Phase transitions, critical behavior and microstructure of the FRW universe in the framework of higher order GUP

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In this paper, we explore the the phase transition, critical behavior and microstructure of the FRW in the framework of a new higher order generalized uncertainty principle. Our initial step involves deriving the equation of state by defining the work density W from GUP-corrected Friedmann equations as the thermodynamic pressure P. Based on the modified equation of state, we conduct an analysis of the $P-V$ phase transition in the FRW universe. Subsequently, we obtain the critical exponents and coexistence curves for the small and large phases of the FRW universe around the critical point. Finally, employing Ruppeiner geometry, we derive the thermodynamic curvature scalar R_N, investigating its sign-changing curve and spinodal curve. The results reveal distinctive thermodynamic properties for FRW universes with positive and negative GUP parameters β. In the case of $\beta > 0$, the phase transition, critical behavior and microstructure of FRW universe are consistent with those of Van der Waals fluids. Conversely, for $\beta < 0$, the results resemble those obtained through effective scalar field theory. These findings underscore the capacity of quantum gravity to induce phase transitions in the universe, warranting further in-depth exploration.

I. INTRODUCTION

As a pivotal concept in the realm of physics, thermodynamic phase transitions exert a profound influence on our comprehension of natural phenomena. These phase transition processes exist both at the microscopic atomic scale and throughout the macroscopic realm of astrophysics. Significant changes in energy, entropy, microstructure, and other properties in the transition of matter from one state to another have triggered an in-depth study of thermodynamic phase transitions. It is noteworthy that thermodynamic phase transitions extend beyond commonplace matter, manifesting also within gravitational systems. As an extreme manifestation of the gravitational field, the thermodynamics of black holes and their properties related to phase transitions have captivated the attention of both theoretical and astrophysical physicists [1–4]. Since the formulation of the four laws of black hole thermodynamics in the 1970s [5], considerable strides have been taken in unraveling the intricacies of black hole phase transitions [6–8]. Subsequent to this, Mann et al. identified parallels between the phase transition behavior of AdS spacetime and that of Van der Waals (VdW) fluids, instigating a surge of interest in black hole physics [9–11]. Now, the thermodynamic phase transition of black holes can be used to further study their microstructure and interactions [12–17]. Collectively, these findings underscore that thermodynamic phase transitions can serve as probes, facilitating a more nuanced analysis of gravity's characteristics.

On the other hand, the universe itself serves as an important gravitational system, whose properties expounded through the Friedmann-Robertson-Walker (FRW) metric. This widely embraced model encapsulates a well-defined horizon, a pivotal element supporting a self-consistent thermodynamic framework. Therefore, the universe establishes a profound physical interconnection with thermodynamics. In analogy to the rich thermodynamic phenomena observed in black holes, the FRW universe unfolds numerous intriguing aspects, including the Hawking temperature, Bekenstein-Hawking entropy, quasi-local energy, *et al.* [18–24]. Notably, recent investigations into the thermodynamic properties of the classical FRW universe by Haximjan *et al.* revealed the absence of a $P-V$ phase transition [25]. However, a revelation emerges when shifting the focus to the realm of modified gravity. In Ref. [26], Kong *et al.* utilized the Horndeski gravity to modify the pressure and density of the FRW universe with a perfect fluid and successfully construct a VdW-like equations of state, which allow them to analyze the phase transitions and critical behavior therein. Interestingly, their findings diverged from conventional expectations, demonstrating that the coexisting phase of the $P-V$ transition materializes above a critical temperature-a departure from the anticipated behavior in typical VdW fluids and many black hole systems. Building on this groundwork, subsequent research has delved into the exploration of phase transitions, critical behavior, and microstructure in the FRW universe within the framework of various modified gravity theories [27–31].

The above works suggest that thermodynamic phase transitions may occur in the models of the FRW universe that are built beyond classical general relativity (GR). This raises our curiosity about whether the models of quantum gravity (QG) that also go beyond GR, such as the generalized uncertainty principle (GUP), could instigate thermodynamic phase transitions in the universe? Indeed, there are numerous instances exist where GUP has been employed to modify the properties of the FRW universe [32–42]. For example, in Ref. [43] the GUP cor-

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rected dynamics of the universe is investigated, the results showed that effect of GUP can avoid the Big Bang singularity. Moussa *et al.* [37, 38] used GUP to modify the properties of the early universe and consequently analyze the the stochastic gravitational wave therein, it was argued that this was a way to test the effectiveness of the GUP theory. Besides, the GUP can correct Friedmann equations and thereby change the pressure and density of the universe, providing a feasible solution to the baryon asymmetry. In Ref. [42], the authors studied the impact of GUP and the laws of thermodynamics on Friedmann equations. These investigations reveal that the attributes of the FRW universe within the GUP framework are more intricate than those in the classical scenario. It raises the possibility that thermodynamic phase transitions might manifest in these scenarios, warranting further in-depth exploration.

Although GUP play a prominent role in cosmology, it is necessary to point out that most of the previous works are based on the two most basic types of GUP models, namely, KMM model ($\Delta x \Delta p \geq \hbar\left[1+\beta_0 \ell_p^2 \Delta p^2/\hbar^2\right]/2$ with the GUP parameter β_0) [45] , and the ADV model ($\Delta x \Delta p \geq \hbar\left[1-\alpha_0 \ell_p^2 \Delta p/\hbar+\beta_0 \ell_p^2 \Delta p^2/\hbar^2\right]/2$ with the GUP parameters β_0 and α_0) [46] (the KMM model can be seen as a special case of the ADV model at $\alpha_0=0$). However, more studies have shown some limitations of these two models. First of all, the perturbations of these models are valid only for small values of the GUP parameter. Second, they do not imply the non-commutative geometry. Third, absence of the value of the observed momentum, this leads to the models are not applicable to double special relativity [47]. In order to address these limitations, the higher-order GUP has been proposed. This non-perturbative model is consistent with various proposals for QG and does not contradict double special relativity, and has therefore received the attention [48]. Now, much research is focused on the construction and application of higher-order GUP [49–54].

In light of the aforementioned considerations, it can be seen that the higher-order GUP formulations harbor the potential to instigate phase transitions in the FRW universe. Such transitions offer a pathway for comprehending the thermodynamic intricacies in the universe. Recently, by considering the minimum length is a model-independent feature of QG, a new higher-order GUP as follows [55] (In fact, the expression can be called the extended GUP according to the definitions provided in Refs. [56–59] since it is no longer related to Δp but to Δx. However, for the sake of consistency, we will continue to use the designation in the original literature)

$$\Delta x \Delta p \geq \frac{\hbar}{2} \frac{1}{1+\left(16\beta/\Delta x^2\right)}, \tag{1}$$

where $\beta=\beta_0 \ell_p^2$ with the the deformation parameter (or GUP parameter) and ℓ_p the Planck length. Δx and Δp are the uncertainties for position and momentum, respectively. By analyzing Eq. (1), one can figure out that it has two main characteristics: (i) The deformation parameter are no longer restricted to positive values, but negative values are also allowed. However, Eq. (1) recovers to the conventional Heisenberg uncertainty principle $\Delta x \Delta p \geq \hbar/2$ when $\beta_0=0$. (ii) It leads to a fixed and consistent minimum length for both cases positive and negative deformation parameters, which indicate that this new model has a better parameter adaptability for minimum length than other ones. The validity of the QG effect in the model is ensured by these advantages, which allow us to analyze the effects of both positive and negative deformation parameters on the same physical system. In this present paper we attempt use the new higher-order GUP to investigate the phase transitions, critical behavior and microstructure of the FRW universe in the framework of QG. Combining a new higher-order GUP into the first law of thermodynamics, we derive the modified Friedmann equations of FRW universe. Then, based on these modifications, the thermodynamic pressure and equation of state in the universe can be obtained, which allows us to further analyze the phase transitions and critical behavior therein. Finally, by utilizing Ruppeiner geometry, we discuss the microscopic properties of the FRW universe through the analysis of the behavior of the reduced thermodynamic curvature.

The outline of our paper is as follows. In Section II, we shortly review the new higher-order GUP corrected Friedmann equations, then we derive the corresponding modified thermodynamic equation of state of FRW universe. In Section III, we analyze its phase transition, critical behaviors and the microstructure of the FRW universe in the framework of QG. The paper ends with conclusions in section IV.

To simplify the notation, this research takes the units $G=c=k_B=1$.

II. MODIFIED FRIEDMANN EQUATIONS AND THE CORRESPONDING EQUATION OF STATE OF FRW UNIVERSE

To begin with, we briefly review the new higher-order GUP corrected Friedmann equations, and then derive the corresponding modified equation of state of FRW universe. To the best of our knowledge, there are two valid and feasible schemes for deriving the Friedmann equation. The first, referred to as the classical scheme, involves deriving the energy density and pressure of the FRW universe from the action of a gravity theory, thereby yielding the Friedmann equation. The second scheme is grounded in the assumption that the apparent horizon of the universe is connected to geometric entropy. By applying this entropy to the first law of thermodynamics, one can consequently derive the Friedmann equations [19]. It is noteworthy that both schemes produced equivalent results. However, given the facile modification of thermodynamic quantities within the GUP framework, our previous work [60] leaned towards the second scheme. According to Eq. (1), we derived the GUP corrected geometric entropy as follows

$$S_{\rm GUP}=\frac{A}{4}+4\pi\beta\ln\left(\frac{A}{A_0}\right), \tag{2}$$

where A is the area of the horizon and A_0 is a constant with units of area. In the works on the phase transition of the universe [26–28], it has been found that the entropy of the universe has a similar form of expression as the above equation, which indicates that it is also possible to obtain the P-V phase transition of the universe in the framework of the GUP based on Eq. (2). Therefore, following the second scheme, the corresponding modified Friedmann equations are given by

$$-4\pi(\rho+p)=\dot{H}\left(1+4\beta H^2\right), \tag{3a}$$
$$\frac{8}{3}\pi\rho=H^2\left(1+2\beta H^2\right), \tag{3b}$$

where H, ρ and p represent the Hubble parameter, energy density and pressure of the universe, respectively. The dot stands for the derivative with respect to the cosmic time. According to Eq. (3), the GUP corrected energy density and pressure are be expressed as

$$\rho=\frac{3H^2}{8\pi}+\frac{3H^4\beta}{4\pi}, \tag{4a}$$
$$p=-\frac{\dot{H}}{4\pi}-\frac{3H^2}{8\pi}-\frac{\dot{H}H^2\beta}{\pi}-\frac{3H^4\beta}{4\pi}. \tag{4b}$$

Now, in order to obtained the GUP corrected thermodynamic equation of state of FRW universe, we first need to express the density and pressure in terms of apparent horizon, it can be taken from the spatially flat FRW metric

$$\mathrm{d}s^2=h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right), \tag{5}$$

where $\mu=\nu=0,\ 1$ with $x^\mu=(t,r)$, and $h_{\mu\nu}=\mathrm{diag}\left[-1,a^2(t)\right]$ is the two-dimensional metric with the scale factor $a(t)$. By using the condition $h^{\mu\nu}(\partial_u R)(\partial_\nu R)=0$ [66], the apparent horizon of the FRW universe reads $R_A=1/H$, and its time derivative is $\dot{R}_A=-\dot{H}R_A^2$ [20]. Therefore, Eq. (4) can be rewritten as

$$\rho\left(R_A,\dot{R}_A\right)=\frac{3}{8\pi R_A^2}+\frac{3\beta}{4\pi R_A^4}, \tag{6a}$$
$$p\left(R_A,\dot{R}_A\right)=-\frac{3}{8\pi R_A^2}+\frac{\dot{R}_A}{4\pi R_A^2}-\frac{3\beta}{4\pi R_A^4}+\frac{\dot{R}_A\beta}{\pi R_A^4}. \tag{6b}$$

Then, taking account of the work density of the matter field $W=-h_{\mu\nu}T^{\mu\nu}/2$, one has

$$\begin{aligned}W&=\frac{1}{2}\left[\rho\left(R_A,\dot{R}_A\right)-p\left(R_A,\dot{R}_A\right)\right]\\&=\frac{3}{8\pi R_A^2}-\frac{\dot{R}_A}{8\pi R_A^2}+\frac{3\beta}{4\pi R_A^4}-\frac{\dot{R}_A\beta}{2\pi R_A^4},\end{aligned} \tag{7}$$

and the Misner-Sharp energy is

$$E=\rho\left(R_A,\dot{R}_A\right)V=\frac{R_A^2+2\beta}{2R_A}, \tag{8}$$

where $V=4\pi R_A^3/3$ is the volume of FRW universe.

Next, it is necessary to discuss the temperature of the system. In the FRW universe, the surface gravity on the apparent horizon can be defined as $\kappa=\partial_\mu(\sqrt{-h}h^{\mu\nu}\partial_\mu r)/2\sqrt{-h}\big|_{r=R_A}=-(1-\dot{R}_A/2)/R_A$. With the help of κ, the temperature of spatially flat FRW universe is given by

$$T=\frac{|\kappa|}{2\pi}=\frac{1}{2\pi R_A}\left(1-\frac{\dot{R}_A}{2}\right). \tag{9}$$

It is important to note that the $\dot{R}_A\ll 1$ is a small quantity, which leads to FRW universe has an inner trapping horizon. Besides, in Ref. [19], the authors pointed out it is not necessary to consider the change in temperature in the application of the first law of thermodynamics. Therefore, when employing the first law of thermodynamics and geometric entropy to derive the Friedmann equations, the small quantity $\dot{R}_A$ can be neglected, and the classical Hawking temperature $T=1/2\pi R_A$ is reinstated. However, the above assumptions are only applicable to derive the FRW equations, and for further discussion of the phase transitions of the universe, it should be considered as undergoing slow evolution, meaning its thermodynamic processes are quasi-static. Therefore, $\dot{R}_A$ is retained in the next study. Now, according to Eq. (7)-Eq. (9), the above defined quantities are satisfy the thermodynamic first law $\mathrm{d}E=-T\mathrm{d}S+W\mathrm{d}V$ (see Appendix A for more details about the first law of thermodynamics) [22]. When comparing it with the standard form of the first law of thermodynamics $\mathrm{d}U=-T\mathrm{d}S+P\mathrm{d}V$, it is discerned that the Misner-Sharp energy should be interpreted as $E\equiv -U$, and more importantly, the thermodynamic pressure can be defined as $P\equiv W$. It is evident that neglecting $\dot{R}_A$ would result in a density equal to zero, making it impossible to define the thermodynamic pressure. Therefore, the inclusion of $\dot{R}_A$ is important for the present study. Now, combining Eq. (7) and Eq. (9), the modified thermodynamic equation of state for the FRW universe is expressed as

$$\begin{aligned}P=&\,T\left[\left(\frac{\pi}{6V}\right)^{1/3}+\frac{8\pi\beta}{3V}\right]\\&+\frac{1}{V^{4/3}}\frac{3\times 6^{1/3}V^{2/3}-4\times(6\pi)^{2/3}\beta}{36\pi^{1/3}}.\end{aligned} \tag{10}$$

The above equation illustrates that the influence of QG alters the pressure by adjusting for the volume (or size) and temperature of the FRW universe, thereby creating conditions conducive to undergoing a phase transition. Nevertheless, when ignoring the GUP parameter β, Eq. (10) simplified to the one in Einstein gravity, which has no phase transition, see Ref. [26].

III. THE PHASE TRANSITION, CRITICAL BEHAVIORS AND MICROSTRUCTURE OF THE FRW UNIVERSE IN THE FRAMEWORK OF GUP

In this section, we are going to investigate the phase transitions, critical behaviors and microstructure of FRW universe based on the modified equation of state. As it is well known, if a thermodynamic system under goes a phase transition, it should satisfy the conditions

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0, \tag{11a}$$

$$\left(\frac{\partial P}{\partial R_A}\right)_T = \left(\frac{\partial^2 P}{\partial R_A^2}\right)_T = 0. \tag{11b}$$

Therefore, by substituting Eq. (10) into Eq. (11), one can obtain two nonnegative volumes as follows

$$V_{c1} = \frac{4}{3}\pi R_{c1}^3 = \frac{32}{3}\pi\left[\left(3+2\sqrt{3}\right)\beta\right]^{3/2}, \tag{12a}$$

$$V_{c2} = \frac{4}{3}\pi R_{c2}^3 = \frac{32}{3}\pi\left[\left(3-2\sqrt{3}\right)\beta\right]^{3/2}. \tag{12b}$$

To ensure the above two results are real, it is necessary to constrain the range of β. Eq. (12a) demands $\beta > 0$, while Eq. (12b) requires $\beta < 0$. Fortunately, the GUP parameter of Eq. (1) is just satisfy these requirements. Meanwhile, it is also worth noting that positive and negative values of the GUP parameter generate different QG effects (see Ref. [55] for the details), potentially leading to different properties of the phase transition, criticality, and microstructure of the FRW universe. Consequently, we will discuss these two scenarios separately below

A. The phase transition, critical behaviors and microstructure for $\beta > 0$

By using Eq. (10)-Eq. (12), the critical radius of apparent horizon, temperature, and pressure for $\beta > 0$ are given by

$$R_c = 2\sqrt{\left(3+2\sqrt{3}\right)\beta}, V_c = \frac{4}{3}\pi R_c^3,$$
$$T_c = -\frac{1}{12\pi\sqrt{\left(1+\frac{2}{\sqrt{3}}\right)\beta}}, P_c = \frac{8\sqrt{3}-15}{576\pi\beta}. \tag{13}$$

One special property of FRW universe in the framework the new higher order GUP with $\beta > 0$ is its critical temperature is negative. It should be noted that negative temperature is an exotic thermodynamic result of quantum physics that has been experimentally verified since the 1950s [61]. Recently, this phenomenon has played an important role in studying the evolution of the universe and solving a series of difficult problems in cosmology [62–65]. Therefore, the negative temperature in Eq. (13) is acceptable. Furthermore, according to viewpoints of Refs. [67, 68], the emergence of this negative temperatures can be attributed to the dominance of rigid matter in the universe. Besides, the similar outcomes have been observed in Ref. [69]. The above critical quantities in turn yield a dimensionless constant

$$\chi = \frac{\nu_c P_c}{T_c} = \frac{6-\sqrt{3}}{12} \approx 0.355, \tag{14}$$

where $\nu_c = 2R_c$ the specific volume. It is clear that Eq. (14) is slightly less than that of VdW gas $\chi_{\mathrm{VdW}} = 8/3 \approx 0.375$. For conveniently demonstrates the characteristic behavior of phase transition, one can define the dimensionless volume, temperature and pressure as follows

$$\tilde{V} = \frac{V}{V_c}, \tilde{T} = \frac{T}{T_c}, \tilde{P} = \frac{P}{P_c}, \tag{15}$$

Substituting Eq. (15) into Eq. (10), the modified reduced thermodynamic equation of state is rewritten as

$$\tilde{P} = \frac{3\left(7+4\sqrt{3}\right)}{\left(111+64\sqrt{3}\right)\tilde{V}^{4/3}} + \frac{4\left(12+7\sqrt{3}\right)\tilde{T}}{\left(111+64\sqrt{3}\right)\tilde{V}}$$
$$- \frac{6\left(45+26\sqrt{3}\right)}{\left(111+64\sqrt{3}\right)\tilde{V}^{2/3}} + \frac{12\left(26+15\sqrt{3}\right)\tilde{T}}{\left(111+64\sqrt{3}\right)\tilde{V}^{1/3}}. \tag{16}$$

According to Eq. (16), the reduced pressure as a function of reduced volume for different reduced temperature is depicted in Fig. 1.

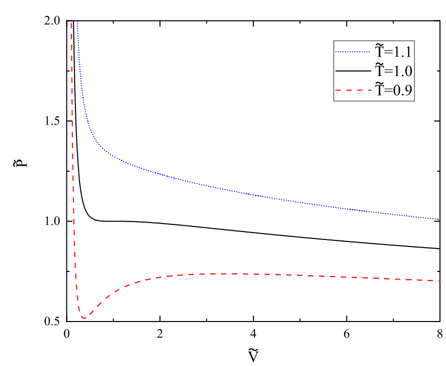


FIG. 1. Variation in reduced pressure with reduced volume for different reduced temperature with $\beta > 0$.

In Fig. 1, we depict the isothermal curves in the $P-V$ plane, with the reduced temperature of isotherms $\tilde{T}$ decreasing from top to bottom. The red curve corresponds to $\tilde{T} > 1$ (or $T > T_c$) and exhibits an ideal gas-like one-phase behavior, indicating the stability of the FRW universe without undergoing a phase transition. These observed behaviors bear similarity to those of VdW gas, implying a profound connection between the new higher-order GUP with a positive deformation parameter and VdW gas.

The same conclusion can also be inferred from the relationship between the Gibbs free energy and pressure. By using the usual thermodynamic relations

$$\left(\frac{\partial \tilde{G}}{\partial \tilde{V}}\right)\Bigg|_{\tilde{T}} = \tilde{V}\left(\frac{\partial \tilde{P}}{\partial \tilde{V}}\right)\Bigg|_{\tilde{T}}, \tag{17}$$

one can obtain the reduced Gibbs free energy $\tilde{G}$. From Eq. (17), we plot the behavior of the reduced Gibbs free energy $\tilde{G}$ as a function as reduced pressure $\tilde{P}$ for different reduced temperature in Fig. 2. It can be observed that the red curve, corresponding to $\tilde{T} < 1$, exhibits swallowtail behavior, with its intersection point signifying a first-order phase transition. However, as $\tilde{T}$ increases above 1 (as seen in the blue dotted curve), the curves gradually become smooth, indicating an absence of phase transition in the system, akin to an ideal gas.

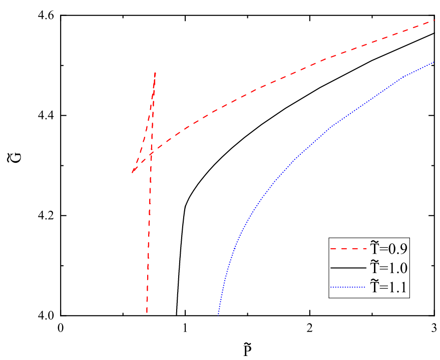


FIG. 2. Variation in reduced Gibbs free energy with reduced pressure for different reduced temperature with $\beta > 0$.

Then, we further discuss the critical exponents since they provide insight into the properties of the phase transition and the behavior of the thermodynamic properties around the critical points. In general, the critical exponents are defined as

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V \propto |\tau|^{-\alpha}, \tag{18a}$$

$$\eta = \frac{V_l - V_s}{V_c} \sim (\omega_l - \omega_s) \propto |\tau|^{\varepsilon}, \tag{18b}$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \propto |\tau|^{-\gamma}, \tag{18c}$$

$$P - 1 \propto \omega^{\delta}, \tag{18d}$$

where $\tau = \tilde{T} - 1$ and $\omega = \tilde{V} - 1$, the labels "s" and "l" represent "small" and "large" states, respectively. It is easy find that the GUP corrected entropy $S_{\mathrm{GUP}} = A/4 + 4\pi\beta \ln(A/A_0) = 6^{2/3}\pi^{1/3}V^{2/3} + 4\pi\beta \ln\left(6^{2/3}\pi^{1/3}V^{2/3}/A_0\right)$ is only a function of the thermodynamic volume V, hence, one has $C_V = 0$, which suggests that $\alpha = 0$. Next, in order to investigate the other three exponents, one needs to expand the equation of state (16) near the critical point, which reads

$$\tilde{P} = 1 + \frac{8}{11}\left(1 + 2\sqrt{3}\right)\tau + \frac{8}{33}\left(3 - 5\sqrt{3}\right)\tau\omega + \frac{4}{297}\left(\sqrt{3} - 5\right)\omega^3 + \mathcal{O}\left(t\omega^2, \omega^4\right), \tag{19}$$

By employing the Maxwell's equal area law $\tilde{P}^*\left(\tilde{V}_s - \tilde{V}_l\right) = \int_l^s \tilde{P} d\tilde{V}$ with the pressure end/starting point of small/large phase $\tilde{P}^*$, one gets

$$\frac{8}{33}\left(3 - 5\sqrt{3}\right)\tau\omega_l + \frac{4}{297}\left(\sqrt{3} - 5\right)\omega_l^3 = \frac{8}{33}\left(3 - 5\sqrt{3}\right)\tau\omega_s + \frac{4}{297}\left(\sqrt{3} - 5\right)\omega_s^3, \tag{20}$$

and

$$\frac{16}{33}\left(3 - 5\sqrt{3}\right)\tau(\omega_l^2 - \omega_s^2) + \frac{12}{297}\left(\sqrt{3} - 5\right)(\omega_l^4 - \omega_s^4) = 0. \tag{21}$$

Solving above equations, the nontrivial solutions are $\omega_s = -3^{5/4}\sqrt{2\left(\tilde{T} - 1\right)}$ and $\omega_l = 3^{5/4}\sqrt{2\left(\tilde{T} - 1\right)}$, which leads to $\omega_l - \omega_s = 2\left(-18\sqrt{3}\tau\right)^{1/2}$, hence, the second critical exponent is $\varepsilon = 1/2$. Moreover, the volumes of the coexistence small and large phases of FRW universe near the critical point can be expressed as

$$\tilde{V}_s = 1 - 3^{5/4}\sqrt{2\left(1 - \tilde{T}\right)}, \tag{22a}$$

$$\tilde{V}_l = 1 + 3^{5/4}\sqrt{2\left(1 - \tilde{T}\right)}. \tag{22b}$$

In Fig. 3, one can see the phase transition behavior of small/large coexisting phases of FRW universe near the critical point. The red dashed curve representing the small coexisting phase terminates precisely where the blue solid curve representing the large coexisting phase commences, forming a peak at $\left(\tilde{T}, \tilde{V}\right) = (1, 1)$. As a result, the two coexisting states persist in the region $\tilde{T} < 1$. These observations will be the basis for our analysis of the behavior of thermodynamic curvature.

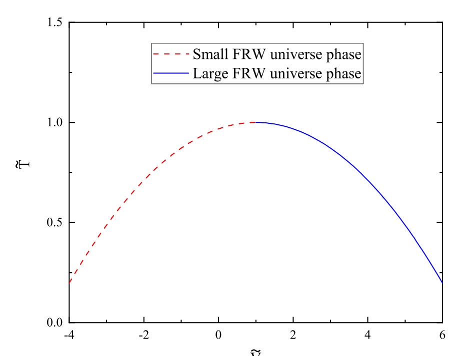


FIG. 3. $\tilde{T}$ as the function as the volumes $\tilde{V}$ of the coexistence small/large phases of FRW universe with $\beta > 0$.

In order to calculate the third critical exponent γ, one should differentiates Eq. (16), which leads to $\partial V/\partial P|_T = -\left[33V_c/8\left(3-5\sqrt{3}\right)P_c\tau\right] + \mathcal{O}\left(\omega\right)$. Therefore, one yields

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\frac{33}{8\left(3-5\sqrt{3}\right)P_c\tau} \propto |\tau|^{-\gamma}, \quad (23)$$

which indicates third exponent is $\gamma = 1$. When considering the $\tau = 0$, that is $T = T_c$, the fourth critical exponent is

$$\tilde{P} - 1 = \frac{4}{297}\left(\sqrt{3}-5\right)\omega^3 \Rightarrow \delta = 3. \quad (24)$$

Moreover, following scaling laws, the four critical exponents $(\alpha, \varepsilon, \gamma, \delta)$ satisfy the two independent relations

$$\begin{aligned} &\alpha + 2\varepsilon + \gamma = 2, \alpha + \varepsilon\left(1+\delta\right) = 2, \\ &\gamma\left(1+\delta\right) = \left(2-\alpha\right)\left(\delta-1\right), \gamma = \varepsilon\left(\delta-1\right), \end{aligned} \quad (25)$$

which coincide with mean field theory.

In the last part, we further discuss the microstructure of the FRW universe using phase transitions. To this end, we employ Ruppeiner geometry, a conceptual framework grounded in Riemannian geometry and rooted in the principles of the fluctuation theory of equilibrium thermodynamics. In this language the entropy plays a vital role, and its line element can be expressed in terms of entropy as follows

$$\mathrm{d}l^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \quad (26)$$

where $\mathrm{d}l^2$ is the distance between two neighbouring fluctuation states, the fluctuation coordinates $x = (U, V)$ are functions of the internal energy U and volume V, $g_{\mu\nu} = -\partial_{\mu,\nu}S$ is the metric element. In the system of FRW universe, the first law of thermodynamics reads

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}U + \frac{P}{T}\mathrm{d}V. \quad (27)$$

By comparing Eq. (26) with Eq. (27), and the conjugate quantities corresponding to x are $y_\mu = \partial S/\partial x^\mu = (1/T, -V/T)$. Therefore, one has the following relations

$$\mathrm{d}\left(\frac{1}{T}\right) = \left(\frac{\partial^2 S}{\partial U^2}\right)\mathrm{d}U + \left(\frac{\partial^2 S}{\partial U\partial V}\right)\mathrm{d}V, \quad (28a)$$

$$\mathrm{d}\left(\frac{P}{T}\right) = \left(\frac{\partial^2 S}{\partial V^2}\right)\mathrm{d}V + \left(\frac{\partial^2 S}{\partial U\partial V}\right)\mathrm{d}U. \quad (28b)$$

Now, the line element (26) can be rewritten as

$$\begin{aligned} \mathrm{d}l^2 &= -\mathrm{d}\left(\frac{1}{T}\right)\mathrm{d}U - \mathrm{d}\left(\frac{P}{T}\right)\mathrm{d}V \\ &= \frac{1}{T^2}\mathrm{d}T\mathrm{d}U - \frac{1}{T}\mathrm{d}P\mathrm{d}V + \frac{P}{T^2}\mathrm{d}T\mathrm{d}V. \end{aligned} \quad (29)$$

Based on the relation $\mathrm{d}U = C_V\mathrm{d}T + \left[T\left(\partial P/\partial T\right)_V - P\right]\mathrm{d}V$ and $\mathrm{d}P = \left(\partial P/\partial T\right)_V\mathrm{d}T + \left(\partial P/\partial T\right)_T\mathrm{d}V$, the line element can be further expressed in term (T, V) as

$$\mathrm{d}l^2 = \frac{C_V}{T^2}\mathrm{d}T^2 - \frac{(\partial_V P)_T}{T}\mathrm{d}V^2. \quad (30)$$

In order to study the thermodynamic geometry as well as the microstructure of the FRW universe, it is necessary to analyze its scalar curvature R. However, it is easy proved that the entropy C_V depends only on the volume, so that the thermodynamic line element (30) is singular, which means that the classical definition of R is always divergent. To avoid this predicament, a new reduced curvature scalar that so called thermodynamic curvature scalar is defined as follows

$$\begin{aligned} R_N &= RC_V \\ &= \frac{(\partial_V P)_T^2 - T^2(\partial_{T,V}P)^2 + 2T^2(\partial_V P)_T\left(\partial_{T,T,V}P\right)}{2\left(\partial_V P\right)_T^2}. \end{aligned} \quad (31)$$

By taking this probe, one can easily analyze the thermodynamic geometry as well as the microstructure of the FRW universe. According to Eq. (16) and Eq. (31), the reduced thermodynamic curvature scalar reads

$$R_N = \frac{\mathcal{A} - 2\tilde{T}\mathcal{B}}{2\mathcal{C}^2}, \quad (32)$$

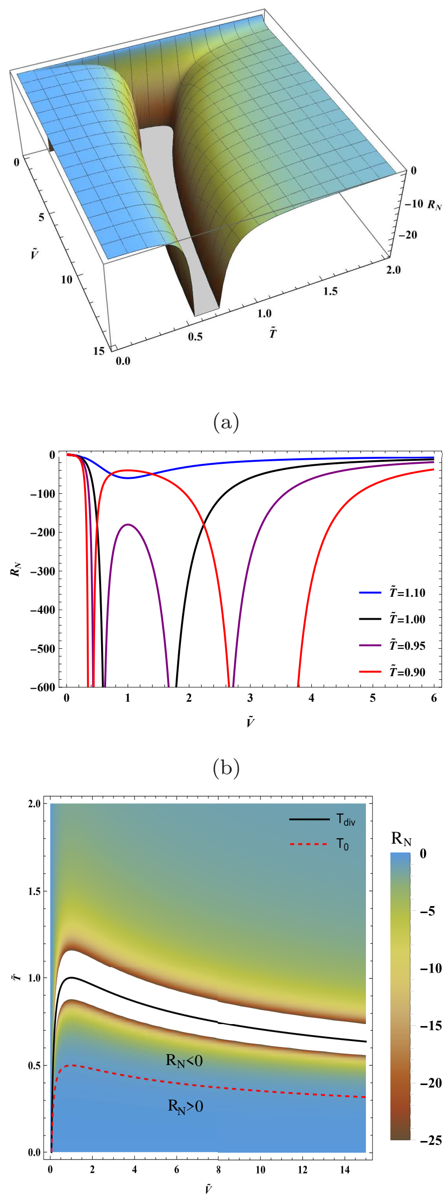
where $\mathcal{A} = 97 + 56\sqrt{3} - 2(627+362\sqrt{3})\tilde{V}^{2/3} + 3(1351+780\sqrt{3})\tilde{V}^{4/3}$, $\mathcal{B} = (724+418\sqrt{3})\tilde{V} + (2340+1351\sqrt{3})\tilde{V}^{5/3} - (168+97\sqrt{3})\tilde{V}^{1/3}$, and $\mathcal{C} = 7 + 4\sqrt{3} + (12+7\sqrt{3})\tilde{T}\tilde{V}^{1/3} - (45+26\sqrt{3})\tilde{V}^{2/3} + (26+15\sqrt{3})\tilde{T}\tilde{V}$. Now, the reduced thermodynamic curvature scalar only related to the $\tilde{V}$ and $\tilde{T}$, rather than the GUP parameter, property is similar to reduced equation of state. Moreover, we depict the behavior of R_N in Fig. 4. One can observe that R_N is near zero for most of the parameter space (see Fig. 4(a)). However, near the temperature

$$\tilde{T}_{\mathrm{div}} = \frac{3\left(2+\sqrt{3}\right)V^{2/3} - \sqrt{3}}{3V^{1/3} + \left(3+2\sqrt{3}\right)V}, \quad (33)$$

the curve of R_N decreases dramatically and eventually reaches negative infinity, indicating that the microstructure of the universe is changing rapidly near $\tilde{T}_{\mathrm{div}}$. As depicted in Fig. 4(b), one can see the divergence behavior of R_N. For $\tilde{T} > 1$, no divergence is observed. For $\tilde{T} = 1$, a single divergence point is present. However, when $\tilde{T}$ falls below 1, this divergence point bifurcates, with the two points moving towards the high and low volume regions, respectively. By solving $R_N = 0$ allows us to deduce the sign-changing curve

$$T_0 = \frac{\left(26+15\sqrt{3}\right)\left(3-2\sqrt{3}+3\tilde{V}^{2/3}\right)}{6\left(7+4\sqrt{3}\right)\tilde{V}^{1/3} + \left(90+52\sqrt{3}\right)\tilde{V}}, \quad (34)$$

which implies a transition between attractive and repulsive interactions in the microstructure. In Fig. 4(c), we illustrate the sign-changing (red dashed) curve and the so-called spinodal (black solid) curve corresponds to the divergent temperature $\tilde{T}_{\mathrm{div}}$. Above the the sign-changing curve, the R_N is negative, which implies that attractive interactions dominate in this region. However, below the sign-changing curve the R_N becomes positive, indicating the repulsive interactions dominate. Therefore, the microstructure of FRW universe exhibits repulsive interactions at low temperature and attractive interactions at high temperature. Besides, one can see that the black solid curve is far from the red dashed curve in cases where the volume is not very small, which suggests that the change in R_N in region below the red dashed curve is very close to zero, and consequently, the attractive interaction there is weak.



(a)

(b)

(c)

FIG. 4. The reduced scalar curvature versus the reduce volume and reduce temperature with $\beta > 0$.

As we know, the reduced thermodynamic curvature scalar along the small-phase and large-phase coexistence curves near the critical points would provide some universal properties. To examine whether the FRW universe in the framework of GUP has these properties, we insert the volumes of the coexistence small and large phases of the FRW universe (22) into Eq. (33), the thermodynamic curvature scalar around the critical point is given by

$$R_N^s = -\frac{1}{8\tau^2} + \frac{\sqrt{27+42\sqrt{3}}}{4(-\tau)^{3/2}} + \mathcal{O}\left(\tau^{-1}\right), \tag{35a}$$

$$R_N^l = -\frac{1}{8\tau^2} - \frac{\sqrt{27+42\sqrt{3}}}{4(-\tau)^{3/2}} + \mathcal{O}\left(\tau^{-1}\right). \tag{35b}$$

Thus, the thermodynamic curvature scalar has a critical exponent 2. When, by ignoring the high orders terms of Eq. (35a) and Eq. (35b), an interesting limiting expression can be obtained that

$$\lim_{\tau\to 0} R_N \tau^2 = -\frac{1}{8}, \tag{36}$$

which indicates that the divergence behavior of thermodynamic curvature scalar of FRW universe in the GUP framework is characterized by a dimensionless constant $-1/8$. This result is consistent with the previous works, such as AdS black holes and VdW fluid.

B. The phase transition, critical behaviors and microstructure for $\beta < 0$ scenario

In this subsection, we examine the properties of the phase transition, criticality and microstructure of the FRW universe with the negative GUP parameter. By using Eq. (10)-Eq. (12), the critical radius of apparent horizon, temperature, and pressure for $\beta < 0$ are

$$R_c = 2\sqrt{\left(3-2\sqrt{3}\right)\beta}, V_c = \frac{4}{3}\pi R_c^3,$$
$$T_c = \frac{1}{4\pi\sqrt{\left(9-6\sqrt{3}\right)\beta}}, P_c = -\frac{15+8\sqrt{3}}{576G\pi\beta}, \tag{37}$$

and the three critical points give the ratio

$$\chi = \frac{\nu_c P_c}{T_c} = \frac{6+\sqrt{3}}{24} \approx 0.64. \tag{38}$$

Unlike the Eq. (14), the critical ratio is much larger than $3/8$, which indicates that the phase transition, criticality and microstructure of the FRW universe with the negative GUP parameter may different from those of VdW fluids. To check this conjecture, it is necessary to derive thermodynamic equation of state. By using Eq. (14), Eq. (15) and Eq. (37), the modified reduced thermodynamic equation of state is provided as follows:

$$\tilde{P} = \frac{6\left(3-2\sqrt{3}\right)\tilde{V}^{2/3} + 4\tilde{T}\left[\sqrt{3}\tilde{V}^{1/3} + 3\left(-2+\sqrt{3}\right)\tilde{V}\right] - 3}{\left(4\sqrt{3}-9\right)\tilde{V}^{4/3}}. \tag{39}$$

According to the equation above, the relationship between $\tilde{P}$ and $\tilde{V}$ for different $\tilde{T}$ is depicted in Fig. 5.

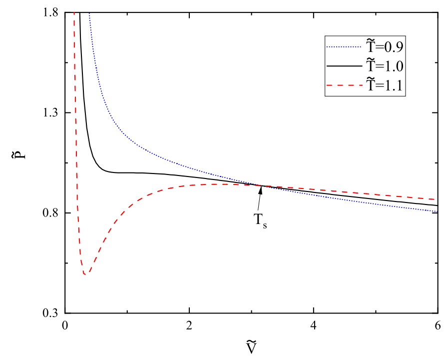


FIG. 5. Variation in reduced pressure with reduced volume for different reduced temperature with $\beta < 0$.

In Fig. 5, the behavior of red dashed curve shows that the phase transition occurs for $\tilde{T} > 1$, while for $\tilde{T} < 1$, the blue isotherm resembles an ideal gas. Those behaviors are in contrast to the trends in Fig. 1. Besides, one can see that all the curves intersect at point T_s, where is characterized by a pressure independent of temperature, and interpreted as a “thermodynamic singularity” [27, 28, 70–72]. All the difference indicates that the GUP with a negative parameter leads to a phase transition behavior that is different from that of VdW fluids and most black hole systems.

Moreover, the observed phase transition phenomena of the FRW universe in the $P - V$ plane, as presented above, can also be expressed through the reduced ”Gibbs free energy-pressure” relation, as shown in Eq. (18).

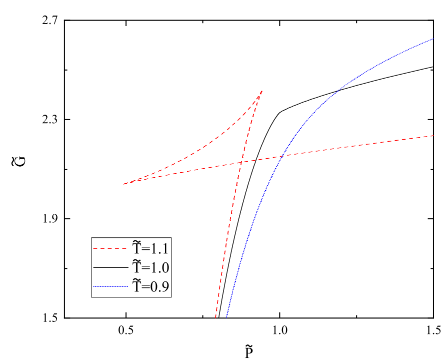


FIG. 6. Variation in reduced Gibbs free energy with reduced pressure for different reduced temperature with $\beta < 0$.

From Fig. 6, it can be observed that the smooth blue dotted curve for $\tilde{T} < 1$ corresponds to the behavior of an ideal gas in the $\tilde{G} - \tilde{P}$ plane, indicating that there is no phase transition in the system. When $\tilde{T} = 1$, the black solid curve is no longer smooth and has a breaking point, meaning that the system is in a critical state. For $\tilde{T} > 1$, the reduced Gibbs free energy exhibits a swallow-tail behavior (red dashed curve) in the $\tilde{G} - \tilde{P}$ plane, which indicates that there is a two-phase coexistence state. Therefore, the intersection represents a first-order phase transition.

Next, by defining $\tau = \tilde{T} - 1$ and $\omega = \tilde{V} - 1$, the equation of state (39) near the critical point can be rewritten as follows

$$\begin{aligned}\tilde{P} = 1 &+ \frac{8}{11}\left(1 - 2\sqrt{3}\right)\tau + \frac{8}{33}\left(3 + 5\sqrt{3}\right)\tau\omega \\ &- \frac{4}{297}\left(5 + \sqrt{3}\right)\omega^3 + \mathcal{O}\left(\tau\omega^2, \omega^4\right).\end{aligned} \tag{40}$$

Based on Eq. (18) and Eq. (40), and then performing simple calculations, the critical exponents are obtained as

$$C_V = 0 \Rightarrow \alpha = 0, \tag{41a}$$

$$\eta = 2\left(-18\sqrt{3}\tau\right)^{1/2} \Rightarrow \varepsilon = \frac{1}{2}, \tag{41b}$$

$$\kappa = -\frac{33}{8\left(3 - 5\sqrt{3}\right)P_c t} \Rightarrow \gamma = 1, \tag{41c}$$

$$\tilde{P} - 1 = \frac{4}{297}\left(\sqrt{3} - 5\right)\omega^3 \Rightarrow \delta = 3, \tag{41d}$$

which are consistent with the predications from the mean field theory, hence, they are also satisfy scaling laws (27). Besides, by using the Maxwell’s equal area law, the volumes of the coexistence small and large phases of FRW universe around the critical point can be expressed as

$$\tilde{V}_s = 1 - 3^{5/4}\sqrt{2\left(\tilde{T} - 1\right)}, \tag{42a}$$

$$\tilde{V}_l = 1 + 3^{5/4}\sqrt{2\left(\tilde{T} - 1\right)}. \tag{42b}$$

According to Eq. (42), we show the temperature $\tilde{T}$ as a function of volumes $\tilde{V}$ in Fig. 7. The red and blue curves represent the small coexisting phase and large coexisting phase, respectively. The two curves intersect at the point $\left(\tilde{T}, \tilde{V}\right) = (1, 1)$ where the volume is minimized, this the opposite of that is shown in Fig. 2. Therefore, the two coexisting states would exist in the region $\tilde{T} > 1$.

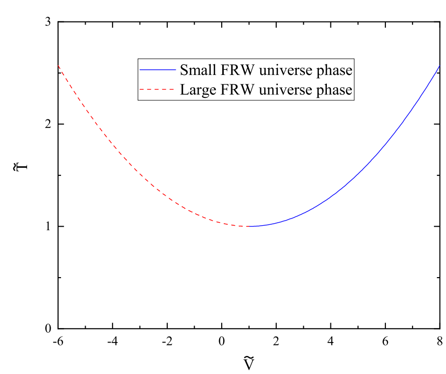


FIG. 7. $\tilde{T}$ as the function as the volumes $\tilde{V}$ of the coexistence small/large phases of FRW universe with $\beta < 0$.

Based on Eq. (26)-Eq. (29), one can express the line element of the Ruppeiner geometry as $\mathrm{d}l^2 = C_V/T^2\mathrm{d}T^2 - (\partial_V P)_T/T\mathrm{d}V^2$, and the corresponding the expression of scalar curvature R_N is the same as Eq. (31). Thus, by putting Eq. (40) into Eq. (31), the scalar curvature has been obtained

$$R_N = \frac{\mathcal{C} + 2\tilde{T}\mathcal{D}}{2\mathcal{E}^2}, \tag{43}$$

where $\mathcal{C} = 1 + \left(4\sqrt{3} - 6\right)\tilde{V}^{2/3} + 3\left(7 - 4\sqrt{3}\right)\tilde{V}^{4/3}$, $\mathcal{D} = 2\left(\sqrt{3} - 2\right)\tilde{V} + \left(7\sqrt{3} - 12\right)\tilde{V}^{5/3} - \sqrt{3}\tilde{V}^{1/3}$, and $\mathcal{E} = \sqrt{3}\tilde{T}\tilde{V}^{1/3} + \left(3 - 2\sqrt{3}\right)\tilde{V}^{2/3} + \left(\sqrt{3} - 2\right)\tilde{T}\tilde{V} - 1$. The result above is only dependent on $\tilde{V}$ and $\tilde{T}$, which is consistent with the conclusion of Eq. (32). However, the two expressions are not the same, which indicate that the positive/negative GUP parameters lead to two different thermodynamic systems.

In Fig. 8, we plot the thermodynamic curvature scalar R_N as a function of $\tilde{T}$ and $\tilde{V}$. Fig. 8(a) shows that the thermodynamic curvature scalar is around 0 for most (T, V) intervals. However, it diverges at temperature

$$\tilde{T}_{\mathrm{div}} = \frac{1 + \left(2\sqrt{3} - 3\right)\tilde{V}^{2/3}}{\sqrt{3}\tilde{V}^{1/3} + \left(\sqrt{3} - 2\right)\tilde{V}}, \tag{44}$$

which means that the interaction of microstructure of FRW universe changes rapidly in the vicinity of $\tilde{T}_{\mathrm{div}}$. Moreover, from Fig. 8(b), it can be seen that there is no divergent point for low temperature $\tilde{T} < 1$. However, the divergent points appear when the reduced temperature is above 1. For $\tilde{T} = 1$, one can see only one negative divergent point at $\tilde{V} = 1$. For $\tilde{T} > 1$, there are two negative divergent points, which are moved toward the high volume region and the low volume region, respectively, as temperature increases. In Fig. 8(c), the red dashed curve represents sign-changing curve, which is expressed as

$$T_0 = \frac{\left(6 - 4\sqrt{3}\right)V^{2/3} + 3\left(-7 + 4\sqrt{3}\right)V^{4/3} - 1}{4\left(\sqrt{3} - 2\right)V + 2\left(7\sqrt{3} - 12\right)V^{5/3} - 2\sqrt{3}V^{1/3}}. \tag{45}$$

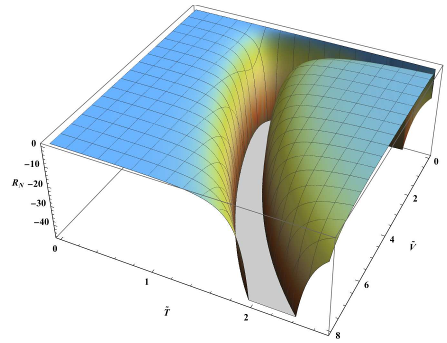
Below the red dashed curve, R_N is larger than 0, meaning that the microstructure of this region is dominated by repulsive interaction. For the region above the red dashed curve, one has $R_N > 0$, indicating that the microstructure in this region is dominated by attractive interaction. However, the red dashed curve is far from the black curve. Therefore, the R_N below the red dashed curve is very close to 0, indicating that the repulsive interaction of the system is very weak.

Finally, by inserting Eq. (42) into Eq. (43), the divergence behavior of thermodynamic scalar curvature along the coexistence small and large phases can be seen from the following expressions

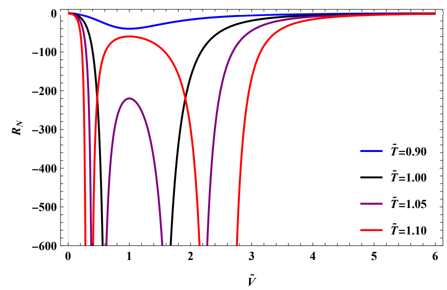
$$R_N^s = -\frac{1}{8\tau^2} + \frac{\sqrt{3\left(14\sqrt{3} - 9\right)}}{4\tau^{3/2}} + \mathcal{O}\left(\tau^{-1}\right), \tag{46a}$$

$$R_N^l = -\frac{1}{8\tau^2} - \frac{\sqrt{3\left(14\sqrt{3} - 9\right)}}{4\tau^{3/2}} + \mathcal{O}\left(\tau^{-1}\right). \tag{46b}$$

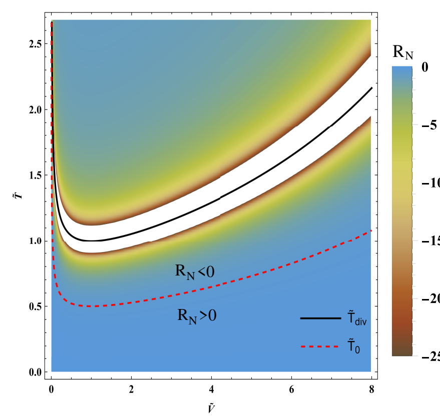
Obviously, the above result explicitly reveal that R_N has a critical exponent 2, and the divergence behavior is characterized by a dimensionless constant $-1/8$ for $\lim_{\tau\to 0} R_N\tau^2$. This results agree with what we obtained in Eq. (35) and Eq. (36), showing that the properties are universal.



(a)



(b)



(c)

FIG. 8. The behavior of R_N along the small and large coexistence phases for the FRW universe with $\beta < 0$.

IV. CONCLUSION

In this paper, we investigated the phase transition, critical behavior and microstructure of the FRW universe in the framework of a new higher order GUP (1). By defining the work density as a thermodynamic pressure ($W := P$), the thermodynamic equation of state for the FRW universe is obtained. Then, with simple calculations, we find that there are two sets of possible critical quantities (Eq. (13) and Eq. (37)) in the system, and they hold for GUP parameter β larger than zero and less than zero, respectively. Considering the flexibility of the GUP to accommodate both $\beta > 0$ and $\beta < 0$, we thereby discuss the thermodynamic properties in these dual scenarios.

We first calculated the ratio of critical quantities χ. As elucidated in Eq. (14), the ratio for $\beta > 0$ is very close to that of VdW fluids, whereas the ratio of critical quantities (38) for $\beta < 0$ is much larger (almost twice as large) than that of VdW fluids. Those imply that the FRW universe with different GUP parameters has different thermodynamic properties. Then, we considered the "Pressure-Volume" relationship near the reduced critical temperature. In the case of $\beta > 0$, the characteristics of the $P - V$ phase transition in the FRW universe are very similar to those of VdW fluids. It occurs at the temperature less than the reduced critical temperature, that is $T < T_c$ or $\tilde{T} < 1$, and becomes to the ideal gas when $T > T_c$. In contrast, for $\beta < 0$, the $P - V$ phase transition in the FRW universe takes place at $T > T_c$, while the ideal gas behavior surfaces at $T < T_c$. What adds a layer of interestingly is the emergence of a "thermodynamic singularity" in this case, a feature absent in VdW fluids. This suggests that there is indeed some difference in the thermodynamic properties between the two scenarios. Furthermore, in order to obtain the specific properties of the $P - V$ phase transitions, we analyzed the "Free Gibbs energy-Temperature" relations. Drawing on the swallowtail behavior evident in Fig. 2 and Fig. 6, these transitions are discerned as first-order phase transitions. Subsequently, according to the equation of state, we derived the critical exponents $(\alpha, \varepsilon, \gamma, \delta)$ around the critical points. The outcomes revealed that the critical exponents remain consistent in both cases, aligning with the expectations of mean-field theory and adhering to the scaling laws. Nevertheless, when contemplating the coexistence curves of small and large phases around the critical points in the two cases, a difference emerges again. In the case of $\beta > 0$, the coexistence curves exhibit a peak, with the small and large phases of the FRW universe exclusively manifesting below this peak. Conversely, for $\beta < 0$, the coexistence curves feature only a singular minimum, signifying that the size phase of the universe exists solely above this minimum. Then, utilizing the Ruppeiner geometry, we derived the thermodynamic curvature scalar R_N, exploring its sign-changing curve and spinodal curve. In the case of $\beta > 0$, it was observed that the sign-changing curve and spinodal curve initially increase with growing volume, reaching a maximum value before decreasing with further volume increase. The R_N is negative above the sign-changing curve, which implies the attractive interaction dominate there. For the region below the red dashed curve, R_N becomes positive. Conversely, in the region below the red dashed curve, R_N becomes positive. However, given the considerable distance from the spinodal curve, the value of R_N in this region approaches zero, indicating the presence of weak repulsive interactions. For $\beta < 0$, an increase in volume leads to a decreasing and then increasing behavior for both the sign-changing curve and the spinodal curve. Above the sign-changing curve, $R_N > 0$, whereas below the sign-changing curve, $R_N < 0$. Consequently, the microstructure of the FRW universe is characterized by attractive interactions in the low-temperature region and repulsive interactions in the high-temperature region. Finally, we verified that the thermodynamic curvature scalar exhibits two universal properties for both $\beta > 0$ and $\beta < 0$. One is a universal critical exponent of 2 along both the small and large phases of FRW universe, the other one is a universal constant $-1/8$ as $\lim\limits_{\tau \to 0} R_N \tau^2$.

In summary, our work showed that for the FRW universe in the GUP framework, the different GUP parameters lead to different phase transition properties. For $\beta > 0$, the phase transitions, critical behaviors and microstructure of FRW universe have many similarities with VdW fluids, while the thermodynamic properties associated with $\beta < 0$ resemble those derived from effective scalar field theory. These findings imply a link between macroscopic thermodynamic phenomena and microscopic QG, warranting in-depth exploration. Moreover, considering the numerous existing GUP models, we anticipate that these models could also induce phase transitions in the universe. Analyzing these phase transitions promises valuable insights into the nature of QG.

Acknowledgements

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Appendix A

In Refs. [20, 25], the unified first law is given by

$$\mathrm{d}\tilde{E} = \tilde{A}\tilde{\Psi} + \tilde{W}\mathrm{d}\tilde{V}, \tag{A.1}$$

where $\tilde{A} = 4\pi R^2$ is the area of a sphere with radius R, and $\tilde{W} = -h^{ab}T_{ab}/2$ is the work density of the matter fields. The energy flux $\tilde{\Psi}$ is defined as

$$\tilde{\Psi} = \tilde{\Psi}_a \mathrm{d}x^a = \left(T_a^b \partial_b R + \tilde{W}\partial_a R\right)\mathrm{d}x^a, \tag{A.2}$$

with the energy-momentum tensor of the perfect fluid $T_{ab}=(\rho+p)u_a u_b+pg_{ab}$. According to above expressions, the energy supply vector can be expressed as follows

$$\tilde{\Psi}_a=\left[-\frac{1}{2}\left(\rho+p\right)HR,\frac{1}{2}\left(\rho+p\right)a\right]. \tag{A.3}$$

Substituting Eq. (A.3) into Eq. (A.2), one has

$$\begin{aligned}\tilde{A}\tilde{\Psi}&=\tilde{A}\tilde{\Psi}_a\mathrm{d}x^a=2\pi R^2\left(\rho-p\right)\left(-HR\mathrm{d}t+a\mathrm{d}r\right)\\&=-\frac{\tilde{A}}{2}\left(\frac{1+4H^2\beta}{4\pi}\right)\dot{H}\left(\mathrm{d}R-2HR\mathrm{d}t\right).\end{aligned} \tag{A.4}$$

At the apparent horizon of the FRW universe R_A, Eq. (A.4) is rewritten as

$$\begin{aligned}\tilde{A}\tilde{\Psi}\Big|_{R=R_A}&=\frac{A}{2}\left(1+\frac{4\beta}{R_A^2}\right)\frac{\dot{R}_A}{4\pi R_A^2}\left(\mathrm{d}R_A-2\mathrm{d}t\right)\\&=\frac{A}{2}\left(1+\frac{4\beta}{R_A^2}\right)\frac{\dot{R}_A}{4\pi R_A^2}\mathrm{d}R_A-\frac{A}{2}\left(1+\frac{4\beta}{R_A^2}\right)\frac{\mathrm{d}R_A}{2\pi R_A^2}\\&=\frac{\dot{R}_A}{2}\left(1+\frac{4\beta}{R_A^2}\right)\mathrm{d}R_A-\left(1+\frac{4\beta}{R_A^2}\right)\mathrm{d}R_A\\&=-\frac{1}{2\pi R_A}\left(1-\frac{\dot{R}_A}{2}\right)\left(2\pi R_A+\frac{8\pi\beta}{R_A}\right)\mathrm{d}R_A.\end{aligned} \tag{A.5}$$

For the second term of the right hand side of Eq. (A.1) at the apparent horizon of the FRW universe becomes

$$\tilde{W}\mathrm{d}\tilde{V}\Big|_{R=R_A}=\frac{4\pi R_A^3}{6}\left(\rho-p\right)=W\mathrm{d}V. \tag{A.6}$$

Now, according to Eq. (A.5) and Eq. (A.6), the unified first law takes the form

$$\mathrm{d}E=-T\mathrm{d}S+W\mathrm{d}V. \tag{A.7}$$

where $\tilde{E}\Big|_{R=R_A}=E$. Obviously, the above equation is the thermodynamic first law.

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