Spectrum occupies pseudospectrum for random matrices with diagonal deformation and variance profile

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Abstract

We consider $n \times n$ non-Hermitian random matrices with independent entries and a variance profile, as well as an additive deterministic diagonal deformation. We show that the support of the asymptotic eigenvalue distribution in the complex plane exactly coincides with the ε-pseudospectrum in the consecutive limits $n \to \infty$ and $\varepsilon \to 0$. Furthermore, we provide a description of this support in terms of a single real-valued function on the complex plane. As a level set of this locally real analytic function, the spectral edge is a real analytic variety of dimension at most one.

1 Introduction

The empirical spectral distribution of a non-Hermitian random matrix typically converges to a non-random probability distribution σ, the limiting spectral measure, on the complex plane as its dimension tends to infinity. The most prominent instance of this phenomenon is the circular law, stating that the eigenvalues of am appropriately normalised matrix X with centered i.i.d. entries converges to the uniform distribution on the complex unit disk [25, 8] (see [47] for optimal moment conditions and [17] for a review).

When a deterministic matrix A is added to X, the associated asymptotic distribution σ and its support, the asymptotic spectrum, depend on A in a complicated manner [33]. This distribution can be realised as the Brown measure [18, 27], which is a generalisation of the spectral measure to non-normal operators, of an element in a W^*-probability space with faithful, tracial state $\langle \cdot \rangle$. In fact, $\sigma = \sigma_{A+\mathfrak{c}}$ is the Brown measure of the sum of an embedding of A into the W^*-probability space and a circular element $\mathfrak{c}$ that is $*$-free from A. In this case, the asymptotic spectrum $\operatorname{supp} \sigma_{A+\mathfrak{c}}$ coincides with the closure of $\mathbb{S} = \{\zeta \in \mathbb{C} : \langle (A-\zeta)^{-1}(A^*-\bar{\zeta})^{-1}\rangle > 1\}$. This observation goes back to [33] in the random matrix setting, has been proven in the infinite dimensional situation in [16] for normal A and extended to general A in [15, 49]. Subsequently, the regularity of $\sigma_{A+\mathfrak{c}}$ has been analysed. The measure is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$ [10] and the density is strictly positive and real analytic on $\mathbb{S}$ [49]. Moreover, the density typically possesses a jump discontinuity at the edge of $\mathbb{S}$ [22].

Instead of adding A to the matrix X with i.i.d. entries we may also introduce more structure into the randomness $X = (x_{ij})_{i,j=1}^n$. When the entries x_{ij} remain independent but admit differing distributions with entry dependent variances $s_{ij} := \mathbb{E}|x_{ij}|^2$, the density σ is still supported on a disk, radially symmetric and has a jump at the edge, but is in general not constant anymore on its support [19, 3]. This remains true when the entries of X are correlated with a decaying correlation structure [6]. In this work we consider a case in which a nontrivial structure of the randomness and a deterministic deformation A are present. Our randomness X has independent entries with variance

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profile $S = (s_{ij})_{i,j=1}^n$ and A is diagonal. In particular, the matrix $X + A$ considered in this work belongs to the class of Kronecker matrices discussed in [5].

Non-normal random matrices and a detailed understanding of their spectra play an important role in many applications, ranging from the stability analysis of food webs [2, 35, 29] and quantum chaotic scattering [24] to investigating the transition to chaos in neuronal networks [44, 38]. A persistent challenge in the analytic study of such matrices X is their spectral instability, i.e. the fact that tiny changes in the matrix entries may lead to large deviations of the eigenvalues. To remedy this issue the ε-pseudospectrum $\mathrm{Spec}_\varepsilon(X)$ is introduced (see e.g. [48] for an overview), which is stable under perturbations, monotonically increasing in $\varepsilon > 0$ and contains the spectrum, namely $\bigcap_{\varepsilon>0} \mathrm{Spec}_\varepsilon(X) = \mathrm{Spec}(X)$. Especially for high dimensional $X = X_n \in \mathbb{C}^{n\times n}$ the dependence of $\mathrm{Spec}_\varepsilon(X_n)$ on ε may be very unstably dependent on n (see e.g. [36, Section 11.6.3] for the example of a shift operator). In particular, the eigenvalues may accumulate in a much smaller area than $\mathrm{Spec}_0^\infty := \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \mathrm{Spec}_\varepsilon(X_n)$. In the case of Töplitz matrices A with a very small added randomness X for example, the spectrum concentrates on curves given by the image of the unit circle by the Toeplitz symbol inside Spec_0^∞ [26, 39, 9, 42].

In contrast, our main result shows that for matrices with independent entries and diagonal deformation the set Spec_0^∞ coincides with the support of the limiting spectral measure σ, i.e., that the spectrum occupies the entire ε-pseudospectrum in the consecutive limits $n \to \infty$ and $\varepsilon \downarrow 0$. Here we assume that $s_{ij} = \frac{1}{n} s(\frac{i}{n}, \frac{j}{n})$ and $A = (a(\frac{i}{n})\delta_{ij})_{i,j=1}^n$ both have piecewise continuous limiting profiles $s : [0, 1]^2 \to \mathbb{R}$ and $a : [0, 1] \to \mathbb{C}$ to ensure the existence of the limit as $n \to \infty$. In particular, Spec_0^∞ stably depends on the expectation profile a and the variance profile s. Furthermore, we show that positivity of the variance profile implies that σ is given by a bounded probability density on the complex plane, which is real analytic and strictly positive on an open domain $\mathbb{S} := \{\beta < 0\} \subset \mathbb{C}$ with boundary $\partial\mathbb{S} = \{\beta = 0\}$, where $\beta : \mathbb{C} \to \mathbb{R}$ is a continuous function that is real analytic in a neighbourhood of $\partial\mathbb{S}$. From this we obtain that $\partial\mathbb{S}$ is a real analytic variety of dimension at most 1. The density σ vanishes outside the closure of $\mathbb{S}$ and typically has a jump discontinuity at the spectral edge $\partial\mathbb{S}$, except at critical points of β, where it vanishes continuously.

These results about the measure σ have been shown in the case of constant variance profile s, e.g. when the entries of X are i.i.d., and for general A in [49, 10, 22]. For constant s our choice of β simplifies to $\beta(\zeta) = \frac{1}{n} \mathrm{Tr}|A - \zeta|^{-2}$, which coincides with the analogous quantity in [49]. In this situation, the study of properties of σ and its relationship to β rely on solving a ζ-dependent family of two coupled scalar equations, called Dyson equation, for two positive functions $v_1, v_2 : \mathbb{S} \to \mathbb{R}$ that vanish at the boundary $\partial\mathbb{S}$. The Dyson equation is a self-consistent equation for the diagonal resolvent entries of the Hermitization of $X + A$ in the $n \to \infty$ limit. In the random matrix setup, the Hermitization idea goes back to [25]. See e.g. [11] for its use in the analysis of Brown measures. For non-constant s, the Dyson equation is no longer finite-dimensional in the $n \to \infty$ limit. Instead, with profiles a and s, it becomes a system of two ζ-dependent equations of the form

$$\frac{1}{v_1(\zeta)} = Sv_2(\zeta) + \frac{|\zeta - a|^2}{S^*v_1(\zeta)}, \qquad \frac{1}{v_2(\zeta)} = S^*v_1(\zeta) + \frac{|\zeta - a|^2}{Sv_2(\zeta)}, \tag{1.1}$$

for two positive functions $v_1, v_2 : \mathbb{S} \to L^\infty[0, 1]$, where $S, S^* : L^\infty[0, 1] \to L^\infty[0, 1]$ are defined through $(Sf)(x) := \int s(x, y)f(y)\mathrm{d}y$ and $(S^*f)(x) := \int s(y, x)f(y)\mathrm{d}y$. From v_1 the probability density σ inside $\mathbb{S}$ is derived through

$$\sigma(\zeta) := -\partial_\zeta \left\langle \frac{v_1(\zeta)(a - \zeta)}{\pi S^*v_1(\zeta)} \right\rangle, \tag{1.2}$$

where $\langle u\rangle := \int u(x)\mathrm{d}x$. Taking the derivative in (1.2) yields a quadratic form of a non-symmetric operator. The main idea for the proof of positivity of σ in the bulk regime, i.e. on $\mathbb{S}$, is to transform the formula for σ into the quadratic form of a strictly positive operator. Near the spectral edge $\partial\mathbb{S}$, the behaviour of σ is governed by the quantity β from the definition of $\mathbb{S}$. In fact, $\beta(\zeta)$ coincides locally around the spectral edge with the isolated eigenvalue of the non-symmetric operator B_ζ that is closest to zero, where $B_\zeta : L^\infty[0, 1] \to L^\infty[0, 1]$ is defined through $B_\zeta f := |a - \zeta|^2 f - Sf$. A consequence that we derive from this insight is that the jump height of the edge discontinuity of σ at the spectral edge is proportional to $|\partial_\zeta\beta|^2$. This requires a careful singular expansion of v_1, v_2 at the spectral edge, where the Dyson equation (1.1) is unstable. A signature of this instability is that B_ζ is singular for $\zeta \in \partial\mathbb{S}$ and the main contributions to v_1 and v_2 near $\partial\mathbb{S}$ point into the singular eigendirections of B_ζ. Owing to the dependence of v_1, v_2 and β on s, treating non-constant s and a is a recurring challenge for the analysis in both regimes.

2 Main results

In this section, we state our assumptions and the main results. In the following, we take $n \in \mathbb{N}$ and write $[\![n]\!]$ for the discrete interval $[\![n]\!] = \{1, \ldots, n\}$.

A1 *Independent, centered entries:* The entries of $X = (x_{ij})_{i,j\in[\![n]\!]}$ are independent and centered, i.e. $\{x_{ij} \colon i, j \in [\![n]\!]\}$ is a family of independent random variables and $\mathbb{E}x_{ij} = 0$.

A2 *Finite moments:* All moments of the entries of $\sqrt{n}X$ are finite, i.e. there is a sequence of positive constants C_ν such that

$$\mathbb{E}\,|x_{ij}|^\nu \le C_\nu n^{-\nu/2}, \tag{2.1}$$

for all $i, j \in [\![n]\!]$ and $\nu \in \mathbb{N}$.

A3 *Anticoncentration of entries:* There is a constant $b \in (0, 1)$ such that, for all $i, j \in [\![n]\!]$,

$$\mathbb{P}\Big(b^{-1} \ge \sqrt{n}|x_{ij} - y_{ij}| \ge b\Big) \ge b,$$

where y_{ij} is an independent copy of x_{ij}.

A4 *Block-continuous variance profile:* For some $K \in \mathbb{N}$, let $I_1, \ldots, I_K \subset [0, 1]$ be disjoint intervals of positive length such that $I_1 \cup \ldots \cup I_K = [0, 1]$. Let $s \colon [0, 1]^2 \to (0, \infty)$ and $a \colon [0, 1] \to \mathbb{C}$ be functions such that $s|_{I_l\times I_k}$ and $a|_{I_l}$ have continuous extensions to the closures $\overline{I_l} \times \overline{I_k}$ and $\overline{I_l}$, respectively, for all l, $k \in [\![K]\!]$. Moreover, we suppose that there is a constant $c > 0$ such that

$$\inf_{x,y\in[0,1]} s(x, y) \ge c. \tag{2.2}$$

The constants in the assumptions **A1** – **A4** are model parameters and independent of n and, therefore, the respective estimates are uniform in n.

The next proposition shows that the empirical spectral measure of non-Hermitian random matrices with independent entries, a variance profile and a diagonal expectation has a deterministic limit as the matrix size tends to infinity. We state and prove this result here, as we did not find it explicitly stated in the literature, although the tools leading to it and closely related results are well-known in the community. The independence of the limit from the entry distributions was shown in [47, Appendix C] and [32, Theorem 1.3]. When X is a Ginibre matrix, the convergence of the empirical spectral measure was proved in [43, Theorem 6] and for an X with i.i.d. entries, in [47, Theorem 1.17].

Proposition 2.1 (Convergence of empirical spectral distribution)**.** *Let the functions $s \colon [0, 1]^2 \to (0, \infty)$ and $a \colon [0, 1] \to \mathbb{C}$ satisfy **A4**. For any $n \in \mathbb{N}$, we set $A_n := (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n\times n}$ with $a_{ij} := a(\frac{i}{n})\delta_{ij}$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random matrices such that, for each $n \in \mathbb{N}$ the random matrix $X_n \in \mathbb{C}^{n\times n}$ with $X_n = (x_{ij})_{i,j\in[\![n]\!]}$ satisfies **A1**, **A2** and **A3** as well as $\mathbb{E}|x_{ij}|^2 = \frac{1}{n}s(\frac{i}{n}, \frac{j}{n})$ for all $i, j \in [\![n]\!]$.*

Then there exists a unique probability measure σ on $\mathbb{C}$ such that the empirical spectral distribution $\frac{1}{n}\sum_{\zeta\in\mathrm{Spec}(X_n+A_n)} \delta_\zeta$ converges to σ weakly in probability as $n \to \infty$, i.e. for every bounded, continuous function $f \colon \mathbb{C} \to \mathbb{C}$ and $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\bigg(\bigg|\frac{1}{n} \sum_{\zeta\in\mathrm{Spec}(X_n+A_n)} f(\zeta) - \int_{\mathbb{C}} f(\zeta)\sigma(\mathrm{d}\zeta)\bigg| > \varepsilon\bigg) = 0.$$

Here the sum $\sum_{\zeta\in\mathrm{Spec}(X_n+A_n)}$ is over all eigenvalues of $X_n + A_n$, counted with multiplicity.

The proof of Proposition 2.1 is presented in Section 7.1 below.

Remark 2.2. *Under the stronger assumptions of Lipschitz-continuity of a and s in Theorem 2.4 below, Proposition 2.1 can be strengthened to a local law. That is, f in Proposition 2.1 can be replaced by $f_{\zeta_0}(\zeta) := n^{2\alpha} f(n^\alpha(\zeta - \zeta_0))$ for $\alpha \in (0, 1/2)$ and $\zeta_0 \in \mathbb{S}$, where $\mathbb{S}$ denotes the spectral bulk introduced in (2.8) below. Moreover, ε can be chosen to tend to zero, when n tends to infinity, i.e. $\varepsilon \equiv \varepsilon_n = n^{-1+2\alpha+o(1)}$.*

The condition **A3** is used to control the smallest singular value of $X + A - \zeta$ for $\zeta \in \mathbb{C}$ through the results from [34, 32], where we omit the index n from our notation in $X = X_n$ and $A = A_n$. For all other aspects of our proofs, the weaker condition $n^{-1} \lesssim \min_{i,j\in[\![n]\!]} \mathbb{E}|x_{ij}|^2$ is sufficient, which is explicitly listed above as (2.2).

Definition 2.3. *The probability measure σ from Proposition 2.1 is called* limiting spectral measure *associated with s and a.*

The next theorem states that the pseudospectrum of the $n \times n$-matrix $X + A$ is asymptotically given by the support of the measure σ from Proposition 2.1 which coincides with the spectrum of $X + A$ by Proposition 2.1 in the limit $n \to \infty$. We first introduce the pseudospectrum of a matrix. For any $\varepsilon > 0$, the *ε-pseudospectrum* of a matrix $R \in \mathbb{C}^{n\times n}$ is defined as the set

$$\operatorname{Spec}_\varepsilon(R) := \{\zeta \in \mathbb{C} \colon \|(R - \zeta)^{-1}\| \ge \varepsilon^{-1}\}. \tag{2.3}$$

Note that $\operatorname{Spec}_\varepsilon(R)$ is monotonically increasing in ε and $\operatorname{Spec}(R) = \cap_{\varepsilon>0} \operatorname{Spec}_\varepsilon(R)$.

Furthermore, for a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of sets we use the customary definitions

$$\liminf_{n\to\infty} \Omega_n := \bigcup_{N\in\mathbb{N}} \bigcap_{n\ge N} \Omega_n, \qquad \limsup_{n\to\infty} \Omega_n := \bigcap_{N\in\mathbb{N}} \bigcup_{n\ge N} \Omega_n.$$

Theorem 2.4 (Spectrum occupies pseudospectrum)**.** *Let $s \colon [0,1]^2 \to (0,\infty)$, $a \colon [0,1] \to \mathbb{C}$ and $(I_k)_{k\in[\![K]\!]}$ be as in **A4**. Suppose that $s|_{I_l\times I_k}$ and $a|_{I_l}$ are Lipschitz-continuous functions for all l, $k \in [\![K]\!]$. Let X_n and A_n be as in Proposition 2.1. Then there exists a monotonically increasing family $(\operatorname{Spec}^\infty_\varepsilon(s,a))_{\varepsilon>0}$ of deterministic subsets of $\mathbb{C}$ such that, almost surely*[1]*,*

$$\limsup_{n\to\infty} \operatorname{Spec}_\varepsilon(X_n + A_n) \subset \operatorname{Spec}^\infty_\varepsilon(s,a) \subset \liminf_{n\to\infty} \operatorname{Spec}_{\varepsilon+\delta}(X_n + A_n) \tag{2.4}$$

hold for all $\varepsilon, \delta > 0$. Moreover, this family is right continuous, i.e. $\cap_{\delta>0} \operatorname{Spec}^\infty_{\varepsilon+\delta}(s,a) = \operatorname{Spec}^\infty_\varepsilon(s,a)$ and the limiting spectral measure σ from Proposition 2.1 satisfies

$$\bigcap_{\varepsilon>0} \operatorname{Spec}^\infty_\varepsilon(s,a) = \operatorname{supp}\sigma. \tag{2.5}$$

The proof of Theorem 2.4 is given in Section 7.2 below. We note that the sets $(\operatorname{Spec}^\infty_\varepsilon(s,a))_{\varepsilon>0}$ are monotonically increasing in ε. In particular, Theorem 2.4 implies that $\operatorname{Spec}(X_n + A_n)$ is eventually almost surely contained in a neighbourhood of $\operatorname{supp}\sigma$.

We now state additional properties of the limiting spectral measure σ and provide a characterisation of $\operatorname{supp}\sigma$ in terms of a and s. The measure σ itself will later also be given by a formula that only depends on a and s (cf. (6.3)). In the following, we write $L^\infty[0,1]$ for the space of essentially bounded functions on $[0,1]$, up to changes of zero measure, when this interval is equipped with the Lebesgue measure. We denote by S the integral operator on $L^\infty[0,1]$ with kernel s, i.e.

$$S : L^\infty[0,1] \to L^\infty[0,1], \quad (Sf)(x) := \int_0^1 s(x,y) f(y) \mathrm{d}y$$

For $u \in L^\infty[0,1]$, let $D_u : L^\infty[0,1] \to L^\infty[0,1]$ be the operator $D_u f := uf$ induced by multiplication with u. Using these definitions, for some bounded and measurable function $a \colon [0,1] \to \mathbb{C}$ and $\zeta \in \mathbb{C}$, we introduce the operator $B \equiv B_\zeta$ on $L^\infty[0,1]$ given by

$$B_\zeta := D_{|\zeta - a|^2} - S. \tag{2.6}$$

[1]We assume that all X_n for $n \in \mathbb{N}$ are realised on the same probability space.

Since B maps real-valued functions to real-valued functions, we obtain a function $\beta\colon \mathbb{C} \to \mathbb{R}$ defined through

$$\beta(\zeta) := \inf_{f>0} \sup_{g>0} \frac{\langle f\,, B_\zeta g\rangle}{\langle f\,, g\rangle} \tag{2.7}$$

for $\zeta \in \mathbb{C}$, where the infimum and supremum are taken over bounded functions $f, g\colon [0,1] \to (0,\infty)$ and

$$\langle f\,, g\rangle := \int_0^1 \overline{f(x)} g(x) \mathrm{d}x$$

is the scalar product on $L^2[0,1]$. The definition of β is motivated by the Birkhoff-Varga formula for the spectral radius of a matrix with positive entries [14]. In terms of β we define the set

$$\mathbb{S} := \{\zeta \in \mathbb{C}\colon \beta(\zeta) < 0\}, \tag{2.8}$$

whose closure coincides with $\operatorname{supp}\sigma$, as stated in the proposition below. We will see in Proposition 5.15 (i) that β is a continuous function and therefore $\mathbb{S}$ is an open set.

Proposition 2.5 (Properties of the limiting spectral measure σ)**.** *Let $s\colon [0,1]^2 \to (0,\infty)$, $a\colon [0,1] \to \mathbb{C}$ and $(I_k)_{k\in[\![K]\!]}$ be as in **A4**. Suppose that $s|_{I_l\times I_k}$ and $a|_{I_l}$ are Lipschitz-continuous functions for all l, $k \in [\![K]\!]$. Then the following holds.*

(i) With respect to the Lebesgue measure $\mathrm{d}^2\zeta$, the measure σ from Proposition 2.1 has a bounded density on $\mathbb{C}$, which we also denote by σ, i.e. $\sigma(\mathrm{d}\zeta) = \sigma(\zeta)\mathrm{d}^2\zeta$.

(ii) On $\mathbb{S}$, the density $\zeta \mapsto \sigma(\zeta)$ is strictly positive and real analytic.

(iii) $\operatorname{supp}\sigma = \overline{\mathbb{S}}$ and this set is bounded. Furthermore $\{a(x) : x \in [0,1]\} \subset \mathbb{S}$.

(iv) $\partial\mathbb{S}$ is a real analytic variety of real dimension at most 1.

(v) There exists a unique continuous extension $\sigma\colon \overline{\mathbb{S}} \to [0,\infty)$ of the density $\sigma|_{\mathbb{S}}$ to $\overline{\mathbb{S}}$ such that $\sigma(\zeta) = g(\zeta)|\partial_\zeta\beta(\zeta)|^2$ for all $\zeta \in \partial\mathbb{S}$, where $g : \partial\mathbb{S} \to (0,\infty)$ is a strictly positive function that can be extended to a real analytic function on a neighbourhood of $\partial\mathbb{S}$.

Proposition 2.5 is a special case of Proposition 6.1 from Section 6, where we used that by Remark 4.2 Assumption **A4** implies Assumption **A6** below.

We now give a brief overview about previous results covering (parts of) Proposition 2.5 for a subclass of the models we analysed. Throughout the following, $\mathfrak{c}$ is a circular element and $\mathfrak{a}$ is an element that is $*$-free from $\mathfrak{c}$. For some examplary choice of $\mathfrak{a}$, the Brown measure of $\mathfrak{a}+\mathfrak{c}$ was computed in [13, Section 5]. In [16, Theorem 1.4], a formula for the support of the Brown measure of $\mathfrak{a}+\mathfrak{c}$, when $\mathfrak{a}$ is a normal operator with a Gaussian spectral density, and its absolute continuity with a smooth density was shown. For general $\mathfrak{a}$, the formula for the support was derived in [15, Proposition 1.2] under an additional assumption. When $\mathfrak{a} = \mathfrak{a}^*$, [31, Theorems 3.13 and 3.14] provided an explicit open set such that the support of the Brown measure of $\mathfrak{a}+\mathfrak{c}$ coincides with the closure of the open set and the Brown measure has a strictly positive density on the open set as well as proved a sharp upper bound on the density. Apart from the sharp upper bound, these results were obtained in [49, Theorems 4.2 and 4.6] for general $\mathfrak{a}$, where the absence of atoms of the Brown measure and the real analyticity of the density were also established. Then [10, Theorem 7.10] proved for general $\mathfrak{a}$ that the Brown measure is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$, i.e. excluding a singular continuous part in the Brown measure, as well as the sharp upper bound on the density. In this setup, the edge behaviour of the Brown measure density was studied in [22, Theorem 2.9] showing a jump discontinuity or a quadratic growth for the density. In particular, [49, Theorems 4.2 and 4.6] and [10, Theorem 7.10] cover Proposition 2.5 (i) – (iii) when s is constant. In this case, [22, Theorem 2.9 and Remark 2.10] yield (iv) and (v). We note that some of the works listed above considered generalisations of circular elements such as elliptic elements in addition. Similar statements about the Brown measure of related models can for example be found in [12, 21, 28].

Remark 2.6 (Necessity of Lipschitz-continuity)**.** *The Lipschitz-continuity assumption on a and s in Proposition 2.5 is needed to ensure that the image of a remains in a positive distance from the boundary $\partial\mathbb{S}$ of the spectrum as stated in Proposition 2.5 (iii). For a counterexample, where the Lipschitz-continuity of a is violated and $\partial\mathbb{S} \cap \{a(x) : x \in [0,1]\} \neq \emptyset$ we refer to [22, Proposition 3.1 (iv)].*

Remark 2.7 (Special cases for $\mathbb{S}$)**.** *In the case when the entries of the random matrix X_n are independent and identically distributed, i.e. when $s = t$ is a constant and $\mathbb{E}|x_{ij}|^2 = \frac{t}{n}$, we recover from* (2.8) *the well-known formula [33, 47]*

$$\mathbb{S} = \left\{ \zeta \in \mathbb{C} : \int_0^1 \frac{\mathrm{d}x}{|a(x) - \zeta|^2} > \frac{1}{t} \right\}. \tag{2.9}$$

If $a = 0$ then $\mathbb{S} = \{\zeta \in \mathbb{C} \colon |\zeta|^2 < \varrho(S)\}$, where S denotes the spectral radius of S. This generalises the corresponding result from [3, 20] to an infinite dimensional setup.

Remark 2.8 (Special behaviours of $\partial\mathbb{S}$)**.** *We note that the boundary $\partial\mathbb{S}$ can have isolated points, i.e. its real dimension can locally be zero (see e.g. [22, Example 3.1 (d)]). Moreover, β can have critical points on the boundary $\partial\mathbb{S}$, i.e. there can be $\zeta \in \partial\mathbb{S}$ such that $\partial_\zeta \beta(\zeta) = 0$ as shown in Example 3.1 below. Even infinitely many critical points of β can occur in $\partial\mathbb{S}$, see Example 3.2. In particular, these examples reveal that a rich class of singularities of σ can occur at the spectral edge $\partial\mathbb{S}$.*

2.1 Notations

We now introduce some notations used throughout. We write $[\![n]\!] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$. For $r > 0$, we denote by $\mathbb{D}_r := \{z \in \mathbb{C} \colon |z| < r\}$ the disk of radius r around the origin in $\mathbb{C}$ and by $\operatorname{dist}(x, A) := \inf\{|x - y| \colon y \in A\}$ the Euclidean distance of a point $x \in \mathbb{C}$ from a set $A \subset \mathbb{C}$.

We use the convention that c and C denote generic constants that may depend on the model parameters, but are otherwise uniform in all other parameters, e.g. n, ζ, etc.. For two real scalars f and g we write $f \lesssim g$ and $g \gtrsim f$ if $f \leq Cg$ for such a constant $C > 0$. In case $f \lesssim g$ and $f \gtrsim g$ both hold, we write $f \sim g$. If the constant C depends on a parameter δ that is not a model parameter, we write $\lesssim_\delta$, $\gtrsim_\delta$ and $\sim_\delta$, respectively. The notation for inequality up to constant is also used for self-adjoint matrices/operators f and g, where $f \leq Cg$ is interpreted in the sense of quadratic forms. For complex f and $g \geq 0$ we write $f = O(g)$ in case $|f| \lesssim g$. Analogously $f = O_\delta(g)$ expresses the fact $|f| \lesssim_\delta g$.

3 Examples

In this section, we present a couple of examples highlighting certain special behaviours of $\partial\mathbb{S}$.

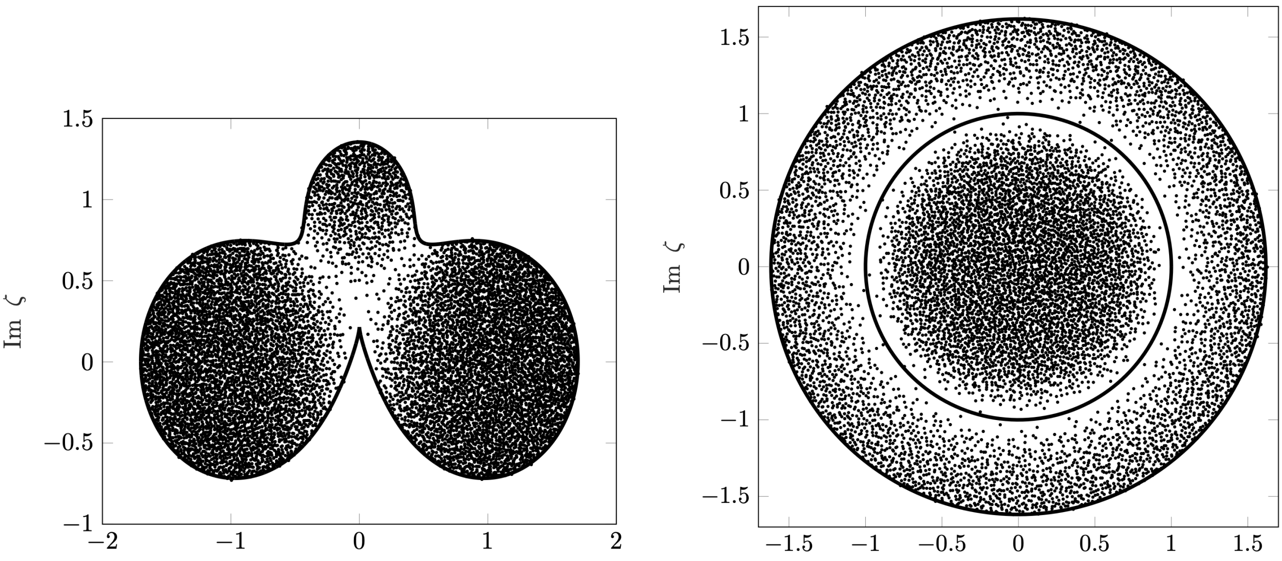
Example 3.1 (Critical points of $\partial\mathbb{S}$)**.** *Let $s \equiv t := \frac{2}{3}(20 - 7\sqrt{7})$ be constant on $[0,1]^2$, $\delta = (-17 + 7\sqrt{7})/8$ and*

$$a \colon [0,1] \to \mathbb{C}, \qquad x \mapsto \begin{cases} 1 & \text{if } x \in [0, 1/(2+\delta)), \\ -1 & \text{if } x \in [1/(2+\delta), 2/(2+\delta)), \\ \mathrm{i} & \text{if } x \in [2/(2+\delta), 1]. \end{cases}$$

We recall that if $s \equiv t$ then $\beta(\zeta) = \frac{1}{t} - \int_0^1 \frac{\mathrm{d}x}{|a(x) - \zeta|^2}$ by (2.9) *and*

$$\mathbb{S} = \left\{ \zeta \in \mathbb{C} \colon \frac{1}{|1 - \zeta|^2} + \frac{1}{|1 + \zeta|^2} + \frac{\delta}{|\zeta - \mathrm{i}|^2} > \frac{2 + \delta}{t} \right\}.$$

We set $y_0 = (\sqrt{7} - 2)/3$ and note that $\beta(\mathrm{i}y_0) = 0$, i.e. $\mathrm{i}y_0 \in \partial\mathbb{S}$ and, moreover, $\beta(\mathrm{i}y_0 + x + \mathrm{i}y) = (c_1 x^2 + c_2 y^3)(1 + o(1))$ for small enough x, $y \in \mathbb{R}$. Here, c_1 and c_2 are two positive constants and $o(1)$ is meant for $x \to 0$ and $y \to 0$. The boundary of $\mathbb{S}$ and sampled eigenvalues of the corresponding (Gaussian) random matrix model are drawn in Figure 1 (a).



(a) Example 3.1

(b) Example 3.2

Figure 1: The solid lines in subfigures (a) and (b) show the boundary of $\mathbb{S}$ from Examples 3.1 and 3.2, respectively. In addition, the black dots show the sampled eigenvalues of $X + A$, where X is an $n \times n$ matrix with i.i.d. $N(0,1)$ standard real normal distributed entries, $n = 10000$ and $A = (\mathrm{diag}(a(i/n))\delta_{ij})_{i,j=1}^n$ is a diagonal matrix and a is chosen as in Examples 3.1 and 3.2, respectively.

We refer to [5, Example 2.6 and Figure 1] for more examples in the spirit of Example 3.1.

Example 3.2 (Infinitely many critical points of $\partial\mathbb{S}$)**.** *Let $s \equiv 1$ be constant on $[0,1]^2$ and*

$$a\colon [0,1] \to \mathbb{C}, \qquad x \mapsto \begin{cases} \sqrt{2}\mathrm{e}^{4\pi \mathrm{i} x} & \text{if } x \in [0,1/2], \\ 0 & \text{if } x \in (1/2,1]. \end{cases}$$

A short computation starting from (2.9) *reveals that*

$$\mathbb{S} = \{\zeta \in \mathbb{C}\colon 0 \leq |\zeta|^2 < 1 \text{ or } 1 < |\zeta|^2 < (3+\sqrt{5})/2\}.$$

Thus, since $\partial\mathbb{D}_1 \cap \mathbb{S} = \emptyset$ while points on both sides of $\partial\mathbb{D}_1$ belong to $\mathbb{S}$, we conclude from the analyticity of β that $\partial_\zeta\beta(\zeta) = \partial_{\bar\zeta}\beta(\zeta) = 0$ for all $\zeta \in \partial\mathbb{D}_1$. Hence, there are infinitely many critical points of β on $\partial\mathbb{S}$. The boundary of $\mathbb{S}$ and sampled eigenvalues of an approximating random matrix model are shown in Figure 1 (b).

Furthermore, using (4.2) *and* (6.3)*, we find that*

$$\sigma(\zeta) = \frac{1}{\pi}\Bigg(1 - \frac{2}{2 + x + \frac{x}{(2x-1)^3}}\bigg(1 + \frac{1}{x^2}\bigg)\Bigg)\mathbf{1}(\zeta \in \mathbb{S}),$$

where $x \in (0,\infty)$ is the unique positive solution of $\frac{1}{x} + \frac{1}{\sqrt{1+4x+x^2-8|z|^2+3}} = 2$.

4 General setup and Dyson equation

In this section, we generalise the setup from the previous section in order to study an analogue of σ when a and s are defined on a probability space $\mathfrak{X}$ and on $\mathfrak{X}^2$ instead of $[0,1]$ and $[0,1]^2$, respectively. Thus, let $(\mathfrak{X}, \mathcal{A}, \mu)$ be a probability space, which represents the labels of the main quantities, e.g. a etc. We denote by $\mathcal{B} = L^\infty(\mathfrak{X}, \mathcal{A}, \mu)$ the measurable essentially bounded functions on $\mathfrak{X}$ up to measure zero with respect to μ.

Throughout the paper, we fix two measurable functions $s\colon \mathfrak{X}\times\mathfrak{X}\to[0,\infty)$ and $a\colon \mathfrak{X}\to\mathbb{C}$ with $a\in\mathcal{B}$. We also assume that

$$\sup_{x\in\mathfrak{X}}\int_{\mathfrak{X}} s(x,y)\mu(\mathrm{d}y)<\infty, \qquad \sup_{y\in\mathfrak{X}}\int_{\mathfrak{X}} s(x,y)\mu(\mathrm{d}x)<\infty.$$

Therefore, the two operators $S\colon\mathcal{B}\to\mathcal{B}$ and $S^*\colon\mathcal{B}\to\mathcal{B}$ defined through

$$(Su)(x)=\int_{\mathfrak{X}} s(x,y)u(y)\mu(\mathrm{d}y), \qquad (S^*u)(x)=\int_{\mathfrak{X}} s(y,x)u(y)\mu(\mathrm{d}y) \tag{4.1}$$

for all $x\in\mathfrak{X}$ and $u\in\mathcal{B}$ are well-defined bounded linear operators.

We consider two coupled equations for functions in $v_1, v_2\in\mathcal{B}$ with $v_1>0$ and $v_2>0$, namely

$$\frac{1}{v_1}=\eta+Sv_2+\frac{|\zeta-a|^2}{\eta+S^*v_1}\,, \tag{4.2a}$$

$$\frac{1}{v_2}=\eta+S^*v_1+\frac{|\zeta-a|^2}{\eta+Sv_2}\,, \tag{4.2b}$$

for all $\eta>0$ and $\zeta\in\mathbb{C}$. Here, η and ζ are interpreted as the constant functions on $\mathfrak{X}$ with the respective value. The equation (4.2) is called the *(vector) Dyson equation.* First, we clarify the existence and uniqueness of its solution.

Lemma 4.1 (Existence and uniqueness)**.** *For each $\eta>0$ and $\zeta\in\mathbb{C}$, there are unique v_1, $v_2\in\mathcal{B}$ such that $v_1>0$ and $v_2>0$ and* (4.2) *holds.*

In Appendix A.1, we present the proof of Lemma 4.1 by inferring it from [30] through a relation of (4.2) to a matrix-valued version.

4.1 Assumptions

Throughout the paper, we will impose some of the following assumptions.

A5 *Flatness of S:* There is a constant $C>0$ such that

$$\frac{1}{C}\le s(x,y)\le C$$

for all x, $y\in\mathfrak{X}$.

We define the function $\Gamma_{a,s}\colon(0,\infty)\to(0,\infty)$ through

$$\Gamma_{a,s}(\tau):=\operatorname*{ess\,inf}_{x\in\mathfrak{X}}\int_{\mathfrak{X}}\frac{1}{\tau^{-1}+|a(x)-a(y)|+d_s(x,y)}\mu(\mathrm{d}y), \tag{4.3}$$

where $d_s(x,y):=\operatorname{ess\,sup}_{q\in\mathfrak{X}}(|s(x,q)-s(y,q)|+|s(q,x)-s(q,y)|)$. Note that $\Gamma_{a,s}$ is strictly monotonically increasing.

A6 *Data regularity:* The data a and s satisfy the regularity assumption

$$\lim_{\tau\to\infty}\Gamma_{a,s}(\tau)=\infty$$

Remark 4.2. *In the case $\mathfrak{X}=[0,1]$ and μ the Lebesgue-measure on $[0,1]$, Assumption **A4** implies Assumption **A6**, since in this case $|a(x)-a(y)|+d_s(x,y)\lesssim|x-y|$ for all x, $y\in I_k$ and all $k\in[\![K]\!]$.*

As we will see in Lemma 5.3 below, **A6** together with **A5** and $a\in\mathcal{B}$ implies that v_1 and v_2 are uniformly bounded in the $\mathcal{B}$-norm on $\mathfrak{X}$. In addition to L^∞, we introduce the usual L^p spaces on $(\mathfrak{X},\mathcal{A},\mu)$. We denote them by $L^p:=L^p(\mathfrak{X},\mathcal{A},\mu)$ and the corresponding norms by $\|\cdot\|_p$. For functions $u\in L^1$ and u_1, $u_2\in L^2$, we define their average and scalar product as

$$\langle u\rangle:=\int u(x)\mu(\mathrm{d}x)\,, \qquad \langle u_1\,,u_2\rangle:=\langle\overline{u}_1u_2\rangle\,,$$

respectively. By normalisation of the probability measure μ on $\mathfrak{X}$ we have $\langle 1\rangle=1$.

5 Solution of the Dyson Equation

In this section, we study various properties of the solution (v_1, v_2) of the Dyson equation (4.2). We start with simple relations and bounds and obtain fine properties and expansions later.

From (4.2), we directly conclude

$$v_2(\eta + S^* v_1) = v_1(\eta + S v_2)\,, \tag{5.1}$$

which also implies

$$\langle v_1 \rangle = \langle v_2 \rangle\,. \tag{5.2}$$

Furthermore, we introduce y defined by

$$y := \frac{v_1\,(\bar{a} - \bar{\zeta})}{\eta + S^* v_1} = \frac{v_2\,(\bar{a} - \bar{\zeta})}{\eta + S v_2}\,, \tag{5.3}$$

where the second step is a consequence of (5.1). We conclude from (4.2) and (5.3) that

$$v_1 \leq \eta^{-1}, \qquad v_2 \leq \eta^{-1}, \qquad |y| \leq |a - \zeta|\eta^{-2} \tag{5.4}$$

for all $\zeta \in \mathbb{C}$ and $\eta > 0$. Furthermore, we have the identity

$$y = \frac{v_1(\bar{a} - \bar{\zeta})}{\eta + S^* v_1} = \frac{1}{a - \zeta}(1 - v_1(\eta + S v_2)) = \frac{1}{a - \zeta} - \frac{v_1 v_2}{\overline{y}} \tag{5.5}$$

for any $\zeta \in \mathbb{C} \setminus \mathrm{Spec}(D_a)$. Here, $\mathrm{Spec}(D_a)$ denotes the spectrum of D_a considered as multiplication operator $\mathcal{B} \to \mathcal{B}$, which coincides with the essential range of a.

Throughout the remainder of this section, we assume that s satisfies **A5**. This implies

$$S^* w \sim S w \sim \langle w \rangle \tag{5.6}$$

for all $w \in \mathcal{B}$ with $w \geq 0$.

5.1 Bounds on the solution

This subsection contains bounds on v_1 and v_2 under varying assumptions on s and a.

Bound in L^2-norm: We start with the following bound with respect to the norm on L^2.

Lemma 5.1. *If s satisfies **A5** then*

$$\langle v_1^2 \rangle + \langle v_2^2 \rangle + \langle |y|^2 \rangle \lesssim 1 \tag{5.7}$$

uniformly for all $\zeta \in \mathbb{C}$ and $\eta > 0$.

Proof. We multiply the first relation in (4.2a) by v_1^2 and estimate $v_1 \geq v_1^2 S v_2 \gtrsim v_1^2 \langle v_2 \rangle = v_1^2 \langle v_1 \rangle$ due to (5.6) and (5.2). Averaging this estimate and using $\langle v_1 \rangle > 0$ yields $\langle v_1^2 \rangle \lesssim 1$. The bound $\langle v_2^2 \rangle \lesssim 1$ is proved analogously. From (5.3), we conclude

$$|y|^2 = \frac{v_1^2 |a - \zeta|^2}{(\eta + S^* v_1)^2} \leq \frac{v_1^2 |a - \zeta|^2}{(\eta + S^* v_1)^2} + \frac{v_1^2(\eta + S v_2)}{\eta + S^* v_1} = \frac{v_1}{\eta + S^* v_1}, \tag{5.8}$$

where we used (4.2) in the last step. Hence, (5.6) implies

$$\langle |y|^2 \rangle \lesssim \frac{\langle v_1 \rangle}{\eta + \langle v_1 \rangle} \leq 1. \qquad \square$$

Corollary 5.2. *Let $a \in \mathcal{B}$ and s satisfy **A5**. Then*

$$\frac{\eta + \langle v_i \rangle}{1 + \eta^2 + |\zeta|^2} \lesssim v_i \lesssim \frac{1}{\eta + \langle v_i \rangle}\,, \qquad i = 1, 2\,.$$

Proof. We only consider the case $i = 1$. The case $i = 2$ follows analogously. For the lower bound, we start from (4.2) and get

$$v_1 = \frac{\eta + S^* v_1}{(\eta + S^* v_1)(\eta + S v_2) + |\zeta - a|^2} \gtrsim \frac{\eta + \langle v_1 \rangle}{\eta^2 + \langle v_1 \rangle^2 + |\zeta|^2 + \|a\|_\infty^2} \gtrsim \frac{\eta + \langle v_1 \rangle}{1 + \eta^2 + |\zeta|^2},$$

using $\langle v_1 \rangle = \langle v_2 \rangle$ by (5.2), (5.7) and (5.6). For the upper bound, (4.2), $\langle v_1 \rangle = \langle v_2 \rangle$ and (5.6) imply

$$v_1 \le \frac{1}{\eta + S v_2} \sim \frac{1}{\eta + \langle v_1 \rangle}. \qquad \square$$

Bound in supremum norm: Under stronger assumptions on s and a, we can also get a bound on v_i in the L^∞-norm. Let $\Gamma_{a,s}$ be as defined in (4.3).

Lemma 5.3. *Let $a \in \mathcal{B}$, s satisfy **A5** and $r > 0$. Then there is a constant $C > 0$ with $C \sim_r 1$ such that $C < \lim_{\tau\to\infty} \Gamma_{a,s}(\tau)$ implies*

$$\max\{\|v_1\|_\infty, \|v_2\|_\infty\} \le \Gamma_{a,s}^{-1}(C)$$

uniformly for $\zeta \in \mathbb{D}_r$ and $\eta > 0$.

Proof. From (4.2), we obtain

$$\begin{aligned}
\frac{1}{v_1(y)} &\le \frac{1}{v_1(x)} + \left| \frac{1}{v_1(x)} - \frac{1}{v_1(y)} \right| \\
&\le \frac{1}{v_1(x)} + |(S v_2)(x) - (S v_2)(y)| + \left| \frac{|\zeta - a(x)|^2}{\eta + (S^* v_1)(x)} - \frac{|\zeta - a(y)|^2}{\eta + (S^* v_1)(y)} \right| \\
&\lesssim_r \frac{1}{v_1(x)} + d_s(x,y)\langle v_1 \rangle + \frac{|a(x) - a(y)|}{\langle v_1 \rangle} + \frac{d_s(x,y)}{\langle v_1 \rangle} \\
&\lesssim \frac{1}{\langle v_1 \rangle} \Big(\frac{1}{v_1(x)} + |a(x) - a(y)| + d_s(x,y) \Big),
\end{aligned}$$

where in the third inequality we used $\|a\|_\infty \lesssim 1$ as $a \in \mathcal{B}$ and $\langle v_1 \rangle + \langle v_2 \rangle \lesssim 1$ by Lemma 5.1, as well as (5.6) and (5.2). We take the inverse on both sides and integrate in y with respect to μ to get $\Gamma_{a,s}(v_1(x)) \lesssim_r 1$ for almost all $x \in \mathfrak{X}$. Together with the analogous calculation with the roles of v_1 and v_2 interchanged, we obtain $\Gamma_{a,s}(\max\{\|v_1\|_\infty, \|v_2\|_\infty\}) \le C$ for some $C \sim_r 1$. Thus, the statement of the lemma follows from the monotonicity of $\Gamma_{a,s}$. $\square$

Corollary 5.4. *Let $a \in \mathcal{B}$. If s and a satisfy **A5** and **A6** then*

$$\|v_1\|_\infty + \|v_2\|_\infty \lesssim 1$$

uniformly for all $\zeta \in \mathbb{C}$ and $\eta > 0$.

Proof. Owing to Lemma 5.3, (5.4) and $a \in \mathcal{B}$, it remains to consider the case $\eta \le 1$ and $|\zeta| \ge \|a\|_\infty + 1$. In this case, we conclude from (4.2a) that $\frac{1}{v_1} \ge \frac{|\zeta - a|^2}{\eta + S^* v_1} \gtrsim 1$ as $S^* v_1 \lesssim \langle v_1 \rangle \lesssim 1$ by (5.6) and Lemma 5.1. Since $v_1 > 0$, we conclude $\|v_1\|_\infty \lesssim 1$. The analogous argument for v_2 completes the proof of Corollary 5.4. $\square$

Remark 5.5. *Let $\mathfrak{X} = [0,1]$ and μ be the Lebesgue measure on $[0,1]$. Let $I_1, \ldots, I_K$ be disjoint intervals in $[0,1]$ such that $I_1 \cup \ldots \cup I_K = [0,1]$. If $s\colon [0,1] \times [0,1] \to [0,\infty)$, $a\colon [0,1] \to \mathbb{C}$ are such that $s|_{I_l \times I_k}$ and $a|_{I_l}$ are Lipschitz-continuous for every l, $k \in [\![K]\!]$ then **A6** is satisfied and $a \in \mathcal{B}$. In particular, if s satisfies **A5** in addition then v_1 and v_2 are bounded in $\|\cdot\|_\infty$ uniformly on $\mathbb{C} \times (0,\infty)$ by Corollary 5.4. In this case **A5** and **A6** all hold.*

Scaling relations

Lemma 5.6. *Let $a \in \mathcal{B}$ and s satisfy **A5** and **A6**. Then*

$$v_1 \sim \langle v_1 \rangle = \langle v_2 \rangle \sim v_2, \qquad |y| \lesssim 1$$

uniformly for all $\zeta \in \mathbb{C}$ and $\eta > 0$. Moreover, for any sufficiently small positive constant $c \sim 1$ the inequalities $\eta + \langle v_1(\zeta,\eta)\rangle \le c$ and $|\zeta| \le 1/c$ imply $|\zeta - a| \sim 1$ and $|y| \sim 1$.

Before going into the proof of Lemma 5.6, we remark that if $a \in \mathcal{B}$ and s satisfies **A5** then

$$\|v_1(\zeta,\eta) - (1+\eta)^{-1}\| \lesssim (1+|\zeta|)\eta^{-2} \tag{5.9}$$

uniformly for $\eta > 0$ and $\zeta \in \mathbb{C}$. Indeed, for $\eta \in (0,1]$, (5.9) is a trivial consequence of (5.4). For $\eta \ge 1$, (5.9) follows by inverting (4.2a), subtracting $(1+\eta)^{-1}$ on both sides and estimating the right-hand side using (5.6) and Lemma 5.1.

Proof. We first prove that $v_1 \sim \langle v_1 \rangle = \langle v_2 \rangle \sim v_2$. As s satisfies **A5**, equation (4.2), v_1, $v_2 > 0$ and (5.6) imply

$$v_1 \sim v_2\,. \tag{5.10}$$

Hence, it suffices to show $v_1 \sim \langle v_1 \rangle$ due to (5.2).

From (5.9), we conclude that $v_1 \sim (1+\eta)^{-1}$ and $\langle v_1 \rangle \sim (1+\eta)^{-1}$ uniformly for $\eta \gtrsim 1$ and $|\zeta| \lesssim 1$. This proves Lemma 5.6 in that regime. If, on the other hand, $|\zeta| \ge \|a\|_\infty + 1$ then $|\zeta - a| \sim |\zeta|$. Hence, for such ζ, we conclude from (4.2a) and (5.6) that

$$\frac{1}{v_1} \sim \eta + \langle v_2 \rangle + \frac{|\zeta|^2}{\eta + \langle v_1 \rangle}.$$

As the right-hand side is a constant function on $\mathfrak{X}$, we obtain $v_1 \sim \langle v_1 \rangle$ if $|\zeta| \ge \|a\|_\infty + 1$.

Hence, it remains to consider $|\zeta| \lesssim 1$ and $\eta \lesssim 1$. In particular, $|\zeta - a| \lesssim 1$ as $a \in \mathcal{B}$. Thus, (4.2a), (5.6) and (5.2) imply

$$\eta + \langle v_1 \rangle \sim v_1((\eta + \langle v_1 \rangle)^2 + |\zeta - a|^2). \tag{5.11}$$

Together with Lemma 5.1, this yields $\eta + \langle v_1 \rangle \lesssim v_1$. We conclude $v_1 \gtrsim \langle v_1 \rangle$ and $v_1 \gtrsim \eta$ as well as $\langle v_1 \rangle \gtrsim \eta$. If $|\zeta - a| \ge c$ for any $c \sim 1$ then $v_1 \lesssim \eta + \langle v_1 \rangle \sim \langle v_1 \rangle$ by (5.11). Therefore we conclude $v_1 \sim \langle v_1 \rangle$ if $|\zeta - a| \ge c$. What remains is the case $|\zeta - a| \le c$ and $\eta \le c$ for some constant $c \sim 1$. As $v_1 \lesssim 1$ by **A6** and Lemma 5.3, we conclude from (5.11) that $1 \gtrsim \eta + \langle v_1 \rangle$ or $1 \gtrsim \frac{|\zeta - a|^2}{\eta + \langle v_1 \rangle}$. In the second case, (5.11) implies $\langle v_1 \rangle \lesssim \eta + \langle v_1 \rangle \sim v_1(|\zeta - a|^4 + |\zeta - a|^2)$. Using $|\zeta - a| \le c$, choosing $c \sim 1$ sufficiently small and averaging $\langle v_1 \rangle \lesssim v_1(|\zeta - a|^4 + |\zeta - a|^2)$ yield a contradiction as $\langle v_1 \rangle > 0$. Hence, $\langle v_1 \rangle \gtrsim 1$ and, thus, $\langle v_1 \rangle \sim 1$ by Lemma 5.1 as well as $1 \gtrsim v_1$ by (5.11) as $\eta \le c$ for some small enough $c \sim 1$. Since $v_1 \lesssim 1$ by Lemma 5.3, this completes the proof of $v_1 \sim \langle v_1 \rangle$ uniformly for $\zeta \in \mathbb{C}$ and $\eta > 0$.

From $v_1 \sim \langle v_1 \rangle$ and (5.8), we conclude $|y| \lesssim 1$ uniformly for $\zeta \in \mathbb{C}$ and $\eta > 0$. Owing to (5.11) and $v_1 \sim \langle v_1 \rangle$, we have $\eta + \langle v_1 \rangle \sim \langle v_1 \rangle (\eta + \langle v_1 \rangle)^2 + \langle v_1 \rangle |\zeta - a|^2$. As $\eta + \langle v_1 \rangle \le c$, by choosing $c \sim 1$ sufficiently small, we can incorporate $\langle v_1 \rangle (\eta + \langle v_1 \rangle)^2$ into the left-hand side and obtain $\langle v_1 \rangle \lesssim |\zeta - a|^2 \langle v_1 \rangle$. Hence, $1 \lesssim |\zeta - a|$ as $\langle v_1 \rangle > 0$. The bound $|y| \sim 1$ follows from (5.5). This proves the additional statement and completes the proof of Lemma 5.6. $\square$

5.2 Relation to Matrix Dyson equation

Let (v_1, v_2) be a solution of (4.2). We now relate (v_1, v_2) to a solution $M \in \mathcal{B}^{2\times 2}$ of a matrix equation. To that end, we define y as in (5.3) and set

$$M := \begin{pmatrix} \mathrm{i}v_1 & \overline{y} \\ y & \mathrm{i}v_2 \end{pmatrix} \in \mathcal{B}^{2\times 2}. \tag{5.12}$$

Then $\operatorname{Im} M := \frac{1}{2\mathrm{i}}(M - M^*)$ is positive definite and inverting the 2×2 matrix M explicitly shows that M satisfies the *Matrix Dyson Equation (MDE)*

$$-M^{-1} = \begin{pmatrix} \mathrm{i}\eta & \zeta - a \\ \overline{\zeta - a} & \mathrm{i}\eta \end{pmatrix} + \Sigma[M]. \tag{5.13}$$

Here, $\Sigma\colon \mathcal{B}^{2\times 2} \to \mathcal{B}^{2\times 2}$ is defined through

$$\Sigma\left[\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\right] = \begin{pmatrix} Sr_{22} & 0 \\ 0 & S^* r_{11} \end{pmatrix} \tag{5.14}$$

for all r_{11}, r_{12}, r_{21}, $r_{22} \in \mathcal{B}$.

On the other hand, if $M \in \mathcal{B}^{2\times 2}$ with $\operatorname{Im} M$ positive definite is a solution of (5.13) then it is easy to see that denoting the diagonal elements of M by $\mathrm{i}v_1$ and $\mathrm{i}v_2$ yields a solution of (4.2).

Matrix Dyson equation with general spectral parameter and measure ρ_ζ More generally, we consider the MDE, where the spectral parameter $\mathrm{i}\eta$ is replaced by $w \in \mathbb{C}$ with $\operatorname{Im} w > 0$, i.e.

$$-M(\zeta, w)^{-1} = \begin{pmatrix} w & \zeta - a \\ \overline{\zeta - a} & w \end{pmatrix} + \Sigma[M(\zeta, w)] \tag{5.15}$$

for $\zeta \in \mathbb{C}$. Then (5.15) has a unique solution $M(\zeta, w) \in \mathcal{B}^{2\times 2}$ under the constraint that $\operatorname{Im} M(\zeta, w) := \frac{1}{2\mathrm{i}}(M(\zeta, w) - M(\zeta, w)^*)$ is positive definite for $\operatorname{Im} w > 0$ [30].

By [4, Proposition 2.1 and Definition 2.2], the map $w \mapsto \langle M(\zeta, w)\rangle$ is the Stieltjes transform of a probability measure on $\mathbb{R}$, where we introduced the short hand notation

$$\langle R\rangle := \frac{1}{2}(\langle r_{11}\rangle + \langle r_{22}\rangle), \qquad R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathcal{B}^{2\times 2}.$$

Definition 5.7. *We denote by ρ_ζ the unique probability measure on $\mathbb{R}$ whose Stieltjes transform is given by $w \mapsto \langle M(\zeta, w)\rangle$. Through ρ_ζ we define*

$$\mathbb{S}_\varepsilon := \{\zeta \in \mathbb{C} : \operatorname{dist}(0, \operatorname{supp} \rho_\zeta) \le \varepsilon\} \tag{5.16}$$

for any $\varepsilon \ge 0$.

In the setup of Theorem 2.4, in particular, μ is the Lebesgue measure on $\mathfrak{X} = [0, 1]$, the measure ρ_ζ from Definition 5.7 is the asymptotic symmetrized empirical singular value distribution of $X_n + A_n - \zeta$ for any $\zeta \in \mathbb{C}$, see [5, Theorem 2.7]. In this case the set $\operatorname{Spec}_\varepsilon^\infty(s, a)$ from Theorem 2.4 is identified in (7.11) below with $\mathbb{S}_\varepsilon$ from (5.16), which is the $n \to \infty$ limit of the ε-pseudospectrum (2.3) for $R = X + A$.

Remark 5.8. *The sets $\mathbb{S}_\varepsilon$ defined in* (5.16) *are monotonously nondecreasing in $\varepsilon \ge 0$, i.e. $\mathbb{S}_{\varepsilon_1} \subset \mathbb{S}_{\varepsilon_2}$ if $\varepsilon_1 \le \varepsilon_2$. Moreover, they are bounded, in fact, $\mathbb{S}_\varepsilon \subset \{\zeta \in \mathbb{C} : |\zeta| \le \varepsilon + \|a\|_\infty + 2(\|S\|_\infty)^{1/2}\}$ for all $\varepsilon \ge 0$ as a consequence of [4, Proposition 2.1]. Here, $\|S\|_\infty$ denotes the operator norm of S viewed as an operator from $\mathcal{B}$ to $\mathcal{B}$.*

5.3 A relation between the derivatives of M

In this subsection, we consider derivatives of M, the solution of (5.13), with respect to η and ζ and establish a useful relation between these in the next lemma.

Lemma 5.9. *Let $a \in \mathcal{B}$ and $s\colon \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be bounded measurable functions. Then*

$$\langle \partial_\eta M_{21}(\zeta, \mathrm{i}\eta)\rangle = 2\mathrm{i}\langle \partial_\zeta M(\zeta, \mathrm{i}\eta)\rangle, \qquad \langle \partial_\eta M_{12}(\zeta, \mathrm{i}\eta)\rangle = 2\mathrm{i}\langle \partial_{\bar\zeta} M(\zeta, \mathrm{i}\eta)\rangle \tag{5.17}$$

for every $\zeta \in \mathbb{C}$ and $\eta > 0$, where we decomposed

$$M(\zeta, \mathrm{i}\eta) = \begin{pmatrix} M_{11}(\zeta, \mathrm{i}\eta) & M_{12}(\zeta, \mathrm{i}\eta) \\ M_{21}(\zeta, \mathrm{i}\eta) & M_{22}(\zeta, \mathrm{i}\eta) \end{pmatrix}.$$

We note that the condition $\sup_{x,y\in\mathfrak{X}} s(x,y) < \infty$ implies that the operators S and S^* from (4.1) can be extended to operators $L^2 \to L^\infty$.

Before proving Lemma 5.9, we prove the differentiability of M with respect to η, ζ and $\bar\zeta$. Let $\mathcal{L}$ be the stability operator of (5.13), defined as

$$\mathcal{L}\colon \mathcal{B}^{2\times 2} \to \mathcal{B}^{2\times 2}, \quad R \mapsto \mathcal{L}[R] := M^{-1}RM^{-1} - \Sigma[R]. \tag{5.18}$$

This operator is invertible for any $\zeta \in \mathbb{C}$ and $\eta > 0$ due to Lemma A.1 below. Therefore, the implicit function theorem applied to (5.13) and simple computations show that

$$\partial_\zeta M = \mathcal{L}^{-1}[E_{12}], \qquad \partial_{\bar\zeta} M = \mathcal{L}^{-1}[E_{21}], \qquad \partial_\eta M = \mathrm{i}\mathcal{L}^{-1}[E_+] \tag{5.19}$$

for all $\eta > 0$ and $\zeta \in \mathbb{C}$, where we used the notations E_{12}, E_{21} and E_+ for the elements of $\mathcal{B}^{2\times 2}$ defined through

$$E_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{21} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_+ := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.20}$$

Proof of Lemma 5.9. As $\langle \partial_\eta M_{21}\rangle = \langle E_{21}^* \partial_\eta M\rangle$, we get from (5.19) that

$$\langle \partial_\eta M_{21}\rangle = 2\mathrm{i}\langle E_{21}^*\mathcal{L}^{-1}[E_+]\rangle = 2\mathrm{i}\langle E_{21}\,, \mathcal{L}^{-1}[E_+]\rangle. \tag{5.21}$$

Since $\langle \partial_\eta M_{12}\rangle = \langle E_{12}^*\partial_\eta M\rangle$, there is an analogous statement for $\langle \partial_\eta M_{12}\rangle$.

We start from the first relation in (5.19), use $(\mathcal{L}[R])^* = \mathcal{L}^*[R^*]$, the invertibility of $\mathcal{L}^*$ and $E_{12}^* = E_{21}$ to obtain

$$\langle \partial_\zeta M\rangle = \langle (\mathcal{L}^{-1}[E_{12}])^*\,, E_+\rangle = \langle (\mathcal{L}^*)^{-1}[E_{12}^*]\,, E_+\rangle = \langle E_{21}\,, \mathcal{L}^{-1}[E_+]\rangle. \tag{5.22}$$

Therefore, combining (5.21) and (5.22) proves the first identity in (5.17). The second one follows analogously. $\square$

5.4 Stability of Dyson equation and analyticity of its solution

In this section we show how the solution v_1, v_2 of (4.2) can be extended to $\eta = 0$. If we stay away from the deterministic analog of the ε-pseudospectrum, then the solution is extended to $\eta = 0$ by setting $v_i = 0$ by the following lemma.

Lemma 5.10. *Let $\varepsilon > 0$. Let $\zeta \in (\mathbb{C}\setminus \mathbb{S}_\varepsilon) \cap \mathbb{D}_{1/\varepsilon}$. Then $v_i(\zeta,\eta) \sim_\varepsilon \eta$ for all $\eta \in (0,1]$ and $i = 1,2$. In particular, v_i is continuously extended to $\zeta \in \mathbb{C}\setminus\mathbb{S}_0$ and $\eta = 0$ by setting $v_i(\zeta,0) := 0$.*

Proof. From (4.2a), we conclude $v_1(|\zeta - a|^2 + (\eta + Sv_2)(\eta + S^*v_1)) = \eta + S^*v_1 \geq \eta$. Thus, $|\zeta| \leq \varepsilon^{-1}$, $a \in \mathcal{B}$, $\eta \leq 1$, (5.6) and Lemma 5.1 imply $v_1 \gtrsim_\varepsilon \eta$ for all $\eta \in (0,1]$. Similarly, $v_2 \gtrsim_\varepsilon \eta$ for all $\eta \in (0,1]$. On the other hand, as $\zeta \in \mathbb{C}\setminus\mathbb{S}_\varepsilon$, the statement (v) of [4, Lemma D.1] holds for $\tau = 0$. Hence, [4, Lemma D.1 (i)] implies $\max\{v_1(\zeta,\eta), v_2(\zeta,\eta)\} \leq \|\mathrm{Im}\, m(\mathrm{i}\eta)\| \lesssim \eta$ for all $\eta \in (0,c]$ for some sufficiently small $c \sim_\varepsilon 1$. If $\eta \in (c,1]$ then the upper bound in Corollary 5.2 yields $v_i \lesssim \eta^{-1} \sim \eta$ for all $\eta \in (c,1]$. This completes the proof. $\square$

The next proposition states that if $\langle v_1\rangle = \langle v_2\rangle$ remains bounded away from zero as $\eta \downarrow 0$, then the solution has an analytic extension to $\eta = 0$.

Proposition 5.11 (Analyticity in the bulk)**.** *Let s satisfy **A5** and $\zeta \in \mathbb{C}$ with $\limsup_{\eta\downarrow 0}\langle v_1(\zeta,\eta)\rangle > 0$. Then $v_1, v_2 : \mathbb{C}\times(0,\infty) \to (0,\infty)$ has an extension to a neighbourhood of $(\zeta, 0)$ in $\mathbb{C}\times\mathbb{R}$ which is real analytic in all variables.*

To prove this proposition, we show that the Dyson equation (4.2) is stable even for $\eta = 0$. However, the equation does not have a unique solution on $\mathcal{B}_+^2$ for $\eta = 0$ without the additional constraint $\langle v_1\rangle = \langle v_2\rangle$. Therefore, we have to reformulate the equation to incorporate this constraint. Proposition 5.11 is proved at the end of this subsection.

We recall that $\mathcal{B}_+ := \{w \in \mathcal{B} : w > 0\}$ and set

$$e_- = (1,-1) \in \mathcal{B}^2 := \mathcal{B}\oplus\mathcal{B}\,, \qquad e_-^\perp := \{h = (h_1,h_2) \in \mathcal{B}^2 : \langle h_1\rangle = \langle h_2\rangle\}.$$

For $\eta > 0$ and $\zeta \in \mathbb{C}$, we define $J \equiv J_{\zeta,\eta}\colon e_-^\perp \cap \mathcal{B}_+^2 \to e_-^\perp, (w_1, w_2) \mapsto (J_1(w_1,w_2), J_2(w_1,w_2))$ through

$$J_1(w_1,w_2) := (\eta + Sw_2)\bigg(w_1 - \frac{\eta + S^*w_1}{(\eta+S^*w_1)(\eta+Sw_2) + |a-\zeta|^2}\bigg),$$
$$J_2(w_1,w_2) := (\eta + S^*w_1)\bigg(w_2 - \frac{\eta + Sw_2}{(\eta+S^*w_1)(\eta+Sw_2) + |a-\zeta|^2}\bigg).$$

Then (4.2) takes the form $J(v) = 0$ with $v = (v_1, v_2) \in \mathcal{B}_+^2$.

On $\mathcal{B}^2$, we introduce a average and a scalar product defined through

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle := \frac{1}{2}(\langle x_1 \rangle + \langle x_2 \rangle), \qquad \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle := \frac{1}{2}(\langle \overline{x}_1 y_1 \rangle + \langle \overline{x}_2 y_2 \rangle) \tag{5.23}$$

for x_1, x_2, y_1, $y_2 \in \mathcal{B}$. For $x \in \mathcal{B}^2$, we write $\|x\|_2 := \sqrt{\langle x\,, x\rangle}$.

For the rest of this section we will assume that s satisfies **A5**. Until the proof of Proposition 5.11, we fix $\zeta \in \mathbb{C}$ such that $\limsup_{\eta\downarrow 0}\langle v_1(\zeta,\eta)\rangle \geq \delta$ for some $\delta > 0$. Under these conditions, J remains well defined on $e_-^\perp \cap \mathcal{B}_+^2$ even for $\eta = 0$ and we set $J_0 := J_{\zeta,\eta=0}$. We now pick candidates for $v_1(\zeta, 0)$ and $v_2(\zeta,0)$ by choosing weakly convergent subsequences in the limit $\eta \downarrow 0$. By Lemma 5.1, there are $v_0 \in (L^2)^2 := L^2 \oplus L^2$ and a monotonically decreasing sequence $\eta_n \downarrow 0$ in $(0,1]$ such that $v_n = v(\zeta, \eta_n)$ is weakly convergent to v_0 in $(L^2)^2$, i.e. for any $h \in (L^2)^2$, $\langle h\,, v_n - v_0\rangle \to 0$ in the limit $n \to \infty$. We recall that $L^2 = L^2(\mathfrak{X}, \mathcal{A}, \mu)$.

Lemma 5.12. *Then $v_0 \in \mathcal{B}_+^2 \cap e_-^\perp$ and $\delta \lesssim v_0 \lesssim \frac{1}{\delta}$. Furthermore, v_0 satisfies* (4.2) *for $\eta = 0$, i.e. $J_0(v_0) = 0$.*

For the following arguments, we introduce the operators S_o and S_d on $\mathcal{B}^2$ defined through

$$S_o := \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}, \quad S_d := \begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix}. \tag{5.24}$$

Owing to the upper bound in **A5**, S_o and S_d can be extended naturally to operators on $(L^2)^2$.

Proof. Since $v_n \to v_0$ weakly and $\langle e_- v_n\rangle = 0$, we conclude $v_0 \perp e_-$. Furthermore, for any $h \in \mathcal{B}_+^2$ we get $\langle hv_0\rangle = \lim_{n\to\infty}\langle hv_n\rangle \gtrsim \delta\langle h\rangle$ because of Corollary 5.2 and $\limsup_{n\to\infty}\langle v_n\rangle \geq \delta$. From this we conclude $v_0 \gtrsim \delta$. Similarly, Corollary 5.2 implies $v_0 \lesssim \frac{1}{\delta}$ and thus $v_0 \in \mathcal{B}_+^2$.

The natural extensions of S and S^* to operators on L^2 are Hilbert-Schmidt operators because $s \in L^2(\mathfrak{X} \times \mathfrak{X}, \mu \otimes \mu)$ due to the upper bound in **A5**. In particular, S and S^* are compact operators on L^2 and, thus, $S_o v_n \to S_o v_0$ and $S_d v_n \to S_d v_0$ in $(L^2)^2$. The bounds $\delta \lesssim v_n \lesssim \delta^{-1}$ then imply that $J_{\zeta,\eta_n}(v_n) \to J_0(v_0)$ weakly in $(L^2)^2$. Consequently, $J_0(v_0) = 0$. $\square$

For the formulation of the next lemma, we note that $\|T\|_\infty$ denotes the operator norm of an operator $T\colon \mathcal{B}^2 \to \mathcal{B}^2$ and, analogously, $\|T\|_2$ is the operator norm if $T\colon (L^2)^2 \to (L^2)^2$.

Lemma 5.13. *Let v_0 be a weak limit of a sequence $v_n = v(\zeta, \eta_n)$ as above. Then*

$$\|(\nabla J_0|_{w=v_0})^{-1}\|_2 + \|(\nabla J_0|_{w=v_0})^{-1}\|_\infty \lesssim_\delta 1.$$

Proof of Lemma 5.13. Within this proof we will make use of some results from [3]. Therefore we introduce notations that match the ones from [3], namely

$$\tau := (|\zeta - a|^2, |\zeta - a|^2) \tag{5.25}$$

and recall the definitions of S_o and S_d from (5.24). In [3] the setup $a = 0$ was treated and therefore $\tau = |\zeta|^2$ was constant. Here $\tau = (\tau_1, \tau_2) \in \overline{\mathcal{B}_+} \oplus \overline{\mathcal{B}_+}$ satisfies $\tau_1 = \tau_2$, which ensures that the necessary computations from [3] remain applicable. Using the notations (5.24) and (5.25), we write J in the form

$$J(w) = (\eta + S_o w)\bigg(w - \frac{1}{\eta + S_o w + \frac{\tau}{\eta + S_d w}}\bigg).$$

Now we take the directional derivative $\nabla_h J$ of J in the direction $h \in \mathcal{B}^2$ with $h \perp e_-$, i.e. $\langle he_- \rangle = 0$, and evaluate at the solution $v = (v_1, v_2)$. Thus, we find

$$\nabla_h J|_{w=v} = (\eta + S_o v)\Big(h + v^2 S_o h - \frac{v^2 \tau}{(\eta + S_d v)^2} S_d h\Big) = (\eta + S_o v) Lh\,, \tag{5.26}$$

where we used $J(v) = 0$ and introduced the linear operator $L \equiv L_{\zeta,\eta}(v) \colon \mathcal{B}^2 \to \mathcal{B}^2$ as

$$Lh := h + v^2 S_o h - \frac{v^2 \tau}{(\eta + S_d v)^2} S_d h \tag{5.27}$$

to guarantee the last equality. We now restrict our analysis to $\eta = 0$ and use the following lemma that provides a resolvent estimate for $L_0 = L_{\zeta,0}(v_0)$, the operator evaluated on the weak limit v_0.

Lemma 5.14. *There is $\varepsilon_* \sim_\delta 1$ such that for any $\varepsilon \in (0, \varepsilon_*)$ we have the bound*

$$\sup\Big\{\|(L_0 - \omega)^{-1}\|_\# : \omega \notin \mathbb{D}_\varepsilon \cup ((1 + \mathbb{D}_{1+\varepsilon}) \setminus \mathbb{D}_{2\varepsilon})\Big\} \lesssim_{\delta,\varepsilon} 1 \tag{5.28}$$

for $\# = 2, \infty$. Here, $\mathbb{D}_\varepsilon$ contains the single isolated eigenvalue 0 of L_0 with corresponding right and left eigenvectors $v_- := e_- v_0$ and $S_o v_-$, i.e.

$$\mathbb{D}_\varepsilon \cap \operatorname{Spec}(L_0) = \{0\}\,, \qquad \ker L_0^2 = \operatorname{Span}(v_-)\,, \qquad L_0 v_- = 0\,, \qquad L_0^* S_o v_- = 0\,.$$

Here, L_0^* is the adjoint of L with respect to the L^2-scalar product introduced in (5.23).

The proof of Lemma 5.14 follows a strategy similar to the one used to prove stability of the Dyson equation in [3], where the case $a = 0$ was treated. For completeness we present the proof in Appendix A.2 below. Using Lemma 5.14 we now show that

$$\|L_0^{-1}|_{(S_o v_-)^\perp}\|_\# \lesssim_\delta 1\,, \qquad \# = 2, \infty\,, \tag{5.29}$$

from which the claim of Lemma 5.13 immediately follows due to (5.26), **A5** and $v_0 \gtrsim \delta$. To see (5.29), we will apply [6, Lemma 4.6] to CL_0 for some appropriately large positive constant $C \sim_\delta 1$. The lemma was formulated for $\mathfrak{X} = \{1, \dots, d\}$ with the normalized counting measure, i.e. $\mathcal{B} = \mathbb{C}^d$. However, its proof is uniform in the underlying dimension d and it therefore translates to the current general setup. We now check the assumptions of the [6, Lemma 4.6]. Note that L_0 maps $e_-^\perp$ to $(S_o v_-)^\perp$. By Lemma 5.14 the right and left eigenvectors of L_0 corresponding to the eigenvalue 0 are v_- and $S_o v_-$, respectively. Moreover, $\langle v_-\,, e_- \rangle \gtrsim_\delta 1$ as $v_0 \gtrsim \delta$, $|\langle e_-\,, w \rangle| \le \|w\|_\#$ and that $\|L_0 w\|_\# \gtrsim_\delta \|w\|_\#$ for any $w \perp S_o v_-$ due to Lemma 5.14. By [6, Lemma 4.6] we get $\|L_0 w\|_\# \gtrsim_\delta \|w\|_\#$ for any $w \perp e_-$. Thus, (5.29) is shown. □

Now we use the stability at $\eta = 0$ to finish the proof of the main result of this subsection.

Proof of Proposition 5.11. Let $\zeta_0 \in \mathbb{C}$ be such that $\limsup_{\eta \downarrow 0} \langle v_1(\zeta_0, \eta) \rangle > 0$. Let $\eta_n \downarrow 0$ such that $v_n = v(\zeta_0, \eta_n)$ is weakly convergent in $(L^2)^2$. This is possible, because the family $v(\zeta_0, \eta)$ with $\eta \in (0, 1]$ is bounded in $(L^2)^2$ due to Lemma 5.1. By Lemma 5.12 the weak limit $v_0 = \lim_{n\to\infty} v_n$ satisfies the Dyson equation, $J_{\zeta,0}(v_0) = 0$, and by Lemma 5.13 the Dyson equation is stable at $v = v_0$ and $\eta = 0$. By the implicit function theorem we find a real analytic function $\widetilde{v}$, defined on a neighbourhood U of $(\zeta_0, 0)$ in $\mathbb{C} \times \mathbb{R}$, such that $\widetilde{v}(\zeta, \eta)$ solves (4.2) and $\widetilde{v}(\zeta_0, 0) = v_0$. Since $v_0 \gtrsim \delta$ according to Lemma 5.12, $\widetilde{v}(\zeta, \eta) > 0$ on U if the neighbourhood U is chosen sufficiently small. By uniqueness of the solution to the Dyson equation we conclude $\widetilde{v}(\zeta, \eta) = v(\zeta, \eta)$ for all $(\zeta, \eta) \in U$. □

5.5 Characterisation of $\mathbb{S}$

Throughout this section we assume that $a \in \mathcal{B}$ and s satisfies **A5** and **A6**. To generalize (2.6), (2.7) and (2.8) to the setup introduced in Section 4, we define an operator $B_\zeta \colon \mathcal{B} \to \mathcal{B}$, a function $\beta \colon \mathbb{C} \to \mathbb{R}$ and a subset $\mathbb{S} \subset \mathbb{C}$ through

$$\beta(\zeta) := \inf_{x \in \mathcal{B}_+} \sup_{y \in \mathcal{B}_+} \frac{\langle x\,, B_\zeta\, y\rangle}{\langle x\,, y\rangle}\,, \qquad B_\zeta := D_{|a-\zeta|^2} - S\,, \qquad \mathbb{S} := \{\zeta \in \mathbb{C} : \beta(\zeta) < 0\}\,. \tag{5.30}$$

We also set

$$\rho_\zeta(0) := \frac{1}{\pi} \lim_{\eta \downarrow 0} \langle v_1(\zeta, \eta)\rangle\,. \tag{5.31}$$

This limit exists, because either $\limsup_{\eta \to 0}\langle v_1(\zeta,\eta)\rangle > 0$, in which case v_1 can be analytically extended to $\eta = 0$ by Proposition 5.11, or $\limsup_{\eta\to 0}\langle v_1(\zeta,\eta)\rangle = 0$ in which case the limit equals zero as well. As explained after Definition 5.7 we can interpret $\rho_\zeta(0)$ as the asymptotic singular value density of $X + A - \zeta$ at zero in case $(\mathfrak{X}, \mu) = ([0,1], \mathrm{d}x)$.

In the following we will denote by $\lambda_{\mathrm{PF}}(T)$ the spectral radius of a compact and positivity preserving operator T, i.e. $\lambda_{\mathrm{PF}}(T)$ is the Perron-Frobenius eigenvalue of T. In particular the operators S and S^* are compact as mentioned in the proof of Lemma 5.12 and therefore so are $D_x S D_y$ and $D_x S^* D_y$ for $x, y \in \mathcal{B}$. We use this fact in the statement of the following proposition.

Proposition 5.15. *The following relations between β, $\mathbb{S}$, $\mathbb{S}_\varepsilon$ and ρ_ζ apply.*

(i) The function $\mathbb{C} \ni \zeta \mapsto \beta(\zeta)$ is continuous and satisfies $\lim_{\zeta\to\infty} \beta(\zeta) = +\infty$. *In particular, $\mathbb{S}$ is bounded.*

(ii) The spectrum of D_a lies inside $\mathbb{S}$, i.e.

$$\mathrm{Spec}(D_a) \subset \mathbb{S}\,. \tag{5.32}$$

(iii) The sign of β satisfies

$$\mathrm{sign}(\beta(\zeta)) = \mathrm{sign}\Big(1 - \lambda_{\mathrm{PF}}\Big(S D^{-2}_{|a-\zeta|}\Big)\Big)\,, \qquad \zeta \in \mathbb{C}\,. \tag{5.33}$$

(iv) For any $\zeta \in \mathbb{C}$ with $\beta(\zeta) > 0$ the operator B_ζ is invertible. Furthermore, all such ζ are characterised by

$$\{\zeta \in \mathbb{C} : \beta(\zeta) > 0\} = \{\zeta \in \mathbb{C} : \mathrm{dist}(0, \mathrm{supp}\,\rho_\zeta) > 0\} = \mathbb{C} \setminus \mathbb{S}_0\,. \tag{5.34}$$

(v) The set $\mathbb{S}$ is characterised by having a positive singular value density at the origin, i.e.

$$\mathbb{S} = \{\zeta \in \mathbb{C} : \rho_\zeta(0) > 0\}\,. \tag{5.35}$$

Proof. Proof of (i): The continuity of $\zeta \mapsto \beta(\zeta) = \beta$ with $B = B_\zeta$ is a consequence of the bound

$$\left| \inf_{x\in\mathcal{B}_+} \sup_{y \in \mathcal{B}_+} \frac{\langle x\,, (B + D_w) y\rangle}{\langle x\,, y\rangle} - \beta \right| \le \sup_{x\in\mathcal{B}_+} \sup_{y\in\mathcal{B}_+} \frac{\langle x\,, D_{|w|} y\rangle}{\langle x\,, y\rangle} \le \|w\|_\infty$$

for any real valued $w \in \mathcal{B}$. The statement $\beta(\zeta) \to +\infty$ as $\zeta \to \infty$ is obvious.

Before we start with the proof of other individual statements of the proposition, we show that $\mathbb{S}$ can be classified in terms of the Perron-Frobenius eigenvalue of $S D^{-2}_{|a-\zeta|}$ in the sense that

$$\mathbb{S} = \{\zeta \in \mathbb{C} : \lambda(\zeta) > 1\}\,, \tag{5.36}$$

where we introduced $\lambda : \mathbb{C} \to [0, \infty]$ as the limit of a strictly increasing sequence via

$$\lambda(\zeta) := \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon(\zeta)\,, \qquad \lambda_\varepsilon(\zeta) := \lambda_{\mathrm{PF}}\big(S(\varepsilon + D_{|a-\zeta|^2})^{-1}\big).$$

To show (5.36) let $\varepsilon \in (0,1)$, $D := D_{|\zeta-a|^2}$, $\lambda_\varepsilon = \lambda_\varepsilon(\zeta)$ and $C > 0$ such that $1 + |\zeta - a|^2 \leq C$. For $\zeta \in \mathbb{C}$ with $\beta(\zeta) \geq 0$ we get

$$\beta + \varepsilon \leq C \inf_{x \in \mathcal{B}_+} \sup_{y \in \mathcal{B}_+} \frac{\langle x\,, (\varepsilon + B)y\rangle}{\langle x\,, (\varepsilon + D)y\rangle} = C\Big(1 - \sup_{x \in \mathcal{B}_+} \inf_{y \in \mathcal{B}_+} \frac{\langle x\,, S(\varepsilon + D)^{-1}y\rangle}{\langle x\,, y\rangle}\Big) = C(1 - \lambda_\varepsilon),$$

where we used $\varepsilon + |\zeta - a|^2 \leq C$ in the first inequality, and conclude

$$\beta \leq C(1 - \lambda) \qquad \text{if } \beta \geq 0\,. \tag{5.37}$$

For $\zeta \in \mathbb{C}$ with $\beta(\zeta) < 0$ we use

$$-\beta - \varepsilon = \sup_{x \in \mathcal{B}_+} \inf_{y \in \mathcal{B}_+} \frac{-\langle x\,, (\varepsilon + B)y\rangle}{\langle x\,, y\rangle}$$

for sufficiently small $\varepsilon > 0$ and find analogously that

$$\beta \geq C(1 - \lambda) \qquad \text{if } \beta < 0\,. \tag{5.38}$$

From (5.37) and (5.38) we conclude (5.36).

We also show that

$$\operatorname{Spec}(D_a) \subset \{\zeta \in \mathbb{C} : \beta(\zeta) \leq 0\}. \tag{5.39}$$

We will improve this to (5.32) below. Let $\zeta \in \operatorname{Spec}(D_a)$. Then $\operatorname{ess\,inf}|\zeta - a| = 0$. Thus, for any $\varepsilon > 0$ we find $x \in \mathcal{B} \setminus \{0\}$ with $x \geq 0$ such that $|\zeta - a|^2 x \leq \varepsilon x$. In the definition of β from (5.30) we can take the supremum over all $x \in \overline{\mathcal{B}}_+$, Thus, we get

$$\beta \leq \varepsilon - \inf_{y \in \mathcal{B}_+} \frac{\langle x\,, Sy\rangle}{\langle x\,, y\rangle} \leq \varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrarily small, we conclude $\beta \leq 0$.

Proof of (iv): Let $\zeta \in \mathbb{C}$ such that $\beta(\zeta) = \beta > 0$. Then (5.37) implies $\lambda_{\mathrm{PF}}(SD^{-1}) < 1$ with $D = D_{|\zeta-a|^2}$. Here, D is invertible because $\operatorname{ess\,inf}|\zeta - a| > 0$ due to (5.39). In particular, $B = (1 - SD^{-1})D$ is invertible.

Now we show $\operatorname{dist}(0, \operatorname{supp}\rho_\zeta) > 0$ to see one inclusion in the characterisation (5.34). The Dyson equation in the matrix representation, (5.15) is solved by

$$M_0 := \begin{pmatrix} 0 & (\overline{a - \zeta})^{-1} \\ (a - \zeta)^{-1} & 0 \end{pmatrix} \tag{5.40}$$

at $w = 0$. Furthermore, the associated stability operator (cf. (5.18))

$$\mathcal{L}_0 \colon \mathcal{B}^{2\times 2} \to \mathcal{B}^{2\times 2}, \quad R \mapsto M_0^{-1} R M_0^{-1} - \Sigma R = \begin{pmatrix} |a - \zeta|^2 r_{22} - S r_{22} & (a - \zeta)^2 r_{21} \\ (\overline{a - \zeta})^2 r_{12} & |a - \zeta|^2 r_{11} - S^* r_{11} \end{pmatrix} \tag{5.41}$$

is invertible because B_ζ is invertible and $\operatorname{ess\,inf}|a - \zeta| > 0$. Therefore (5.15) can be uniquely solved for sufficiently small w as an analytic function $w \mapsto M(\zeta, w)$ with $M(\zeta, 0) = M_0$ and we get

$$\mathcal{L}_0[\partial_w M(\zeta, w)|_{w=0}] = 1 \quad \text{and} \quad \partial_\eta M(\zeta, \mathrm{i}\eta)|_{\eta=0} = \mathrm{i}\mathcal{L}_0^{-1}[1]$$

In particular, $\operatorname{Im} M(\zeta, \mathrm{i}\eta)$ is positive definite for sufficiently small $\eta > 0$ because $\mathcal{L}_0^{-1}$ is positivity preserving, as can be seen from a Neumann series expansion using that $\lambda_{\mathrm{PF}}(SD^{-1}) < 1$. Therefore $M(\zeta, \mathrm{i}\eta)$ is the unique solution of (5.15) with $m_{11}(\zeta, \mathrm{i}\eta) = \mathrm{i}v_1(\zeta, \eta)$ and $m_{22}(\zeta, \mathrm{i}\eta) = \mathrm{i}v_2(\zeta, \eta)$. Since $m_{11}|_{\eta=0} = m_{22}|_{\eta=0} = 0$ we conclude that $\rho_\zeta(0) = \langle v_1(\zeta, 0)\rangle = 0$. The invertibility of $\mathcal{L}_0$ also implies analyticity of $M(\zeta, w)$ in w in a small neighbourhood of zero. Thus, $\rho_\zeta([-\varepsilon, \varepsilon]) = 0$ for $\varepsilon > 0$ sufficiently small and $\operatorname{dist}(0, \operatorname{supp}\rho_\zeta) > 0$.

To see the other inclusion in (5.34), let $\zeta \in \mathbb{C}$ be such that $\operatorname{dist}(0, \operatorname{supp} \rho_\zeta) \geq \delta$ for some $\delta > 0$. From [4, Lemma D.1 (iv)] we know that $M = M(\zeta, \mathrm{i}\eta)$ is locally a real analytic function of η with an expansion $M = M_0 + \mathrm{i}\eta M_1 + O(\eta^2)$, where $M_0 = M_0^*$. Taking the imaginary part of (5.15) at $w = \mathrm{i}\eta$, dividing both sides by η shows that

$$(M^*)^{-1} K_\eta M^{-1} = 1 + \Sigma[K_\eta]\,, \tag{5.42}$$

where $K_\eta = \frac{1}{\eta} \operatorname{Im} M = \operatorname{Re} M_1 + O(\eta)$. In particular, $K_0 := \lim_{\eta \downarrow 0} K_\eta$ exists and since $\operatorname{Im} M(\zeta, \mathrm{i}\eta) \sim_\delta \eta$ by Lemma 5.10 we get $K_0 \sim_\delta 1$. Evaluating (5.42) at $\eta = 0$ yields

$$M_0(\Sigma K_0) M_0 = K_0 - 1\,. \tag{5.43}$$

Taking the scalar product of (5.43) with the left Perron-Frobenius eigenvector of $R \mapsto M_0(\Sigma R) M_0$ and using $K_0 \sim_\delta 1$ we see that $\lambda_{\mathrm{PF}}(R \mapsto M_0(\Sigma R) M_0) < 1$. This is equivalent to $\lambda := \lambda_{\mathrm{PF}}(S D^{-1}) < 1$ with $D := D_{|a-\zeta|^2}$. Now let $u \in \mathcal{B}_+$ be the Perron-Frobenius eigenvector of SD^{-1}. Since $\varepsilon := \operatorname{ess\,inf} |a - \zeta| > 0$ we get with $y_0 := D^{-1} u$ that

$$(D - S) y_0 = (1 - \lambda) D y_0 > 0\,.$$

Thus,

$$\beta = \inf_{x > 0} \sup_{y > 0} \frac{\langle x\,, By \rangle}{\langle x\,, y \rangle} \geq \inf_{x > 0} \frac{\langle x\,, B y_0 \rangle}{\langle x\,, y_0 \rangle} \geq (1 - \lambda) \varepsilon^2 > 0\,.$$

This finishes the proof of (5.34), i.e. of (iv).

Proof of (iii): We have now collected enough information to improve (5.36) to (5.33). Indeed, by (5.37) and (5.38) it remains to show that $\lambda < 1$ implies $\beta > 0$. Due to (5.36) we already know $\beta \geq 0$ in case $\lambda < 1$. Now let $\beta = 0$ and $\lambda \leq 1$. Then we show that $\lambda = 1$. Indeed by the characterisation (5.34) we have $0 \in \operatorname{supp} \rho_\zeta$. Now we consider the identity

$$B_\zeta v_2 = \eta - v_2 (\eta + S^* v_1)(\eta + S v_2)$$

which follows from (4.2b). For some $\varepsilon > 0$ we add εv_2 to both sides and apply the inverse of $\varepsilon + D$ with $D = D_{|a-\zeta|^2}$. Then we take the scalar product with the right Perron-Frobenius eigenvector $x_\varepsilon \in \mathcal{B}_+$ of $S^*(\varepsilon + D)^{-1}$ corresponding to its Perron-Frobenius eigenvalue $\lambda_\varepsilon > 0$. Note that the Perron-Frobenius eigenvalues of $S^*(\varepsilon + D)^{-1}$, $(\varepsilon + D)^{-1} S$ and $S(\varepsilon + D)^{-1}$ all coincide. Thus we get

$$(1 - \lambda_\varepsilon) \langle x_\varepsilon v_2 \rangle = \eta \langle (\varepsilon + D)^{-1} x_\varepsilon \rangle - \langle x_\varepsilon (\varepsilon + D)^{-1} (v_2 (\eta + S^* v_1)(\eta + S v_2) - \varepsilon v_2) \rangle\,. \tag{5.44}$$

From [4, Corollary D.2] and $\langle v_i \rangle \sim v_i$ by Lemma 5.6 we see that $\eta / \langle v_i \rangle \to 0$ for $\eta \downarrow 0$. Thus, dividing (5.44) by $\langle v_2 \rangle$, taking the limit $\eta \downarrow 0$ and using (5.6) reveals

$$(1 - \lambda_\varepsilon) \langle x_\varepsilon \rangle \sim (1 - \lambda_\varepsilon) \langle x_\varepsilon k \rangle = \varepsilon \langle x_\varepsilon (\varepsilon + D)^{-1} k \rangle \sim \varepsilon \langle x_\varepsilon (\varepsilon + D)^{-1} S 1 \rangle = \varepsilon \langle S^* (\varepsilon + D)^{-1} x_\varepsilon \rangle = \varepsilon \lambda_\varepsilon \langle x_\varepsilon \rangle\,,$$

where $k := \limsup_{\eta \downarrow 0} \frac{v_2}{\langle v_2 \rangle} \sim 1$. Letting $\varepsilon \downarrow 0$ shows $\lambda = 1$. Thus, (5.33) is proven.

Proof of (v): By (5.34) we know that $\rho_\zeta(0) > 0$ implies $\beta(\zeta) \leq 0$. Thus, it suffices to show that for $\zeta \in \mathbb{C}$ with $\beta(\zeta) \leq 0$ we get $\beta(\zeta) = 0$ if and only if $\rho_\zeta(0) = 0$. Now let $\beta = \beta(\zeta) \leq 0$. As above, we consider the identity (5.44). First, suppose $\rho_\zeta(0) > 0$, i.e. we can analytically extend v to $\eta = 0$ by Proposition 5.11 and have $v|_{\eta=0} > 0$. Then in the limit $\eta \downarrow 0$ we find

$$(\lambda_\varepsilon - 1) \langle x_\varepsilon v_2 \rangle = \langle x_\varepsilon (\varepsilon + D)^{-1} (v_2 (S^* v_1 S v_2 - \varepsilon)) \rangle$$

Using $v_2 \sim \langle v_2 \rangle \sim \rho_\zeta(0)$ for small enough $\varepsilon > 0$ the right hand side satisfies

$$\langle x_\varepsilon (\varepsilon + D)^{-1} (v_2 (S^* v_1 S v_2 - \varepsilon)) \rangle \sim \rho_\zeta(0)^3 \langle x_\varepsilon (\varepsilon + D)^{-1} S 1 \rangle \sim \lambda_\varepsilon \rho_\zeta(0)^3 \langle x_\varepsilon \rangle.$$

Since $\langle x_\varepsilon v_2 \rangle \sim \rho_\zeta(0) \langle x_\varepsilon \rangle$ we infer $\lambda_\varepsilon - 1 \sim \lambda_\varepsilon \rho_\zeta(0)^2$. Thus, $\lambda > 1$ and by (5.33) therefore $\beta < 0$.

Conversely, let $v|_{\eta=0} = 0$. Then we know from Lemma 5.6 that $\delta := \operatorname{ess\,inf} |a - \zeta| > 0$. Since $\beta(\zeta) \leq 0$ the characterisation (5.34) implies $0 \in \operatorname{supp} \rho_\zeta$ and by (5.33) we have $\lambda \geq 1$. Since $\eta / \langle v \rangle \to 0$

for $\eta \downarrow 0$ by [4, Corollary D.2] we get, dividing (5.44) by $\langle v_2 \rangle$ and taking the limit $\eta \downarrow 0$, the scaling behaviour

$$1 - \lambda_\varepsilon \sim \varepsilon \lambda_\varepsilon \,.$$

This implies $\lambda_\varepsilon \leq 1$, thus $\lambda = 1$, and completes the proof of (v).

Proof of (ii): Let $\zeta \in \mathrm{Spec}(D_a)$. By (5.39) we know $\beta(\zeta) \leq 0$. Suppose $\beta(\zeta) = 0$. Then (5.35) would imply $\rho_\zeta(0) = 0$. However, this contradicts $\zeta \in \mathrm{Spec}(D_a)$ because of Lemma 5.6. This finishes the proof of the proposition. □

5.6 Expansion of v_1 and v_2 at the spectral edge

In this section we expand the solution v_1, v_2 of (4.2) around any edge point $\zeta_0 \in \mathbb{C}$. We will see later in Proposition 6.1 that points in the boundary of the support of the Brown measure σ, a probability measure in the complex plane associated with our data a and s and defined in that proposition, satisfy $\beta(\zeta_0) = 0$. Therefore we consider in this section a fixed $\zeta_0 \in \mathbb{C}$ with $\beta(\zeta_0) = 0$. The expansion of v_1, v_2 around ζ_0 is based on analytic perturbation theory for β. Throughout this section we will always assume $|\zeta - \zeta_0| + \eta \leq c$ for some sufficiently small positive constant $c \sim 1$, i.e. we assume that (ζ, η) lies within a small neighbourhood of $(\zeta_0, 0)$. We will see in Corollary 5.18 below that the function $\zeta \mapsto \beta(\zeta)$, used to define $\mathbb{S}$ in (5.30), coincides locally around ζ_0 with the isolated non-degenerate eigenvalue of B_ζ closest to zero. We assume **A5** and **A6** throughout the remainder of this section.

To shorten notation, we denote $v_i = v_i(\zeta, \eta)$. The identities

$$B_\zeta v_2 = \eta - v_2(\eta + S^* v_1)(\eta + S v_2), \qquad B_\zeta^* v_1 = \eta - v_1(\eta + S^* v_1)(\eta + S v_2), \tag{5.45}$$

which follow from (4.2a) and (4.2b), respectively, are used to expand v_1 and v_2 in a neighbourhood of ζ_0. We denote by $b = b_\zeta \in \mathcal{B}_+$ and $\ell = \ell_\zeta \in \mathcal{B}_+$ the right and left eigenvectors of $B = B_\zeta$, corresponding to the eigenvalue $\beta = \beta(\zeta)$ with normalisation $\langle b \rangle = \langle \ell \rangle = 1$, i.e.

$$Bb = \beta b, \qquad B^* \ell = \beta \ell \,. \tag{5.46}$$

The existence and uniqueness of b and ℓ is a consequence of analytic perturbation theory and Lemma 5.17 below. This lemma also implies that $\zeta \mapsto b_\zeta$ and $\zeta \mapsto \ell_\zeta$ are real analytic functions. The main result of this section is the following proposition.

Proposition 5.16. *Let s and a satisfy **A5** and **A6**. Furthermore, let $\zeta_0 \in \mathbb{C}$ such that $\beta(\zeta_0) = 0$. Then there is an open neighbourhood $U \subset \mathbb{C} \times \mathbb{R}^2$ of $(\zeta_0, 0, 0)$, an open neighbourhood $V \subset \mathbb{C} \times \mathbb{R}$ of $(\zeta_0, 0)$ and real analytic functions $\widetilde{w}_1, \widetilde{w}_2 \colon U \to \mathcal{B}$ such that*

$$v_1(\zeta, \eta) = \vartheta(\zeta, \eta) \ell_\zeta + \widetilde{w}_i(\zeta, \eta, \vartheta(\zeta, \eta)) \,, \qquad v_2(\zeta, \eta) = \vartheta(\zeta, \eta) b_\zeta + \widetilde{w}_i(\zeta, \eta, \vartheta(\zeta, \eta))$$

for $(\zeta, \eta) \in V$ and $\eta > 0$. Furthermore, $\vartheta = \vartheta(\zeta, \eta)$ satisfies

$$\vartheta^3 \langle \ell b (S^* \ell)(Sb) \rangle + \beta \vartheta \langle \ell b \rangle - \eta = g(\zeta, \eta, \vartheta) \,, \tag{5.47}$$

for all $(\zeta, \eta) \in V$ where $\ell = \ell_\zeta$, $b = b_\zeta, \beta = \beta(\zeta)$ and $g \colon U \to \mathbb{R}$ is a real analytic function, such that

$$g(\zeta, \eta, x) = O(|\eta x|^2 + |x|^5) \,, \qquad (\zeta, \eta, x) \in U \,.$$

The proof of Proposition 5.16 is the content of the remainder of this section and will be summarised at its end. We remark that as a solution to the cubic equation (5.47) the quantity ϑ and with it v_1, v_2 are not analytic at $\zeta = \zeta_0$ and $\eta = 0$.

The following lemma collects spectral properties of B_{ζ_0}. These properties yield corresponding properties of B_ζ for sufficiently small $|\zeta - \zeta_0|$, using analytic perturbation theory. We will use this idea throughout the remainder of this section after the statement of Lemma 5.17.

Lemma 5.17 (Properties of B)**.** *Let $\zeta_0 \in \mathbb{C}$ with $\beta(\zeta_0) = 0$ and $B_0 := B_{\zeta_0}$. Then there is a constant $\varepsilon > 0$ with $\varepsilon \sim 1$ such that*

$$\sup\Big\{\|(B_0-\omega)^{-1}\|_{\#} + \|(B_0^*-\omega)^{-1}\|_{\#} : \omega \in \mathbb{D}_{2\varepsilon} \setminus \mathbb{D}_\varepsilon\Big\} \lesssim 1 \tag{5.48}$$

for $\# = 2, \infty$. Here $\mathbb{D}_\varepsilon$ contains a single isolated non-degenerate eigenvalue 0 of B_0, i.e,

$$\mathbb{D}_\varepsilon \cap \operatorname{Spec}(B_0) = \{0\}\,, \qquad \dim\ker B_0^2 = 1\,. \tag{5.49}$$

Moreover, the right and left eigenvectors, $b_0 \in \mathcal{B}_+$ and $\ell_0 \in \mathcal{B}_+$, corresponding to this eigenvalue with normalisation $\langle b_0\rangle = \langle \ell_0\rangle = 1$ satisfy the bounds $\ell_0 \sim b_0 \sim 1$. Furthermore, if

$$P_0 := \frac{\langle \ell_0\,\cdot\rangle}{\langle \ell_0 b_0\rangle} b_0\,, \qquad Q_0 := 1 - P_0$$

denote the associated spectral projections then

$$\|B_0^{-1}Q_0\|_{\#} + \|(B_0^*)^{-1}Q_0^*\|_{\#} \lesssim 1\,. \tag{5.50}$$

Proof. Here we present the proofs of the bounds (5.48) and (5.50) for B_0. The corresponding bounds for B_0^* follow analogously. From Proposition 5.15 (ii) and since $\mathbb{S}$ is bounded we know that $|a-\zeta_0| \sim 1$. Thus, b_0 is the right eigenvector of $D^{-1}S$ with eigenvalue 1 and ℓ_0 is the right eigenvector of $D^{-1}S^*$ with eigenvalue 1, where $D := D_{|a-\zeta_0|^2}$. In particular, $b_0, \ell_0 \in \mathcal{B}_+$ by the Krein-Rutman theorem and the geometric multiplicity of the eigenvalue 0 of B_0 is 1. Furthermore, the non-degeneracy of the eigenvalue 0 is a consequence of $b_0, \ell_0 \in \mathcal{B}_+$. Indeed, suppose we had $\dim\ker B^2 > 1$. Then there would be a generalised eigenvector x with $Bx = b_0$ and $\langle \ell_0 b_0\rangle = \langle \ell_0 B_0 x\rangle = 0$ which contradicts $\ell_0 > 0$ and $b_0 > 0$. This proves (5.49), which together with (5.48) implies (5.50). The relation $b_0 \sim \ell_0 \sim 1$ is a direct consequence of $|a-\zeta_0| \sim 1$ and (5.6).

We are left with proving (5.48). Instead of controlling the resolvent of B_0, it suffices to bound the inverse of $1 - SD^{-1} - \omega D^{-1}$ because

$$\frac{1}{B_0-\omega} = \frac{1}{D}\left(\frac{1}{1-SD^{-1}-\omega D^{-1}}\widetilde{Q}_\omega + \frac{1}{1-SD^{-1}-\omega D^{-1}}\widetilde{P}_\omega\right), \tag{5.51}$$

where $\widetilde{P}_\omega$ and $\widetilde{Q}_\omega := 1 - \widetilde{P}_\omega$ are the analytic spectral projections associated with $SD^{-1} - \omega D^{-1}$ such that

$$\widetilde{P}_0 = \frac{\langle \ell_0\,\cdot\rangle}{\langle \ell_0 D b_0\rangle} D b_0\,.$$

Analytic perturbation theory can be applied to SD^{-1} because of Lemma B.2, which shows that the resolvent of the operator SD^{-1} is bounded in annulus around its isolated eigenvalue 1. Consequently, the first summand in (5.51) is bounded for sufficiently small $|\omega|$. The second summand admits the expansion

$$\frac{1}{1-SD^{-1}-\omega D^{-1}}\widetilde{P}_\omega = \frac{1}{\widetilde{\beta}(\omega)}\widetilde{P}_\omega\,, \qquad \widetilde{\beta}(\omega) = -\omega\frac{\langle \ell_0 b_0\rangle}{\langle \ell_0 D b_0\rangle} + O(|\omega|^2)$$

by standard analytic perturbation formulas, see e.g. [4, Lemma C.1]. Therefore the second summand is bounded for $\omega \in \mathbb{C} \setminus \mathbb{D}_\varepsilon$ for sufficiently small ε. $\square$

Corollary 5.18. *Let $\zeta_0 \in \mathbb{C}$ with $\beta(\zeta_0) = 0$. Then $0 \in \operatorname{Spec}(B_{\zeta_0})$, $\operatorname{ess\,inf}|a-\zeta_0| > 0$ and $\lambda_{\mathrm{PF}}(SD^{-1}_{|a-\zeta_0|^2}) = 1$. Furthermore, there is $\varepsilon > 0$ such that $\beta(\zeta)$ is an isolated non-degenerate eigenvalue of B_ζ for all $\zeta \in \zeta_0 + \mathbb{D}_\varepsilon$. In particular $\mathbb{D}_\varepsilon \ni \zeta \mapsto \beta(\zeta)$ is real analytic and has the expansion*

$$\begin{aligned}\beta(\zeta) = -2\mathrm{Re}\Big[\frac{\langle \ell_0 b_0 (a-\zeta_0)\rangle}{\langle \ell_0 b_0\rangle}\overline{(\zeta-\zeta_0)}\Big] + \Big(1 - 2\,\mathrm{Re}\Big[\frac{\langle \ell_0 \overline{(a-\zeta_0)} B_0^{-1} Q_0[b_0(a-\zeta_0)]\rangle}{\langle \ell_0 b_0\rangle}\Big]\Big)|\zeta-\zeta_0|^2 \\ - 2\,\mathrm{Re}\Big[\frac{\langle \ell_0 (a-\zeta_0) B_0^{-1} Q_0[b_0(a-\zeta_0)]\rangle}{\langle \ell_0 b_0\rangle}\overline{(\zeta-\zeta_0)}^2\Big] + O(|\zeta-\zeta_0|^3)\,,\end{aligned} \tag{5.52}$$

which implies the formulas

$$\partial_\zeta \beta(\zeta_0) = -\frac{\langle \ell_0 b_0 \overline{(a-\zeta_0)}\rangle}{\langle \ell_0 b_0\rangle}\,, \qquad \partial_\zeta \partial_{\bar\zeta} \beta(\zeta_0) = 1 - 2\,\mathrm{Re}\Big[\frac{\langle \ell_0 \overline{(a-\zeta_0)} B_0^{-1} Q_0[b_0(a-\zeta_0)]\rangle}{\langle \ell_0 b_0\rangle}\Big] \tag{5.53}$$

for the derivatives of β at $\zeta = \zeta_0$.

Proof. Let $\zeta_0 \in \mathbb{C}$ be such that $\beta(\zeta_0) = 0$. By Lemma 5.17 we have $0 \in \mathrm{Spec}(B_{\zeta_0})$ and by Proposition 5.15 (ii) we get $\operatorname{ess\,inf}|a-\zeta_0| > 0$. The fact that $\lambda_{\mathrm{PF}}(SD^{-1}_{|\zeta_0 - a|^2}) = 1$ was shown in (5.33).

Now we show that $\beta(\zeta)$ is an eigenvalue of B_ζ for sufficiently small $|\zeta - \zeta_0|$. Using analytic perturbation theory, let $b(\zeta)$ and $\ell(\zeta)$ be the right and left eigenvectors of B_ζ corresponding to the isolated non-degenerate eigenvalue $\widetilde\beta(\zeta)$ with $\widetilde\beta(\zeta_0) = 0$ that depends real analytically on ζ. As $\widetilde\beta(\zeta_0)$ is a real isolated eigenvalue and B_{ζ_0} as well as $B_\zeta - B_{\zeta_0}$ are invariant under complex conjugation, $\widetilde\beta(\zeta)$, $b(\zeta)$ and $\ell(\zeta)$ are also real. Since $\ell(\zeta_0) \sim b(\zeta_0) \sim 1$ we have $b(\zeta), \ell(\zeta) \in \mathcal{B}_+$ for sufficiently small $|\zeta - \zeta_0|$. Therefore

$$\widetilde\beta(\zeta) = \inf_{x>0} \frac{\langle x\,, B_\zeta b(\zeta)\rangle}{\langle x\,, b(\zeta)\rangle} \le \beta(\zeta) \le \sup_{y>0} \frac{\langle \ell(\zeta)\,, B_\zeta y\rangle}{\langle \ell(\zeta)\,, y\rangle} = \widetilde\beta(\zeta)\,,$$

which proves $\widetilde\beta = \beta$.

The expansion (5.52) is now a direct consequence of analytic perturbation theory, as we see e.g. by using [4, Lemma C.1] with $B = B_0 + E$ and $E = D_{|a-\zeta|^2} - D_{|a-\zeta_0|^2} = D_{|\zeta-\zeta_0|^2 - 2\mathrm{Re}\,(\overline{(a-\zeta_0)}(\zeta-\zeta_0))}$. $\square$

Due to analytic perturbation theory with ζ in a small neighbourhood of ζ_0 and by Lemma 5.17 we have $b \sim \ell \sim 1$. We split v_1 and v_2 according to the spectral decompositions of B^* and B, namely

$$v_1 = \vartheta_1 \ell + \widetilde v_1, \qquad v_2 = \vartheta_2 b + \widetilde v_2 \tag{5.54}$$

with the contributions $\vartheta_i = \vartheta_i(\zeta,\eta)$ to the eigendirections ℓ and b of B^* and B as well as their complements $\widetilde v_i = \widetilde v_i(\zeta,\eta)$ given as

$$\vartheta_1 := \frac{\langle b v_1\rangle}{\langle \ell b\rangle}\,, \qquad \vartheta_2 := \frac{\langle \ell v_2\rangle}{\langle \ell b\rangle}\,, \qquad \widetilde v_1 := Q^* v_1\,, \qquad \widetilde v_2 := Q v_2\,, \qquad Q := 1 - \frac{\langle \ell\,\cdot\,\rangle}{\langle \ell b\rangle} b\,.$$

To quantify the error terms we introduce

$$\alpha := \|v_1\|_\infty + \|v_2\|_\infty\,.$$

Projecting the identities (5.45) with Q and Q^*, respectively, leads to

$$B_\zeta \widetilde v_2 = O(\eta + \alpha^3), \qquad B_\zeta^* \widetilde v_1 = O(\eta + \alpha^3)\,. \tag{5.55}$$

Using $\|B^{-1}Q\|_\infty \lesssim 1$, a consequence of (5.50) and analytic perturbation theory , we find

$$\|\widetilde v_1\|_\infty + \|\widetilde v_2\|_\infty = O(\eta + \alpha^3)\,. \tag{5.56}$$

Because of $\langle v_1\rangle = \langle v_2\rangle$, i.e. by (5.2), (5.54) and the normalisation $\langle b\rangle = \langle \ell\rangle = 1$, (5.56) implies

$$\vartheta_1 = \vartheta_2 + O(\eta + \alpha^3)\,. \tag{5.57}$$

Inserting the decomposition (5.54) into (5.45) and using (5.56), as well as (5.57), leads to

$$\begin{aligned}
\beta\vartheta_2 b + B\widetilde v_2 &= \eta - \vartheta^3 b (Sb)(S^*\ell) + O(\eta\alpha^2 + \alpha^5)\,,\\
\beta\vartheta_1 \ell + B^*\widetilde v_1 &= \eta - \vartheta^3 \ell (Sb)(S^*\ell) + O(\eta\alpha^2 + \alpha^5)\,,
\end{aligned}$$

where we set $\vartheta := \frac{1}{2}(\vartheta_1 + \vartheta_2)$. Now we average the first equation against ℓ and the second equation against b, use $\langle b\rangle = \langle \ell\rangle = 1$ and then take the arithmetic mean of the resulting equations to find

$$\vartheta^3 \langle \ell b (S^*\ell)(Sb)\rangle + \beta\vartheta\langle \ell b\rangle - \eta = O(\eta\alpha^2 + \alpha^5)\,. \tag{5.58}$$

From this approximate cubic equation we conclude the scaling behaviours

$$\alpha \sim \vartheta \sim \sqrt{\max\{0,-\beta\}} + \frac{\eta}{\eta^{2/3}+|\beta|}\,, \qquad \|\widetilde{v}_1\|_\infty + \|\widetilde{v}_2\|_\infty \sim \eta + \max\{0,-\beta\}^{3/2}, \tag{5.59}$$

in the regime of sufficiently small α, where we used $\vartheta \geq 0$ and $\vartheta > 0$ for $\eta > 0$ to choose the correct branch of the solution. The corresponding argument is summarised in Lemma B.1 in the appendix.

To apply this lemma we absorb the $O(\alpha^5)$-term on the right hand side of (5.58) into the cubic term in ϑ on the left hand side, i.e. we write $O(\alpha^5) = \gamma\vartheta^3$ for some $\gamma = O(\alpha^2)$, which we absorb into the coefficient of the ϑ^3-term. Such rewriting is possible since $\alpha = O(\vartheta)$ in the regime where α is sufficiently small. This holds because $\vartheta \gtrsim \eta$ and $\alpha = O(\vartheta + \eta + \alpha^3)$ by (5.54), (5.57) and (5.56). Now we see that α is indeed small for (ζ, η) in a neighbourhood of $(\zeta_0, 0)$. Due to the characterisation of $\mathbb{S}$ in (5.35) we have $\lim_{\eta\downarrow 0} \alpha|_{\zeta=\zeta_0} = 0$. With $\beta(\zeta_0) = 0$ and because α is a continuous function of η when $\eta > 0$, the scaling (5.59) implies $\alpha|_{\zeta=\zeta_0} \sim \eta^{1/3}$. Since $\zeta \mapsto \beta(\zeta)$ is continuous by Proposition 5.15 (i) and α is a continuous function of ζ for any $\eta > 0$ the behaviour (5.59) holds as long as $\eta + |\zeta - \zeta_0|$ is sufficiently small.

We now summarise our insights by finishing the proof of Proposition 5.16.

Proof of Proposition 5.16. Following the computation leading to (5.55) we easily see that the right hand side of these equations are real analytic functions of ϑ, η, ζ and $\widetilde{v}_i$. By the implicit function theorem and the invertibility of B on the range of Q the $\widetilde{v}_i$ are real analytic functions of ϑ, η and ζ. Similarly, the right hand side of (5.58) is a real analytic function of ϑ, η and ζ. Together, we have proved Proposition 5.16 with

$$\widetilde{v}_i(\zeta,\eta) = \widetilde{w}_i(\vartheta(\zeta,\eta),\eta,\zeta)\,. \qquad \square$$

6 Properties of the Brown measure σ

We now present our main result about the existence and properties of the measure σ in the general setup introduced in Section 4. Here, we introduce the measure σ as a distributional derivative of the function L defined through

$$L(\zeta) := \int_0^\infty \Big(\langle v_1(\zeta,\eta)\rangle - \frac{1}{1+\eta}\Big)\mathrm{d}\eta \tag{6.1}$$

for each $\zeta \in \mathbb{C}$, where v_1 is the solution of the Dyson equation (4.2). The existence of this integral in the Lebesgue sense will be established in Lemma 6.5 below.

In the proof of Proposition 2.1 in Section 7.1 below, we relate this definition to the limiting measure of the empirical spectral distribution. In particular, we refer to (7.2), (7.4) and Proposition 7.1 below.

Proposition 6.1 (Properties of σ, general setup)**.** *Let $a \in \mathcal{B}$ and s satisfy **A5** and **A6**. If $L\colon \mathbb{C} \to \mathbb{R}$ is defined as in* (6.1) *then the following holds.*

(i) There is a unique probability measure on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} f(\zeta)\sigma(\mathrm{d}\zeta) = -\frac{1}{2\pi}\int_{\mathbb{C}} \Delta f(\zeta) L(\zeta)\mathrm{d}^2\zeta \tag{6.2}$$

for all $f \in C_0^2(\mathbb{C})$, where $\mathrm{d}^2\zeta$ denotes the Lebesgue measure on $\mathbb{C}$.

(ii) With respect to the Lebesgue measure, the measure σ from (6.2) *has a bounded density on $\mathbb{C}$, which we also denote by σ, i.e. $\sigma(\mathrm{d}\zeta) = \sigma(\zeta)\mathrm{d}^2\zeta$.*

(iii) On $\mathbb{S}$, the density $\zeta \mapsto \sigma(\zeta)$ is strictly positive and real analytic.

(iv) $\operatorname{supp}\sigma = \overline{\mathbb{S}}$ *and this set is bounded. Furthermore* $\operatorname{Spec}(D_a) \subset \mathbb{S}$.

(v) $\partial\mathbb{S}$ is a real analytic variety of (real) dimension at most 1.

(vi) There exists a unique continuous extension $\sigma\colon \overline{\mathbb{S}} \to [0,\infty)$ of the density $\sigma|_{\mathbb{S}}$ to $\overline{\mathbb{S}}$ such that $\sigma(\zeta) = g(\zeta)|\partial_\zeta\beta(\zeta)|^2$ for all $\zeta \in \partial\mathbb{S}$, where $g : \partial\mathbb{S} \to (0,\infty)$ is a strictly positive function that can be extended to a real analytic function on a neighbourhood of $\partial\mathbb{S}$.

The proof of Proposition 6.1 is presented at the end of this section, Section 6. In the next subsection, we first establish the existence of the measure σ via (6.2).

We recall that σ was called limiting spectral measure (cf. Definition 2.3) in the random matrix setup of Section 2. In the present general setup, we rename σ according to the next definition. The motivation for the naming of σ originates from the Brown measure of operators in von Neumann algebras and is explained by Proposition D.1 in Appendix D.

Definition 6.2. *We call the probability measure σ on $\mathbb{C}$, defined by* (6.2)*, the Brown measure associated with a and s.*

6.1 Representation of the Brown measure σ

For the next result, we recall the definition of $\mathbb{S}_\varepsilon$ from (5.16).

Proposition 6.3. *Let $a \in \mathcal{B}$ and s satisfy **A5** and let L be defined as in* (6.1)*. Then there is a unique probability measure σ on $\mathbb{C}$ such that* (6.2) *holds for all $f \in C_0^2(\mathbb{C})$. Moreover,* $\operatorname{supp}\sigma \subset \mathbb{S}_0$*. The measure σ satisfies the identity*

$$\sigma(\zeta) = -\lim_{\eta\downarrow 0} \frac{1}{\pi}\partial_{\bar\zeta}\langle y(\zeta,\eta)\rangle \tag{6.3}$$

in the sense of distributions, where y is the $(2,1)$ *component of M from* (5.12)*.*

The main idea of the proof of Proposition 6.3 will be to show that $-L$ is subharmonic and, therefore, the distribution $-\frac{1}{2\pi}\Delta L$ is induced by a measure. Before we present this proof, we establish a few necessary ingredients. The next lemma will, in particular, imply that L is well-defined.

Lemma 6.4 (Integrating $\langle v_1\rangle$ with respect to η)**.** *Let $a \in \mathcal{B}$ and s satisfy **A5**. Then, uniformly for $\zeta \in \mathbb{C}$ and $\eta > 0$, we have*

$$0 \le \langle v_1(\zeta,\eta)\rangle \lesssim \frac{1}{1+\eta}. \tag{6.4}$$

Furthermore, uniformly for any $T > 0$ and $\zeta \in \mathbb{C}$, we have

$$\int_0^T \left|\langle v_1(\zeta,\eta)\rangle - \frac{1}{1+\eta}\right|\mathrm{d}\eta \lesssim \min\Big\{T, \sqrt{1+|\zeta|}\Big\}, \qquad \int_T^\infty \left|\langle v_1(\zeta,\eta)\rangle - \frac{1}{1+\eta}\right|\mathrm{d}\eta \lesssim \frac{1+|\zeta|}{T}. \tag{6.5}$$

Proof. From (5.4) and Lemma 5.1, we immediately conclude (6.4). The bounds in (6.5) follow directly from (6.4) and (5.9). □

In the next lemma, we truncate the lower integration bound in the definition of L and, thus, obtain L_ε. It is an approximate version of L which is more regular in ζ and its derivative with respect to ζ is given by $\langle y(\zeta,\varepsilon)\rangle/2$.

Lemma 6.5 (Definition and derivatives of L_ε)**.** *Let $a \in \mathcal{B}$ and s satisfy **A5**. Then the following holds.*

(i) For each $\varepsilon \ge 0$, the integral

$$L_\varepsilon(\zeta) := \int_\varepsilon^\infty \Big(\langle v_1(\zeta,\eta)\rangle - \frac{1}{1+\eta}\Big)\mathrm{d}\eta \tag{6.6}$$

exists in Lebesgue sense for every $\zeta \in \mathbb{C}$ and the map $\mathbb{C} \to \mathbb{R}$, $\zeta \mapsto L_\varepsilon(\zeta)$ is continuous.

(ii) When $\varepsilon \downarrow 0$ then $L_\varepsilon \to L_0$ uniformly on $\mathbb{C}$.

(iii) For each $\varepsilon > 0$, L_ε is infinitely often continuously differentiable with respect to ζ and $\bar\zeta$ on $\mathbb{C}$.

(iv) For each $\varepsilon > 0$, $y(\cdot,\varepsilon)$ is infinitely often continuously differentiable with respect to ζ and $\bar\zeta$ and, for each $\zeta \in \mathbb{C}$,

$$\partial_\zeta L_\varepsilon(\zeta) = \frac{1}{2}\langle y(\zeta,\varepsilon)\rangle, \qquad \partial_{\bar\zeta} L_\varepsilon(\zeta) = \frac{1}{2}\langle \overline{y(\zeta,\varepsilon)}\rangle. \tag{6.7}$$

Proof. The bounds in (6.5) imply that $L_\varepsilon(\zeta)$ is well-defined for every $\varepsilon \geq 0$ and every $\zeta \in \mathbb{C}$. Moreover, $\zeta \mapsto L_\varepsilon(\zeta)$ is continuous on $\mathbb{C}$ for every $\varepsilon \geq 0$. The first bound in (6.5) implies that $L_\varepsilon \to L_0$ uniformly on $\mathbb{C}$ when $\varepsilon \downarrow 0$. To prove the differentiability of L_ε and $y(\cdot, \varepsilon)$, we conclude from (5.19) and the bound (A.2) in Lemma A.1 that for some constant $C \equiv C_\zeta > 0$, $\|\partial_\zeta v_1(\zeta, \mathrm{i}\eta)\| + \|\partial_{\bar\zeta} v_1(\zeta, \mathrm{i}\eta)\| + \|\partial_\eta v_1(\zeta, \mathrm{i}\eta)\| \leq C \max\{\eta^{-C}, \eta^{-2}\}$ for every $\eta > 0$ and every fixed $\zeta \in \mathbb{C}$. Therefore, L_ε is differentiable with respect to ζ and $\bar\zeta$ and we can take ∂_ζ and $\partial_{\bar\zeta}$ derivatives of $L_\varepsilon(\zeta)$ by exchanging the derivative and the integral in the definition of $L_\varepsilon(\zeta)$. Hence, we conclude from (5.17) that $\partial_\zeta L_\varepsilon(\zeta) = \langle M_{21}(\zeta, \mathrm{i}\varepsilon)\rangle/2 = \langle y(\zeta, \mathrm{i}\varepsilon)\rangle/2$ for all $\zeta \in \mathbb{C}$ and $\varepsilon > 0$, which implies (6.7). Furthermore, note that $\mathcal{L}$ defined in (5.18) below is the stability operator of the Matrix Dyson equation, (5.13). As it is invertible when $\eta > 0$ by Lemma A.1, the implicit function theorem implies that $y(\cdot, \varepsilon)$ is infinitely often differentiable. Therefore, the same holds of L_ε due to (6.7), which completes the proof of Lemma 6.5. ☐

Together with (6.7) in Lemma 6.5, the next proposition implies that $-\Delta L_\varepsilon \geq 0$.

Proposition 6.6. *Let $a \in \mathcal{B}$ and s satisfy **A5**. Then $-\partial_{\bar\zeta}\langle y(\zeta, \eta)\rangle > 0$ for all $\zeta \in \mathbb{C}$ and $\eta > 0$.*

The proof of Proposition 6.6 is presented in Section 6.2 below. We now have all ingredients for the proof of Proposition 6.3.

Proof of Proposition 6.3. First, we show that $-L$ is subharmonic since this implies the existence of a measure σ on $\mathbb{C}$ such that (6.2) holds. We know from Lemma 6.5 (i) that $L = L_0$ is a continuous function on $\mathbb{C}$. We now verify that $-L$ satisfies the circle average inequality. Since $L_\varepsilon \to L_0 = L$ uniformly on $\mathbb{C}$ for $\varepsilon \downarrow 0$ and L_ε is twice differentiable by Lemma 6.5, it suffices to show that $-\Delta L_\varepsilon \geq 0$ on $\mathbb{C}$. The latter implies that $-L_\varepsilon$ satisfies the circle average inequality and consequently, $-L$ satisfies it as well. From (6.7) in Lemma 6.5 and Proposition 6.6, we conclude $-\Delta_\zeta L_\varepsilon = -4\partial_{\bar\zeta}\partial_\zeta L_\varepsilon \geq 0$ on $\mathbb{C}$. Therefore, $-L$ is subharmonic on $\mathbb{C}$ and there exists a positive measure σ on $\mathbb{C}$ such that (6.2) holds.

Next, we prove that $\operatorname{supp}\sigma \subset \mathbb{S}_0$. Owing to [4, Lemma D.1], Section 5.2 and (5.19), derivatives of v_1 and v_2 with respect to ζ and $\bar\zeta$ are bounded locally uniformly for $\zeta \in \mathbb{C} \setminus \mathbb{S}_0$ and uniformly for $\eta \in (0, \infty)$. Therefore, $\mathbb{C} \setminus \mathbb{S}_0$ is open and Proposition 5.15 (ii) and (iv) imply $\operatorname{Spec}(D_a) \subset \mathbb{S}_0$. Moreover, L is twice continuously differentiable on $\mathbb{C} \setminus \mathbb{S}_0$ and for any $\zeta \in \mathbb{C} \setminus \mathbb{S}_0$, the second identity in (5.5), (6.7) with $\varepsilon \downarrow 0$ and $\|v_1(\zeta, \eta)\| + \|v_2(\zeta, \eta)\| \to 0$ for $\eta \downarrow 0$ and such ζ imply

$$\partial_\zeta L(\zeta) = \frac{1}{2}\Big\langle \frac{1}{a - \zeta} \Big\rangle, \qquad \partial_{\bar\zeta} L(\zeta) = \frac{1}{2}\Big\langle \frac{1}{\bar a - \bar\zeta} \Big\rangle \tag{6.8}$$

for all $\zeta \in \mathbb{C} \setminus \mathbb{S}_0$. In particular, $\Delta L = 4\partial_\zeta\partial_{\bar\zeta} L = 0$ on $\mathbb{C} \setminus \mathbb{S}_0$ and, therefore, $\operatorname{supp}\sigma \subset \mathbb{S}_0$ by (6.2).

What remains is to show that σ is a probability measure. The identities in (6.8) yield $L(\zeta) = -\langle \log|a - \zeta|\rangle + C$ for $\zeta \in \mathbb{C} \setminus \mathbb{S}_0$, where $C \in \mathbb{R}$ is independent of ζ. (One can check that $C = 0$ with some extra effort by expanding v_1 for large $|\zeta|$ using (4.2).) Owing to Remark 5.8, it is possible to choose $\varphi \sim 1$ such that $\mathbb{S}_0$ is contained in $\mathbb{D}_\varphi$. Note that φ can be chosen such that it depends only on the upper bounds in **A5** and any upper bound on $\|a\|_\infty$. In particular, $\operatorname{supp}\sigma \subset \mathbb{D}_\varphi$. In order to show that σ is a probability measure on $\mathbb{C}$, we pick a rotationally symmetric function $f \in C_0^\infty(\mathbb{C})$ such that $\operatorname{ran} f \subset [0, 1]$, $f \equiv 1$ on $\mathbb{D}_\varphi$ and $f \equiv 0$ on $\mathbb{C} \setminus \mathbb{D}_{2\varphi}$. Thus, (6.2) and $\operatorname{supp}\sigma \subset \mathbb{D}_\varphi$ imply

$$\sigma(\mathbb{C}) = \int_{\mathbb{C}} \sigma(\mathrm{d}\zeta) = -\frac{1}{2\pi}\int_{\mathbb{D}_{3\varphi}\setminus\mathbb{D}_\varphi} \Delta f(\zeta) L(\zeta)\mathrm{d}^2\zeta = -\frac{1}{2\pi}\int_{\mathbb{D}_{3\varphi}\setminus\mathbb{D}_\varphi} (\nabla f) \cdot (\nabla\langle \log|a - \cdot|\rangle)\mathrm{d}^2\zeta,$$

where in the second step we used Green's first identity and that the boundary terms vanish as $\nabla f \equiv 0$ on $\partial\mathbb{D}_{3\varphi} \cup \partial\mathbb{D}_\varphi$. We change to polar coordinates and obtain

$$\begin{aligned} -\frac{1}{2\pi}\int_{\mathbb{D}_{3\varphi}\setminus\mathbb{D}_\varphi} (\nabla f) \cdot (\nabla\langle \log|a - \cdot|\rangle) &= -\frac{1}{2\pi}\int_\varphi^{3\varphi} \partial_r f(r) \partial_r \Big(\int_0^{2\pi} \langle \log|a - r\mathrm{e}^{\mathrm{i}\theta}|\rangle \mathrm{d}\theta \Big) r\mathrm{d}r \\ &= -\int_\varphi^{3\varphi} \partial_r f(r)\mathrm{d}r = f(\varphi) = 1, \end{aligned}$$

where the second step follows from $\int_0^{2\pi} \log|w - r\mathrm{e}^{\mathrm{i}\theta}|\mathrm{d}\theta = 2\pi\log r$ for all $w \in \mathbb{D}_r$ (see e.g. [40, Example 5.7]). This shows that σ is a probability measure on $\mathbb{C}$ and, thus, completes the proof of Proposition 6.3. ☐

6.2 Strict positivity in the bulk

In this subsection we show that the Brown measure has strictly positive density in the bulk, i.e. inside $\mathbb{S}$ as defined in (5.30).

Proposition 6.7 (Strict positivity of Brown measure on $\mathbb{S}$)**.** *Let $a \in \mathcal{B}$ and s satisfy **A5** and **A6**. Then the Brown measure σ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$ and on $\mathbb{S}$ its density is strictly positive and real analytic.*

For the proof of Proposition 6.7 we compute the Brown measure through the formula (6.3), i.e. the distributional identity $\pi\sigma = -2\lim_{\eta\downarrow 0}\overline{\partial}_\zeta\langle E_{21}\,, M\rangle$. First we will see that the right hand side in (6.3) is non-negative and is in fact positive when evaluated at $\eta > 0$, i.e. we prove Proposition 6.6. After that we will see that under assumption **A6** the right hand side can be continuously extended to $\eta = 0$ away from $\partial\mathbb{S}$ and remains a bounded function of ζ, i.e. σ has a density.

Proof of Proposition 6.6. For $\eta > 0$ and $\zeta \in \mathbb{C}$, we start from the second identity in (5.19) and compute

$$\partial_{\overline{\zeta}}\langle E_{21}\,, M\rangle = \Big\langle E_{21}\,,\, \mathcal{L}^{-1}E_{21}\Big\rangle = \Big\langle \mathcal{C}_M^* E_{21}\,,\, E_{21}\Big\rangle + \Big\langle \mathcal{C}_M^* E_{21}\,,\, (1-\Sigma\mathcal{C}_M)^{-1}\Sigma\mathcal{C}_M E_{21}\Big\rangle,$$

where $\mathcal{C}_M R := MRM$. With

$$\Sigma\mathcal{C}_M E_{21} = \begin{pmatrix} \mathrm{i}S(v_2\overline{y}) & 0 \\ 0 & \mathrm{i}S^*(v_1\overline{y}) \end{pmatrix}, \qquad \mathcal{C}_{M^*}E_{21} = \begin{pmatrix} -\mathrm{i}v_1\overline{y} & \overline{y}^2 \\ -v_1v_2 & -\mathrm{i}v_2\overline{y} \end{pmatrix},$$

and the action of $\Sigma\mathcal{C}_M$ on diagonal matrices in $\mathcal{B}^{2\times 2}$ given by

$$\Sigma\mathcal{C}_M \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} S(|y|^2 r_1 - v_2^2 r_2) & 0 \\ 0 & S^*(-v_1^2 r_1 + |y|^2 r_2) \end{pmatrix}$$

this simplifies to

$$-\partial_{\overline{\zeta}}\langle E_{21}\,, M\rangle = \frac{1}{2}\langle v_1 v_2\rangle + \Big\langle \begin{pmatrix} v_1\overline{y} \\ v_2\overline{y} \end{pmatrix},\, (1-Y)^{-1}\begin{pmatrix} S(v_2\overline{y}) \\ S^*(v_1\overline{y}) \end{pmatrix}\Big\rangle, \tag{6.9}$$

where the scalar product on $\mathcal{B}^2$ is the one from (5.23) and $Y\colon \mathcal{B}^2 \to \mathcal{B}^2$ is defined as

$$Y\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} S(|y|^2 r_1 - v_2^2 r_2) \\ S^*(-v_1^2 r_1 + |y|^2 r_2) \end{pmatrix} = \begin{pmatrix} SD_{|y|}^2 & -SD_{v_2}^2 \\ -S^*D_{v_1}^2 & S^*D_{|y|}^2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

Now we introduce a symmetrisation of Y. For this purpose we define $\widehat{v} \in \mathcal{B}$ via

$$\widehat{v} := \sqrt{v_1(\eta + Sv_2)} = \sqrt{v_2(\eta + S^*v_1)}, \tag{6.10}$$

where the second identity is due to (5.1), and $V, F, T \in \mathcal{B}^{2\times 2}$ as

$$T := \begin{pmatrix} -\hat{v}^2 & |a-\zeta|^2\frac{v_1v_2}{\hat{v}^2} \\ |a-\zeta|^2\frac{v_1v_2}{\hat{v}^2} & -\hat{v}^2 \end{pmatrix}, \quad V := \begin{pmatrix} \frac{\hat{v}}{v_1} & 0 \\ 0 & \frac{\hat{v}}{v_2} \end{pmatrix}, \quad F := V^{-1}S_oV^{-1} \tag{6.11}$$

analogous to [3, (3.27)]. Then $VFTV^{-1} = Y$ and represented in terms of F and T the formula (6.9) reads

$$-2\partial_{\overline{\zeta}}\langle E_{21}\,, M\rangle = \Big\langle \begin{pmatrix} \widehat{v}\overline{y} \\ \widehat{v}\overline{y} \end{pmatrix},\, \Big(\frac{1}{X} + \frac{2}{1-FT}F\Big)\begin{pmatrix} \widehat{v}\overline{y} \\ \widehat{v}\overline{y} \end{pmatrix}\Big\rangle, \tag{6.12}$$

where we introduced

$$X := \begin{pmatrix} D(\frac{\hat{v}^2}{v_1v_2}|y|^2) & 0 \\ 0 & D(\frac{\hat{v}^2}{v_1v_2}|y|^2) \end{pmatrix}. \tag{6.13}$$

In particular, (6.9) is the quadratic form of a self-adjoint operator, evaluated on a vector in the subspace of $\mathcal{B}^2$ with identical entries in the first and second component. With the orthogonal projection onto this subspace represented by

$$E := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}^{2\times 2},$$

we have $TE = ET$, $XE = EX$ and

$$T = -1 + 2XE\,. \tag{6.14}$$

The representation (6.14) of T holds because of

$$1 = \widehat{v}^2 + \frac{\widehat{v}^2}{v_1 v_2}|y|^2\,,$$

which follows directly from (4.2), (5.3) and the definition of $\widehat{v}$ in (6.10).

Inserting (6.14) into (6.12), we see that proving positivity of the right hand side of (6.12) reduces to proving that the operator

$$\begin{aligned} E\Big(\frac{1}{X} + \frac{2}{1+F-2FXE}F\Big)E &= E\frac{1}{\sqrt{X}}\Big(1 + \frac{2}{1+\widetilde{F}\frac{1}{X} - 2\widetilde{F}E}\widetilde{F}\Big)\frac{1}{\sqrt{X}}E \\ &= E\frac{1}{\sqrt{X}}\Big(1 - \frac{2}{1+\widetilde{F}\frac{1}{X}}\widetilde{F}E\Big)^{-1}\frac{1}{\sqrt{X}}E = E\frac{1}{\sqrt{X}}\frac{1}{1-K}\frac{1}{\sqrt{X}}E \end{aligned} \tag{6.15}$$

on $\mathcal{B}^2$ is positive definite on the image of E, where we introduced

$$\widetilde{F} := \sqrt{X}\,F\sqrt{X} \qquad \text{and} \qquad K := E\frac{2}{1+\widetilde{F}\frac{1}{X}}\widetilde{F}E = E\sqrt{X}\frac{2F}{1+F}\sqrt{X}E \tag{6.16}$$

in the calculation and used $E^2 = E$ as well as $EX = XE$. Indeed, this is the case since

$$1 - K = \sqrt{X}\Big(\frac{1}{X} - \frac{2F_+}{1+F_+} + \frac{2F_-}{1-F_-}\Big)\sqrt{X} \geq \sqrt{X}\Big(1 - \frac{2F_+}{1+F_+}\Big)\sqrt{X} > 0\,, \tag{6.17}$$

where we first split the self-adjoint operator $F = F_+ - F_-$ into its positive and negative parts, as well as used $0 < X \leq 1$ and $\|F\| < 1$ for the first inequality. The final inequality follows from $\frac{2F_+}{1+F_+} < 1$ because $0 \leq F_+ \leq \|F\| < 1$. This completes the proof of Proposition 6.6. ☐

In the proof of Proposition 6.6 we have seen that the Brown measure admits the representation (cf. (6.3), (6.12), (6.14), (6.13) and (6.15))

$$\pi\sigma = \lim_{\eta\downarrow 0}\Big\langle \begin{pmatrix} \overline{e}_y\sqrt{v_1v_2} \\ \overline{e}_y\sqrt{v_1v_2} \end{pmatrix}, \frac{1}{1-K}\begin{pmatrix} \overline{e}_y\sqrt{v_1v_2} \\ \overline{e}_y\sqrt{v_1v_2} \end{pmatrix}\Big\rangle \tag{6.18}$$

in a distributional sense. Here, $e_y := \frac{y}{|y|} \in \mathcal{B}$ and K is defined in (6.16).

Under the additional assumption **A6** we get strict positivity of the density of the Brown measure inside $\mathbb{S}$.

Proof of Proposition 6.7. For the proof of analyticity of σ, we recall the definition of y from (5.3). We conclude from Proposition 5.15 (v), (5.31) and Proposition 5.11 that $\mathbb{S} \to \mathbb{C}$, $\zeta \mapsto y(\zeta, 0)$ is real analytic. Therefore, (6.3) implies that σ is real analytic on $\mathbb{S}$.

To prove a lower bound on σ, we use (6.18) and see that $1-K$ remains bounded on the image of E as $\eta \downarrow 0$. Indeed by the identity in (6.17) the only contribution to K that may potentially be unbounded is the one associated with F_-. However, $EF_-E|_{\eta=0} \leq 1-\varepsilon$ for some $\varepsilon > 0$ because of the spectral gap of F above -1 in Lemma A.2 and the fact that $(\widehat{v}, -\widehat{v})$, the eigenvector corresponding to eigenvalue -1, is mapped to zero by E. ☐

6.3 Edge behaviour of the Brown measure

Here we show that σ can be continuously extended to the boundary of $\mathbb{S}$ and compute its boundary values. Throughout this subsection we assume **A5** and **A6**.

Proposition 6.8 (Boundary values of σ)**.** *There is a unique continuous extension of $\sigma|_{\mathbb{S}}$ to $\overline{\mathbb{S}}$. This extension satisfies*

$$\sigma(\zeta_0) = \frac{1}{\pi}\frac{|\langle (a-\zeta_0)\ell_0 b_0\rangle|^2}{\langle |a-\zeta_0|^4\ell_0^2 b_0^2\rangle} = \frac{\langle \ell_0 b_0\rangle^2 |\partial_\zeta \beta(\zeta_0)|^2}{\pi\langle |a-\zeta_0|^4\ell_0^2 b_0^2\rangle} \tag{6.19}$$

for any $\zeta_0 \in \partial\mathbb{S}$, where $\ell_0 := \ell|_{\zeta=\zeta_0}$ and $b_0 := b|_{\zeta=\zeta_0}$.

Proof. We use the identity (6.3) to compute σ at some ζ in a neighbourhood of ζ_0 in terms of y. We expand y in terms of v_1, v_2 with the help of (5.5) and expand v_1, v_2 in terms of β.

For $\eta = 0$ equations (5.59) and (5.58) imply that either $v = 0$ for $\beta \geq 0$ or

$$\vartheta^2\langle \ell b (S^*\ell)(Sb)\rangle + \beta\langle \ell b\rangle = O(\vartheta^4) \tag{6.20}$$

for $\beta < 0$, or equivalently for $\zeta \in \mathbb{S}$. Note that according to Proposition 5.16 the right hand side of (6.20) is a real analytic function of ζ and ϑ. In particular, we can write $\vartheta = \vartheta(\zeta, 0)$ as

$$\vartheta = \sqrt{\frac{-\beta\langle \ell b\rangle}{\langle \ell^2 b^2 |a-\zeta|^4\rangle}}\,(1 + \beta h(\zeta, \sqrt{-\beta}))\,\mathbb{1}(\zeta \in \mathbb{S}) \tag{6.21}$$

for some real analytic function h, where we used (5.46).

According to (6.21) the leading order behaviour of ϑ is determined by β, whose local expansion is given in (5.52). To express y in terms of v_1 and v_2 we recall (5.5). From (5.5), (5.54), (5.56) and (5.57), we obtain the expansion

$$y = \frac{1}{a-\zeta} - \overline{(a-\zeta)}v_1 v_2 + O(\alpha^4) = \frac{1}{a-\zeta} - \vartheta^2\overline{(a-\zeta)}\ell b + O(\alpha^4 + \eta\alpha)\,.$$

By Proposition 5.16 and (6.21) it is easy to see that the error term is a real analytic function of ϑ and ζ in the regime $\zeta \in \mathbb{S}$ and for $\eta = 0$, i.e.

$$y|_{\eta=0} = \frac{1}{a-\zeta} - \vartheta^2\overline{(a-\zeta)}\ell b(1 + \vartheta^2 f(\zeta, \vartheta))\,\mathbb{1}(\zeta \in \mathbb{S})$$

with a real analytic f. Now we differentiate with respect to $\overline{\zeta}$ and use (6.21), in particular $|\partial_{\overline{\zeta}}\vartheta^2| \lesssim 1$, to get

$$\partial_{\overline{\zeta}} y|_{\eta=0} = -\overline{(a-\zeta)}\ell b\,\partial_{\overline{\zeta}}\vartheta^2 + O(\vartheta^2) = \frac{\overline{(a-\zeta)}\ell b\langle \ell b\rangle}{\langle |a-\zeta|^4\ell^2 b^2\rangle}\partial_{\overline{\zeta}}\beta + O(|\beta|^{1/2}) \tag{6.22}$$

for $\zeta \in \mathbb{S}$. The right hand side can be continuously extended to $\overline{\mathbb{S}}$. Indeed, by the definition 5.30 of $\mathbb{S}$ and the continuity of β from Proposition 5.15 (i) we have $\beta(\zeta_0) = 0$ for any $\zeta_0 \in \partial\mathbb{S}$ and thus (6.22) holds for $\zeta \in \mathbb{S}$ in a neighbourhood of such ζ_0 and the error term vanishes as $\zeta \to \zeta_0$. Inserting the formula (5.53) for $\partial_{\overline{\zeta}}\beta|_{\zeta=\zeta_0} = \overline{\partial_\zeta\beta}|_{\zeta=\zeta_0}$ into (6.22) and using (6.3) shows the claim (6.19). ☐

Lemma 6.9. *Let $\zeta \in \mathbb{C}$ such that $\beta(\zeta) = 0$ and $\partial_\zeta\beta(\zeta) = 0$. Then $\Delta\beta(\zeta) < 0$. In particular,*

$$\partial\mathbb{S} = \{\zeta \in \mathbb{C} : \beta(\zeta) = 0\}. \tag{6.23}$$

Proof. Let $\zeta \in \mathbb{C}$ with $\beta(\zeta) = 0$ and $\partial_\zeta\beta(\zeta) = 0$. From (5.53) we read off

$$\langle \ell b(a-\zeta)\rangle = 0\,, \qquad \partial_\zeta\partial_{\overline{\zeta}}\beta = 1 - 2\,\mathrm{Re}\Big[\frac{\langle \ell\overline{(a-\zeta)}B^{-1}b(a-\zeta)\rangle}{\langle \ell b\rangle}\Big]\,, \tag{6.24}$$

where we evaluated the expressions in (5.53) at $\zeta_0 = \zeta$ and omitted the projection Q_0 in the formula for $\partial_\zeta \partial_{\overline{\zeta}} \beta$ since $\langle \ell b(a-\zeta)\rangle = 0$ implies $Q_0[b(a-\zeta)] = b(a-\zeta)$.

We write $\partial_\zeta \partial_{\overline{\zeta}} \beta$ in terms of

$$K := D\Big(\frac{\sqrt{\ell}}{|a-\zeta|\sqrt{b}}\Big) S D\Big(\frac{\sqrt{b}}{|a-\zeta|\sqrt{\ell}}\Big), \qquad x := \sqrt{\ell b}\frac{a-\zeta}{|a-\zeta|}$$

and arrive at

$$\partial_\zeta \partial_{\overline{\zeta}} \beta = 1 - 2\,\mathrm{Re}\Big[\frac{\langle \overline{x}(1-K)^{-1}x\rangle}{\langle |x|^2\rangle}\Big] = -\frac{1}{\langle |x|^2\rangle}\Big\langle \frac{1}{1-K}x \,,\, (1-K^*K)\frac{1}{1-K}x\Big\rangle.$$

Here we used that $(1-K)^{-1}x$ is well-defined since $x \perp k$ due to (6.24), where $k := |a-\zeta|\sqrt{\ell b}$ is the right and left Perron-Frobenius eigenvector of K, i.e. $(1-K)k = 0 = (1-K^*)k$ due to (5.46) with $\beta = 0$. Furthermore, $(1-K^*K)k = 0$ implies that k is the Perron-Frobenius eigenvector of K^*K and thus $1 - K^*K$ is strictly positive definite on $k^\perp$, implying $\Delta\beta < 0$. Since β is real analytic in a neighbourhood of ζ with $\beta(\zeta) = \partial_\zeta\beta(\zeta) = 0$ according to Corollary 5.18 and such ζ cannot be a local minimum of β due to $\Delta\beta(\zeta) < 0$ we infer (6.23). ☐

As a consequence of Lemma 6.9 the definition of $\mathbb{S}$ in (5.30) and Proposition 5.15 (iv) yields

$$\mathbb{S}_0 = \overline{\mathbb{S}}. \tag{6.25}$$

We now have all ingredients to prove Proposition 6.1.

Proof of Proposition 6.1. Part (i) is Proposition 6.3. Items (ii) and (iii) are proved in Proposition 6.7. For the proof of (iv), we conclude $\operatorname{supp}\sigma \subset \mathbb{S}_0 = \overline{\mathbb{S}}$ from Proposition 6.3 and (6.25). Moreover, $\mathbb{S} \subset \operatorname{supp}\sigma$ follows from (iii), which completes the proof of (iv). Note that $\partial\mathbb{S}$ is a real analytic variety due to (6.23) and Corollary 5.18. The dimension of $\partial\mathbb{S}$ is at most one as $\Delta\beta(\zeta) \neq 0$ if $\partial_\zeta\beta(\zeta) = 0$ by Lemma 6.9. This shows (v). Part (vi) follows from Proposition 6.8 and the fact that g is real analytic by Corollary 5.18. ☐

7 Proof of main results – Proposition 2.1 and Theorem 2.4

This section is devoted to the proofs of our main results, Proposition 2.1 and Theorem 2.4. They are derived from the results in the previous sections as well as some inputs from [5, 34, 32]. The underlying idea for both derivations is the Hermitization approach going back to Girko [25] which allows to understand the eigenvalue density of $X + A$ by understanding the spectra of the Hermitian matrices $(H_\zeta)_{\zeta\in\mathbb{C}}$ defined through

$$H_\zeta := \begin{pmatrix} 0 & X + A - \zeta \\ (X+A-\zeta)^* & 0 \end{pmatrix}. \tag{7.1}$$

The usefulness of H_ζ becomes apparent from the following properties. A complex number $\zeta \in \mathbb{C}$ is an eigenvalue of $X + A$ if and only if H_ζ has a nontrivial kernel. Furthermore, the spectrum of H_ζ is symmetric around zero and its non-negative eigenvalues coincide with the singular values of $X + A - \zeta$ (with multiplicities).

7.1 Proof of Proposition 2.1

After this general explanation, we now focus on the proof of Proposition 2.1. To that end, we now explain in detail how the empirical spectral distribution of $X + A$ is expressed in terms of the family $(H_\zeta)_{\zeta\in\mathbb{C}}$.

First, as $\log|\cdot|$ is the fundamental solution for the Laplace operator on $\mathbb{C}$, we obtain

$$\frac{1}{n}\sum_{\xi\in\mathrm{Spec}(X+A)} f(\xi) = \frac{1}{2\pi n}\sum_{\xi\in\mathrm{Spec}(X+A)}\int_{\mathbb{C}} \Delta f(\zeta)\log|\xi-\zeta|\mathrm{d}^2\zeta = \frac{1}{4\pi n}\int_{\mathbb{C}}\Delta f(\zeta)\log|\det H_\zeta|\mathrm{d}^2\zeta, \tag{7.2}$$

where the last step follows from

$$\sum_{\xi \in \operatorname{Spec}(X+A)} \log|\xi - \zeta| = \log|\det(X + A - \zeta)| = \frac{1}{2} \log|\det H_\zeta|. \tag{7.3}$$

We can now express the log-determinant of H_ζ as an integral of the normalised trace of the resolvent $G(\zeta, \mathrm{i}\eta) := (H_\zeta - \mathrm{i}\eta)^{-1}$ of H_ζ on the imaginary axis; this expression reads as

$$\log|\det H_\zeta| = -2n \int_0^T \operatorname{Im} \langle G(\zeta, \mathrm{i}\eta) \rangle \mathrm{d}\eta + \log|\det(H_\zeta - \mathrm{i}T)| \tag{7.4}$$

for any $T > 0$ (see [46] for an application of (7.4) in a similar context). Here and in the following, for a $K \times K$-matrix $R \in \mathbb{C}^{K \times K}$, we denote by $\langle R \rangle = \frac{1}{K} \operatorname{Tr} R$ the normalized trace of R.

For the proof of Proposition 2.1, we follow the strategy of [6, proof of Theorem 2.3], which is presented in [6, Section 3.2].

The next proposition, which follows directly from results in [5], shows that $\langle G(\zeta, \mathrm{i}\eta) \rangle$ is approximately deterministic. Given (7.2) and (7.4), this explains the origin of the definition of σ via (6.2) and (6.1).

In the next proposition and throughout this section, we use the following notion of high probability events. We say that a sequence of events $(\Omega_n)_{n \in \mathbb{N}}$ occurs *with very high probability* if for each $\nu \in \mathbb{N}$, there is a constant $C_\nu > 0$ (i.e. C_ν does not depend on n) such that $\mathbb{P}(\Omega_n) \geq 1 - C_\nu n^{-\nu}$ for all $n \in \mathbb{N}$.

Proposition 7.1 (Deterministic approximation of resolvent of H_ζ, averaged version)**.** *Let X and $A = D(a)$ for some $a = (a_i)_{i=1}^n \in \mathbb{C}^n$ satisfy **A1**, **A2** and $\|a\|_\infty = \max_{i=1}^n |a_i| \lesssim 1$. Let $(v_1^{(n)}, v_2^{(n)})$ be the solution of (4.2) with $\mathfrak{X} = [\![n]\!]$, μ the normalized counting measure on $[\![n]\!]$ as well as a and s with $s(i,j) := n\mathbb{E}|x_{ij}|^2$ for $i, j \in [\![n]\!]$ interpreted as functions on $[\![n]\!]$ and $[\![n]\!]^2$, respectively. Let $\varphi > 0$ be fixed. Then there are universal constants $\delta > 0$ and $P \in \mathbb{N}$ such that*

$$|\langle G(\zeta, \mathrm{i}\eta) \rangle - \mathrm{i}\langle v_1^{(n)}(\zeta, \eta) \rangle| \leq \frac{n^{P\delta}}{(1 + \eta^2)n}$$

with very high probability uniformly for all $n \in \mathbb{N}$, $\eta \in [n^{-\delta}, \infty)$ and $\zeta \in \mathbb{D}_\varphi$.

Proof. The matrix $X + A$ is a Kronecker matrix according to [5, Definition 2.1] with the choices $L = 1$, $\ell = 1$, $\widetilde{\alpha}_1 = 1$, $X_1 = X$, $\beta_1 = 0$, $Y_1 = 0$ and $\widetilde{a}_i = a_i$ for all $i \in [\![n]\!]$. In particular, the Hermitization H_ζ defined in (7.1) is also a Kronecker matrix. Moreover, H_ζ satisfies the assumptions of [5, Lemma B.1 (ii)] due to **A1**, **A2** and $\|a\|_\infty \lesssim 1$. Since the Hermitized matrix Dyson equation from [5, eq.s (2.2) – (2.6)] coincides with the matrix Dyson equation, (5.13), associated with (4.2) for $(v_1^{(n)}, v_2^{(n)})$ (see Section 5.2 for more explanations), [5, eq. (B.5)] and [5, eq. (4.46)] imply Proposition 7.1. $\square$

To control the integral in (7.4), we need to ensure that the smallest singular value $\mathrm{s}_{\min}(X + A - \zeta)$ of $X + A - \zeta$, i.e. the smallest, in modulus, eigenvalue of H_ζ, is not too small. This is ensured by the next proposition. It follows easily from the main results of [34] in the real case, and [32] in the complex case.

Proposition 7.2 (Smallest singular value of $X + A - \zeta$)**.** *Let $X \in \mathbb{C}^{n \times n}$ satisfy **A1**, **A2** for $\nu = 2$ and all $i, j \in [\![n]\!]$ and **A3**. Then the following holds.*

(i) For any constant $K > 0$, there are constants $C > 0$ and $c > 0$ such that

$$\mathbb{P}\big(\mathrm{s}_{\min}(X + A - \zeta) \leq \varepsilon n^{-1}\big) \leq C\varepsilon + 2\mathrm{e}^{-cn}$$

for all $\varepsilon \geq 0$, $\zeta \in \mathbb{D}_K$ and $A \in \mathbb{C}^{n \times n}$ satisfying $\|A\|_{\mathrm{hs}}^2 \leq Kn$.

(ii) Let K, A and ζ be as in (i). For any $\delta > 0$, the event $\{\mathrm{s}_{\min}(X + A - \zeta) \geq \mathrm{e}^{-n^\delta}\}$ holds with very high probability.

Proof. We check the conditions of [32, Theorem 1.1] for $A_n = \sqrt{n}(X + A - \zeta)$. The assumptions of Proposition 7.2 on A and ζ as well as **A2** with $\nu = 2$ imply $\mathbb{E}\|X + A - \zeta\|_{\mathrm{hs}}^2 \lesssim n$. The second condition of [32, Theorem 1.1] is identical to **A3**. Therefore, Proposition 7.2 follows from [32, Theorem 1.1]. $\square$

In addition to the control on the smallest singular value of $X + A - \zeta$, we also need to bound the number of its small singular values. This is the content of the next lemma, which follows simply from Proposition 7.1 and an upper bound on $|\langle M(\zeta, \mathrm{i}\eta)\rangle|$.

Lemma 7.3 (Number of small singular values of $X + A - \zeta$)**.** *Let X satisfy **A1** and **A2** and let $A = D(a)$ for some $a \in \mathbb{C}^n$ with $\|a\|_\infty \lesssim 1$. Let $\varphi > 0$ be fixed. Then there is a universal constant $\delta > 0$ such that*

$$|\mathrm{Spec}(H_\zeta) \cap [-\eta, \eta]| \lesssim n\eta$$

with very high probability uniformly for all $\eta \in [n^{-\delta}, \infty)$ and $\zeta \in \mathbb{D}_\varphi$.

Proof. Proposition 7.1 and (6.4) imply that the trace of $G(\zeta, \mathrm{i}\eta)$ is bounded by n with very high probability. More precisely, $|\mathrm{Tr}\, G(\zeta, \mathrm{i}\eta)| \lesssim n$ with very high probability uniformly for all $\eta \in [n^{-\delta}, \infty)$ and $\zeta \in \mathbb{D}_\varphi$. Hence, we set $\Sigma_\eta := \mathrm{Spec}(H_\zeta) \cap [-\eta, \eta]$ and estimate

$$\frac{|\Sigma_\eta|}{2\eta} \leq \sum_{\lambda \in \Sigma_\eta} \frac{\eta}{\lambda^2 + \eta^2} \leq \mathrm{Im}\ \mathrm{Tr}\, G(\zeta, \mathrm{i}\eta) \lesssim n. \qquad \square$$

We apply the previous results, i.e. Proposition 7.1, Proposition 7.2 and Lemma 7.3, to the right-hand side of (7.7) by discretizing the integral in ζ through the next lemma.

Lemma 7.4 (Monte Carlo Sampling)**.** *Let $\Omega \subset \mathbb{C}$ be bounded and of positive Lebesgue measure. Let μ be the normalised Lebesgue measure on Ω and $F \colon \Omega \to \mathbb{C}$ square-integrable with respect to μ. Let $m \in \mathbb{N}$ and $\xi_1, \ldots, \xi_m$ be independent random variables distributed according to μ. Then, for any $\delta > 0$, we have*

$$\mathbb{P}\bigg(\bigg|\frac{1}{m}\sum_{i=1}^m F(\xi_i) - \int_\Omega F \mathrm{d}\mu\bigg| \leq \frac{1}{\sqrt{m\delta}}\Big(\int_\Omega \Big|F - \int_\Omega F\mathrm{d}\mu\Big|^2\Big)^{1/2}\bigg) \geq 1 - \delta.$$

Lemma 7.4 is a special case of [46, Lemma 36]. For the convenience of the reader, we present the very short proof here.

Proof. Each of the i.i.d. random variables $F(\xi_1), \ldots, F(\xi_m)$ has expectation $\int_\Omega F \mathrm{d}\mu$ and variance $\int_\Omega |F - \int_\Omega F \mathrm{d}\mu|^2 \mathrm{d}\mu$. Hence, Chebysheff's inequality yields Lemma 7.4. $\square$

The final ingredient for the proof of Proposition 2.1 is the following remark which asserts that all eigenvalues of $X + A$ are contained in $\mathbb{S}_\varepsilon$ defined in (5.16) with very high probability.

Remark 7.5 (No outlier eigenvalues of $X + A$)**.** *If X satisfies **A1** and **A2** and $A = D(a)$ for some $a \in \mathbb{C}^n$ with $\|a\|_\infty \lesssim 1$ then, for every $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$, all eigenvalues of $X + A$ are contained in $\mathbb{S}_\varepsilon$ with very high probability, i.e. for each $\nu > 0$, there is a constant $C \equiv C_{\varepsilon,\delta,\nu} > 0$ such that*

$$\mathbb{P}\big(\mathrm{Spec}(X + A) \subset \mathrm{Spec}_{\varepsilon-\delta}(X + A) \subset \mathbb{S}_\varepsilon\big) \geq 1 - Cn^{-\nu}$$

for all $n \in \mathbb{N}$. This follows directly from [5, Lemma 6.1]. Here, we used that $X + A$ is a Kronecker matrix according to [5, Definition 2.1] and that the Dyson equation (5.13) *and [5, eq. (2.6)] coincide as explained in the proof of Proposition 7.1.*

We have now collected all ingredients for the proof of Proposition 2.1, which we present next.

Proof of Proposition 2.1. Let $(v_1^{(n)}, v_2^{(n)})$ and $\sigma^{(n)}$ be defined[2] as in Corollary C.2. Because of Corollary C.2, it suffices to show that for each $f \in C_b(\mathbb{C})$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\Big|\frac{1}{n}\sum_{\xi\in\operatorname{Spec}(X+A)} f(\xi) - \int_{\mathbb{C}} f\mathrm{d}\sigma^{(n)}\Big| > \varepsilon\Big) = 0. \tag{7.5}$$

The main part of the proof will be to show the existence of a constant $\delta > 0$ such that

$$\Big|\frac{1}{n}\sum_{\zeta\in\operatorname{Spec} X} f(\zeta) - \int_{\mathbb{C}} f(\zeta)\sigma^{(n)}(\mathrm{d}\zeta)\Big| \le n^{-\delta}\|\Delta f\|_{\mathrm{L}^1} + n^{-10}\|\Delta f\|_{\mathrm{L}^3} \tag{7.6}$$

with very high probability uniformly for all $f \in C_0^2(\mathbb{C})$ satisfying $\operatorname{supp} f \subset \mathbb{D}_\varphi$ for any fixed constant $\varphi > 0$.

We now explain how (7.6) implies (7.5), i.e. Proposition 2.1. If $f \in C_0^2(\mathbb{C})$ then this is obvious. Let $f \in C_b(\mathbb{C}) \setminus C_0^2(\mathbb{C})$. Owing to Remark 7.5, we know that $\operatorname{Spec}(X + A) \subset \mathbb{S}_1$ with very high probability. We note that $\mathbb{S}_1 \subset \mathbb{D}_\varphi$ for some $\varphi \sim 1$ by Remark 5.8. By possibly increasing $\varphi \sim 1$, we also have $\operatorname{supp}\sigma^{(n)} \subset \mathbb{D}_\varphi$ due to Corollary C.2 (ii). Therefore, it suffices to consider $f \in C_b(\mathbb{C})$ with $\operatorname{supp} f \subset \mathbb{D}_{\varphi+1}$. Then we find $f_\varepsilon \in C_0^2(\mathbb{C})$ such that $\|f - f_\varepsilon\|_{\mathrm{L}^\infty} \le \varepsilon/2$, $\operatorname{supp} f_\varepsilon \subset \mathbb{D}_{\varphi+1}$ and $\|\Delta f_\varepsilon\|_{\mathrm{L}^1} + \|\Delta f_\varepsilon\|_{\mathrm{L}^3} \lesssim_\varepsilon 1$. Hence, approximating f by f_ε in (7.5) and using (7.6) for f_ε shows that (7.6) implies (7.5).

It remains to show (7.6). We fix a constant $\varphi > 0$ and set $\Omega = \mathbb{D}_\varphi$. For any $T > 0$, we conclude from (7.2), (7.4), Corollary C.2 (ii) and the second bound in (6.5) that

$$\frac{1}{n}\sum_{\xi\in\operatorname{Spec}(X+A)} f(\xi) - \int_{\mathbb{C}} f(\zeta)\sigma^{(n)}(\mathrm{d}\zeta) = \int_\Omega F(\zeta)\frac{\mathrm{d}^2\zeta}{|\Omega|} + O(T^{-1}\|\Delta f\|_{\mathrm{L}^1}), \tag{7.7}$$

where

$$F(\zeta) := \frac{|\Omega|}{\pi}(\Delta f(\zeta))h(\zeta), \qquad h(\zeta) := \frac{1}{n}\sum_{\xi\in\operatorname{Spec}(X+A)} \log|\xi - \zeta| + \int_0^T \Big(\langle v_1^{(n)}(\zeta,\eta)\rangle - \frac{1}{1+\eta}\Big)\mathrm{d}\eta.$$

Note that h and, thus, F depend on the choice of T.

We now estimate $\int_\Omega F(\zeta)\frac{\mathrm{d}^2\zeta}{|\Omega|}$ by applying Lemma 7.4 to it. Since $\zeta \mapsto \log|\xi - \zeta|$ lies in $\mathrm{L}^p(\Omega)$ for every $p \in [1,\infty)$, the first bound in (6.5) implies that, for every $p \in [1,\infty)$, $\|h\|_{\mathrm{L}^p(\Omega)} \lesssim_p 1$ uniformly for $T > 0$. Therefore, $F \in L^2(\Omega)$ and Lemma 7.4 with $\delta = n^{-\nu}$ and $m = n^{\nu-130}$ yields

$$\Big|\int_\Omega F(\zeta)\frac{\mathrm{d}^2\zeta}{|\Omega|} - \frac{1}{m}\sum_{i=1}^m F(\xi_i)\Big| \le n^{-10}\|F\|_{\mathrm{L}^2} \lesssim n^{-11}\|\Delta f\|_{\mathrm{L}^3} \tag{7.8}$$

with very high probability, where $\xi_1, \ldots, \xi_m$ are independent random variables distributed according to the normalized Lebesgue measure on Ω.

What remains for the proof of (7.6) is to bound $\frac{1}{m}\sum_{i=1}^m F(\xi_i)$. To that end, we set $T = n^{100}$ and show in the following that, for all small enough $\delta > 0$,

$$|F(\zeta)| \le n^{-\delta}|\Delta f(\zeta)| \tag{7.9}$$

with very high probability uniformly for all $\zeta \in \Omega$. We set $\eta_* := n^{-\delta}$ and introduce

$$h_1(\zeta) := \int_{\eta_*}^T (\langle v_1^{(n)}(\zeta,\eta)\rangle - \operatorname{Im}\langle G(\zeta,\mathrm{i}\eta)\rangle)\mathrm{d}\eta, \qquad h_2(\zeta) := -\int_0^{\eta_*} \operatorname{Im}\langle G(\zeta,\mathrm{i}\eta)\rangle\mathrm{d}\eta$$

$$h_3(\zeta) := \frac{1}{4n}\sum_{\lambda\in\operatorname{Spec}(H_\zeta)} \log\Big(1 + \frac{\lambda^2}{T^2}\Big) - \log\Big(1 + \frac{1}{T}\Big), \qquad h_4(\zeta) := \int_0^{\eta_*} \langle v_1^{(n)}(\zeta,\eta)\rangle\mathrm{d}\eta.$$

[2] We note that $(v_1^{(n)}, v_2^{(n)})$ from Corollary C.2 can be naturally identified with $(v_1^{(n)}, v_2^{(n)})$ from Proposition 7.1. In particular, $\langle v_1^{(n)}\rangle$ yields the same result for either definition and this is the only quantity derived from $(v_1^{(n)}, v_2^{(n)})$ that plays a role in the following.

Hence, owing to (7.3), (7.4) and $\int_0^T (1+\eta)^{-1}\mathrm{d}\eta = \log(1+T)$, we obtain the decomposition $h(\zeta) = h_1(\zeta) + h_2(\zeta) + h_3(\zeta) + h_4(\zeta)$.

Next, we estimate the terms $h_1, \ldots, h_4$ individually. For h_1, Proposition 7.1, a union bound and a continuity argument in η imply $|h_1(\zeta)| \le n^{-1+P\delta+\delta}$ with very high probabilityfootnote 2. A simple computation shows that

$$-h_2(\zeta) = \frac{1}{4n} \sum_{\lambda \in \mathrm{Spec}(H_\zeta)} \log\left(1 + \frac{\eta_*^2}{\lambda^2}\right) \le \frac{1}{4n} \sum_{\lambda \in \mathrm{Spec}(H_\zeta) \cap [-\eta_*^{1/2}, \eta_*^{1/2}]} \log\left(1 + \frac{\eta_*^2}{\lambda^2}\right) + \eta_*,$$

where in the last step we used that $\log(1+\eta_*^2\lambda^{-2}) \le \log(1+\eta_*) \le \eta_*$ if $|\lambda| > \eta_*^{1/2}$. To estimate the remaining sum, we conclude from Proposition 7.2 and Lemma 7.3 that

$$\frac{1}{4n} \sum_{\lambda \in \mathrm{Spec}(H_\zeta) \cap [-\eta_*^{1/2}, \eta_*^{1/2}]} \log\left(1+\frac{\eta_*^2}{\lambda^2}\right) \lesssim \frac{\log \eta_* + |\log \min_{\lambda \in \mathrm{Spec}(H_\zeta)} |\lambda||}{n} |\mathrm{Spec}(H_\zeta) \cap [-\eta_*^{1/2}, \eta_*^{1/2}]| \lesssim n^\varepsilon \eta_*^{1/2}$$

with very high probability for any $\varepsilon > 0$. Therefore, $|h_2(\zeta)| \lesssim n^{-\delta/2+\varepsilon}$, which yields $|h_2(\zeta)| \le n^{-\delta}$ by shrinking δ. To estimate h_3, we use $\log(1+x) \le x$ and obtain

$$|h_3(\zeta)| \le \frac{1}{4nT^2} \mathrm{Tr}(H_\zeta)^2 + T^{-1} = \frac{1}{2nT^2} \sum_{i,j=1}^n (\overline{x_{ji}} + (\bar{a}_i - \bar{\zeta})\delta_{ji})(x_{ij} + (a_i - \zeta)\delta_{ij}) + T^{-1} \lesssim T^{-1}$$

since $|x_{ij}| \le n^{-1/2+\varepsilon}$ with very high probability due to Assumption **A2** and $|a_i| + |\zeta| \lesssim 1$ as $\|a\|_\infty \lesssim 1$ and $\zeta \in \mathbb{D}_\varphi$. Since $a^{(n)}$ and $s^{(n)}$ from (C.1) satisfy $\|a^{(n)}\|_\infty \lesssim 1$ and **A5** with the same constants as a and s, Lemma 5.1 implies $|h_4(\zeta)| \lesssim \eta_*$ uniformly for all $n \in \mathbb{N}$. This completes the proof of (7.9).

The bound (7.9) implies

$$\frac{1}{m} \sum_{i=1}^m |F(\xi_i)| \le \frac{n^{-\delta}}{m} \sum_{i=1}^m |\Delta f(\xi_i)| \le n^{-\delta} \|\Delta f\|_{\mathrm{L}^1} + n^{-11} \|\Delta f\|_{\mathrm{L}^2} \tag{7.10}$$

with very high probability, where the second step follows from Lemma 7.4 with $F = \Delta f$.

Finally, we combine (7.7), (7.8) and (7.10) and, thus, obtain (7.6), which completes the proof of Proposition 2.1. □

7.2 Proof of Theorem 2.4

We recall the definition of $\mathbb{S}_\varepsilon$ from (5.16) and specialise to the case $\mathfrak{X} = [0,1]$ with μ the Lebesgue-measure on $[0,1]$. Then we set

$$\mathrm{Spec}_\varepsilon^\infty(s, a) := \mathbb{S}_\varepsilon. \tag{7.11}$$

With this definition, (2.5) follows from Proposition 6.1 (iv) and $\cap_{\varepsilon>0} \mathrm{Spec}_\varepsilon^\infty(s,a) = \mathbb{S}_0 = \overline{\mathbb{S}}$ due to Remark 5.8 and (6.25).

Now we verify (2.4). First we see that for any $\varepsilon, \delta > 0$ the inclusion

$$\limsup_{n\to\infty} \mathrm{Spec}_\varepsilon(X_n + A_n) \subset \mathrm{Spec}_{\varepsilon+\delta}^\infty(s,a) = \mathbb{S}_{\varepsilon+\delta}$$

holds almost surely by Remark 7.5 and the Borel-Cantelli lemma. Since $\cap_{\delta>0} \mathbb{S}_{\varepsilon+\delta} = \mathbb{S}_\varepsilon$ by definition this shows the first inclusion in (2.4).

The second inclusion in (2.4) follows from

$$\mathbb{S}_\varepsilon \subset \mathrm{Spec}_{\varepsilon+\delta}(X_n + A_n) = \{\zeta \in \mathbb{C} : \mathrm{dist}(0, \mathrm{Spec}(H_\zeta)) \le \varepsilon + \delta\} \tag{7.12}$$

eventually almost surely for any $\varepsilon, \delta > 0$. Here H_ζ is the Hermitisation of $X_n + A_n$ from (7.1). To prove (7.12) we see that the global law from [5, Theorem 2.7] holds almost surely when all random matrices in the statement are realised on the same probability space. This can be seen easily from its proof. Indeed, the global law is an immediate consequence of [5, (B.5)], which holds with very high probability. Thus, the Borel-Cantelli lemma ensures almost sure convergence in

$$\frac{1}{2n} \mathrm{Tr}\, f(H_\zeta) \to \int_{\mathbb{R}} f(\tau) \rho_\zeta(\mathrm{d}\tau)$$

for every compactly supported continuous function f. □

A Existence and uniqueness of solution to Dyson equation

A.1 Proof of Lemma 4.1

Owing to the identification of solutions to (4.2) and (5.13) in Section 5.2, we can now infer the existence and uniqueness of the solution to (4.2) to the existence and uniqueness of the solution to (5.13). Indeed, the latter is a very simple case of the general existence and uniqueness result [30, Theorem 2.1]. This proves Lemma 4.1. □

A.2 Stability operator of Matrix Dyson equation

Let M be the solution of (5.13). The stability operator of (5.13) is given by (5.18).

Lemma A.1. *If a and s satisfy the assumptions of Lemma 5.9, then the following holds.*

(i) For all $\eta > 0$ and $\zeta \in \mathbb{C}$, we have

$$\|M\| \leq \eta^{-1}. \tag{A.1}$$

(ii) For all $\eta > 0$ and $\zeta \in \mathbb{C}$, the stability operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$ are invertible. Moreover, for any constant $K > 0$, there is a constant $C \geq 2$ such that

$$\|\mathcal{L}^{-1}\|_2 + \|\mathcal{L}^{-1}\|_\infty \leq C \max\{\eta^{-2}, \eta^{-C}\} \tag{A.2}$$

for all $\eta > 0$, $a \in \mathcal{B}$ and $\zeta \in \mathbb{C}$ satisfying $|\zeta| + \|a\|_\infty \leq K$.

Proof. For a proof of (A.1), we refer to [1, eq. (4.1)], where the proof was carried out in the finite dimensional setting. In our setting the proof follows the same argument. The invertibility of $\mathcal{L}$ and its adjoint $\mathcal{L}^*$ as well as the bound (A.2) are obtained by translating the proofs of [5, Lemmas 3.4 and 3.7] to the present setup using (A.1). □

Proof of Lemma 5.14. We recall that $\zeta \in \mathbb{C}$ is fixed such that $\limsup_{\eta \downarrow 0} \langle v_1(\zeta, \eta) \rangle \geq \delta$ for some $\delta > 0$ and that $v_n = v(\zeta, \eta_n) \to v_0$ weakly in $(L^2)^2$, where $v_0 \sim_\delta 1$. First we use the identities

$$Lv_- = -\eta \frac{\tau v^2}{(\eta + S_d v)^2}\,, \qquad L^*(e_-(\eta + S_o v)) = \eta e_-$$

with $v_- = ve_-$ and $v = v_n$. In the limit $\eta \downarrow 0$ we see $L_0 v_- = 0$ and $L_0^* S_o v_- = 0$. Here we used that v satisfies the Dyson equation. For the rest of this proof we drop the 0-index from our notation. We introduce T, V, $F : \mathcal{B}^2 \to \mathcal{B}^2$ as in (6.11), evaluated at $v = v_0$ and $\eta = 0$. In terms of T, F and V we obtain

$$L = V^{-1}(1 - TF)V\,. \tag{A.3}$$

We consider the natural extensions $F\colon (L^2)^2 \to (L^2)^2$ and $T\colon (L^2)^2 \to (L^2)^2$. These operators are self-adjoint. We import results about their spectral properties from [3, Lemma 3.4, Lemma 3.6, (3.44) and (3.45)]. The proofs of these properties translate immediately to the current setup, although originally formulated for $\mathcal{B} = \mathbb{C}^d$, since they are uniform in the dimension d. Furthermore, the proofs from [3] are not affected by the changed definition of $\tau = (\tau_1, \tau_2)$ from (5.25) since we still have $\tau_1 = \tau_2$.

Lemma A.2 (Spectral properties of F and T)**.** *The Hermitian operator $F\colon (L^2)^2 \to (L^2)^2$ satisfies the following properties:*

(i) F has non-degenerate isolated eigenvalues at ± 1 and a spectral gap $\varepsilon \sim_\delta 1$, i.e.

$$\mathrm{Spec}(F) \subset \{-1\} \cup [-1 + \varepsilon, 1 - \varepsilon] \cup \{1\}\,.$$

(ii) The eigenvectors corresponding to the eigenvalues ± 1 are

$$FVv = Vv\,, \qquad FVv_- = -Vv_-\,,$$

where $v = (v_1, v_2)$ and $v_- = (v_1, -v_2)$.

The Hermitian operator $T\colon (L^2)^2 \to (L^2)^2$ satisfies the following properties:

(iii) The spectrum is bounded away from 1 *by a gap of size $\varepsilon \sim_\delta 1$, i.e.*

$$\mathrm{Spec}(T) \subset [-1, 1-\varepsilon]\,.$$

(iv) For $x \in \{(y,-y) : y \in \mathcal{B}\}$ we have $Tx = -x$.

We now prove Lemma 5.14 by tacitly using the properties of F and T from Lemma A.2. Since $f_- := Vv_- \in \{(y,-y) : y \in \mathcal{B}\}$ we see that F and T both leave the subspace $(f_-)^\perp$ invariant. Now, using (A.3), we rewrite the resolvent of L as

$$\frac{1}{L-\omega} = V^{-1}\frac{1}{1-\omega-TF}V\,.$$

The operator V and its inverse satisfy the bound $\|V\|_\# + \|V^{-1}\|_\# \lesssim_\delta 1$ for $\# = 2, \infty$, where we used $\delta \lesssim v \lesssim \frac{1}{\delta}$ from Lemma 5.12. Thus, it suffices to show (5.28) with L replaced by $1-TF$. Furthermore, we can restrict to the case $\# = 2$, since [6, Lemma 4.5] is applicable because

$$\|TF\|_\infty + \|TF\|_{\infty\to 2}\|TF\|_{2\to\infty} \lesssim_\delta 1\,.$$

Here, $\|TF\|_{\infty\to2}$ and $\|TF\|_{2\to\infty}$ denote the operator norms of TF viewed as operator $\mathcal{B}^2 \to (L^2)^2$ and $(L^2)^2 \to \mathcal{B}^2$, respectively.

Since $\|TF\|_2 \le 1$ we have the bound $\|(1-TF-\omega)^{-1}\|_2 \lesssim_{\varepsilon,\delta} 1$ for any $\omega \notin 1+\mathbb{D}_{1+\varepsilon}$ and any $\varepsilon > 0$. Now we can decompose

$$\frac{1}{1-TF-\omega} = (1-P)\frac{1}{1-TF-\omega}(1-P) - \frac{1}{\omega}P\,, \tag{A.4}$$

where P is the orthogonal projection onto the span of f_-. Provided $\varepsilon > 0$ is chosen sufficiently small, the first summand in (A.4) is bounded for $\omega \in \mathbb{D}_{2\varepsilon}$. Indeed $1-TF$ has a bounded inverse on $f_-^\perp$ by applying a generalisation of [3, Lemma 3.7] from the finite dimensional case $\mathcal{B} = \mathbb{C}^d$ to the general setup here. The proof of [3, Lemma 3.7] is not affected by this generalisation. Finally, the second summand in (A.4) is bounded on $\mathbb{C}\setminus\mathbb{D}_\varepsilon$. Altogether the bound (5.28) is proven. The non-degeneracy of the eigenvector v_- of L also follows from the decomposition (A.4). $\square$

B Auxiliary results

Lemma B.1. *Let $y > 0$ be the unique solution to the equation $y^3 + \beta y = x$ for $x > 0$ and $\beta \in \mathbb{R}$. Then*

$$y \sim \sqrt{\max\{0,-\beta\}} + \frac{x}{x^{2/3} + |\beta|}\,.$$

Proof. First we consider the case $\beta \ge 0$. Then clearly

$$y \sim \frac{x}{x^{2/3}+|\beta|}\,.$$

Now let $\beta < 0$, then we must have $y = \sqrt{-\beta}(1+\varepsilon)$ for some $\varepsilon > 0$ since $y^2 + \beta y > 0$. For this ε we get the equation $\varepsilon^3 + 3\varepsilon^2 + 2\varepsilon = x|\beta|^{-3/2}$. Thus, we have the scaling

$$\varepsilon \sim \min\Big\{\frac{x}{|\beta|^{3/2}}, \frac{x^{1/3}}{|\beta|^{1/2}}\Big\}\,.$$

Therefore we conclude

$$y = \sqrt{|\beta|} + \min\Big\{\frac{x}{|\beta|}, x^{1/3}\Big\} \sim \sqrt{|\beta|} + \frac{x}{x^{2/3}+|\beta|}\,,$$

which is the claim of the lemma. $\square$

Lemma B.2. *Let $S\colon \mathcal{B} \to \mathcal{B}$ be an integral operator as in* (4.1) *with a kernel $s : \mathfrak{X}^2 \to (0,\infty)$ that satisfies the bounds $\varepsilon \le s(x,y) \le \frac{1}{\varepsilon}$ for all x, $y \in \mathfrak{X}$ and some constant $\varepsilon > 0$ and such that the spectral radius of S is normalised to $\varrho(S) = 1$. Then there are constants $\delta, C > 0$, depending only on ε, such that*

$$\sup\Big\{\|(S-z)^{-1}\|_2\colon z \notin D_{1-\delta}(0) \cup D_\delta(1)\Big\} \le C$$

and $\operatorname{Spec}(S) \cap D_\delta(1) = \{1\}$ *is non-degenerate.*

Proof. We follow line by line the arguments from the proof of [23, Lemma A.1], where the finite dimensional case with $|\mathfrak{X}| < \infty$ and μ the counting measure is carried out. □

C Discretizing the Dyson equation

In order to formulate the next lemma under the weakest assumptions, we first relax the condition **A4** by avoiding the lower bound on s in the following assumption.

A7 *Piecewise continuity of s and a:* Let $I_1, \ldots, I_K \subset [0,1]$ be disjoint intervals with some $K \in \mathbb{N}$ such that $I_1 \cup \ldots \cup I_K = [0,1]$. Let $s\colon [0,1]^2 \to [0,\infty)$ and $a\colon [0,1] \to \mathbb{C}$ be functions such that $s|_{I_l \times I_k}$ and $a|_{I_l}$ have continuous extensions to $\overline{I_l} \times \overline{I_k}$ and $\overline{I_l}$, respectively, for all $l, k \in [\![K]\!]$.

Throughout this section, we write $\mathbb{C}_+ := \{w \in \mathbb{C}\colon \operatorname{Im} w > 0\}$. Let (v_1, v_2) be the solution of (4.2) on $\mathfrak{X} = [0,1]$ and with μ the Lebesgue measure on $[0,1]$, M the unique solution of (5.15) and ρ_ζ the associated probability measure from Definition 5.7.

Lemma C.1. *Let s and a satisfy **A7**. For $n \in \mathbb{N}$, define the functions $a^{(n)}$ on $[0,1]$ and $s^{(n)}$ on $[0,1]^2$ through*

$$a^{(n)} := \sum_{i=1}^n a\Big(\frac{i}{n}\Big)\mathbf{1}_{[(i-1)/n, i/n)}, \qquad s^{(n)} := \sum_{i,j=1}^n \frac{1}{n} s\Big(\frac{i}{n}, \frac{j}{n}\Big)\mathbf{1}_{[(i-1)/n, i/n) \times [(j-1)/n, j/n)}, \tag{C.1}$$

where $\mathbf{1}_\Omega$ denotes the indicator function of the set Ω. Let $\Sigma^{(n)}$ be defined analogously to (5.14) *with s replaced by $s^{(n)}$. If $M^{(n)}$ is the unique solution of* (5.15) *with $a^{(n)}$ and $\Sigma^{(n)}$ instead of a and Σ, $\delta > 0$ is constant and $\zeta \in \mathbb{C}$ is fixed, then*

$$\lim_{n\to\infty} \|M^{(n)}(\zeta, w) - M(\zeta, w)\|_2 = 0$$

uniformly for all $w \in \mathbb{C}_+$ satisfying $\operatorname{dist}(w, \operatorname{supp}\rho_\zeta) \ge \delta$. *Here, $\|R\|_2 := \|\operatorname{Tr}(R^*R)\|_1^{1/2}/\sqrt{2}$ for any $R \in \mathcal{B}^{2\times 2}$, where $\operatorname{Tr}(R^*R)$ is considered as a function on $[0,1]$, and $\|f\|_p$ is the $L^p([0,1],\mu)$-norm for $f\colon [0,1] \to \mathbb{C}$.*

If $(v_1^{(n)}, v_2^{(n)})$ is the solution of (4.2) *on $[0,1]$ with the Lebesgue measure μ and a and s replaced by $a^{(n)}$ and $s^{(n)}$ from* (C.1)*, then for any (fixed) $\zeta \in \mathbb{C}$ and $\eta > 0$, we have*

$$\lim_{n\to\infty} \max\{\|v_1^{(n)}(\zeta,\eta) - v_1(\zeta,\eta)\|_2, \|v_2^{(n)}(\zeta,\eta) - v_2(\zeta,\eta)\|_2\} = 0.$$

Throughout the remainder of this section, a operators appear that map $\mathcal{B}^{2\times 2}$ to $\mathcal{B}^{2\times 2}$. We write $\|\cdot\|_{*\to\#}$ with $*, \# \in \{2,\infty\}$ for the operator norm if the definition space is equipped with the norm $\|\cdot\|_*$ and the target space with $\|\cdot\|_\#$. If $* = \#$ then we simply write $\|\cdot\|_*$ for the corresponding operator norm.

Proof. Owing to the explanations in Section 5.2, especially, (5.12), it suffices to show that $\|M^{(n)}(\zeta,w) - M(\zeta,w)\|_2 \to 0$ if $n \to \infty$.

Fix $\delta > 0$ and $\zeta \in \mathbb{C}$. We introduce the matrices $A \in \mathcal{B}^{2\times 2}$ and $A^{(n)} \in \mathcal{B}^{2\times 2}$ through

$$A := \begin{pmatrix} 0 & a \\ \overline{a} & 0 \end{pmatrix}, \qquad A^{(n)} := \begin{pmatrix} 0 & a^{(n)} \\ \overline{a^{(n)}} & 0 \end{pmatrix}.$$

For $w \in \mathbb{C}_+$ satisfying $\mathrm{dist}(w, \mathrm{supp}\,\rho_\zeta) \geq \delta$ and $t \geq 0$, we set $M^{(n)} = M^{(n)}(\zeta, w+\mathrm{i}t)$, $M = M(\zeta, w+\mathrm{i}t)$ and $L[R] := R - M\Sigma[R]M$ for all $R \in \mathcal{B}^{2\times 2}$. With $\Delta := M^{(n)} - M$, a short computation starting from (5.15) and the analogous relation with $M^{(n)}$, $a^{(n)}$ and $\Sigma^{(n)}$ yields

$$L[\Delta] = M\Sigma[\Delta]\Delta + M(\Sigma^{(n)} - \Sigma)[M^{(n)}]M^{(n)} + M(A - A^{(n)})M^{(n)}. \tag{C.2}$$

We now invert L and estimate the resulting relation in $\|\cdot\|_2$. We collect a few auxiliary bounds. From [5, eq.s (3.22), (3.11a), (3.11c)], we conclude the existence of a constant $C_1 > 0$, depending only on δ but independent of w and t, such that $\|L^{-1}\|_2 \leq C_1$ for all $w \in \mathbb{C}_+$ with $\mathrm{dist}(w, \mathrm{supp}\,\rho_\zeta) \geq \delta$ and $t \geq 0$. As $\|M(\zeta, w)\| \leq (\mathrm{dist}(w, \mathrm{supp}\,\rho_\zeta))^{-1}$ by [5, eq. (3.11a)][3], we have $\|M\| \leq (\max\{\delta, t\})^{-1}$ and $\|M^{(n)}\| \leq (\max\{\delta, t\})^{-1}$ for all $t \geq 0$. Owing to [4, Lemma B.2(i)], the upper bound on s following from its piecewise continuity implies that there is a constant $C_2 \geq 1$ such that $\|\Sigma\|_{2\to\infty} \leq C_2$. Therefore, there is a constant $C > 0$ depending only on δ but not on w or t such that

$$\|\Delta\|_2 \leq C(\|\Delta\|_2^2 + \Psi_n), \qquad \Psi_n := \|\Sigma^{(n)} - \Sigma\|_2 + \|A - A^{(n)}\|_2 \tag{C.3}$$

for all $w \in \mathbb{C}_+$ with $\mathrm{dist}(w, \mathrm{supp}\,\rho_\zeta) \geq \delta$ and all $t \geq 0$. Here, $\Delta \equiv \Delta(\zeta, w + \mathrm{i}t)$.

Owing to **A7**, we get $\Psi_n \to 0$ as $n \to \infty$. Thus, we find $n_0 \in \mathbb{N}$ such that $2\Psi_n C^2 \leq 1/4$ for all $n \geq n_0$. Fix $w \in \mathbb{C}_+$ with $\mathrm{dist}(w, \mathrm{supp}\,\rho_\zeta) \geq \delta$. We set $t_* := \sup\{t \geq 0 \colon \|\Delta(\zeta, w + \mathrm{i}t)\|_2 \geq 2C\Psi_n\}$. Since $\|M^{(n)}\| + \|M\| \to 0$ for $t \to \infty$, we obtain $t_* < \infty$. Next, we conclude $t_* = 0$. Suppose $t_* > 0$. Hence, $\|\Delta(\zeta, w + \mathrm{i}t_*)\|_2 = 2C\Psi_n$ by continuity. As $2\Psi_n C^2 \leq 1/4$, we conclude from (C.3) that $\|\Delta(\zeta, w + \mathrm{i}t_*)\|_2 \leq 3C\Psi_n/2 < 2C\Psi_n = \|\Delta(\zeta, w + \mathrm{i}t_*)\|_2$. This contradiction implies $t_* = 0$. Note that this holds for any $w \in \mathbb{C}_+$ as long as $\mathrm{dist}(w, \mathrm{supp}\,\rho_\zeta) \geq \delta$ and $n \geq n_0$. Thus, for $n \geq n_0$, we obtain $\|M^{(n)}(\zeta, w) - M(\zeta, w)\|_2 = \|\Delta(\zeta, w)\|_2 \leq 2C\Psi_n$ for all $w \in \mathbb{C}_+$ with $\mathrm{dist}(w, \mathrm{supp}\,\rho_\zeta) \geq \delta$, which concludes the proof of Lemma C.1 as $\Psi_n \to 0$ with $n \to \infty$. $\square$

Corollary C.2. *Let s and a satisfy **A4** and $(v_1^{(n)}, v_2^{(n)})$ be as in Lemma C.1. Then the following holds.*

(i) For each $n \in \mathbb{N}$, replacing v_1 by $v_1^{(n)}$ in (6.1) *yields a well-defined continuous function $L^{(n)} \colon \mathbb{C} \to \mathbb{R}$.*

(ii) For each $n \in \mathbb{N}$, there exists a probability measure $\sigma^{(n)}$ on $\mathbb{C}$ such that (6.2) *holds with σ and L replaced by $\sigma^{(n)}$ and $L^{(n)}$, respectively. Furthermore, there is $\varphi \sim 1$ such that $\mathrm{supp}\,\sigma^{(n)} \subset \mathbb{D}_\varphi$ for all $n \in \mathbb{N}$.*

(iii) Moreover, $\sigma^{(n)}$ converges to σ weakly as n tends to infinity.

Proof. Clearly, $a^{(n)}$ and $s^{(n)}$ from (C.1) satisfy $\|a^{(n)}\|_\infty \lesssim 1$ and **A5** (with the same constants as a and s). Hence, Lemma 6.5, Proposition 6.3 and Remark 5.8 imply the well-definedness of $L^{(n)}$ and the existence of probability measures $\sigma^{(n)}$ satisfying (6.2) for $L^{(n)}$ as well as $\mathrm{supp}\,\sigma^{(n)} \subset \mathbb{D}_\varphi$, respectively.

Since $\sigma^{(n)}$ for all $n \in \mathbb{N}$ and σ are probability measures on $\mathbb{C}$, for the weak convergence it suffices to show $\int_{\mathbb{C}} f \mathrm{d}\sigma^{(n)} \to \int_{\mathbb{C}} f \mathrm{d}\sigma$ as $n \to \infty$ for all $f \in C_0^2(\mathbb{C})$. Fix $f \in C_0^2(\mathbb{C})$. As $a^{(n)}$ and $s^{(n)}$ from (C.1) satisfy $\|a^{(n)}\|_\infty \lesssim 1$ and **A5** with the same constants as a and s, we conclude from Lemma 5.1 and (5.9) that $|\Delta f(\zeta)(\langle v_1^{(n)}(\zeta, \eta)\rangle - \frac{1}{1+\eta})| \lesssim \frac{|\Delta f(\zeta)|}{1+\eta^2}$ uniformly $\eta > 0$, $\zeta \in \mathbb{C}$ and $n \in \mathbb{N}$. That is the implicit constant hidden by $\lesssim$ does not depend on η, ζ and n. Owing to the integrability of the right-hand side with respect to ζ and η over $\mathbb{C} \times (0, \infty)$, we obtain from (6.2), Fubini and dominated convergence that

$$\begin{aligned} \int_{\mathbb{C}} f \mathrm{d}\sigma^{(n)} &= \int_{\mathbb{C}} \int_0^\infty \Delta f(\zeta) \Big(\langle v_1^{(n)}(\zeta, \eta)\rangle - \frac{1}{1+\eta} \Big) \mathrm{d}\eta\, \mathrm{d}^2\zeta \\ &\longrightarrow \int_{\mathbb{C}} \int_0^\infty \Delta f(\zeta) \Big(\langle v_1(\zeta, \eta)\rangle - \frac{1}{1+\eta} \Big) \mathrm{d}\eta\, \mathrm{d}^2\zeta = \int_{\mathbb{C}} f \mathrm{d}\sigma \end{aligned}$$

as $n \to \infty$. This completes the proof of Corollary C.2. $\square$

[3] The proof in [5] is given in the finite dimensional setup; the proof in the setup of this article is identical.

Corollary C.3. *Let s and a satisfy **A7** and let $M^{(n)}$ be as in Lemma C.1. Fix $\zeta \in \mathbb{C}$. Let $\rho_\zeta^{(n)}$ be the self-consistent density of states associated with the Hermitization of $A_n + X_n$, i.e.*

$$\int_{\mathbb{R}} \frac{\rho_\zeta^{(n)}(\mathrm{d}\tau)}{\tau - w} = \frac{1}{2} \operatorname{Tr}\langle M^{(n)}(\zeta, w)\rangle \tag{C.4}$$

for all $w \in \mathbb{C}_+$. Here, Tr denotes the trace of 2×2 matrices in $\mathbb{C}^{2\times 2}$ and the average $\langle \cdot \rangle$ is taken entrywise. Then

(i) $\rho_\zeta^{(n)} \to \rho_\zeta$ *weakly as $n \to \infty$.*

(ii) $\limsup_{n\to\infty} \operatorname{supp} \rho_\zeta^{(n)} \subset \operatorname{supp} \rho_\zeta$.

Proof. Item (i) follows directly from the convergence of the Stieltjes transforms, i.e. for all $w \in \mathbb{C}_+$, $\langle \operatorname{Tr} M^{(n)}(\zeta, w)\rangle \to \langle \operatorname{Tr} M(\zeta, w)\rangle$ as $n \to \infty$, due to Lemma C.1.

For the proof of (ii), we fix $\delta > 0$ and show that $\operatorname{supp} \rho_\zeta^{(n)} \subset \operatorname{supp} \rho_\zeta + (-\delta, \delta)$ for all sufficiently large n. If $\tau \in \mathbb{R}$ satisfies $\operatorname{dist}(\tau, \operatorname{supp} \rho_\zeta) \geq \delta$ then, by [4, Lemma D.1], $M = M(\zeta, \tau) = \lim_{\eta\downarrow 0} M(\zeta, \tau + \mathrm{i}\eta)$ exits and is self-adjoint. Moreover, $\|L^{-1}\|_2 + \|M\| \lesssim_\delta 1$ uniformly for $\eta \geq 0$ and $\tau \in \mathbb{R}$ with $\operatorname{dist}(\tau, \operatorname{supp} \rho_\zeta) \geq \delta$. We recall the definition $L[R] = R - M\Sigma[R]M$ for $R \in \mathcal{B}^{2\times 2}$ from the proof of Lemma C.1. As $\|M\| \lesssim_\delta 1$, Lemma C.1 implies $\|M^{(n)}(\zeta, \tau + \mathrm{i}\eta)\|_2 \lesssim_\delta 1$ uniformly for all $\eta > 0$, $\tau \in \mathbb{R}$ with $\operatorname{dist}(\tau, \operatorname{supp} \rho_\zeta) \geq \delta$ and all sufficiently large n. Arguing similarly as in the proof of Lemma 5.3, we conclude $\|M^{(n)}(\zeta, \tau + \mathrm{i}\eta)\| \lesssim_\delta 1$ uniformly for η, τ and n as before. We set $M^{(n)} := M^{(n)}(\zeta, \tau + \mathrm{i}\eta)$ and $L^{(n)}[R] = R - M^{(n)}\Sigma^{(n)}[R]M^{(n)}$ for $R \in \mathcal{B}^{2\times 2}$. For such η, τ and n, we obtain $\|(L^{(n)})^{-1}\|_2 \lesssim_\delta 1$ by perturbation theory from $\|M\| + \|M^{(n)}\| + \|L^{-1}\|_2 \lesssim_\delta 1$, $\|\Sigma^{(n)}\|_{2\to\infty} + \|\Sigma\|_{2\to\infty} \lesssim 1$ and $\|M^{(n)} - M\|_2 \to 0$ for $n \to \infty$. Hence, by the implicit function theorem, for all sufficiently large n, the function $\eta \mapsto M^{(n)}(\zeta, \tau + \mathrm{i}\eta)$ is continuous on $[\eta_0 - \varepsilon, \eta_0 + \varepsilon]$ for some $\varepsilon > 0$ independent of $\eta_0 > 0$. In particular, we can extend $M^{(n)}$ continuously to $\eta = 0$ in a unique way.

Let $\tau \in \mathbb{R}$ with $\operatorname{dist}(\tau, \operatorname{supp} \rho_\zeta) \geq \delta$. For $M = M(\zeta, \tau)$, we now consider the relation

$$L[\Delta] = \frac{1}{2}(K_n(\Delta, \widetilde{\Sigma}, \widetilde{A}) + K_n(\Delta^*, \widetilde{\Sigma}, \widetilde{A})^*), \quad K_n(\Delta) := M\Sigma[\Delta]\Delta + M\widetilde{\Sigma}[M+\Delta](M+\Delta) + M\widetilde{A}(M+\Delta),$$

with variables $\Delta \in \mathcal{B}^{2\times 2}$, $\widetilde{A} = \widetilde{A}^* \in \mathcal{B}^{2\times 2}$, $\widetilde{\Sigma} \colon \mathcal{B}^{2\times 2} \to \mathcal{B}^{2\times 2}$ such that $\widetilde{\Sigma}[R]^* = \widetilde{\Sigma}[R^*]$ for all $R \in \mathcal{B}^{2\times 2}$. Since $\|L^{-1}\|_2 \lesssim_\delta 1$, by the implicit function theorem, this relation has a unique solution Δ as long as $\|\widetilde{\Sigma}\|_2$ and $\|\widetilde{A}\|_2$ are sufficiently small, as $L[0] = 0$ and $K_n(0, 0, 0) = 0$. Moreover, this solution satisfies $\Delta = \Delta^*$ as $L[R]^* = L[R^*]$ for all $R \in \mathcal{B}^{2\times 2}$ due to $M^* = M$. Owing to (C.2) and $M = M^*$, we have $L[M^{(n)} - M] = (K_n(M^{(n)} - M, \Sigma^{(n)} - \Sigma, A - A^{(n)}) + K_n((M^{(n)} - M)^*, \Sigma^{(n)} - \Sigma, A - A^{(n)}))/2$ with $M^{(n)} = M^{(n)}(\zeta, \tau)$. Hence, as $\|\Sigma^{(n)} - \Sigma\|_2 + \|A - A^{(n)}\|_2 \to 0$ for $n \to \infty$ by the proof of Lemma C.1, we get $\Delta = M^{(n)} - M$ for all sufficiently large n and, therefore, $M^{(n)} = (M^{(n)})^*$ for such n. Since this holds for any $\delta > 0$ we conclude that $\operatorname{Im} M^{(n)}(\zeta, \tau + \omega) = 0$ for sufficiently small $|\omega|$ with $\omega \in \mathbb{R}$. Because of (C.4) this implies that $\tau \notin \operatorname{supp} \rho_\zeta^{(n)}$. ☐

D Representation of σ as Brown measure

In Proposition D.1 of this appendix, we represent σ from Proposition 6.1 as the Brown measure of an operator in a von Neumann algebra. This is the motivation behind Definition 6.2.

The definition of a Brown measure is given after the next proposition, which is formulated in the language of operator-valued free probability[4].

Proposition D.1 (Representation of σ as Brown measure)**.** *Let $a \in \mathcal{B}$ and s satisfy **A5**. Then there is an operator-valued probability space $(\mathcal{A}, E, \mathcal{B})$ and an operator $\mathfrak{c} \in \mathcal{A}$ such that*

- *$E \colon \mathcal{A} \to \mathcal{B}$ is a positive conditional expectation.*
- *$(\mathcal{A}, \langle E[\,\cdot\,]\rangle)$ is a tracial W^*-probability space.*

[4]For the necessary definitions in free probability, we refer to the recent monograph [36].

- *the Brown measure of $a+\mathfrak{c}$ on $(\mathcal{A}, \langle E(\cdot)\rangle)$ coincides with σ.*
- $\begin{pmatrix} 0 & \mathfrak{c} \\ \mathfrak{c}^* & 0 \end{pmatrix}$ *is an operator-valued semi-circular element in the operator-valued probability space* $(\mathcal{A}^{2\times 2}, \mathrm{id}\otimes E, \mathcal{B}^{2\times 2})$*, where we identified* $\mathcal{A}^{2\times 2} = \mathbb{C}^{2\times 2}\otimes\mathcal{A}$*, whose covariance is given by* Σ *from* (5.14)*. In particular,*

$$M(\zeta, \mathrm{i}\eta) = \mathrm{id}\otimes E\bigg[\begin{pmatrix} -\mathrm{i}\eta & \mathfrak{c}+a-\zeta \\ (\mathfrak{c}+a-\zeta)^* & -\mathrm{i}\eta \end{pmatrix}^{-1}\bigg] \tag{D.1}$$

  satisfies the Matrix-Dyson equation, (5.13)*.*
- $E[\mathfrak{c}b\mathfrak{c}] = 0$ *for all* $b\in\mathcal{B}$*.*

Before we show Proposition D.1, we introduce the notion of a Brown measure. Let $(\mathcal{A},\tau)$ be a tracial W^*-probability space. The Brown measure is a generalisation of the spectral distribution of normal operators to non-normal ones. Let $\mathfrak{a}\in\mathcal{A}$. The *Brown measure* $\mu_{\mathfrak{a}}$ of $\mathfrak{a}$ is the unique compactly supported probability measure on $\mathbb{C}$ with

$$\int_{\mathbb{C}} \log|\zeta-\xi|\mu_{\mathfrak{a}}(\mathrm{d}\xi) = \log D(\mathfrak{a}-\zeta) \tag{D.2}$$

for all $\zeta\in\mathbb{C}$. Here, $D(\mathfrak{a}-\zeta)$ denotes the *Fuglede-Kadison determinant* of $\mathfrak{a}-\zeta$. The Fuglede-Kadison determinant of an arbitrary $\mathfrak{b}\in\mathcal{A}$ is defined by

$$D(\mathfrak{b}) := \lim_{\varepsilon\downarrow 0}\exp(\tau(\log(\mathfrak{b}^*\mathfrak{b}+\varepsilon)^{1/2}))\in[0,\infty).$$

Originally introduced in [18], the Brown measure was revived in [27]. An introduction to the Brown measure and the Fuglede-Kadison determinant can be found in [36, Chapter 11].

Proof of Proposition D.1. As $\mathcal{B}$ is a commutative C*-algebra, it is a standard result that S and S^* are completely positive maps (see e.g. [37, Theorem 3.9 or Theorem 3.11]). Hence, $\Sigma\colon\mathcal{B}^{2\times 2}\to\mathcal{B}^{2\times 2}$ from (5.14) is also a completely positive map. From the constructions in [45, Sections 4.3 and 4.6], we obtain a von Neumann algebra $\widehat{\mathcal{A}}$ such that $\mathcal{B}^{2\times 2}\subset\widehat{\mathcal{A}}$ is a sub-von Neumann algebra with the same unit as well as a positive conditional expectation $\widehat{E}\colon\widehat{\mathcal{A}}\to\mathcal{B}^{2\times 2}$ and an operator-valued semicircular element $\mathfrak{H}=\mathfrak{H}^*\in\widehat{\mathcal{A}}$ such that

$$\widehat{E}[\mathfrak{H}B\mathfrak{H}] = \Sigma[B] \tag{D.3}$$

for all $B\in\mathcal{B}^{2\times 2}$. A concise summary of this construction is given in [41, Section 3.5][5].

We recall the definitions of E_{12} and $E_{21}\in\mathcal{B}^{2\times 2}$ from (5.20) and define E_{11} and $E_{22}\in\mathcal{B}^{2\times 2}$ analogously. Let

$$\mathcal{A} := \mathrm{vN}(E_{1i}\mathfrak{A}E_{j1}\colon i,j=1,2\,,\ \mathfrak{A}\in\widehat{\mathcal{A}})$$

be the sub-von Neumann algebra of $\widehat{\mathcal{A}}$ generated by $E_{1i}\mathfrak{A}E_{j1}$ for i, $j=1$, 2 and $\mathfrak{A}\in\widehat{\mathcal{A}}$. Note that $E_{11}\in\mathcal{A}$ is the unit of $\mathcal{A}$. We introduce the map

$$\Phi\colon\mathcal{A}^{2\times 2}\to\widehat{\mathcal{A}}, \qquad \begin{pmatrix}\mathfrak{a} & \mathfrak{b}\\ \mathfrak{c} & \mathfrak{d}\end{pmatrix}\mapsto E_{11}\mathfrak{a}E_{11}+E_{11}\mathfrak{b}E_{12}+E_{21}\mathfrak{c}E_{11}+E_{21}\mathfrak{d}E_{12},$$

which is clearly an injective *-algebra homomorphism. Surjectivity follows from

$$\Phi\begin{pmatrix} E_{11}\mathfrak{A}E_{11} & E_{11}\mathfrak{A}E_{21}\\ E_{12}\mathfrak{A}E_{11} & E_{12}\mathfrak{A}E_{21}\end{pmatrix} = E_{11}\mathfrak{A}E_{11}+E_{11}\mathfrak{A}E_{22}+E_{22}\mathfrak{A}E_{11}+E_{22}\mathfrak{A}E_{22}=\mathfrak{A}$$

as $E_{11}+E_{22}$ is the unit of $\mathcal{B}^{2\times 2}\subset\widehat{\mathcal{A}}$. Moreover, Φ maps the unit of $\mathcal{A}^{2\times 2}$ to the unit $\widehat{\mathcal{A}}$. Hence, Φ is bijective, unital *-algebra homomorphism between unital C*-algebras, which is consequently also

[5] We also refer to Lemma 8.2 and its proof in the first arXiv-version of [4], which can be found at arXiv:1804.07752v1.

isometric. We define $E\colon \mathcal{A} \to \mathcal{B}$ through setting $E[\mathfrak{a}]$ as the $(1,1)$-entry of $\widehat{E}[\mathfrak{a}] \in \mathcal{B}^{2\times 2}$ for all $\mathfrak{a} \in \mathcal{A}$. Then $E\colon \mathcal{A} \to \mathcal{B}$ is a positive conditional expectation and, hence, $(\mathcal{A}, E, \mathcal{B})$ is an operator-valued probability space. It follows that $(\mathcal{A}, \langle E[\,\cdot\,]\rangle)$ is a tracial W^*-probability space[6].

Using Φ implicitly, we now identify $\mathfrak{H}$ with an element in $\mathcal{A}^{2\times 2}$, i.e. we find $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c} \in \mathcal{A}$ satisfying $\mathfrak{a} = \mathfrak{a}^*$ and $\mathfrak{b} = \mathfrak{b}^*$ such that

$$\mathfrak{H} = \begin{pmatrix} \mathfrak{a} & \mathfrak{c} \\ \mathfrak{c}^* & \mathfrak{b} \end{pmatrix}.$$

Moreover, by identifying $\mathcal{A}^{2\times 2}$ with $\mathbb{C}^{2\times 2} \otimes \mathcal{A}$, we obtain

$$\mathrm{id} \otimes E\left[\begin{pmatrix} \mathfrak{a} & \mathfrak{c} \\ \mathfrak{c}^* & \mathfrak{b} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} \mathfrak{a} & \mathfrak{c} \\ \mathfrak{c}^* & \mathfrak{b} \end{pmatrix} \right] = \widehat{E}\left[\mathfrak{H} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \mathfrak{H} \right] = \begin{pmatrix} S[r_{22}] & 0 \\ 0 & S^*[r_{11}] \end{pmatrix}$$

for all r_{11}, r_{12}, r_{21}, $r_{22} \in \mathcal{B}$. This implies $E[\mathfrak{a}\mathfrak{a}] = 0$ and $E[\mathfrak{b}\mathfrak{b}] = 0$, hence, $\mathfrak{a} = \mathfrak{b} = 0$ as $\mathfrak{a} = \mathfrak{a}^*$ and $\mathfrak{b} = \mathfrak{b}^*$. Moreover, $E[\mathfrak{c} r_{21} \mathfrak{c}] = 0$ for all $r_{21} \in \mathcal{B}$.

Owing to the standard relation between the covariance of an operator-valued semicircular element and its R-transform, see e.g. [36, Theorem 11 of Chapter 9], we conclude that $M(\zeta, \mathrm{i}\eta)$ as defined in (D.1) satisfies (5.13).

Given the above construction, it remains to show that the Brown measure of $a + \mathfrak{c}$ coincides with σ. This proof proceeds analogously to [6, proof of Proposition 2.9]. We explain the necessary replacements. First, analogously to [6, proof of (5.28)], we obtain

$$-L(\zeta) = \log D(a + \mathfrak{c} - \zeta)$$

for all $\zeta \in \mathbb{C}$.

Proposition 6.3 and standard results from potential theory (cf. [7, Chapter 4.3]) imply

$$\int_{\mathbb{C}} \log|\zeta - \xi| \sigma(\mathrm{d}\xi) = -L(\zeta) + h(\zeta)$$

for all $\zeta \in \mathbb{C}$ and some harmonic function $h\colon \mathbb{C} \to \mathbb{C}$. In the proof of Proposition 6.3, we showed that $L(\zeta) = -\langle \log|a - \zeta| \rangle + C$ for all sufficiently large $\zeta \in \mathbb{C}$ with some constant $C \in \mathbb{R}$, independent of ζ. An expansion of v_1 starting from (4.2) for large $|\zeta|$ reveals that $C = 0$. Hence, $h(\zeta) \to 0$ for $\zeta \to \infty$ and, therefore, $h \equiv 0$ as it is harmonic. $\square$

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