Spontaneous disentanglement and thermalisation

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The problem of quantum measurement can be partially resolved by incorporating a process of spontaneous disentanglement into quantum dynamics. We propose a modified master equation, which contains a nonlinear term giving rise to both spontaneous disentanglement and thermalisation. We find that the added nonlinear term enables limit cycle steady states, which are prohibited in standard quantum mechanics. This finding suggests that an experimental observation of such a limit cycle steady state can provide an important evidence supporting the spontaneous disentanglement hypothesis.

I. INTRODUCTION

The problem of quantum measurement [1, 2] is considered as one of the most important open questions in physics. This problem arguably originates from a self-inconsistency in quantum theory [3–5]. This long-standing problem has motivated some proposals for nonlinear extensions to quantum theory [6–11]. Moreover, processes giving rise to spontaneous collapse have been explored [12–21]. For some cases, however, nonlinear quantum dynamics may give rise to conflicts with well-established physical principles, such as causality [22–27] and separability [23, 28, 29]. In addition, some predictions of standard quantum mechanics (QM), which have been experimentally confirmed to very high accuracy, are inconsistent with some of the proposed extensions.

A modified Schrödinger equation having a nonlinear term that gives rise to suppression of entanglement (i.e. disentanglement) has been recently proposed [30]. This nonlinear extension partially resolves the self-inconsistency associated with the measurement problem by making the collapse postulate of QM redundant. The proposed modified Schrödinger equation can be constructed for any physical system whose Hilbert space has finite dimensionality, and it does not violate norm conservation of the time evolution. The nonlinear term added to the Schrödinger equation has no effect on product (i.e. disentangled) states. The spontaneous disentanglement generated by the modified Schrödinger equation gives rise to a process similar to state vector collapse.

The nonlinear extension that was proposed in Ref. [30] is applicable only for bipartite systems, and only for pure states. To allow incorporating spontaneous disentanglement for more general cases, we propose here a modified master equation for the time evolution of the density operator ρ. The modified master equation [see Eq. (2) below] contains a nonlinear term that gives rise to spontaneous disentanglement. For a multipartite system, disentanglement between any pair of subsystems can be introduced by the added nonlinear term. Moreover, thermalisation can be incorporated by an additional nonlinear term added to the master equation (2).

In contrast to the modified master equation (2), which is nonlinear in ρ, in standard QM the time evolution of ρ is governed by the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation [18, 31, 32], which is linear in ρ. This linear dependency excludes any nonlinear dynamics in the time evolution of ρ (see appendix B of Ref. [33]), and, in particular, it excludes a limit cycle steady state for any quantum system having a Hilbert space of finite dimensionality and time independent Hamiltonian. On the other hand, as is demonstrated below, the modified master equation (2) yields rich nonlinear dynamics. In particular, both a Hopf bifurcation and a limit cycle steady state may occur. Note, however, that, nonlinearity in ρ does not necessarily imply that dynamical instabilities are possible, as was demonstrated in Ref. [34].

II. MODIFIED MASTER EQUATION

Consider a modified Schrödinger equation for the ket vector $|\psi\rangle$ having the form [35]

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \left(-i\hbar^{-1}\mathcal{H} - \Theta + \langle\psi|\Theta|\psi\rangle\right)|\psi\rangle \ , \tag{1}$$

where $\hbar$ is the Planck's constant, $\mathcal{H} = \mathcal{H}^\dagger$ is the Hamiltonian, and the operator $\Theta = \Theta^\dagger$ is allowed to depend on $|\psi\rangle$. Note that the norm conservation condition $0 = (\mathrm{d}/\mathrm{d}t)\langle\psi|\psi\rangle$ is satisfied by the modified Schrödinger equation (1), provided that $|\psi\rangle$ is normalized, i.e. $\langle\psi|\psi\rangle = 1$.

The modified Schrödinger equation (1) for the ket vector $|\psi\rangle$ yields a master equation for the pure state density operator $\rho = |\psi\rangle\langle\psi|$ given by [19, 36]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = i\hbar^{-1}[\rho, \mathcal{H}] - \Theta\rho - \rho\Theta + 2\langle\Theta\rangle\rho \ , \tag{2}$$

where $\langle\Theta\rangle = \langle\psi|\Theta|\psi\rangle = \mathrm{Tr}(\Theta\rho)$. Note that $\mathrm{d}\,\mathrm{Tr}\,\rho/\mathrm{d}t = 0$ provided that $\mathrm{Tr}\,\rho = 1$ (i.e. ρ is normalized), and that $\mathrm{d}\,\mathrm{Tr}\,\rho^2/\mathrm{d}t = 0$, provided that $\rho^2 = \rho$ (i.e. ρ represents a pure state) [see Eq. (2)].

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For the case $\mathcal{H}=0$, and for a fixed operator Θ, the modified master equation (2) yields an equation of motion for $\langle\Theta\rangle$ given by

$$\frac{\mathrm{d}\langle\Theta\rangle}{\mathrm{d}t}=-2\left\langle(\Theta-\langle\Theta\rangle)^2\right\rangle . \tag{3}$$

The above result (3) implies for this case that the expectation value $\langle\Theta\rangle$ monotonically decreases with time. Hence, the nonlinear term in the modified master equation (2) can be employed to suppress a given physical property, provided that $\langle\Theta\rangle$ quantifies that property.

Here the operator Θ is assumed to be given by $\Theta=\gamma_{\mathrm{H}}\mathcal{Q}^{(\mathrm{H})}+\gamma_{\mathrm{D}}\mathcal{Q}^{(\mathrm{D})}$, where both rates γ_{H} and γ_{D} are positive, and both operators $\mathcal{Q}^{(\mathrm{H})}$ and $\mathcal{Q}^{(\mathrm{D})}$ are Hermitian. The first term $\gamma_{\mathrm{H}}\mathcal{Q}^{(\mathrm{H})}$, which gives rise to thermalisation [37, 38], is discussed below in section III, whereas section IV is devoted to the second term $\gamma_{\mathrm{D}}\mathcal{Q}^{(\mathrm{D})}$, which gives rise to disentanglement.

III. THERMALISATION

Consider the master equation (2) for the case where $\mathcal{H}$ is time independent, $\gamma_{\mathrm{D}}=0$ (i.e. no disentanglement), and $\mathcal{Q}^{(\mathrm{H})}=\beta\mathcal{U}_{\mathrm{H}}$, where $\mathcal{U}_{\mathrm{H}}=\mathcal{H}+\beta^{-1}\log\rho$ is the Helmholtz free energy operator, $\beta=1/(k_{\mathrm{B}}T)$ is the thermal energy inverse, k_{B} is the Boltzmann's constant, and T is the temperature. For this case, the thermal equilibrium density matrix ρ_0, which is given by

$$\rho_0=\frac{e^{-\beta\mathcal{H}}}{\mathrm{Tr}\left(e^{-\beta\mathcal{H}}\right)} , \tag{4}$$

is a steady state solution of the master equation (2), for which the Helmholtz free energy $\langle\mathcal{U}_{\mathrm{H}}\rangle$ is minimized [34, 37–39]. The rate γ_{H} represents the thermalisation inverse time. Note that γ_{H} needs not be a constant.

IV. DISENTANGLEMENT

Consider the case where $\gamma_{\mathrm{H}}=0$ (i.e. no thermalisation). As can be seen from Eq. (3), disentanglement can be generated by the term proportional to γ_{D} in the Schrödinger equation (1), and by the term proportional to γ_{D} in the master equation (2), provided that the operator $\mathcal{Q}^{(\mathrm{D})}$ is chosen such that $\left\langle\mathcal{Q}^{(\mathrm{D})}\right\rangle$ quantifies entanglement [40–50]. In Ref. [30] the operator $\mathcal{Q}^{(\mathrm{D})}$ was chosen to be equal to $\mathcal{Q}^{(\mathrm{S})}$, where $\mathcal{Q}^{(\mathrm{S})}$ is constructed using the Schmidt decomposition. This allowed deriving a modified Schrödinger equation having the form given by Eq. (1), which contains a nonlinear term that gives rise to pure state bipartite disentanglement. However, the Schmidt decomposition is inapplicable for both mixed states and for multipartite systems. Here we employ an alternative operator (henceforth denoted as $\mathcal{Q}^{(\mathrm{D})}$), which can be used to derive a modified master equation having the form given by Eq. (2), and which is applicable for a general multipartite case [35], and for a general mixed state.

Consider a multipartite system composed of three subsystems labeled as 'a', 'b' and 'c'. The Hilbert space of the system $H=H_{\mathrm{a}}\otimes H_{\mathrm{b}}\otimes H_{\mathrm{c}}$ is a tensor product of subsystem Hilbert spaces H_{a}, H_{b} and H_{c}. The dimensionality of the Hilbert space H_{L} of subsystem L, which is denoted by d_{L}, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b},\mathrm{c}\}$, is assumed to be finite. A general observable of subsystem L can be expanded using the set of generalized Gell-Mann matrices $\left\{\lambda_1^{(\mathrm{L})},\lambda_2^{(\mathrm{L})},\cdots,\lambda_{d_{\mathrm{L}}^2-1}^{(\mathrm{L})}\right\}$.

Entanglement between subsystems a and b can be characterized by the matrix $D_{\mathrm{ab}}\equiv\rho_{\mathrm{ab}}-\rho_{\mathrm{a}}\otimes\rho_{\mathrm{b}}$, where ρ_{ab} is the reduced density matrix of the combined a and b subsystems, and ρ_{a} (ρ_{b}) is the reduced density matrix of subsystem a (b). The following holds [see Eq. (A4) of appendix A]

$$D_{\mathrm{ab}}=\sum_{a=1}^{d_{\mathrm{a}}^2-1}\sum_{b=1}^{d_{\mathrm{b}}^2-1}\frac{\left\langle\mathcal{C}\left(\lambda_a^{(\mathrm{a})},\lambda_b^{(\mathrm{b})}\right)\right\rangle\lambda_a^{(\mathrm{a})}\otimes\lambda_b^{(\mathrm{b})}\otimes I_{\mathrm{c}}}{4} , \tag{5}$$

where for any given observable $O_{\mathrm{a}}=O_{\mathrm{a}}^{\dagger}$ of subsystem a, and a given observable $O_{\mathrm{b}}=O_{\mathrm{b}}^{\dagger}$ of subsystem b, the observable $\mathcal{C}(O_{\mathrm{a}},O_{\mathrm{b}})$ is defined by

$$\mathcal{C}(O_{\mathrm{a}},O_{\mathrm{b}})=O_{\mathrm{a}}\otimes O_{\mathrm{b}}\otimes I_{\mathrm{c}}-\langle O_{\mathrm{a}}\otimes I_{\mathrm{b}}\otimes I_{\mathrm{c}}\rangle\langle I_{\mathrm{a}}\otimes O_{\mathrm{b}}\otimes I_{\mathrm{c}}\rangle , \tag{6}$$

where I_{L} is the $d_{\mathrm{L}}\times d_{\mathrm{L}}$ identity matrix, and where $\mathrm{L}\in\{\mathrm{a},\mathrm{b},\mathrm{c}\}$.

The above result (5) suggests that entanglement between subsystems a and b can be quantified by the nonnegative variable τ_{ab}, which is given by $\tau_{\mathrm{ab}}=\left\langle\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}\right\rangle$, where the operator $\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$ is given by

$$\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}=\eta_{\mathrm{ab}}\,\mathrm{Tr}\left(C^{\mathrm{T}}\langle C\rangle\right) , \tag{7}$$

and where η_{ab} is a positive constant. The (a,b) entry of the $\left(d_{\mathrm{a}}^2-1\right)\times\left(d_{\mathrm{b}}^2-1\right)$ matrix C is the observable $\mathcal{C}\left(\lambda_a^{(\mathrm{a})},\lambda_b^{(\mathrm{b})}\right)$, and the (a,b) entry of the $\left(d_{\mathrm{a}}^2-1\right)\times\left(d_{\mathrm{b}}^2-1\right)$ matrix $\langle C\rangle$ is its expectation value $\left\langle\mathcal{C}\left(\lambda_a^{(\mathrm{a})},\lambda_b^{(\mathrm{b})}\right)\right\rangle$, and thus τ_{ab} can be expressed as [compare to Eq. (31) of Ref. [40]]

$$\tau_{\mathrm{ab}}=\eta_{\mathrm{ab}}\sum_{a=1}^{d_{\mathrm{a}}^2-1}\sum_{b=1}^{d_{\mathrm{b}}^2-1}\left\langle\mathcal{C}\left(\lambda_a^{(\mathrm{a})},\lambda_b^{(\mathrm{b})}\right)\right\rangle^2 . \tag{8}$$

In a similar way, the entanglement between subsystems b and c, which is denoted by τ_{bc}, and the entanglement between subsystems c and a, which is denoted by τ_{ca}, can be defined. Deterministic disentanglement between subsystems L' and L'' can be generated by the modified master equation (2), provided that the operator $\mathcal{Q}^{(\mathrm{D})}$ in Eq. (2) is replaced by the operator $\mathcal{Q}_{\mathrm{L}',\mathrm{L}''}^{(\mathrm{D})}$.

The entanglement variable τ_{ab} is invariant under any single subsystem unitary transformation [40]. Under such a transformation, the matrix C is transformed according to $C \to C' = T_{\mathrm{a}} C T_{\mathrm{b}}^{\mathrm{T}}$. A completeness relation, which is satisfied by the generalized Gell-Mann matrices [see Eq. (8.177) or Ref. [51]], can be used to show that both the $\left(d_{\mathrm{a}}^2 - 1\right) \times \left(d_{\mathrm{a}}^2 - 1\right)$ matrix T_{a} and the $\left(d_{\mathrm{b}}^2 - 1\right) \times \left(d_{\mathrm{b}}^2 - 1\right)$ matrix T_{b} are orthonormal, i.e. $T_{\mathrm{a}}^{\mathrm{T}} T_{\mathrm{a}} = T_{\mathrm{a}} T_{\mathrm{a}}^{\mathrm{T}} = 1$ and $T_{\mathrm{b}}^{\mathrm{T}} T_{\mathrm{b}} = T_{\mathrm{b}} T_{\mathrm{b}}^{\mathrm{T}} = 1$, and thus, as can be seen from Eq. (7), τ_{ab} is invariant [see Eq. (8.881) of Ref. [51]]. The invariance of τ_{bc} and τ_{ca} can be shown in a similar way.

V. BIPARTITE PURE STATE DISENTANGLEMENT

To gain some insight into the disentanglement process, the relatively simple case of a bipartite system in a pure state $|\psi\rangle$ is considered. Subsystems are labeled as 'a' and 'b'. With the help of the Schmidt decomposition, $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_{l=1}^{d_{\mathrm{m}}} q_l \, |l, l\rangle \ , \tag{9}$$

where $d_{\mathrm{m}} = \min(d_{\mathrm{a}}, d_{\mathrm{b}})$, the coefficients q_l are non-negative real numbers, the tensor product $|l\rangle_{\mathrm{a}} \otimes |l\rangle_{\mathrm{b}}$ is denoted by $|l, l\rangle$, and $\{|l\rangle_{\mathrm{a}}\}$ $(\{|l\rangle_{\mathrm{b}}\})$ is an orthonormal basis spanning the Hilbert space H_{a} (H_{b}) of subsystem a (b). The normalization condition reads $\langle \psi \, |\psi\rangle = L_2 = 1$, where the n'th moment L_n is defined by

$$L_n = \sum_{l=1}^{d_{\mathrm{m}}} q_l^n \ . \tag{10}$$

For a product state, for which $q_l = \delta_{l,l_0}$, where $l_0 \in \{1, 2, \cdots, d_{\mathrm{m}}\}$, τ_{ab} obtains its minimum value of $\tau_{\mathrm{ab}} = 0$ [see Eqs. (6) and (8)]. The maximum value of τ_{ab}, which is given by $\tau_{\mathrm{ab}} = \eta_{\mathrm{ab}} \left(d_{\mathrm{m}}^2 - 1\right) (2/d_{\mathrm{m}})^2$, is obtained when $q_l = d_{\mathrm{m}}^{-1/2}$ [maximum entropy state, see Eq. (8)]. The constant η_{ab} is chosen to be given by

$$\eta_{\mathrm{ab}} = \frac{d_{\mathrm{m}}^2}{4\left(d_{\mathrm{m}}^2 - 1\right)} \ . \tag{11}$$

For this choice τ_{ab} is bounded between zero and unity.

Consider for simplicity the case where the Hamiltonian vanishes, i.e. $\mathcal{H} = 0$. For that case the modified Schrödinger equation (1) for $d_{\mathrm{m}} \geq 3$ yields

$$\frac{\mathrm{d} \log q_l}{\mathrm{d}t} = 4\gamma_{\mathrm{D}} \eta_{\mathrm{ab}} K_l^{(3)} \ , \tag{12}$$

where the so-called capitalistic function $K_l^{(m)}$ is given by $K_l^{(m)} = q_l^{2(m-1)} - L_{2m}$. For $d_{\mathrm{m}} = 2$ the factor 4 in Eq. (12) is replaced by 12. The identity $K_l^{(m)} = \partial H^{(m)} / \partial q_l$, where the potential function $H^{(m)}$ is given by

$$H^{(m)} = \frac{1 + m\left(1 - L_2\right)}{2m} L_{2m} \ , \tag{13}$$

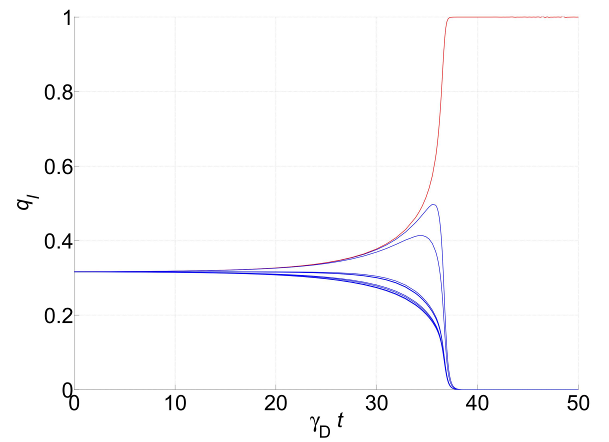

FIG. 1: Pure bipartite disentanglement. The time evolution of the coefficients q_l is calculated using Eq. (12) for the case $d_{\mathrm{m}} = 10$ and $\gamma\eta_{\mathrm{ab}} = 1$. The coefficients q_{l_0}, which initially, at time $t = 0$, is the largest one, i.e. $q_{l_0} = \max\{q_l\}$, is represented by the red curve.

implies that L_{2m} (i.e. L_6 for $m = 3$) monotonically increases in time (recall the normalization condition $L_2 = 1$). A similar derivation, based on the operator $\mathcal{Q}^{(\mathrm{S})}$, leads to a set of equations of motion similar to (12), but with $m = 2$ [30]. For that case L_4 monotonically increases in time [see Eq. (13)].

The following holds [see Eq. (12)]

$$\frac{\mathrm{d} \log \frac{q_{l'}}{q_{l''}}}{\mathrm{d}t} = 4\gamma_{\mathrm{D}} \eta_{\mathrm{ab}} \left(q_{l'}^{2(m-1)} - q_{l''}^{2(m-1)} \right) \ . \tag{14}$$

For both cases $\mathcal{Q}^{(\mathrm{S})}$ (for which $m = 2$) and $\mathcal{Q}^{(\mathrm{D})}$ (for which $m = 3$), time evolution governed by Eq. (14) gives rise to disentanglement. Consider the case where initially, at time $t = 0$, $q_{l_0} = \max\{q_l\}$ for a unique positive integer $l_0 \in \{1, 2, \cdots, d_{\mathrm{m}}\}$. As can be seen from Eq. (14), for this case $|\psi\rangle$ evolves into the product state $|l_0, l_0\rangle$ in the long time limit, i.e. $q_l \to \delta_{l,l_0}$ for $t \to \infty$. This behavior is demonstrated by the plot shown in Fig. 1.

VI. TWO SPIN 1/2

While thermalisation increases entropy, disentanglement decreases it (as is demonstrated by the plot shown in Fig. 1) [52]. The interplay between thermalisation and disentanglement is explored below using a relatively simple system composed of two spins 1/2. For this case $d_{\mathrm{a}} = d_{\mathrm{b}} = 2$, and $\eta_{\mathrm{ab}} = 1/3$ [see Eq. (11)]. The angular momentum vector operator of spin L is denoted by $\mathbf{S}_{\mathrm{L}} = (S_{\mathrm{Lx}}, S_{\mathrm{Ly}}, S_{\mathrm{Lz}})$, where $\mathrm{L} \in \{\mathrm{a}, \mathrm{b}\}$.

Consider first the case where the system is in a pure state $|\psi\rangle$ given by $|\psi\rangle = q_{00} \, |00\rangle + q_{01} \, |01\rangle + q_{10} \, |10\rangle + q_{11} \, |11\rangle$, where the ket vector $|\sigma_{\mathrm{b}} \sigma_{\mathrm{a}}\rangle$ is an eigenvector of $(1/2)\left(1 - (2/\hbar) S_{\mathrm{az}}\right)$ and of $(1/2)\left(1 - (2/\hbar) S_{\mathrm{bz}}\right)$, with eigenvalues $\sigma_{\mathrm{a}} \in \{0,1\}$ and $\sigma_{\mathrm{b}} \in \{0,1\}$, respectively. For this case Eq. (8) yields (hereafter it is assumed that disentanglement is generated by the operator $\mathcal{Q}^{(\mathrm{D})}$)

$$\tau_{\mathrm{ab}} = \frac{8\left|\mathcal{D}\right|^2 \left(1+2\left|\mathcal{D}\right|^2\right)}{3} , \tag{15}$$

where $\mathcal{D} = q_{00}q_{11} - q_{01}q_{10}$ (note that $\left|D\right|^2 \leq 1/4$ [53]).

To explore the interplay between thermalisation and disentanglement, the system's state is henceforth allowed to be mixed. Consider the case where the Hamiltonian is given by $\mathcal{H} = -\hbar\omega_{\mathrm{B}}P_{\mathrm{B}}$, where ω_{B} is a positive constant, and $P_{\mathrm{B}} = \left|\psi_{\mathrm{B}}\right\rangle\left\langle\psi_{\mathrm{B}}\right|$ is a projection operator associated with the fully entangled Bell singlet state $\left|\psi_{\mathrm{B}}\right\rangle = 2^{-1/2}\left(\left|01\right\rangle - \left|10\right\rangle\right)$. As can be verified using Eq. (2), the modified master equation for this case has a fixed point given by

$$\rho_{\mathrm{s}} = \frac{1+\kappa\left(1-4P_{\mathrm{B}}\right)}{4} , \tag{16}$$

where the real variable κ is found by solving

$$\log\frac{1-3\kappa}{1+\kappa} = \hbar\omega_{\mathrm{B}}\beta + \frac{4\eta_{\mathrm{ab}}\gamma_{\mathrm{D}}\kappa}{\gamma_{\mathrm{H}}} . \tag{17}$$

In the limit $\hbar\omega_{\mathrm{B}}\beta \gg 1$ where thermalisation dominates, Eq. (17) yields $\kappa \simeq -1+4\exp\left(-\hbar\omega_{\mathrm{B}}\beta + 4\gamma_{\mathrm{D}}/\gamma_{\mathrm{H}}\right)$ (i.e. $\rho_{\mathrm{s}} \simeq P_{\mathrm{B}}$). For this limit the ground state $\left|\psi_{\mathrm{B}}\right\rangle$, which is fully entangled, is nearly fully occupied, and the density matrix ρ_{s} represents a nearly pure state. In the opposite limit $\hbar\omega_{\mathrm{B}}\beta \ll 1$ where disentanglement dominates, Eq. (17) yields $\kappa \simeq -(1/4)\hbar\omega_{\mathrm{B}}\beta\left(1+\gamma_{\mathrm{D}}/\gamma_{\mathrm{H}}\right)^{-1}$ (i.e. $\rho_{\mathrm{s}} \simeq 1/4$). In this limit the density matrix ρ_{s} represents a nearly fully mixed and fully disentangled state.

An example for time evolution, which is obtained by numerically integrating the modified master equation (2), is shown in Fig. 2. Assumed parameters' values are listed in the figure caption.

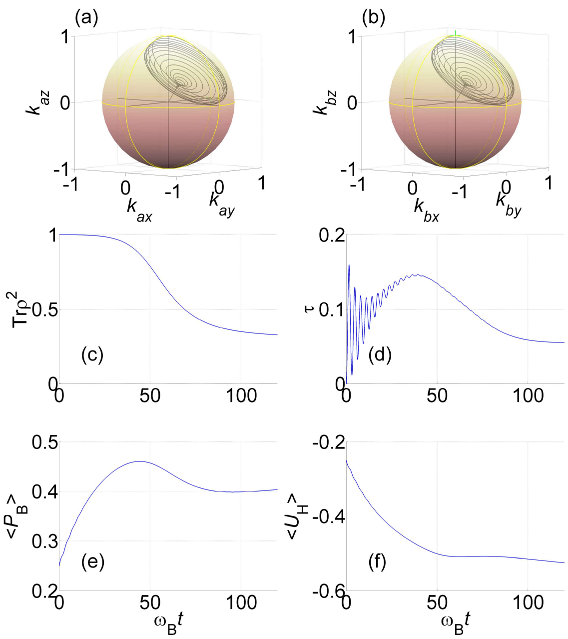

FIG. 2: Bell singlet state. The time evolution of the single spin Bloch vectors $\mathbf{k}_{\mathrm{a}}$ and $\mathbf{k}_{\mathrm{b}}$ is shown in (a) and (b), respectively. Initial values for $\mathbf{k}_{\mathrm{a}}$ and $\mathbf{k}_{\mathrm{b}}$ are denoted by green cross symbols. (c) The purity $\mathrm{Tr}\,\rho^2$. (d) The entanglement variable τ. (e) The expectation value $\langle P_{\mathrm{B}}\rangle$ of the projection $P_{\mathrm{B}} = \left|\psi_{\mathrm{B}}\right\rangle\left\langle\psi_{\mathrm{B}}\right|$. (f) The expectation value $\langle\mathcal{U}_{\mathrm{H}}\rangle$ of the Helmholtz free energy. Assumed parameters' values are $\gamma_{\mathrm{H}}/\omega_{\mathrm{B}} = 0.005$, $\gamma_{\mathrm{D}}/\omega_{\mathrm{B}} = 0.05$ and $\hbar\omega_{\mathrm{B}}\beta = 10$.

VII. TRUNCATION APPROXIMATION

The simplest physical system suitable for the exploration of disentanglement is the above-discussed two spin 1/2 system. For some cases, further simplification can be achieved by implementing a truncation approximation.

Let $\mathcal{H}$ be the Hamiltonian of a two spin 1/2 system. The matrix representation of $\mathcal{H}$ in the basis $\{\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle\}$ is assumed to be given by

$$\hbar^{-1}\mathcal{H} \doteq \Omega = \begin{pmatrix} \frac{\omega_{\mathrm{s}}}{2} & 0 & 0 & 0 \\ 0 & \Omega_{22} & \Omega_{23} & 0 \\ 0 & \Omega_{32} & \Omega_{33} & 0 \\ 0 & 0 & 0 & \frac{\omega_{\mathrm{s}}}{2} \end{pmatrix} , \tag{18}$$

where the central 2×2 block of Ω is given by

$$\begin{pmatrix} \Omega_{22} & \Omega_{23} \\ \Omega_{32} & \Omega_{33} \end{pmatrix} = \frac{\boldsymbol{\omega}_{\mathrm{E}}\cdot\boldsymbol{\sigma}}{2} , \tag{19}$$

both the scalar ω_{s} and the vector $\boldsymbol{\omega}_{\mathrm{E}} = \left(\omega_{\mathrm{E}x}, \omega_{\mathrm{E}y}, \omega_{\mathrm{E}z}\right)$ are real, and $\boldsymbol{\sigma} = \left(\sigma_x, \sigma_y, \sigma_z\right)$ is the Pauli matrix vector. Consider the case where $\hbar\omega_{\mathrm{s}}\beta \gg 1$. For this case, for which the population of the states $\left|00\right\rangle$ and $\left|11\right\rangle$ is low, a truncation approximation can be employed. In this approximation it is assumed that the system's Hilbert space is spanned by the vector states $\left|01\right\rangle$ and $\left|10\right\rangle$. Note that in the truncation approximation, τ_{ab} for pure states is given by Eq. (15), with $\mathcal{D} = q_{01}q_{10}$.

The truncated density matrix is expressed as

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{20}$$

where the central 2×2 block of ρ is given by

$$\begin{pmatrix} \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} \end{pmatrix} = \frac{1+\mu\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}{2} , \tag{21}$$

where $\mu \in [0,1]$ is real, and $\hat{\mathbf{n}} = \left(n_x, n_y, n_z\right)$ is a unit vector (i.e. $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$). Note that $\mathrm{Tr}\,\rho^2 = (1/2)\left(1+\mu^2\right)$. As can be seen from Eq. (21), in the truncation approximation the system's state is describe by a single real 3-dimensionla Bloch vector $\mathbf{k} = \mu\hat{\mathbf{n}}$.

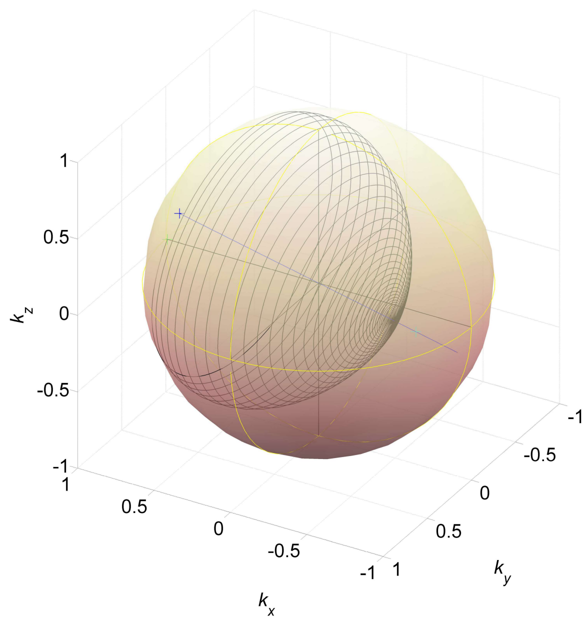

FIG. 3: Truncation approximation. The green cross symbol represents the initial value of the Bloch vector $\mathbf{k} = \mu\hat{\mathbf{n}}$, the blue line is parallel to $\boldsymbol{\omega}_{\mathrm{E}}$, the blue cross symbol represents the unit vector $\omega_{\mathrm{E}}^{-1}\boldsymbol{\omega}_{\mathrm{E}}$, and the cyan cross symbol represents the point $-\omega_{\mathrm{E}}^{-1}\tanh(\beta\hbar\omega_{\mathrm{E}}/2)\,\boldsymbol{\omega}_{\mathrm{E}}$ (steady state thermal equilibrium in the absence of disentanglement). Assumed parameters' values are $\gamma_{\mathrm{D}}/\gamma_{\mathrm{H}} = 1$, $\boldsymbol{\omega}_{\mathrm{E}}/\gamma_{\mathrm{H}} = 10^2(1,1,1)$ and $\beta\hbar\omega_{\mathrm{E}} = 1$.

With the help of the truncated density matrix ρ (20), one finds that the matrix $\Theta^{(\mathrm{D})} = \gamma_{\mathrm{D}}\left(\mathcal{Q}^{(\mathrm{D})}\rho + \rho\mathcal{Q}^{(\mathrm{D})} - 2\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle \rho\right)$ is given by

$$\Theta^{(\mathrm{D})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Theta^{(\mathrm{D})}_{22} & \Theta^{(\mathrm{D})}_{23} & 0 \\ 0 & \Theta^{(\mathrm{D})}_{32} & \Theta^{(\mathrm{D})}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{22}$$

where the central 2×2 block of $\Theta^{(\mathrm{D})}$ is given by

$$\begin{pmatrix} \Theta^{(\mathrm{D})}_{22} & \Theta^{(\mathrm{D})}_{23} \\ \Theta^{(\mathrm{D})}_{32} & \Theta^{(\mathrm{D})}_{33} \end{pmatrix} = \mathbf{k}^{(\mathrm{D})}\cdot\boldsymbol{\sigma} , \tag{23}$$

$\mathbf{k}^{(\mathrm{D})} = -(2/3)\,\gamma_{\mathrm{D}}\mu\left(n_x\left(N_\perp^2-1\right), n_y\left(N_\perp^2-1\right), n_z N_\perp^2\right)$, and $N_\perp^2 = \mu^2\left(1-n_z^2\right)$. The following holds

$$\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle = \frac{1+2\mu^2+\mu^2 n_z^2\left(\mu^2 n_z^2-4\right)}{3} , \tag{24}$$

thus for a given μ, the expectation value $\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle = \mathrm{Tr}\left(\mathcal{Q}^{(\mathrm{D})}\rho\right)$ is bounded by $\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle \in \left[(1/3)\left(1-\mu^2\right)^2, (1/3)\left(1+2\mu^2\right)\right]$. The expectation value $\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle$ obtains its minimum value $\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle = 0$ for $n_z^2 = 1$ and $\mu = 1$. These two points (north and south poles of the Bloch sphere) represent fully disentangled states.

The entropy matrix $S = -\log\rho$ is given by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & s_{22} & s_{23} & 0 \\ 0 & s_{32} & s_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{25}$$

where the central 2×2 block of S is given by

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} = -\frac{1-\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}{2}\log\frac{1-\mu}{2} - \frac{1+\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}{2}\log\frac{1+\mu}{2} . \tag{26}$$

Note that the expectation value $\langle S\rangle$, which is given by

$$\langle S\rangle = -\frac{1-\mu}{2}\log\frac{1-\mu}{2} - \frac{1+\mu}{2}\log\frac{1+\mu}{2} , \tag{27}$$

is bounded by $\langle S\rangle \in [0,\log 2]$. With the help of Eq. (26) one finds that the central 2×2 block of $(S\rho+\rho S-2\langle S\rangle\rho)$ is given by $\mathbf{k}^{(\mathrm{S})}\cdot\boldsymbol{\sigma}$, where $\mathbf{k}^{(\mathrm{S})} = -\left(1-\mu^2\right)\left(\tanh^{-1}\mu\right)\hat{\mathbf{n}}$ [recall the identity $\log\left((1-\mu)/(1+\mu)\right) = -2\tanh^{-1}\mu$].

The central 2×2 block of $(\mathcal{H}\rho+\rho\mathcal{H}-2\langle\mathcal{H}\rangle\rho)$ is given by $\mathbf{k}^{(\Omega)}\cdot\boldsymbol{\sigma}$, where $\mathbf{k}^{(\Omega)} = (\hbar/2)\left(\boldsymbol{\omega}_{\mathrm{E}} - \mu^2\left(\boldsymbol{\omega}_{\mathrm{E}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}\right)$ [see Eq. (19), and recall the identity $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$], thus the central 2×2 block of $\Theta^{(\mathrm{H})} = \gamma_{\mathrm{H}}\left(\mathcal{Q}^{(\mathrm{H})}\rho + \rho\mathcal{Q}^{(\mathrm{H})} - 2\left\langle \mathcal{Q}^{(\mathrm{H})}\right\rangle\rho\right)$ is given by $\mathbf{k}^{(\mathrm{H})}\cdot\boldsymbol{\sigma}$, where

$$\frac{\mathbf{k}^{(\mathrm{H})}}{\gamma_{\mathrm{H}}} = \frac{\beta\hbar\boldsymbol{\omega}_{\mathrm{E}}}{2} + \frac{2\left(1-\mu^2\right)\tanh^{-1}\mu - \beta\hbar\mu^2\left(\boldsymbol{\omega}_{\mathrm{E}}\cdot\hat{\mathbf{n}}\right)}{2}\hat{\mathbf{n}} . \tag{28}$$

The expectation value $\langle\Theta\rangle$ is given by $\langle\Theta\rangle = \gamma_{\mathrm{H}}\left\langle \mathcal{Q}^{(\mathrm{H})}\right\rangle + \gamma_{\mathrm{D}}\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle$, where

$$\left\langle \mathcal{Q}^{(\mathrm{H})}\right\rangle = \frac{\mu\beta\hbar\left(\boldsymbol{\omega}_{\mathrm{E}}\cdot\hat{\mathbf{n}}\right)}{2} + \frac{1-\mu}{2}\log\frac{1-\mu}{2} + \frac{1+\mu}{2}\log\frac{1+\mu}{2} , \tag{29}$$

and $\left\langle \mathcal{Q}^{(\mathrm{D})}\right\rangle$ is given by Eq. (24). The expectation value $\left\langle \mathcal{Q}^{(\mathrm{H})}\right\rangle$ is minimized at thermal equilibrium, for which $\boldsymbol{\omega}_{\mathrm{E}}\cdot\hat{\mathbf{n}} = -\omega_{\mathrm{E}}$ (i.e. $\hat{\mathbf{n}}$ is anti-parallel to $\boldsymbol{\omega}_{\mathrm{E}}$) and $\mu = \tanh(\beta\hbar\omega_{\mathrm{E}}/2)$ [see Eq. (29)].

The modified master equation (2) yields an equation of motion for the 3-dimensional Bloch vector $\mathbf{k} = \mu\hat{\mathbf{n}}$ given by

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = \boldsymbol{\omega}_{\mathrm{E}}\times\mathbf{k} - 2\left(\mathbf{k}^{(\mathrm{H})} + \mathbf{k}^{(\mathrm{D})}\right) . \tag{30}$$

An example of numerical integration of Eq. (30) is shown in Fig. 3. This example demonstrates a disentanglement-induced shift of a steady state fixed point away from thermal equilibrium (which is represented by a cyan cross symbol). Assumed parameters are listed in the figure caption.

VIII. MEAN FIELD APPROXIMATION

When disentanglement is sufficiently efficient (i.e. γ_{D} is sufficiently large), the equations of motion generated by the modified master equation (2) can be simplified by employing the mean field approximation (MFA).

To demonstrate the MFA for the two spin 1/2 system, consider the case where spin a has a relatively low angular Larmor frequency ω_{a}, in comparison with the angular Larmor frequency ω_{b} of spin b, which is externally driven. For this case the Hamiltonian $\mathcal{H}$ of the closed system is assumed to be given by

$$\mathcal{H} = \omega_{\mathrm{a}} S_{\mathrm{az}} + \omega_{\mathrm{b}} S_{\mathrm{bz}} + \frac{\omega_1 \left(S_{\mathrm{b}+} + S_{\mathrm{b}-}\right)}{2} + V \ , \tag{31}$$

where the driving amplitude and angular frequency are denoted by ω_1 and $\omega_{\mathrm{p}} = \omega_{\mathrm{b}} + \Delta$, respectively ($\Delta$ is the driving detuning), the operators $S_{\mathrm{a}\pm}$ are given by $S_{\mathrm{a}\pm} = S_{\mathrm{ax}} \pm i S_{\mathrm{ay}}$, and the rotated operators $S_{\mathrm{b}\pm}$ are given by $S_{\mathrm{b}\pm} = \left(S_{\mathrm{bx}} \pm i S_{\mathrm{by}}\right) e^{\pm i \omega_{\mathrm{p}} t}$. The dipolar coupling term V is given by $V = g\hbar^{-1}\left(S_{\mathrm{a}+} + S_{\mathrm{a}-}\right) S_{\mathrm{bz}}$, where g is a coupling rate. The largest effect of dipolar coupling occurs when the Hartmann–Hahn matching condition $\omega_{\mathrm{a}} = \omega_{\mathrm{R}}$ is satisfied, where $\omega_{\mathrm{R}} = \sqrt{\omega_1^2 + \Delta^2}$ is the Rabi angular frequency [33, 54, 55].

Instead of employing the (nonlinear in ρ) operator $\mathcal{Q}^{(\mathrm{H})}$, damping for this case is taken into account by adding on the right hand side of the modified master equation (2) a Lindblad superoperator $\mathcal{L}$, which is linear in ρ, and which is given by [56]

$$\begin{aligned}\mathcal{L} = \sum_{\mathrm{L}\in\{\mathrm{a},\mathrm{b}\}} & \frac{\left(\hat{n}_0^{(\mathrm{L})} + 1\right)\Gamma_1^{(\mathrm{L})}}{4} \mathcal{D}_\rho\left(\sigma_-^{(\mathrm{L})}\right) + \frac{\hat{n}_0^{(\mathrm{L})}\Gamma_1^{(\mathrm{L})}}{4}\mathcal{D}_\rho\left(\sigma_+^{(\mathrm{L})}\right) \\ & + \frac{\left(2\hat{n}_0^{(\mathrm{L})} + 1\right)\Gamma_\varphi^{(\mathrm{L})}}{2}\mathcal{D}_\rho\left(\sigma_z^{(\mathrm{L})}\right) \ ,\end{aligned} \tag{32}$$

where the Lindbladian $\mathcal{D}_\rho(X)$ for an operator X is given by

$$\mathcal{D}_\rho(X) = X\rho X^\dagger - \frac{X^\dagger X\rho + \rho X^\dagger X}{2} \ , \tag{33}$$

the matrices σ_- and σ_+ are given by $\sigma_\pm = \sigma_x \pm i\sigma_y$, and σ_x, σ_y and σ_z are Pauli matrices. The positive damping rates $\Gamma_1^{(\mathrm{L})}$ and $\Gamma_\varphi^{(\mathrm{L})}$, and the thermal occupation factor $\hat{n}_0^{(\mathrm{L})}$, are related to the longitudinal $T_1^{(\mathrm{L})}$ and the transverse $T_2^{(\mathrm{L})}$ relaxation times, and to the thermal equilibrium spin polarization $k_{z0}^{(\mathrm{L})}$, by $1/T_1^{(\mathrm{L})} = \Gamma_1^{(\mathrm{L})}\left(2\hat{n}_0^{(\mathrm{L})}+1\right)$, $1/T_2^{(\mathrm{L})} = \left(\Gamma_1^{(\mathrm{L})}/2 + \Gamma_\varphi^{(\mathrm{L})}\right)\left(2\hat{n}_0^{(\mathrm{L})}+1\right)$ and $-1/k_{z0}^{(\mathrm{L})} = 2\hat{n}_0^{(\mathrm{L})} + 1$.

Equations of motion for the single spin Bloch vectors $\mathbf{k}_{\mathrm{a}}$ and $\mathbf{k}_{\mathrm{b}}$ are derived from the modified master equation (2) using Eq. (A6) of appendix A. In the MFA Eq. (A7) of appendix A yields

$$\frac{\mathrm{d}\mathbf{k}_{\mathrm{a}}}{\mathrm{d}t} = \begin{pmatrix} -\omega_{\mathrm{a}} k_{\mathrm{ay}} - \frac{k_{\mathrm{ax}}}{T_{2\mathrm{a}}} \\ \omega_{\mathrm{a}} k_{\mathrm{ax}} - g k_{\mathrm{az}} k_{\mathrm{bz}} - \frac{k_{\mathrm{ay}}}{T_{2\mathrm{a}}} \\ g k_{\mathrm{ay}} k_{\mathrm{bz}} - \frac{k_{\mathrm{az}} - k_{\mathrm{z0a}}}{T_{1\mathrm{a}}} \end{pmatrix} \ , \tag{34}$$

and

$$\frac{\mathrm{d}\mathbf{k}_{\mathrm{b}}}{\mathrm{d}t} = \begin{pmatrix} -\Delta k_{\mathrm{by}} - g k_{\mathrm{ax}} k_{\mathrm{by}} - \frac{k_{\mathrm{bx}}}{T_{2\mathrm{b}}} \\ \Delta k_{\mathrm{bx}} - \omega_1 k_{\mathrm{bz}} + g k_{\mathrm{ax}} k_{\mathrm{bx}} - \frac{k_{\mathrm{by}}}{T_{2\mathrm{b}}} \\ \omega_1 k_{\mathrm{by}} - \frac{k_{\mathrm{bz}} - k_{\mathrm{z0b}}}{T_{1\mathrm{b}}} \end{pmatrix} \ . \tag{35}$$

The plots shown in Fig. 4 exhibit the time evolution of the single spin Bloch vectors $\mathbf{k}_{\mathrm{a}}$ and $\mathbf{k}_{\mathrm{b}}$, where $\mathbf{k}_{\mathrm{L}} = \left((1/2)\langle S_{\mathrm{L}+} + S_{\mathrm{L}-}\rangle, (-i/2)\langle S_{\mathrm{L}+} - S_{\mathrm{L}-}\rangle, \langle S_{\mathrm{Lz}}\rangle\right)$, and where $\mathrm{L} \in \{\mathrm{a}, \mathrm{b}\}$. The case $g = 0$ (i.e. no dipolar coupling) is represented by the plots (a1) and (b1). For this case the steady state is a fixed point, which is labeled by a red cross symbol in Fig. 4(a1) and (b1). On the other hand, in the presence of sufficiently strong dipolar coupling, the steady state becomes a limit cycle, as is demonstrated by the plots shown in Fig. 4(a2) and (b2), for which $g/\omega_{\mathbf{a}} = 0.1$. The limit cycle angular frequency is close to ω_{a}.

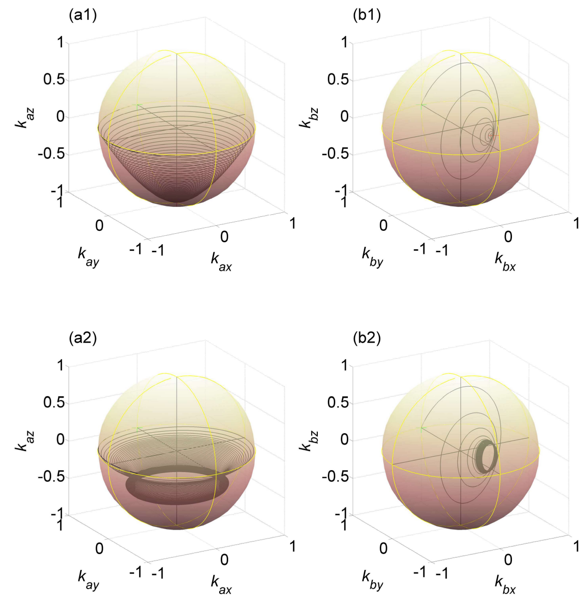

FIG. 4: Mean field approximation. The time evolution of the single spin Bloch vector $\mathbf{k}_{\mathrm{a}}$ ($\mathbf{k}_{\mathrm{b}}$) is shown in the plots labeled by the letter a (b). For plots (a1) and (b1) the spins are decoupled (i.e. $g = 0$), whereas $g/\omega_{\mathrm{a}} = 0.1$ for plots (a2) and (b2). Other assumed parameters' values are $\Delta/\omega_{\mathrm{a}} = \sin(\pi/8)$, $\omega_1/\omega_{\mathrm{a}} = \cos(\pi/8)$ (note that the Hartmann–Hahn matching condition is satisfied), $\Gamma_1^{(\mathrm{a})}/\omega_{\mathrm{a}} = 10^{-2}$, $\Gamma_\varphi^{(\mathrm{a})}/\Gamma_1^{(\mathrm{a})} = 10^{-1}$, $\Gamma_1^{(\mathrm{b})}/\Gamma_1^{(\mathrm{a})} = 10$, $\Gamma_\varphi^{(\mathrm{b})}/\Gamma_\varphi^{(\mathrm{a})} = 10$, $\hat{n}_0^{(\mathrm{a})} = 0.005$ and $\hat{n}_0^{(\mathrm{b})} = 0.0001$. Initial values for $\mathbf{k}_{\mathrm{a}}$ and $\mathbf{k}_{\mathrm{b}}$ are denoted by green cross symbols.

IX. DISCUSSION AND SUMMARY

The above-discussed MFA greatly simplifies the analysis. For the two spin 1/2 system, the modified master equation (2) leads to a set [see Eq. (A6)] of $4^2-1=15$ real equations of motion (generalized Bloch equation) for 15 real variables (generalized Bloch vector). On the other hand, for the same system, the MFA leads to a set of $2^2-1+2^2-1=6$ real equations of motion for 6 real variables [see Eqs. (34) and (35)].

As is demonstrated by the plots shown in Fig. 4, the nonlinear terms in Eqs. (34) and (35) can give rise to a limit cycle steady state. Such limit cycle steady states cannot be obtained from the GKSL master equation, which is linear in ρ (see appendix B of Ref. [33]). On the other hand, in spite of the linear dependency of the GKSL equation on ρ, some theoretical studies have revealed nonlinear dynamics that is derived from this GKSL master equation [57–61]. However, the origin of such nonlinearity is the assumption that entanglement between subsystems can be disregarded. It has remained unclear how such an assumption can be justified in the framework of standard QM. On the other hand, this assumption represents a limiting case for the modified master equation (2), for which disentanglement is sufficiently efficient.

The limit cycle steady state shown in Fig. 4(a2) and (b2) can occur only when the driving detuning Δ is positive (i.e. driving is blue-detuned). This behavior demonstrates that disentanglement can give rise to detuning asymmetry. On the other hand, in the absence of disentanglement, the system's response is theoretically expected to be an even function of the detuning Δ [e.g. see Eq. (4) of Ref. [62]]. Many examples can be found in the published literature for a profound detuning asymmetry observed in spin systems under nutation driving or dynamical decoupling. In most papers the presented asymmetry is not discussed, however, a paper from 1955 [63], and another one from 2005 [64], explicitly state that the observed asymmetry is theoretically unexpected. Moreover, limit cycle steady states are experimentally observed in systems of correlated spins [65]. Further study is needed to explore possible connections between experimentally observed nonlinear dynamics in spin systems [66] and disentanglement.

In summary, the spontaneous disentanglement hypothesis is inherently falsifiable, because it yields predictions, which are experimentally distinguishable from predictions obtained from standard QM. In particular, as was discussed above, the experimental observation of a limit cycle steady state in a system having a Hilbert space of finite dimensionality (i.e. a spin system) may provide a supporting evidence for the spontaneous disentanglement hypothesis. Moreover, such an experimental observation may yield some insight related to the question 'what determines the disentanglement rate γ_{D}?', which has remained entirely open.

Appendix A: Hilbert space factorization

Consider a d_{H}-dimensional Hilbert space, where $d_{\mathrm{H}} \in \{2,3,\cdots\}$ is finite. The generalized Gell-Mann set $G=\{\lambda_l\}$, which spans the $\mathrm{SU}(d_{\mathrm{H}})$ Lie algebra, contains d_{H}^2-1 square $d_{\mathrm{H}}\times d_{\mathrm{H}}$ Hermitian matrices. For the case $d_{\mathrm{H}}=2$ ($d_{\mathrm{H}}=3$) the 3 (8) set elements are called Pauli (Gell-Mann) matrices. The Generalized Gell-Mann matrices are traceless, i.e. $\mathrm{Tr}\,\lambda_l=0$, and they satisfy the orthogonality relation

$$\frac{\mathrm{Tr}\left(\lambda_{l'}\lambda_{l''}\right)}{2}=\delta_{l',l''}\ . \tag{A1}$$

Unless d_{H} is prime, it can be factored as $d_{\mathrm{H}}=d_{\mathrm{a}}d_{\mathrm{b}}$, where $d_{\mathrm{a}}>1$ and $d_{\mathrm{b}}>1$ are both integers. The two subsystems corresponding to the factorization [67] are labelled as 'a' and 'b', respectively. The generalized Gell-Mann $d_{\mathrm{L}}\times d_{\mathrm{L}}$ matrices corresponding to subsystem L, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b}\}$, are denoted by $\lambda_l^{(\mathrm{L})}$, where $l\in\left\{1,2,\cdots,d_{\mathrm{L}}^2-1\right\}$. For a given factorization, consider the set of d_{H}^2-1 matrices $G^{(\mathrm{ab})}=\left\{\Gamma_a^{(\mathrm{a})}\otimes\Gamma_b^{(\mathrm{b})}\right\}-\left\{\Gamma_0^{(\mathrm{a})}\otimes\Gamma_0^{(\mathrm{b})}\right\}$, where $a\in\left\{0,1,2,\cdots,d_{\mathrm{a}}^2-1\right\}$ and $b\in\left\{0,1,2,\cdots,d_{\mathrm{b}}^2-1\right\}$. For subsystem L, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b}\}$, the matrix $\Gamma_0^{(\mathrm{L})}$ is defined by $\Gamma_0^{(\mathrm{L})}=\left(2^{1/4}/d_{\mathrm{L}}^{1/2}\right)I_{\mathrm{L}}$, where I_{L} is the $d_{\mathrm{L}}\times d_{\mathrm{L}}$ identity matrix, and for $l\in\left\{1,2,\cdots,d_{\mathrm{L}}^2-1\right\}$ the matrix $\Gamma_l^{(\mathrm{L})}$ is defined by $\Gamma_l^{(\mathrm{L})}=2^{-1/4}\lambda_l^{(\mathrm{L})}$.

With the help of the Kronecker matrix product identities $\mathrm{Tr}\,(A\otimes B)=\mathrm{Tr}\,A\,\mathrm{Tr}\,B$ and $(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$, one finds that the set $G^{(\mathrm{ab})}$ shares two properties with the Gell-Mann set G of the d_{H}-dimensional Hilbert space. The first one is tracelessness $\mathrm{Tr}\,G_{a,b}=0$ for any $G_{a,b}\equiv\Gamma_a^{(\mathrm{a})}\otimes\Gamma_b^{(\mathrm{b})}\in G^{(\mathrm{ab})}$ [recall that $G_{0,0}\notin G^{(\mathrm{ab})}$], and the second one is orthogonality [see Eq. (A1)]

$$\frac{\mathrm{Tr}\left(G_{a',b'}G_{a'',b''}\right)}{2}=\delta_{a',a''}\delta_{b',b''}\ . \tag{A2}$$

The set $G^{(\mathrm{ab})}$ can be used to expand the entire system density matrix ρ (which is assume to be normalized, i.e. $\mathrm{Tr}\,\rho=1$) as

$$\rho=\sum_{a=0}^{d_{\mathrm{a}}^2-1}\sum_{b=0}^{d_{\mathrm{b}}^2-1}\frac{\langle G_{a,b}\rangle\,G_{a,b}}{2}\ , \tag{A3}$$

where $\langle O\rangle=\mathrm{Tr}\,(O\rho)$ for a given observable O. Partial trace is used to derive the reduced density matrices $\rho_{\mathrm{a}}=\mathrm{Tr}_{\mathrm{b}}\,\rho=2^{-1/2}\sum_{a=0}^{d_{\mathrm{a}}^2-1}\left\langle\Gamma_a^{(\mathrm{a})}\otimes I_{\mathrm{b}}\right\rangle\Gamma_a^{(\mathrm{a})}$ and $\rho_{\mathrm{b}}=\mathrm{Tr}_{\mathrm{a}}\,\rho=2^{-1/2}\sum_{b=0}^{d_{\mathrm{b}}^2-1}\left\langle I_{\mathrm{a}}\otimes\Gamma_b^{(\mathrm{b})}\right\rangle\Gamma_b^{(\mathrm{b})}$ [recall the identities $\mathrm{Tr}_{\mathrm{A}}\,(A\otimes B)=\mathrm{Tr}\,(A)\,B$ and $\mathrm{Tr}_{\mathrm{B}}\,(A\otimes B)=\mathrm{Tr}\,(B)\,A$]. Level of entanglement can be characterized

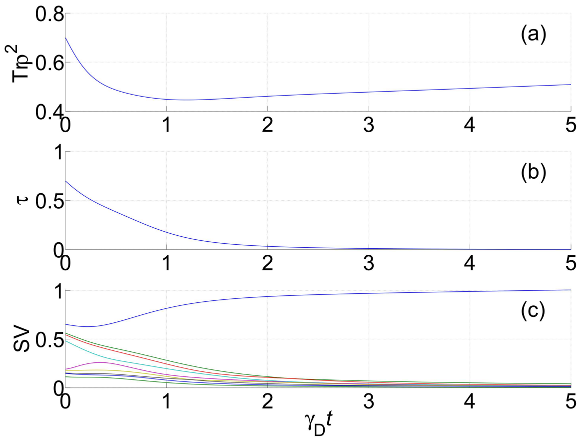

FIG. 5: The Bloch matrix $\langle G_{a,b}\rangle$. For this example, $\mathcal{H}=0$, $\gamma_{\mathrm{H}}=0$, $d_{\mathrm{a}}=3$ and $d_{\mathrm{b}}=4$. (a) Purity $\operatorname{Tr}\rho^2$. (b) Entanglement τ. (c) SV of the Bloch matrix $\langle G_{a,b}\rangle$.

using the matrix $D=\rho-\rho_{\mathrm{a}}\otimes\rho_{\mathrm{b}}$, which is given by

$$\begin{aligned} D &= \sum_{a=0}^{d_{\mathrm{a}}^2-1}\sum_{b=0}^{d_{\mathrm{b}}^2-1}\frac{\left(\langle G_{a,b}\rangle-\left\langle\Gamma_a^{(\mathrm{a})}\otimes I_{\mathrm{b}}\right\rangle\left\langle I_{\mathrm{a}}\otimes\Gamma_b^{(\mathrm{b})}\right\rangle\right)G_{a,b}}{2} \\ &= \sum_{a=1}^{d_{\mathrm{a}}^2-1}\sum_{b=1}^{d_{\mathrm{b}}^2-1}\frac{\left(\langle \lambda_{a,b}\rangle-\left\langle\lambda_a^{(\mathrm{a})}\otimes I_{\mathrm{b}}\right\rangle\left\langle I_{\mathrm{a}}\otimes\lambda_b^{(\mathrm{b})}\right\rangle\right)\lambda_{a,b}}{4}\ , \end{aligned} \tag{A4}$$

where $\lambda_{a,b}=\lambda_a^{(\mathrm{a})}\otimes\lambda_b^{(\mathrm{b})}$. For any product state $D=0$. Alternatively, as can be seen from Eq. (A4), the density matrix ρ represents a product state if and only if rank $\langle G_{a,b}\rangle=1$ [this is proved by showing that the assumption rank $\langle G_{a,b}\rangle=1$, which implies that the $d_{\mathrm{a}}^2\times d_{\mathrm{b}}^2$ matrix $\langle G_{a,b}\rangle$ equals an outer product, leads to $\left\langle\lambda_a^{(\mathrm{a})}\otimes\lambda_b^{(\mathrm{b})}\right\rangle/\left(\left\langle\lambda_a^{(\mathrm{a})}\otimes I_{\mathrm{b}}\right\rangle\left\langle I_{\mathrm{a}}\otimes\lambda_b^{(\mathrm{b})}\right\rangle\right)=1$].

The $d_{\mathrm{a}}^2\times d_{\mathrm{b}}^2$ matrix $\langle G_{a,b}\rangle$ is henceforth referred to as the Bloch matrix. By expanding the master equation as [see Eq. (A2)]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\sum_{a=0}^{d_{\mathrm{a}}^2-1}\sum_{b=0}^{d_{\mathrm{b}}^2-1}\frac{\operatorname{Tr}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}G_{a,b}\right)G_{a,b}}{2}\ , \tag{A5}$$

one finds that the time evolution of the Bloch matrix is governed by [see Eq. (A3)]

$$\frac{\mathrm{d}\langle G_{a,b}\rangle}{\mathrm{d}t}=\operatorname{Tr}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}G_{a,b}\right)\ . \tag{A6}$$

In the mean field approximation (MFA) it is assumed that

$$\langle G_{a,b}\rangle=\langle G_{a,0}\rangle\langle G_{0,b}\rangle\ . \tag{A7}$$

To demonstrate the effect of spontaneous disentanglement on the time evolution of the Bloch matrix $\langle G_{a,b}\rangle$, the case where the Hamiltonian $\mathcal{H}$ vanishes, and $\gamma_{\mathrm{H}}=0$ (i.e. no thermalisation) is considered. For the plots shown in Fig. 5, $d_{\mathrm{a}}=3$ and $d_{\mathrm{b}}=4$. The time evolution of (a) the purity $\operatorname{Tr}\rho^2$, (b) the entanglement τ, and (c) the singular values (SV) of the Bloch matrix $\langle G_{a,b}\rangle$, is evaluated by numerically integrating the master equation (2). Note that, in the long time limit $\gamma_{\mathrm{D}}t\to\infty$, for which entanglement if fully suppressed, i.e. $\tau=0$, the Bloch matrix $\langle G_{a,b}\rangle$ has a single non-zero SV (i.e. its rank becomes unity).

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