Cyclic extensions of prime power degree

We study the extended genus field $K_{\mathfrak{gex}}$ of a finite cyclic extension K/k of degree l^n with l a prime number and $n \geq 1$. We will assume that the extension K/k is geometric, that is, the field of constants of K is $\mathbb{F}_q$. We consider four type of primes l:

(1) $l=p$, where p is the characteristic of k, the Artin-Schreier-Witt case,
(2) $l\neq p$ and $l\nmid q-1$,
(3) $l^n|q-1$, the Kummer case,
(4) $l^\rho|q-1$ wiith $1\leq \rho<n$ and $l^n\nmid q-1$, the "semi-Kummer" case.

5.1. **The Artin-Schreir-Witt case:** $l=p$**.** Since $\mathcal{H}$ is a subgroup of the inertia group of $\mathfrak{p}_\infty$ in E/k and the order of this last group is a divisor of $q-1$, the order of $\mathcal{H}$ is relative prime to p. Hence $\mathcal{H}=\{\mathrm{Id}\}$. It follows that $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}}=E_{\mathfrak{gex}}$.

5.2. **Case** $l\neq p$ **and** $l\nmid q-1$**.** By the same reason of Subsection 5.1, the order of $\mathcal{H}$ is relative prime to l. Thus $\mathcal{H}=\{\mathrm{Id}\}$ and $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}}=E_{\mathfrak{gex}}$.

5.3. **The Kummer case:** $l^n|q-1$**.**

5.3.1. *The genus field in the cyclotomic case.* Let $K=F=k\big(\sqrt[l^n]{D^*}\big)$ with $D=P_1^{\alpha_1}\cdots P_r^{\alpha_r}$ be a Kummer cyclic extension of k. Let $X=\langle\chi\rangle$ be the group of Dirichlet characters associated to F. Note that for any $\nu\in\mathbb{N}$ relatively prime to l, the field associated to χ^ν is F since $X=\langle\chi^\nu\rangle$. The above corresponds to the fact that $F=k\big(\sqrt[l^n]{(D^\nu)^*}\big)$.

When $D=P\in R_T^+$ we have that the character associated to F is $\left(\frac{\ }{P}\right)_{l^n}$, the Legendre symbol which is defined as follows: if P is of degree d, then for any $N\in R_T$ with $P\nmid N$, $N \bmod P\in (R_T/\langle P\rangle)^*\cong \mathbb{F}_{q^d}^*$. Then $\left(\frac{N}{P}\right)_{l^n}$ is defined as the unique element of $\mathbb{F}_{q^d}^*$ such that $N^{\frac{q^d-1}{l^n}}\equiv \left(\frac{N}{P}\right)_{l^n} \bmod P$. We have that $\left(\frac{\ }{P}\right)_{l^n}$ is the character associated to $k\big(\sqrt[l^n]{P^*}\big)$ (see [9, Proposición 9.6.1]). Let us denote $\chi_P=\left(\frac{\ }{P}\right)_{l^n}$. Then χ_P^ν is the character associated to $k\big(\sqrt[l^n]{(P^\nu)^*}\big)$.

Hence, if χ_D is the character associated to $k\big(\sqrt[l^n]{D^*}\big)$, then $\chi_D=\prod_{j=1}^r\chi_{P_j}^{\alpha_j}$.

Remark 5.1. Let l be a prime number different to p, the characteristic of K, such that $l^\kappa|q-1$, $\kappa\geq 1$. We have that $-1\in(\mathbb{F}_q^*)^{l^\kappa}$ for all l and all κ except when $l=2$ and $2^{\kappa+1}\nmid q-1$.

Proof. If l is odd, $(-1)^{l^\kappa}=-1$ for all $\kappa\in\mathbb{N}$. Let $l=2$. If $2^{\kappa+1}|q-1$, $\mathbb{F}_q$ contains a primitive $2^{\kappa+1}$ root of unity ξ. Let $\mu:=\xi^{2^\kappa}$. Then $\mu\neq 1$ and $\mu^2=1$. Thus $\mu=-1$.

On the other hand, if $2^{\kappa+1}\nmid q-1$, $\xi\notin\mathbb{F}_q^*$. Now if we had $\mu=-1=\rho^{2^\kappa}$ for some $\rho\in\mathbb{F}_q^*$, then ρ is a primitive $2^{\kappa+1}$ root of unity, contrary to our hypothesis. $\square$

Corollary 5.2. *For any prime number l such that $l^\kappa|q-1$, with $\kappa\in\mathbb{N}$, and $D\in R_T$ we have $k\big(\sqrt[l^\kappa]{D^*}\big)=k\big(\sqrt[l^\kappa]{D}\big)$, except when $\deg D$ is odd, $l=2$, and $2^{\kappa+1}\nmid q-1$. We also have that if $\gamma\in\mathbb{F}_q^*$ and $\varepsilon=(-1)^{\deg D}\gamma$, then $\mathbb{F}_q\big(\sqrt[l^\kappa]{\varepsilon}\big)=\mathbb{F}_q\big(\sqrt[l^\kappa]{\gamma}\big)$ with the same exception.*

Proof. If $\deg D$ is even, $(-1)^{\deg D}=1$. If $\deg D$ is odd, $(-1)^{\deg D}=-1\in(\mathbb{F}_q^*)^{l^\kappa}$ except when $l=2$ and $2^{\kappa+1}\nmid q-1$. The same for $\mathbb{F}_q\big(\sqrt[l^\kappa]{\varepsilon}\big)$ and $\mathbb{F}_q\big(\sqrt[l^\kappa]{\gamma}\big)$. $\square$

In general, for a radical extension, we have:

Theorem 5.3. *Let $F = k\big(\sqrt[s]{\gamma D}\big)$ be a geometric separable extension of k, $\gamma \in \mathbb{F}_q^*$, and let $D = P_1^{\alpha_1} \cdots P_r^{\alpha_r} \in R_T$. Then*

$$e_{F/k}(P_j) = \frac{s}{\gcd(\alpha_j, s)}, \quad 1 \le j \le r \quad \textit{and}$$
$$e_\infty(F|k) := e_{F/k}(\mathfrak{p}_\infty) = \frac{s}{\gcd(\deg D, s)}.$$

Proof. See [6, §5.1]. □

As a consequence we obtain the following result for a cyclic cyclotomic Kummer extension $F = k(\sqrt[l^n]{D^*})$. Let X be the group of Dirichlet characters associated to F and let $Y = \prod_{P \in R_T^+} X_P$ be the group associated to M, the maximal cyclotomic extension of F unramified at the finite primes.

Let $P = P_j$, $X = X_P = \langle \chi_P \rangle$ and let F_P be the field associated to X_P. Then F_P is cyclotomic, P is the only ramified prime in F_P/k and P is tamely ramified in F_P/k. This implies that $F_P \subseteq k(\Lambda_P)$ and $\operatorname{Gal}(k(\Lambda_P)/k) \cong C_{q^{d_P}-1}$ with $d_P := \deg P$. Therefore F_P is the only field of degree $o(\chi_P) =: l^{\beta_P}$ over k. Since F_P/k is a Kummer extension, it follows that $F_P = k\big(\sqrt[l^{\beta_P}]{P^*}\big)$.

Theorem 5.4. *The maximal unramified cyclotomic extension of $F = k\big(\sqrt[l^n]{D^*}\big)$ at the finite primes is $M := k\big(\sqrt[l^n]{(P_1^{\alpha_1})^*}, \ldots, \sqrt[l^n]{(P_r^{\alpha_r})^*}\big)$. In other words,*

$$F_{\mathfrak{ger}} = \big(\sqrt[l^n]{(P_1^{\alpha_1})^*}, \ldots, \sqrt[l^n]{(P_r^{\alpha_r})^*}\big).$$

Proof. It follows since M corresponds to the group of Dirichlet characters $Y = \prod_{P \in R_T^+} X_P$ and the field associated to X_P is $F_P = k\big(\sqrt[l^{\beta_P}]{P^*}\big)$ for each $P \in R_T^+$. The result follows. □

Remark 5.5. Let $\alpha = l^a b$ with $\gcd(b, l) = 1$ and $a < n$. Then $k\big(\sqrt[l^n]{(P^\alpha)^*}\big) = k\big(\sqrt[l^{n-a}]{P^*}\big)$ and

$$F_{\mathfrak{ger}} = k\big(\sqrt[l^{n-a_1}]{P_1^*}, \ldots, \sqrt[l^{n-a_r}]{P_r^*}\big) = F_1 \cdots F_r,$$

with $F_j = k\big(\sqrt[l^{n-a_j}]{P_j^*}\big)$, $1 \le j \le r$.

Another proof of Theorem 5.4 is using Abhyankar's Lemma. On the one hand we have that

$$[M : k] = \prod_{P \in R_T^+} |X_P| = \prod_{j=1}^{r} |X_{P_j}| = \prod_{j=1}^{r} e_{M/k}(P_j) = \prod_{j=1}^{r} l^{n-a_j}.$$

On the other hand if, $F_j = k\big(\sqrt[l^{n-a_j}]{(P_j)^*}\big)$, from Abyankar's Lemma, FF_j/F is unramified at every finite prime, so $FF_1 \cdots F_r/F$ is unramified at the finite primes and $F \subseteq F_1 \cdots F_r$. Hence $F_1 \cdots F_r \subseteq F_{\mathfrak{ger}}$ and $[F_1 \cdots F_r : k] = [M : k]$. Therefore $M = F_1 \cdots F_r$.

The genus field of a cyclotomic cyclic extension, is given by the following theorem.

Theorem 5.6. *Let $E = k(\sqrt[l^n]{D^*})$, with $D = P_1^{\alpha_1} \cdots P_r^{\alpha_r}$, $1 \le \alpha_j \le l^n - 1$, $\alpha_j = b_j l^{a_j}$ with $\gcd(b_j, l) = 1$, $1 \le j \le r$, $P_1, \ldots, P_r \in R_T^+$ different monic irreducible polynomials with $\deg P_j = c_j l^{d_j}$, $\gcd(c_j, l) = 1$, $1 \le j \le r$. We order the polynomials $P_1, \ldots, P_r$ such that $0 = a_1 \le \cdots \le a_r \le n - 1$.*

Let $E_{\mathfrak{ger}} := E_1 \cdots E_r$ *with* $E_j = k(\sqrt[l^{n-a_j}]{P_j^*})$, $1 \le j \le r$. *Let*

$$e_\infty(E|k) = l^t \text{ with } t = n - \min\{n, v_l(\deg D)\},$$

$$\begin{aligned} e_\infty(E_{\mathfrak{ger}}|k) = l^m \text{ with } m &= \max_{1\le j\le r} v_l(e_\infty(E_j|k)) \\ &= \max\{n - a_j - \min\{n - a_j, d_j\} \quad 1 \le j \le r\}. \end{aligned}$$

Let i_0, $1 \le i_0 \le r$, *be such that* $n - a_{i_0} - \min\{n - a_{i_0}, d_{i_0}\} = m$ *and* $n - a_j - d_j < m$ *for* $j > i_0$. *For* $m > 0$ *we have* $\gcd(\deg P_{i_0}, l^n) = l^{d_{i_0}}$, *and therefore there exist* $a, b \in \mathbb{Z}$ *such that* $a \deg P_{i_0} + bl^n = l^{d_{i_0}}$. *For* $j < i_0$, *we have* $d_{i_0} \le d_j$. *Let* $z_j := -ac_j l^{d_j - d_{i_0}}$. *For* $j > i_0$, *let* $y_j \equiv -c_j c_{i_0}^{-1} \mod l^n \in \mathbb{Z}$.

Then

$$E_{\mathfrak{ge}} = F_1 \cdots F_r,$$

where $F_j = E_j$ *with* $1 \le j \le r$ *if* $m = t$, *i.e*, $E_{\mathfrak{ge}} = E_{\mathfrak{ger}}$, *and if* $m > t \ge 0$, *then*

$$F_j := \begin{cases} k\left(\sqrt[l^{n-a_j}]{P_j P_{i_0}^{z_j}}\right) & \text{if } j < i_0, \\ k\left(\sqrt[l^{d_{i_0}+t}]{P_{i_0}^*}\right) & \text{if } j = i_0, \\ k\left(\sqrt[l^{n-a_j}]{P_j P_{i_0}^{y_j l^{d_j - d_{i_0}}}}\right) & \text{if } j > i_0 \text{ and } d_j \ge d_{i_0}, \\ k\left(\sqrt[l^{n-a_j+d_{i_0}-d_j}]{P_j^{l^{d_{i_0}-d_j}} P_{i_0}^{y_j}}\right) & \text{if } j > i_0 \text{ and } d_{i_0} > d_j. \end{cases} \tag{5.1}$$

Proof. See [7, Theorem 3.2]. □

Remark 5.7. When $m = t$, we may also use the description of $K_{\mathfrak{ge}}$ given in the case $m > t$.

5.3.2. *The genus field in the general case.* The genus field of a general l^n cyclic extension of k is given by the following theorem.

Theorem 5.8. *Let* $K = k(\sqrt[l^n]{\gamma D}) \subseteq k(\Lambda_D)_u$, *with* $\gamma \in \mathbb{F}_q^*$, $D = P_1^{\alpha_1} \cdots P_r^{\alpha_r}$, $1 \le \alpha_j \le l^n - 1$, $\alpha_j = b_j l^{a_j}$ *with* $\gcd(b_j, l) = 1$, $1 \le j \le r$, $P_1, \ldots, P_r \in R_T^+$ *different polynomials and some* $u \in \mathbb{N}$. *We order the polynomials* $P_1, \ldots, P_r$ *so that* $0 = a_1 \le \cdots \le a_r \le n - 1$. *Let* $E = K_u \cap k(\Lambda_D)$, t *as in Theorem* 5.6 *and* $\alpha = v_l(|\mathcal{H}|)$. *Let* $\mathcal{H}' := \mathcal{H}\,|_{E_{\mathfrak{ge}}}$. *Then* $E_{\mathfrak{ge}}^{\mathcal{H}'} = F_1 \cdots F_{i_0-1} F_{i_0+1} \cdots F_r(\sqrt[l^{d_{i_0}+(t-\alpha)}]{P_{i_0}^*})$ *where* F_j *are given in* (5.1) *for all* j. *Thus*

$$K_{\mathfrak{ge}} = E_{\mathfrak{ge}}^{\mathcal{H}'} K = \prod_{\substack{i=1 \\ i \ne i_0}}^{r} F_i K(\sqrt[l^{d_{i_0}+(t-\alpha)}]{P_{i_0}^*}).$$

Further, if $d = \min\{n, v_l(\deg D)\}$, *we have*

$$|\mathcal{H}| = l^\alpha = [\mathbb{F}_q(\sqrt[l^n]{(-1)^{\deg D}\gamma}) : \mathbb{F}_q(\sqrt[l^d]{(-1)^{\deg D}\gamma})].$$

Proof. See [7, Theorem 4.1]. □

The general structure of $K_{\mathfrak{ger}}$ when K/k is a finite l–Kummer extension for a prime number l, is given by $K_{\mathfrak{ger}} = DK$ with D a field satisfying $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{ger}} \subseteq D \subseteq E_{\mathfrak{ger}}$.

With notations given above, particularly in Theorem 5.6, we consider first the case $m > t$.

Proposition 5.9. *If $m > t$ then $i \geq 2$ and there exists $j < i$ such that $m = n - a_j - d_j = n - a_i - d_i$.*

Proof. First assume that $i \geq 2$. Suppose that for all $1 \leq j \leq i-1$ we have $n - a_j - d_j < n - a_i - d_i = m$.

We have that for all $j \neq i$, $n - a_i - d_i > n - a_j - \min\{n - a_j, d_j\} \geq n - a_j - d_j$. Thus

$$n - a_j - d_j < n - a_i - d_i, \quad \text{so that} \quad a_i + d_i < a_j + d_j \quad \text{for all} \quad j \neq i. \tag{5.2}$$

We have

$$\begin{aligned}\deg D &= \sum_{j=1}^{r} \alpha_j \deg P_j = \sum_{j=1}^{r} b_j l^{a_j} c_j l^{d_j} \\ &= b_i c_i l^{a_i+d_i} + l^{a_i+d_i+1}\Big(\sum_{\substack{j=1\\ j\neq i}}^{r} b_j c_j l^{a_j+d_j-a_i-d_i-1}\Big).\end{aligned}$$

Hence $v_l(\deg D) = l^{a_i+d_i}$. It follows that

$$\begin{gathered} n - \min\{n, v_l(\deg D)\} \geq n - v_l(\deg D) = n - a_i - d_i = m \quad \text{and} \\ t \geq n - \min\{n, v_l(\deg D)\} \geq m \geq t.\end{gathered}$$

Therefore $m = t$ contrary to our assumption. Thus, there exists $1 \leq j \leq i-1$ with $n - a_j + d_j = n - a_i - d_i = m$.

The same argument shows that if $i = 1$, then $m = t$. $\square$

From Theorems 5.6 and 5.8 we have that for all $j \neq i$, we have $e_{P_j}(E_{\mathfrak{ge}}^{\mathcal{H}}|k) = e_{P_j}(K|k) = e_{P_j}(E|k)$. In the case when there exists $1 \leq j \leq i-1$ such that $n - a_j - d_j = n - a_i - d_i$ we obtain that

$$\begin{aligned} e_{P_i}(E_j|k) &= e_{P_i}\big(k\big(\sqrt[l^{n-a_j}]{P_j P_i^{-ac_j l^{d_j-d_i}}}\big)|k\big) = l^{n-a_j-d_j+d_i} \\ &= l^{n-a_i} = e_{P_i}(E|k) = e_{P_i}(K|k).\end{aligned}$$

Hence

$$e_{P_i}(E_{\mathfrak{ge}}^{\mathcal{H}}|k) = e_{P_i}(K|k) = e_{P_i}(E|k).$$

From the above, the following result is immediate.

Proposition 5.10. *If there exists $1 \leq j \leq i-1$ such that $n - a_j - d_j = n - a_i - d_i$, in particular when $m > t$, then $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} = E_{\mathfrak{gex}}$.* $\square$

The first main result on extended genus fields, is the following:

Theorem 5.11. *With the above notations we have that $K_{\mathfrak{gex}} = E_{\mathfrak{gex}} K$, except in the following case:*

(a).- $K \neq E$,
(b).- $\mathcal{H} \neq \{\mathrm{Id}\}$,
(c).- $t = m > 0$,
(d).- $m = n - a_i - \min\{n - a_i, d_i\} > n - a_j - \min\{n - a_j, d_j\}$ *for all $j \neq i$,*

Proof. If $\mathcal{H} = \{\mathrm{Id}\}$ the result follows from [11, Theorem 4.5]. If $E = K$, K is cyclotomic and therefore $\mathcal{H} = \{\mathrm{Id}\}$. If $m > t$, then from Proposition 5.9, we have that $e_{P_j}(E^{\mathcal{H}}_{\mathfrak{ge}}|k) = e_{P_j}(E_{\mathfrak{ge}}|k)$ for all $1 \leq j \leq r$ and therefore $(E^{\mathcal{H}}_{\mathfrak{ge}})_{\mathfrak{gex}} = E_{\mathfrak{gex}}$. The result follows from [11, Theorem 4.5]. If $n - a_i - \min\{n - a_i, d_i\} = n - a_j - \min\{n - a_j, d_j\}$ for some $j \neq i$, then $e_{P_j}(E^{\mathcal{H}}_{\mathfrak{ge}}|k) = e_{P_j}(E_{\mathfrak{ge}}|k)$ for all $1 \leq j \leq r$ and therefore $(E^{\mathcal{H}}_{\mathfrak{ge}})_{\mathfrak{gex}} = E_{\mathfrak{gex}}$. If $t = 0$, we have that $\mathfrak{p}_\infty$ in unramified in E/k so that $\mathcal{H} = \{\mathrm{Id}\}$. ☐

5.3.3. *The especial case.* We now consider the especial case, that is, the exception given in Theorem 5.11.

Let $K = k\big(\sqrt[l^n]{\gamma D},\big)$ be a geometric separable extension of k, with $\gamma \in \mathbb{F}_q^*$ and let $D = P_1^{\alpha_1} \cdots P_r^{\alpha_r} \in R_T$, with $P_1, \ldots, P_r \in R_T^+$ distinct, $1 \leq \alpha_j \leq l^n - 1$, $1 \leq j \leq r$. Let $\alpha_j = l^{a_j} b_j$, $l \nmid b_j$, $\deg P_j = c_j l^{d_j}$, $l \nmid c_j$. We assume that we have the exception given in Theorem 5.11. Let $E = k\big(\sqrt[l^n]{D^*}\big)$, and $\deg D = l^\delta c$ with $l \nmid c$. Then $e_\infty(K|k) = l^{n-\delta} = l^t = l^m = l^{n-a_i-d_i}$, so that $\delta = a_i + d_i$.

Since $m = t > 0$, we have $m = n - a_i - \min\{n - a_i, d_i\} = n - a_i - d_i$ and $n - a_i - d_i > n - a_j - d_j$ for all $j \neq i$. We also have that $\varepsilon := (-1)^{\deg D}\gamma \notin (\mathbb{F}_q^*)^l$.

Lemma 5.12. *We have $E_{\mathfrak{ge}} = E^{\mathcal{H}}_{\mathfrak{ge}}E$. It also holds that EK/E and EK/K are extensions of constants and $EK = E\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big) = K\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big)$. That is,*

$$EK = E\big(\sqrt[l^n]{\varepsilon}\big) = K\big(\sqrt[l^n]{\varepsilon}\big).$$

We also have that $E_{\mathfrak{ge}}K/K_{\mathfrak{ge}}$ and $E_{\mathfrak{ge}}K/E_{\mathfrak{ge}}$ are extensions of constants. Furthermore, $E_{\mathfrak{ge}}K = K_{\mathfrak{ge}}\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big) = K_{\mathfrak{ge}}\big(\sqrt[l^n]{\varepsilon}\big)$ and $E_{\mathfrak{ge}}K = E_{\mathfrak{ge}}\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big) = E_{\mathfrak{ge}}\big(\sqrt[l^n]{\varepsilon}\big)$.

Proof. The extension $E_{\mathfrak{ge}}/E^{\mathcal{H}}_{\mathfrak{ge}}$ is fully ramified at the infinite prime $\mathfrak{p}_\infty$. Since $E^{\mathcal{H}}_{\mathfrak{ge}} \subseteq E^{\mathcal{H}}_{\mathfrak{ge}}E \subseteq E_{\mathfrak{ge}}$ and since $e_\infty(E|k) = e_\infty(E_{\mathfrak{ge}}|k)$, it follows that $E_{\mathfrak{ge}} = E^{\mathcal{H}}_{\mathfrak{ge}}E$.

Now $EK = k\big(\sqrt[n]{\gamma D}\big)k\big(\sqrt[n]{(1)^{\deg D}D}\big) = E\big(\sqrt[l^n]{\varepsilon}\big) = K\big(\sqrt[l^n]{\varepsilon}\big)$.

We also have $E_{\mathfrak{ge}}K = E^{\mathcal{H}}_{\mathfrak{ge}}EK = E^{\mathcal{H}}_{\mathfrak{ge}}KEK = K_{\mathfrak{ge}}K\big(\sqrt[l^n]{\varepsilon}\big) = K_{\mathfrak{ge}}\big(\sqrt[l^n]{\varepsilon}\big)$. Therefore

$$E_{\mathfrak{ge}}K = E_{\mathfrak{ge}}EK = E_{\mathfrak{ge}}\big(\sqrt[l^n]{\varepsilon}\big).$$

☐

Corollary 5.13. *The field of constants of $E_{\mathfrak{ge}}K$ is $\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big)$.* ☐

Theorem 5.14. *In the exceptional case given in Theorem* 5.11, *we have that $E_{\mathfrak{ge}} = E_{\mathfrak{gex}}$, the field of constants of $K_{\mathfrak{ge}}$ is $\mathbb{F}_{q^{\deg_K \mathfrak{p}_\infty}}$, the field of constants of $E_{\mathfrak{gex}}K$ is $\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big)$.*

Proof. Since $m = t$, we have $E_{\mathfrak{ge}} = E_{\mathfrak{gex}}$. ☐

Later on, we will see that the field of constants of $K_{\mathfrak{ge}}$ is $\mathbb{F}_q\big(\sqrt[l^\delta]{\gamma}\big)$.

We fix an infinite prime $\mathfrak{P}$ of K and we denote $K_\infty := K_{\mathfrak{P}}$.

Since $K_\infty = k_\infty\big(\sqrt[l^n]{\gamma D}\big)$, $\deg D = d = l^\delta c$ with $l \nmid c$, we have

$$\begin{aligned} D(T) &= T^d + a_{d-1}T^{d-1} + \cdots + a_1 T + a_0 \\ &= T^d\Big(1 + a_{d-1}\big(\frac{1}{T}\big) + \cdots a_1\big(\frac{1}{T}\big)^{d-1} + a_0\big(\frac{1}{T}\big)^d\Big) = T^d D_1(1/T). \end{aligned}$$

We have that $D_1(1/T) \in U_\infty^{(1)}$ and since l is different to the characteristic, it follows that $(U_\infty^{(1)})^{l^n} = U_\infty^{(1)}$. Therefore

$$K_\infty = k_\infty\big(\sqrt[l^n]{\gamma D}\big) = k_\infty\big(\sqrt[l^n]{\gamma T^d D_1(1/T)}\big) = k_\infty\big(\sqrt[l^n]{\gamma T^{l^\delta c}}\big).$$

Since $\gcd(l,c)=1$, there exists $c_1 \in \mathbb{Z}$ such that $cc_1 \equiv -1 \bmod l^n$. Thus

$$K_\infty = k_\infty\big(\sqrt[l^n]{\gamma^{c_1} T^{l^\delta c c_1}}\big) = k_\infty\big(\sqrt[l^n]{\gamma^{c_1}(1/T)^{l^\delta}}\big) = k_\infty\big(\sqrt[l^n]{\gamma^{c_1}\pi_\infty^{l^\delta}}\big).$$

Now $[K_\infty : k_\infty] = e_\infty(K|k) f_\infty(K|k) = l^{n-\delta}\deg_K \mathfrak{P}$. Set $K_0 = k\big(\sqrt[l^\delta]{\gamma D}\big) \subseteq K$. We have that $e_\infty(K_0|k) = \frac{l^\delta}{\gcd(\deg D, l^\delta)} = \frac{l^\delta}{\gcd(l^\delta c, l^\delta)} = \frac{l^\delta}{l^\delta} = 1$ and $e_\infty(K|K_0) = e_\infty(K|k) = l^{n-\delta} = [K : K_0]$. Therefore $\mathfrak{p}_\infty$ is fully ramified in K/K_0.

$$\begin{array}{ccc}
K = k\big(\sqrt[l^n]{\gamma D}\big) & \qquad & K_\infty \\
\Big| {\scriptstyle l^{n-\delta}} & & \Big| {\scriptstyle e_\infty(K|k) = l^{n-\delta}} \\
K_0 = k\big(\sqrt[l^\delta]{\gamma D}\big) & & K_{0,\infty} \\
\Big| & & \Big| {\scriptstyle f_\infty(K|k) = \deg_K \mathfrak{P} = l^\lambda} \\
k & & k_\infty
\end{array}$$

We have $f_\infty(K|k) = f_\infty(K_0|k) = f_\infty(K_{0,\infty}|k_\infty) = \deg_K \mathfrak{P}$ and

$$K_{0,\infty} = k_\infty\big(\sqrt[l^\delta]{\gamma D}\big) = k_\infty\big(\sqrt[l^\delta]{\gamma T^d}\big) = k_\infty\big(\sqrt[l^\delta]{\gamma T^{l^\delta c}}\big) = k_\infty\big(\sqrt[l^\delta]{\gamma}\big).$$

Lemma 5.15. *The field of constants of $K_{\mathfrak{ge}}$ is $\mathbb{F}_q\big(\sqrt[l^\delta]{\gamma}\big)$.* $\square$

Let

$$f_\infty(K|k) = \deg_K \mathfrak{P} = [K_{0,\infty} : k_\infty] = [\mathbb{F}_q\big(\sqrt[l^\delta]{\gamma}\big) : \mathbb{F}_q] =: l^\lambda.$$

We also have

$$EK = E\big(\sqrt[l^n]{\varepsilon}\big) = K\big(\sqrt[l^n]{\varepsilon}\big).$$

Then, EK/E is an extension of constants and, since $\deg_E \mathfrak{p}_\infty = 1$, it follows that

$$f_\infty(EK|k) = f_\infty(EK|E) = [EK : E] = [\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big) : \mathbb{F}_q] =: l^\nu.$$

Now, $|\mathcal{H}| = f_\infty(EK|K) = \frac{f_\infty(EK|k)}{f_\infty(K|k)} = \frac{l^\nu}{l^\lambda} = l^{\nu-\lambda} =: l^u$. The field of constants of $K_{\mathfrak{ge}}$ is $\mathbb{F}_{q^{l^\lambda}}$ and the field of constants of $E_{\mathfrak{ge}}K$ is $\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big) = \mathbb{F}_{q^{f_\infty(EK|k)}} = \mathbb{F}_{q^{l^\nu}}$. We have $\mathbb{F}_q\big(\sqrt[l^\delta]{\gamma}\big) = \mathbb{F}_q\big(\sqrt[l^\delta]{\gamma^{c_1}}\big) = \mathbb{F}_{q^{l^\lambda}}$. Therefore $[\mathbb{F}_q\big(\sqrt[l^\delta]{\gamma^{c_1}}\big) : \mathbb{F}_q] = l^\lambda$. From Theorem 3.5, we obtain that $\gamma^{c_1} \in (\mathbb{F}_q^*)^{l^{\delta-\lambda}} \setminus (\mathbb{F}_q^*)^{l^{\delta-\lambda+1}}$.

Let $\gamma^{c_1} = \theta^{l^{\delta-\lambda}}$, with $\theta \in \mathbb{F}_q^*$ and $\theta \notin (\mathbb{F}_q^*)^l$. Then

$$K_\infty = k_\infty\big(\sqrt[l^n]{\gamma^{c_1}\pi_\infty^{l^\delta}}\big) = k_\infty\big(\sqrt[l^n]{\theta^{l^{\delta-\lambda}}\pi_\infty^{l^\delta}}\big) = k_\infty\big(\sqrt[l^{n-\delta+\lambda}]{\theta\pi_\infty^{l^\lambda}}\big).$$

The element $\xi := \sqrt[l^{n-\delta+\lambda}]{\theta\pi_\infty^{l^\lambda}}$ satisfies $\xi^{l^{n-\delta+\lambda}} = \theta\pi_\infty^{l^\lambda}$, that is, ξ is a root of $X^{l^{n-\delta+\lambda}} - \theta\pi_\infty^{l^\lambda} \in k_\infty[X]$. Since $[K_\infty : k_\infty] = l^{n-\delta+\lambda}$, the polynomial

$$X^{l^{n-\delta+\lambda}} - \theta\pi_\infty^{l^\lambda}$$

is irreducible. We also have $K_{0,\infty} = k_\infty\big(\sqrt[l^\delta]{\gamma}\big) = k_\infty \mathbb{F}_{q^{l^\lambda}}$ and K/K_0 is fully ramified at $\mathfrak{p}_\infty$.

Set $\tilde{\Pi} := \xi = \sqrt[l^{n-\delta+\lambda}]{\theta\pi_\infty^{l^\lambda}}$. Then $\tilde{\Pi}^{l^{n-\delta+\lambda}} = \theta\pi_\infty^{l^\lambda}$ and

$$\begin{aligned} v_{\mathfrak{P}_\infty}\big(\tilde{\Pi}^{l^{n-\delta+\lambda}}\big) &= l^{n-\delta+\lambda} v_{\mathfrak{P}_\infty}(\tilde{\Pi}) = e_\infty(K_\infty|k_\infty) v_\infty\big(\theta\pi_\infty^{l^\lambda}\big) \\ &= e_\infty(K|k) l^\lambda = l^{n-\delta} l^\lambda = l^{n-\delta+\lambda}. \end{aligned}$$

Hence $v_{\mathfrak{P}_\infty}(\tilde{\Pi}) = 1$ and $\tilde{\Pi}$ is prime element of K_∞. We also have

$$\deg_{K_\infty} \tilde{\Pi} = \deg_{K_\infty} \mathfrak{P}_\infty v_{\mathfrak{P}_\infty}(\tilde{\Pi}) = \deg_K \mathfrak{p}_\infty \cdot 1 = l^\lambda.$$

Now $\theta \notin (\mathbb{F}_q^*)^l$. Let $\zeta_{l^{n-\delta+\lambda}}$ be a primitive $l^{n-\delta+\lambda}$-th root of unity and set $\mathrm{N}_\infty := \mathrm{N}_{K_\infty|k_\infty}$. We have

$$\mathrm{Irr}(\tilde{\Pi}, X, k_\infty) = X^{l^{n-\delta+\lambda}} - \theta\pi_\infty^{l^\lambda} = \prod_{j=0}^{l^{n-\delta+\lambda}-1} \big(X - \zeta_{l^{n-\delta+\lambda}}^j \tilde{\Pi}\big).$$

Thus

$$\begin{aligned} \mathrm{N}_\infty \tilde{\Pi} &= \prod_{j=0}^{l^{n-\delta+\lambda}-1} \big(\zeta_{l^{n-\delta+\lambda}}^j \tilde{\Pi}\big) = (-1)^{l^{n-\delta+\lambda}} \prod_{j=0}^{l^{n-\delta+\lambda}-1} \big(-\zeta_{l^{n-\delta+\lambda}}^j \tilde{\Pi}\big) \\ &= (-1)^{l^{n-\delta+\lambda}}\big(-\theta\pi_\infty^{l^\lambda}\big) = (-1)^{l^{n-\delta+\lambda}+1}\theta\pi_\infty^{l^\lambda}. \end{aligned}$$

Now we consider a generic element of $Y \in K_\infty^*$:

$$Y = \tilde{\Pi}^s \Lambda w, \quad \text{with} \quad s \in \mathbb{Z}, \quad \Lambda \in \mathbb{F}_{q^{l^\lambda}}, \quad \text{and} \quad w \in U_{K_\infty}^{(1)}.$$

Then

$$\begin{aligned} &\mathrm{N}_\infty \tilde{\Pi}^s = \big(\mathrm{N}_\infty \tilde{\Pi}\big)^s = (-1)^{(l^{n-\delta+\lambda}+1)s}\theta^s\pi_\infty^{l^\lambda s}, \\ &\mathrm{N}_\infty \Lambda = \mathrm{N}_{K_{0,\infty}|k_\infty}\big(\mathrm{N}_{K_\infty|K_{0,\infty}} \Lambda\big) = \mathrm{N}_{K_{0,\infty}|k_\infty}(\Lambda^{l^{l^{n-\delta}}}) = \big(\mathrm{N}_{K_{0,\infty}|k_\infty} \Lambda\big)^{l^{n-\delta}}, \\ &\mathrm{N}_\infty w = v \in U_\infty^{(1)}. \end{aligned}$$

It follows that

$$\begin{aligned} \phi_{\mathfrak{P}_\infty}(Y) &= \phi_\infty(\mathrm{N}_\infty(Y)) = \phi_\infty\big((-1)^{(l^{n-\delta+\lambda}+1)s}\theta^s\pi_\infty^{l^\lambda s}(\mathrm{N}_{K_{0,\infty}|k_\infty}\Lambda)^{l^{n-\delta}} v\big) \\ &= (-1)^{(l^{n-\delta+\lambda}+1)s}\theta^s\big(\mathrm{N}_{K_{0,\infty}|k_\infty}\Lambda\big)^{l^{n-\delta}} \\ &= (-\theta)^s\big[(-1)^{l^\lambda}\big(\mathrm{N}_{K_{0,\infty}|k_\infty}\Lambda\big)\big]^{l^{n-\delta}}. \end{aligned}$$

Therefore $Y \in \ker \phi_{\mathfrak{P}_\infty} \iff$ there exists $\Lambda \in \mathbb{F}_{q^{l^\lambda}}$ such that

$$(-\theta)^s\big[(-1)^{l^\lambda}\big(\mathrm{N}_{K_{0,\infty}|k_\infty}\Lambda\big)\big]^{l^{n-\delta}} = 1.$$

Now, $\mathrm{N}_{K_{0,\infty}|k_\infty}\mathbb{F}_{q^{l^\lambda}} = \mathbb{F}_q$, thus $\mathrm{N}_\infty \mathbb{F}_{q^{l^\lambda}}^* = \big(\mathbb{F}_q^*\big)^{l^{n-\delta}}$ and $\Lambda \in \mathbb{F}_{q^{l^\lambda}}$. Hence $-\theta^s \in (\mathbb{F}_q^*)^{l^{n-\delta}}$.

5.3.4. *Case* $n = 1$. This case was considered in [4].

5.3.5. *Case* $n \geq 2$. . We now assume that $n > 1$. We always have that, since $\theta \notin (\mathbb{F}_q^*)^l$, then $-\theta \notin (\mathbb{F}_q^*)^l$ because $n \geq 2$ and therefore $-1 \in (\mathbb{F}_q^*)^l$ (see Remark 5.1). Hence $-\theta^s \in (\mathbb{F}_q^*)^{l^{n-\delta}} \iff l^{n-\delta}|s$. That is $\ker \phi_{\mathfrak{P}_\infty} = \{Y = \tilde{\Pi}^s \Lambda w \mid l^{n-\delta}|s\}$.

Because $\deg Y = \deg\big(\tilde{\Pi}^s \Lambda w\big) = \deg \tilde{\Pi} \cdot v_{\mathfrak{P}_\infty}(Y) = l^\lambda \cdot s$, it follows that

$$\min\{\kappa \in \mathbb{N} \mid \text{there exists } \tilde{\tilde{\alpha}} \in B \text{ and } \deg \tilde{\tilde{\alpha}} = \kappa\} = l^{n-\delta+\lambda},$$

and that the field of constants of K_{H^+} is $\mathbb{F}_{q^{l^{n-\delta+\lambda}}}$.

We have that $l^\lambda = [\mathbb{F}_q\big(\sqrt[l^\delta]{\gamma}\big) : \mathbb{F}_q]$, so that, from Theorem 3.5, we obtain

$$\gamma \in (\mathbb{F}_q^*)^{l^{\delta-\lambda}} \setminus (\mathbb{F}_q^*)^{l^{\delta-\lambda+1}}. \tag{5.3}$$

On the other hand, the field of constants of $K_{\mathfrak{ge}}E$ is $\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big)$ and $[\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big) : \mathbb{F}_q] = l^\nu$. Again, from Theorem 3.5, we obtain that

$$\varepsilon \in (\mathbb{F}_q^*)^{l^{n-\nu}} \setminus (\mathbb{F}_q^*)^{l^{n-\nu+1}}.$$

We have $u = \nu - \lambda \geq 1$ so that $\nu \geq \lambda + 1$ and $n - \nu \leq n - 1$. Therefore $(-1)^{\deg D} = \pm 1 \in (\mathbb{F}_q^*)^{l^{n-\nu}}$. Hence $\gamma = (-1)^{\deg D}\varepsilon \in (\mathbb{F}_q^*)^{l^{n-\nu}}$. It follows from (5.3) that $n - \nu \leq \delta - \lambda$ and $n - \delta \leq \nu - \lambda$.

Thus, $e_\infty(K|k) = l^{n-\delta}|l^{\nu-\lambda} = l^u = |\mathcal{H}|$. Since $|\mathcal{H}||e_\infty(K|k) = e_\infty(E|k)$, $l^u|l^{n-\delta}$. Therefore, $\nu - \lambda \leq n - \delta$, so that $\nu - \lambda = n - \delta$ and $\nu = n - \delta + \lambda$. In particular, $|\mathcal{H}| = l^u = l^{n-\delta} = e_\infty(K|k)$.

It follows that the field of constants of $E_{\mathfrak{ge}}K = E_{\mathfrak{gex}}K$ is $\mathbb{F}_q\big(\sqrt[l^n]{\varepsilon}\big) = \mathbb{F}_{q^{l^\nu}} = \mathbb{F}_{q^{l^{n-\delta+\lambda}}}$. In short, the field of constants of both, K_{H^+} and $E_{\mathfrak{ge}}K$, is $\mathbb{F}_{q^{l^{n-\delta+\lambda}}} = \mathbb{F}_{q^{l^{u+\lambda}}}$.

Thus $E_{\mathfrak{ge}}K \subseteq K_{H^+}$ and $E_{\mathfrak{ge}}K \subseteq K_{\mathfrak{gex}} \subseteq E_{\mathfrak{gex}}K = E_{\mathfrak{ge}}K$. Therefore $E_{\mathfrak{gex}}K \subseteq K_{H^+}$. Since $K_{\mathfrak{gex}} \subseteq E_{\mathfrak{gex}}K$, we finally obtain that $K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K$.

Theorem 5.16. *In the especial case of Theorem* 5.11*, we obtain* $K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K$. □

Corollary 5.17. *For any cyclic Kummer extension* $k\big(\sqrt[l^n]{\gamma D}\big)$ *of* k*, we have* $K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K$. □

5.4. **Semi-Kummer case:** $l^\rho|q-1$**,** $\rho \geq 1$ **and** $l^n \nmid q-1$**.** Recall that we only need to consider the case $(E_{\mathfrak{ge}}^{\mathcal{H}})_{\mathfrak{gex}} \neq E_{\mathfrak{gex}}$, or, equivalently, there exists a finite prime P_j ramified in $E_{\mathfrak{gex}}/E_{\mathfrak{ge}}^{\mathcal{H}}$, and therefore $E_{\mathfrak{gex}} = E_{\mathfrak{ge}}$ (see Proposition 4.4). The prime $\mathfrak{p}_\infty$ is fully ramified in $E_{\mathfrak{gex}}/E_{\mathfrak{ge}}^{\mathcal{H}}$.

Lemma 5.18. *We have that* $f_\infty(K|k) = \deg_K \mathfrak{p}_\infty$*.*

Proof. Since the extension K/k is geometric, and $\deg_k \mathfrak{p}_\infty = 1$, we have $f_\infty(K|k) = f_\infty(K|k)\deg_k \mathfrak{p}_\infty = \deg_K \mathfrak{p}_\infty$. □

We use the following notation. Let $|\mathcal{H}| := l^u = f_\infty(EK|K)$. Since $\mathcal{H}$ is a quotient of I, it follows that $l^u|l^\rho$ since $|I| = e_\infty(E_{\mathfrak{gex}}|k)|q^{\deg_K \mathfrak{p}_\infty} - 1 = q - 1$. In particular $u \leq \rho$.

Set $l^\lambda := \deg_K \mathfrak{p}_\infty = f_\infty(K|k)$. We have

$$\begin{tikzcd}
 & E \arrow[r, dash] \arrow[d, dash] & EK \arrow[d, dash] \\
 & E \cap K \arrow[r, dash] & K \\
k \arrow[ur, dash] & &
\end{tikzcd}$$

$$\begin{aligned} f_\infty(EK|E) &= f_\infty(EK|k) \\ &= f_\infty(EK|K) f_\infty(K|k) \\ &= |\mathcal{H}| \cdot \deg_K \mathfrak{p}_\infty = l^{u+\lambda}. \end{aligned}$$

Since $e_P(K|k) = e_P(E|k)$ for all $P \in R_T^+ \cup \{\infty\}$, from Abhyankar's Lemma we obtain that $e_P(EK|E) = 1$ for all $P \in R_T^+ \cup \{\infty\}$, that is, EK/E is an unramified extension. We will show that it is an extension of constants.

It is easy to see that $[K : k] = [E : k]$. We have, on the one hand

$$[K : E \cap K] = \frac{[K : k]}{[E \cap K : k]} = \frac{[E : k]}{[E \cap K : k]} = [E : E \cap K].$$

On the other hand

$$[EK : E] = [K : E \cap K] \quad \text{and} \quad [EK : K] = [E : E \cap K].$$

Therefore

$$[EK : K] = [EK : E] = [E : E \cap K] = [K : E \cap K].$$

Because EK/E and EK/K are unramified extensions,

$$e_P(E|E \cap K) = e_P(K|E \cap K) \quad \text{for all} \quad P \in R_T^+ \cup \{\infty\}.$$

We have that $E_{\mathfrak{ge}}K/K_{\mathfrak{ge}} = E_{\mathfrak{ge}}^{\mathcal{H}}K$ is an extension of constants of degree $|\mathcal{H}| = l^u = [E_{\mathfrak{ge}}K : K_{\mathfrak{ge}}]$ ([2, Theorem 2.2]). We also have that the field of constants of $K_{\mathfrak{ge}}$ is $\mathbb{F}_{q^{\deg_K \mathfrak{p}_\infty}}$. Hence, the field of constants of $E_{\mathfrak{ge}}K = E_{\mathfrak{gex}}K$ is $\mathbb{F}_{q^\psi}$ where $\psi = \deg_K \mathfrak{p}_\infty \cdot |\mathcal{H}| = l^{\lambda+u} = f_\infty(EK|k)$.

Let see that the field of constants of EK is also $\mathbb{F}_{q^\psi} = \mathbb{F}_{q^{l^{\lambda+u}}}$, the same of $E_{\mathfrak{ge}}K$.

Since K/k is tamely ramified, the conductor of constants ([2, Theorem 3.1]) is the minimum η such that $K \subseteq k(\Lambda_N)_\eta$. In the notation of Theorem 3.1 and Remark 3.2 of [2], we have that $\eta = td$ where $t = f_\infty(K|k) = f_\infty(K|J) = \deg_K \mathfrak{p}_\infty = l^\lambda$, $d = f_\infty(EK|K) = f_\infty(E_{\mathfrak{ge}}K|K_{\mathfrak{ge}}) = |\mathcal{H}| = l^u$, and $J = K \cap {}_n k(\Lambda_N) = K \cap k(\Lambda_N) = K \cap E$. Therefore

$$\eta = l^\lambda \cdot l^u = l^{\lambda+u} = \psi.$$

Furthermore, in the same Theorem 3.1 of [2], we have

$$\eta = [K : J] = [K : K \cap E](= [E : K \cap E] = [EK : E] = [EK : K]).$$

We have, see [5]

$$
\begin{tikzcd}
E \arrow[rr, dash] \arrow[ddd, dash] & & EK \arrow[r, dash] \arrow[dl, dash] \arrow[dddd, dash, dashed] & (EK)_\eta = E_\eta = K_\eta \arrow[ddd, dash] \\
& K \arrow[ddl, dash] & & \\
& & & \\
E \cap K \arrow[rrr, dash] \arrow[d, dash] & & & (E \cap K)_\eta \arrow[d, dash] \\
k = k_\eta \cap E \arrow[rr, dash] & & k_\sigma \arrow[r, dash] & k_\eta
\end{tikzcd}
$$

If $EK \cap k_\eta = k_\sigma \subseteq k_\eta$, then $K \subseteq EK = k_\sigma E \subseteq k_\sigma k(\Lambda_N) = k(\Lambda_N)_\sigma$. Since η is the minimum, it follows that $\eta = \sigma$, and $EK = (EK)_\eta = E_\eta = K_\eta$. Therefore, the field of constants of EK is $\mathbb{F}_{q^\psi} = \mathbb{F}_{q^{l^{\lambda+u}}}$.

Proposition 5.19. *The field of constants of either $E_{\mathfrak{ge}}K$ or EK is $\mathbb{F}_{q^{l^{\lambda+u}}}$, where $l^\lambda = \deg_K \mathfrak{p}_\infty$ and $l^u = |\mathcal{H}|$.* $\square$

Furthermore, $[E_\eta : E] = \eta = \psi = l^{\lambda+u} = [K : J] = [K : E \cap K] = [E : E \cap K] = [EK : K] = [EK : E]$.

$$
\begin{tikzcd}
E \arrow[r, dash, bend left, "\eta"] \arrow[d, dash, "\eta"'] & EK = E_\eta = K_\eta \arrow[d, dash, "\eta"] \\
E \cap K \arrow[r, dash, "\eta"'] & K
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
EK \arrow[d, dash, "l^u"', "\mathcal{H}"] \arrow[dd, dash, bend left=60, "\eta = l^{\lambda+u}"] \\
E^{\mathcal{H}}K \arrow[d, dash, "l^\lambda"] \\
K
\end{tikzcd}
$$

We have that EK/E is an extension of constants of degree η, and the same is true is for the extension EK/K. The field of constants of $E^{\mathcal{H}}K$ is $\mathbb{F}_{q^{l^\lambda}}$, that is, the same of the field $K_{\mathfrak{ge}} = E^{\mathcal{H}}_{\mathfrak{ge}}K$.

Let $\mathfrak{P}_\infty$ be a prime above $\mathfrak{p}_\infty$ and denote $K_\infty := K_{\mathfrak{P}_\infty}$, $k_\infty := k_{\mathfrak{p}_\infty}$. Let $e_{\mathfrak{p}_\infty}(K|k) = l^\tau | q-1$, that is, $l^\tau | l^\rho$ and $\tau \leq \rho$. We have

$$[K_\infty : k_\infty] = e_\infty(K|k) f_\infty(K|k) = l^\tau \cdot l^\lambda = l^{\tau+\lambda}.$$

Let $k_\infty \subseteq F \subseteq K_\infty$ be the inertia field of K_∞/k_∞, that is, $F := K_\infty^{I_\infty(K_\infty/k_\infty)}$ and $[F : k_\infty] = l^\lambda$. We have that F/k_∞ is unramified.

For each local field, there exists a unique unramified extension of each degree (see [9, Teorema 17.3.37]). Therefore $F = k_\infty \mathbb{F}_{q^{l^\lambda}}$, that is, F/k_∞ is an "extension of constants" of degree l^λ. More precisely,

$$\mathcal{O}_{\mathfrak{P}_\infty}/\mathfrak{P}_\infty \cong \mathbb{F}_{q^{l^\lambda}} \quad \text{and} \quad F^* = \langle \pi_F \rangle \times \mathbb{F}^*_{q^{l^\lambda}} \times U_F^{(1)},$$

where $\pi_F = \pi_\infty = 1/T$ is a prime element of F^*.

$$\begin{tikzcd}[arrows=dash] K_\infty \arrow[d] \\ F = k_\infty \mathbb{F}_{q^{l^\lambda}} \arrow[d] \\ k_\infty \end{tikzcd}$$

Let $K_\infty^* = \langle \tilde{\Pi} \rangle \times \mathbb{F}^*_{q^{l^\lambda}} \times U^{(1)}_{K_\infty}$ where $\tilde{\Pi}$ is a prime element of K_∞^*. Now, (see [9, Teorema 17.3.14]) we have

$$1 = v_{K_\infty}(\tilde{\Pi}) = \frac{e_\infty(K|k)}{[K_\infty : k_\infty]} v_{k_\infty}(\mathrm{N}_{K_\infty/k_\infty}(\tilde{\Pi})) = \frac{1}{f_\infty(K|k)} v_{k_\infty}(\mathrm{N}_{K_\infty/k_\infty}(\tilde{\Pi})).$$

Hence $v_{k_\infty}(\mathrm{N}_{K_\infty/k_\infty}(\tilde{\Pi})) = l^\lambda$.

We have

$$\begin{tikzcd}[arrows=dash] K_\infty \arrow[d, "e_\infty = l^\tau"] \\ F \arrow[d, "f_\infty = l^\lambda"] \\ k_\infty \end{tikzcd}$$

$e_\infty = l^\tau | l^\rho$ and $l^\rho | q-1$. Therefore the l^τ-th primitive the unity ζ_{l^τ} belongs to F. Since $\mathcal{H} \subseteq I_\infty(E_{\mathfrak{gex}}/k)$, we also have $u \leq \tau$.

It follows that K_∞/F is a Kummer extension, say $K_\infty = F(\sqrt[l^\tau]{Y})$ for some $Y \in F^* = \langle \pi_\infty \rangle \times \mathbb{F}^*_{q^{l^\lambda}} \times U^{(1)}_F$.

Let $Y = \pi_\infty^s \Lambda w$, with $s \in \mathbb{Z}$, $\Lambda \in \mathbb{F}^*_{q^{l^\lambda}}$, and $w \in U^{(1)}_F$. Since $\gcd(l,p) = 1$, we have $U^{(1)}_F = (U^{(1)}_F)^{l^\tau}$. We write $s = \alpha l^\tau + r$, with $0 \leq r < l^\tau$. Then, if $w_0^{l^\tau} = w$, we have

$$K_\infty = F\big(\sqrt[l^\tau]{\pi_\infty^{\alpha l^\tau + r} \Lambda w_0^{l^\tau}}\big) = F\big(\sqrt[l^\tau]{\pi_\infty^r \Lambda}\big).$$

Let $r = l^b r_0$, with $0 \leq b < \tau$ and $\gcd(l, r_0) = 1$. Set $F_1 := F\big(\sqrt[l^b]{\pi_\infty^{l^b r_0} \Lambda}\big) = F\big(\sqrt[l^b]{\Lambda}\big)$. Thus F_1/F is unramified, $F \subseteq F_1 \subseteq K_\infty$ and K_∞/F is totally ramified. It follows that $F_1 = F$ and that $b = 0$, that is, $\gcd(r, l) = 1$.

Therefore $K_\infty = F\big(\sqrt[l^\tau]{\pi_\infty \theta}\big)$ for some $\theta \in \mathbb{F}^*_{q^{l^\lambda}}$. Set $\phi := \sqrt[l^\tau]{\pi_\infty \theta}$. Then $\phi^{l^\tau} = \pi_\infty \theta$. Hence

$$l^\tau v_{K_\infty}(\phi) = v_{K_\infty}(\phi^{l^\tau}) = v_{K_\infty}(\pi_\infty \theta) = v_{K_\infty}(\pi_\infty) = e(K_\infty|F) v_\infty(\pi_\infty) = l^\tau \cdot 1.$$

It follows that $v_{K_\infty}(\phi) = 1$. Therefore we may take $\phi = \tilde{\Pi}$ as a prime element of F.

Now we consider $E_\infty = k_\infty\big(\sqrt[l^\tau]{\pi_\infty\mu}\big)$ for some $\mu \in \mathbb{F}_q^*$. We have

$$\begin{tikzcd}
E_\infty \arrow[r, dash, "\eta=l^{\lambda+u}"] \arrow[d, dash, "l^u"'] \arrow[dd, dash, bend right=60, "l^\tau"'] & (EK)_\infty = E_\infty K_\infty \arrow[d, dash, "l^u"] \\
E_\infty \cap K_\infty \arrow[r, dash, "l^{\lambda+u}"] \arrow[d, dash, "l^{\tau-u}"'] & K_\infty \\
k_\infty \arrow[ur, dash, "l^{\tau+\lambda}"'] &
\end{tikzcd}$$

Because $E_{\mathfrak{ger}} \subseteq k(\Lambda_N)$ is cyclotomic, the field of constants of E_{H^+} is also $\mathbb{F}_q$.

As before, $\vartheta := \sqrt[l^\tau]{\pi_\infty\mu}$ is a prime element of E_∞. We have

$$X^{l^\tau} - \pi_\infty\mu = \prod_{i=0}^{l^\tau-1}\big(X - \zeta_{l^\tau}^i\vartheta\big).$$

Hence

$$\prod_{i=0}^{l^\tau-1}\big(-\zeta_{l^\tau}^i\vartheta\big) = (-1)^{l^\tau}\prod_{i=0}^{l^\tau-1}\big(\zeta_{l^\tau}^i\vartheta\big) = (-1)^{l^\tau}\,\mathrm{N}_{E_\infty/k_\infty}\vartheta = -\pi_\infty\mu.$$

Thus

$$\mathrm{N}_{E_\infty/k_\infty}\vartheta = (-1)^{l^\tau+1}\mu\pi_\infty.$$

Since the field of constants of E_{H^+} is $\mathbb{F}_q$, then, from Theorem 4.6, there exists an element of degree 1 in E_∞^* satisfying

$$\phi_{E_\infty}(X) = \phi_\infty(\mathrm{N}_{E_\infty/k_\infty}(X)) = 1.$$

Set $X = \vartheta^s\alpha w$ with $s \in \mathbb{Z}, \alpha \in \mathbb{F}_q^*, w \in U_{E_\infty}^{(1)}$. Since $\deg X = \deg \pi_\infty v_\infty(X) = 1\cdot s = s$, it follows that $s = 1$. Furthermore $\mathrm{N}_{E_\infty/k_\infty}(w) \in U_\infty^{(1)} = \big(U_\infty^{(1)}\big)^{l^\tau}$ and $\mathrm{N}_{E_\infty/k_\infty}(\alpha) = \alpha^{l^\tau}$. Therefore

$$1 = \phi_{E_\infty}(X) = \phi_\infty(\mathrm{N}_{E_\infty/k_\infty}(X)) = \phi_\infty\Big(\big((-1)^{l^\tau+1}\mu\pi_\infty\big)\alpha^{l^\tau}u^{l^\tau}\Big) = (-1)^{l^\tau+1}\mu\alpha^{l^\tau},$$

where $u \in U_\infty^{(1)}$. Therefore $-\mu = (-\alpha)^{-l^\tau} \in \big(\mathbb{F}_q^*\big)^{l^\tau}$ and since $\tau < n$, it follows that $-1 \in \big(\mathbb{F}_q^*\big)^{l^\tau}$. Thus $\mu \in \big(\mathbb{F}_q^*\big)^{l^\tau}$ and $E_\infty = k_\infty\big(\sqrt[l^\tau]{\pi_\infty\mu}\big) = k_\infty\big(\sqrt[l^\tau]{\pi_\infty}\big)$.

We obtain

$$E_\infty K_\infty = K_\infty k_\infty\big(\sqrt[l^\tau]{\pi_\infty}\big) = F\big(\sqrt[l^\tau]{\pi_\infty\theta},\ \sqrt[l^\tau]{\pi_\infty}\big) = F\big(\sqrt[l^\tau]{\pi_\infty\theta},\ \sqrt[l^\tau]{\theta}\big) = K_\infty\big(\sqrt[l^\tau]{\theta}\big).$$

The field of the constants of EK is $\mathbb{F}_{q^{l^{\lambda+u}}}$, therefore $[\mathbb{F}_{q^{l^\lambda}}\big(\sqrt[l^\tau]{\theta}\big) : \mathbb{F}_{q^{l^\lambda}}] = l^u$. From Theorem 3.5 we have that $\theta \in \big(\mathbb{F}_{q^{l^\lambda}}^*\big)^{l^{\tau-u}} \setminus \big(\mathbb{F}_{q^{l^\lambda}}^*\big)^{l^{\tau-u+1}}$.

In short, $K_\infty = F\big(\sqrt[l^\tau]{\pi_\infty\theta}\big)$ with $\theta \in \big(\mathbb{F}_{q^{l^\lambda}}^*\big)^{l^{\tau-u}} \setminus \big(\mathbb{F}_{q^{l^\lambda}}^*\big)^{l^{\tau-u+1}}$, $F = k_\infty\mathbb{F}_{q^{l^\lambda}}$, and $\tilde{\Pi} = \sqrt[l^\tau]{\theta\pi_\infty}$.

The irreducible polynomial of $\tilde{\Pi}$ over F is $X^{l^\tau} - \theta\pi_\infty \in F[X]$. Then $X^{l^\tau} - \theta\pi_\infty = \prod_{j=0}^{l^\tau - 1}(X - \zeta_{l^\tau}^j \tilde{\Pi})$ and

$$\begin{aligned}
\mathrm{N}_{K_\infty/F}\, \tilde{\Pi} &= \prod_{j=0}^{l^\tau-1} \zeta_{l^\tau}^j \tilde{\Pi} = (-1)^{l^\tau} \prod_{j=0}^{l^\tau-1} (-\zeta_{l^\tau}^j \tilde{\Pi}) = (-1)^{l^\tau}(-\theta\pi_{\pi_\infty}) = (-1)^{l^\tau+1}\theta\pi_\infty,\\
\mathrm{N}_{K_\infty/k_\infty}\, \tilde{\Pi} &= \mathrm{N}_{F/k_\infty}(\mathrm{N}_{K_\infty/F}\, \tilde{\Pi}) = \mathrm{N}_{F/k_\infty}((-1)^{l^\tau+1}\theta\pi_\infty)\\
&= (-1)^{(l^\tau+1)l^\lambda}(\mathrm{N}_{F/k_\infty}\,\theta)\pi_\infty^{l^\lambda}.
\end{aligned}$$

Now $\mathrm{N}_{F/k_\infty}\, \mathbb{F}_{q^{l^\lambda}}^* = \mathbb{F}_q^*$. Therefore, $\mathrm{N}_{F/k_\infty}\,\theta \in \left(\mathbb{F}_q^*\right)^{l^{\tau-u}} \setminus \left(\mathbb{F}_q^*\right)^{l^{\tau-u+1}}$.

Let's see the norm of an arbitrary element X of K_∞^*. Let $X = \tilde{\Pi}^s\Lambda\omega$ with $s \in \mathbb{Z}$, $\Lambda \in \mathbb{F}_{q^{l^\lambda}}^*$, and $\omega \in U_{K_\infty}^{(1)}$. Then

$$\begin{aligned}
&\mathrm{N}_{K_\infty/k_\infty}\, \omega = \omega_0 \in U_\infty^{(1)},\\
&\mathrm{N}_{K_\infty/k_\infty}\, \tilde{\Pi}^s = (-1)^{(l^\tau+1)l^\lambda s}\xi^s\pi_\infty^{l^\lambda s} \quad \text{with} \quad \xi = \mathrm{N}_{F/k_\infty}\,\theta \in \left(\mathbb{F}_q^*\right)^{l^{\tau-u}} \setminus \left(\mathbb{F}_q^*\right)^{l^{\tau-u+1}},\\
&\mathrm{N}_{K_\infty/k_\infty}\, \Lambda = \mathrm{N}_{F/k_\infty}(\mathrm{N}_{K_\infty/F}\,\Lambda) = \mathrm{N}_{F/k_\infty}\,\Lambda^{l^\tau} = (\mathrm{N}_{F/k_\infty}\,\Lambda)^{l^\tau},\\
&\qquad \text{and} \quad \mathrm{N}_{K_\infty/k_\infty}\, \mathbb{F}_{q^{l^\lambda}}^* = (\mathbb{F}_q^*)^{l^\tau}.
\end{aligned}$$

Therefore

$$\mathrm{N}_{K_\infty/k_\infty}\, X = (-1)^{(l^\tau+1)l^\lambda s}\xi^s\pi_\infty^{l^\lambda s}\big(\mathrm{N}_{F/k_\infty}\,\Lambda\big)^{l^\tau}\omega_0,$$

and

$$\begin{aligned}
\phi_{K_\infty}(X) = \phi_\infty(\mathrm{N}_{K_\infty/k_\infty}(X)) &= (-1)^{(l^\tau+1)l^\lambda s}\xi^s\big(\mathrm{N}_{F/k_\infty}\,\Lambda\big)^{l^\tau}\\
&= \big((-1)^{l^\lambda}\xi\big)^s\big[(-1)^{l^\lambda s}(\mathrm{N}_{F/k_\infty}\,\Lambda)\big]^{l^\tau}.
\end{aligned}$$

Now

$$X \in \ker\phi_{K_\infty} \iff \big((-1)^{l^\lambda}\xi\big)^s\big[(-1)^{l^\lambda s}(\mathrm{N}_{F/k_\infty}\,\Lambda)\big]^{l^\tau} = 1.$$

In other words,

$$X \in \ker\phi_{K_\infty} \iff \text{there exists } \Lambda \in \mathbb{F}_{q^{l^\lambda}}^* \text{ such that } \big(\mathrm{N}_{F/k_\infty}\,\Lambda\big)^{l^\tau} = \Big(\pm\,\xi^{-1}\Big)^s.$$

Since $l^\rho | q-1$ with $\rho \geq 1$ and $l^n \nmid q-1$, we have $n \geq 2$. Thus $-1 \in (\mathbb{F}_q^*)^{l^{\tau-u+1}}$. Therefore $\xi^s \in (\mathbb{F}_q^*)^{l^\tau}$. Since $\xi \in \left(\mathbb{F}_q^*\right)^{l^{\tau-u}} \setminus \left(\mathbb{F}_q^*\right)^{l^{\tau-u+1}}$, it follows that $l^u | s$ and l^u is the minimum positive integer with this property. For such X, we have

$$\deg_{K_\infty} X = \deg_{K_\infty}\mathfrak{P}_\infty v_{K_\infty}(X) = l^\lambda \cdot l^u = l^{\lambda+u}.$$

Therefore, the field of constants of K_{H^+} is $\mathbb{F}_{l^{\lambda+u}}$, the same of $E_{\mathfrak{ger}}K$.

We have obtained our first main result

Theorem 5.20. *Let K/k be a geometric cyclic extension of degree l^n with l a prime number and $n \geq 1$. Then, if E is given by* (2.1)*, we have*

$$K_{\mathfrak{ger}} = E_{\mathfrak{ger}}K. \qquad \square$$

6. General finite abelian extensions

The main key to obtain the extended genus field of a finite abelian extension, is the following:

Lemma 6.1. *Let E_1 and E_2 be two finite cyclotomic extensions of k and let $E = E_1E_2$. Then $E_{\mathfrak{gex}} = (E_1)_{\mathfrak{gex}}(E_2)_{\mathfrak{gex}}$.*

Proof. See [2, Proposition 6.3]. □

As a consequence we obtain our final main result.

Theorem 6.2. *Let K/k be any geometric finite abelian extension. Then if E is given in* (2.1)*, we have that*

$$K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K.$$

Proof. Let $K = K_1 \cdots K_s$ where each K_j/k is a cyclic extension of prime power degree. Let $E = E_1 \cdots E_s$ with each E_j be given in (2.1). Then, from Lemma 6.1, we obtain

$$E_{\mathfrak{gex}} = (E_1)_{\mathfrak{gex}} \cdots (E_s)_{\mathfrak{gex}}.$$

Therefore, from Theorem 5.20, it follows that

$$K_{\mathfrak{gex}} \subseteq E_{\mathfrak{gex}}K = (E_1)_{\mathfrak{gex}}K_1 \cdots (E_s)_{\mathfrak{gex}}K_s = (K_1)_{\mathfrak{gex}} \cdots (K_s)_{\mathfrak{gex}} \subseteq K_{\mathfrak{gex}}.$$

Hence $K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K$. □

We obtain some consequences from Theorem 6.2. We consider K/k a geometric abelian finite extension. We have $K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K$ and $K_{\mathfrak{ge}} = E_{\mathfrak{ge}}^{\mathcal{H}}K$. Let $K \subseteq {}_nk(\Lambda_N)_m$, $\mathcal{M} = L_nk_m$ and $E = \mathcal{M}K \cap k(\Lambda_N)$. For any finite abelian extension L/J and any prime $\mathfrak{p}$ of J, we denote by $e_{\mathfrak{p}}^*(L|J)$ to the tame ramification index of the prime $\mathfrak{p}$ in the extension L/J, namely, if $e_{\mathfrak{p}}(L|J) = p^{\beta}\alpha$ with $\gcd(\alpha, p) = 1$, then $e_{\mathfrak{p}}^*(L|J) = \alpha$. The set of tame ramification indexes is multiplicative.

Lemma 6.3. *We have $e_\infty(E|k) = e_\infty^*(K|k)$.*

Proof. First note that if $k \subseteq J \subseteq \mathcal{M}$ then $e_\infty^*(\mathcal{M}|J) = e_\infty^*(J|k) = 1$. Now we consider

$$\begin{tikzcd} K \arrow[r, dash, "e_\infty^*=1"] \arrow[d, dash] & K\mathcal{M} \arrow[d, dash] \\ K\cap\mathcal{M} \arrow[r, dash, "e_\infty^*=1"] & \mathcal{M} \end{tikzcd}$$

Since $e_\infty^*(K\mathcal{M}|K) | e_\infty^*(\mathcal{M}|K \cap \mathcal{M})$, it follows that $e_\infty^*(K\mathcal{M}|K) = e_\infty^*(\mathcal{M}|K \cap \mathcal{M}) = 1$. Therefore $e_\infty^*(K|K \cap \mathcal{M}) = e_\infty^*(K\mathcal{M}|\mathcal{M})$.

We have

$$\begin{tikzcd} E \arrow[rrr, dash] \arrow[dd, dash] & & & K\mathcal{M} \arrow[dd, dash] \arrow[dl, dash] \\ & & K \arrow[dll, dash] \arrow[dl, dash] & \\ k \arrow[r, dash] & K\cap\mathcal{M} \arrow[rr, dash] & & \mathcal{M} \end{tikzcd}$$

$$\begin{aligned} e_\infty(K\mathcal{M}|k) &= e_\infty(K\mathcal{M}|\mathcal{M})e_\infty(\mathcal{M}|k) \\ &= e_\infty(K\mathcal{M}|E)e_\infty(E|k). \end{aligned}$$

It follows

$$e_\infty^*(K\mathcal{M}|k) = e_\infty^*(K\mathcal{M}|\mathcal{M}) = e_\infty^*(K|K\cap\mathcal{M}) = e_\infty^*(E|k) = e_\infty(E|k).$$

Now

$$e_\infty^*(K|K\cap\mathcal{M}) = e_\infty^*(K|K\cap\mathcal{M})e_\infty^*(K\cap\mathcal{M}|k) = e_\infty^*(K|k) = e_\infty(E|k).$$

□

Theorem 6.4. *We have* $[K_{\mathfrak{gex}} : K_{\mathfrak{ge}}] = [E_{\mathfrak{gex}} : E_{\mathfrak{ge}}^{\mathcal{H}}] = [E_{\mathfrak{gex}} : E_{\mathfrak{ge}}]\cdot|\mathcal{H}|$. *In particular* $[K_{\mathfrak{gex}} : K_{\mathfrak{ge}}]|q-1$. *It also holds*

$$f_\infty(K_{\mathfrak{gex}} : K_{\mathfrak{ge}}) = |\mathcal{H}|, \quad e_\infty(K_{\mathfrak{gex}} : K_{\mathfrak{ge}}) = [E_{\mathfrak{gex}} : E_{\mathfrak{ge}}].$$

Furthermore, the field of constants of $K_{\mathfrak{ge}}$ *is* $\mathbb{F}_{q^{\deg_K \mathfrak{p}_\infty}}$, $\deg_K \mathfrak{p}_\infty = f_\infty(K|k)$ *and the field of constants of both,* $K_{\mathfrak{gex}}$ *and* K_{H^+} *is* $\mathbb{F}_{q^{|\mathcal{H}|\deg_K \mathfrak{p}_\infty}}$ *and we have* $|\mathcal{H}|\deg_K \mathfrak{p}_\infty = f_\infty(EK|k)$.

Proof. We have

$$\begin{tikzcd}
E \arrow[r, dash] \arrow[d, dash] & EK \arrow[d, dash] \\
E^{\mathcal{H}} \arrow[r, dash] \arrow[d, dash] & E^{\mathcal{H}}K \arrow[d, dash] \\
E\cap K \arrow[r, dash] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{\mathfrak{gex}} \arrow[r, dash] \arrow[d, dash] & K_{\mathfrak{gex}} = E_{\mathfrak{gex}}K \arrow[d, dash] \\
E_{\mathfrak{ge}} \arrow[r, dash] \arrow[d, dash] & E_{\mathfrak{ge}}K \arrow[d, dash] \\
E_{\mathfrak{ge}}^{\mathcal{H}} \arrow[r, dash] & K_{\mathfrak{ge}} = E_{\mathfrak{ge}}^{\mathcal{H}}K
\end{tikzcd}$$

It follows that $[K_{\mathfrak{gex}} : K_{\mathfrak{ge}}] = [E_{\mathfrak{gex}} : E_{\mathfrak{ge}}^{\mathcal{H}}]|q-1$.

Now, since $E_{\mathfrak{ge}}K/(E_{\mathfrak{ge}}K)^{\mathcal{H}} = K_{\mathfrak{ge}}$ is an extension of constants of degree $|\mathcal{H}|$ ([2, Theorem 2.2]), in fact, $|\mathcal{H}| = f_\infty(E_{\mathfrak{ge}}K|K_{\mathfrak{ge}})$, we will see that the extension $K_{\mathfrak{gex}}/E_{\mathfrak{ge}}K$ is totally ramified.

$$\begin{tikzcd}
E_{\mathfrak{gex}} \arrow[rr, dash, "e_\infty^*=1"] \arrow[d, dash, "{e_\infty=[E_{\mathfrak{gex}}:E_{\mathfrak{ge}}]}"'] & & E_{\mathfrak{gex}}K \arrow[d, dash] \\
E_{\mathfrak{ge}} \arrow[rr, dash, "e_\infty^*=1"] \arrow[dd, dash] & & E_{\mathfrak{ge}}K \arrow[dl, dash, "e_\infty=1"] \\
 & K \arrow[dl, dash] & \\
k & &
\end{tikzcd}$$

We have $e_\infty(E_{\mathfrak{ge}}K|k) = e_\infty(K|k)$. Hence $e_\infty^*(E_{\mathfrak{ge}}K|E_{\mathfrak{ge}}) = 1$. Similarly, we obtain $e_\infty^*(E_{\mathfrak{gex}}K|E_{\mathfrak{gex}}) = 1$.

Therefore $e_\infty(E_{\mathfrak{gex}}K|E_{\mathfrak{ge}}K) = e_\infty(E_{\mathfrak{gex}}|E_{\mathfrak{ge}}) = [E_{\mathfrak{gex}} : E_{\mathfrak{ge}}]$ and $K_{\mathfrak{gex}}/E_{\mathfrak{ge}}K$ is totally ramified.

Since $\deg_k \mathfrak{p}_\infty = 1$ and K/k is geometric, we obtain that $f_\infty(K|k) = \deg_K \mathfrak{p}_\infty$. We know that the field of constants of $K_{\mathfrak{ge}}$ is $\mathbb{F}_{q^{\deg_K \mathfrak{p}_\infty}}$.

Finally, $E_{\mathfrak{ge}}K/E_{\mathfrak{ge}}^{\mathcal{H}}K$ is an extension of constants of degree $|\mathcal{H}| = f_\infty(EK|K)$. Hence, the field of constants of both, $K_{\mathfrak{gex}}$ and K_{H^+}, is $\mathbb{F}_{q^{\deg_K \mathfrak{p}_\infty \cdot |\mathcal{H}|}} = \mathbb{F}_{q^{f_\infty(EK|k)}}$. $\square$

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