

A quantum annealing approach to minimum distance problem

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Abstract

Quantum error-correcting codes (QECCs) is at the heart of fault-tolerant quantum computing. As the size of quantum platforms is expected to grow, one of the open questions is to design new optimal codes of ever-increasing size. A related challenge is to “certify” the quality of a given code by evaluating its minimum distance, a quantity characterizing code’s capacity to preserve quantum information. This problem is known to be NP-hard. Here we propose to harness the power of contemporary quantum platforms to address this question, and in this way help design quantum platforms of the future. Namely, we introduce an approach to compute the minimum distance of quantum stabilizer codes by reformulating the problem as a Quadratic Unconstrained Binary Optimization (QUBO) problem and leveraging established QUBO algorithms and heuristics as well as quantum annealing (QA) to address the latter. The reformulation as a QUBO introduces only a logarithmic multiplicative overhead in the required number of variables. We demonstrate practical viability of our method by comparing the performance of purely classical algorithms with the D-Wave Advantage 4.1 quantum annealer as well as hybrid quantum-classical algorithm Qbsolv. We found that the hybrid approach demonstrates competitive performance, on par with the best available classical algorithms to solve QUBO. In a practical sense the QUBO-based approach is currently lagging behind the best deterministic minimal distance algorithms, however this advantage may disappear as the size of the platforms grows.

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I. INTRODUCTION

The field of quantum error correction has been the focus of intense research in recent decades due to its role in the development of fault-tolerant quantum computing. Quantum fault tolerance offers a trade-off between efficiency in storing and manipulating information and robust protection against errors [1]. One of the central theoretical goals is to design QECCs that minimize the required resources while enhancing the protective properties.

Quantum algorithms operate on an abstract Hilbert space, known as the logical Hilbert space. When employing a QECC, we restrict the logical Hilbert space to a specific subspace of the full computational (also referred to as physical) Hilbert space. Errors that send the computation outside the code subspace could be detected and corrected. In contrast, errors that map the code subspace into itself are undetectable. The code's minimum (or Hamming) distance d is defined as the smallest weight of an undetectable error, i.e. how many physical qubits this error affects [2].

A QECC can be characterized by three parameters: the number of physical qubits n , the number of logical states K it protects and its minimum distance d . Corresponding codes are conventionally denoted as $((n, K, d))$. Constructing QECCs of ever-increasing n , with a fixed ratio $\ln K/n$ and the best (largest) d/n is an open problem. Our emphasis will be

on stabilizer codes, the most common class of quantum codes introduced in [2] and widely considered the best strategy to achieve fault tolerance in the near term. A binary stabilizer code, i.e. over qubits, protects the $K = 2^k$ states, where k denotes the number of logical qubits and is denoted by $[[n, k, d]]_2$.

An optimal code is the one with the highest known value of d for given n and k . In practice to certify that code is optimal or close to optimal is a difficult problem. Depending on how the code is defined, determining d could be a computationally challenging task. In fact, finding d for a general (binary) stabilizer code specified by its generating matrix is known to be NP-hard. Even approximating the minimum distance, with a guaranteed precision specified by a given multiplicative or additive factor, is also NP-hard [3]. There exist multiple approaches for computing the minimum distance of quantum codes. One of the best deterministic algorithms was developed by G. White and M. Grassl and has been incorporated into MAGMA [4, 5]. Multiple additional techniques have been also tailored for specific families of quantum codes, such as Low-Density-Parity-Check (qLDPC) codes [6–8].

In this paper, we introduce a new computational technique to evaluate or bound from above the minimum distance of binary quantum stabilizer codes, by reformulating the question as a Quadratic Unconstrained Binary Optimization (QUBO) problem. A QUBO is a general framework to represent combinatorial optimization problem as a problem to minimize quadratic functions of binary variables [9]. To reformulate the minimum distance problem as a QUBO, one needs to introduce auxiliary variables, e.g. to implement mod 2 arithmetic. As a result, our method involves a multiplicative logarithmic overhead in terms of the total number of variables.

Reformulating the problem in terms of QUBO introduces a promising pathway to harness power of quantum computational schemes such as Adiabatic Quantum Computing (AQC) and Quantum Annealing (QA), alongside a spectrum of established classical algorithms and heuristics designed specifically for QUBO problems. There is an ongoing effort to reformulate many NP-hard problems of theoretical and practical value within a unified framework of QUBO, with such problems as SAT, Knapsack, Maximum Cut, and Graph Coloring already being worked out in the literature [9–11]. These problems arise in different fields, from Quantum Information to Machine Learning to RNA folding [12–15]. Our work can be seen as contributing to this effort.

To investigate the efficiency of our approach, implemented on the contemporary quantum simulators, we focus on quantum annealing (QA) as a method to solve QUBO. QA is a quantum heuristic approach, inspired by the principles of Adiabatic Quantum Computing (AQC) [16, 17]. QA provides a rigorous upper bound for optimization problems, albeit without a guarantee on how close the obtained result is to the optimal value. To investigate the viability of QA in our case, we tested our method using both the D-Wave Advantage 4.1 quantum annealer and Qbsolv, a hybrid quantum-classical algorithm developed by D-Wave and running on their platform [18]. As a benchmark, we compared the performance of quantum and quantum-classical methods with simulated annealing (SA), a well-established classical heuristic approach [19]. Whenever available, we compared the results of quantum, classical, or hybrid optimization with the exact results.

Our study has shown that the hybrid quantum/classical approach is already demonstrating performance at the level of purely classical optimization scheme, possibly opening a new avenue for near-future practical applications of quantum annealing.

The paper is organized as follows. In Section II, we introduce the basics of stabilizer

codes. We formulate the main algorithm to evaluate d as a QUBO problem in Section III. Section IV provides a brief overview of AQC and QA. Section V details the comparison of purely classical, quantum, and hybrid approaches. An open-source implementation of our algorithm is available on GitHub [20]. We conclude with a discussion in section VI.

II. THEORETICAL PRELIMINARIES

A. Stabilizer Codes: a brief overview

One of the most frequently used class of QECCs is the stabilizer codes. If S is an Abelian subgroup of the Pauli group \mathcal{P}_n ¹ with $(n-k)$ generators and $-I \notin S$, then the simultaneous +1 eigenspace, V_S , of all elements of S is an $[[n, k, d]]$ quantum stabilizer code, where n , k , and d denote the number of physical qubits, logical qubits, and the code minimum distance, correspondingly. Note that the two conditions on the subgroup, being Abelian and $-I \notin S$, are to ensure that the code space is non-trivial.

To specify the stabilizer code, it is sufficient to provide $(n-k)$ generators of the stabilizer group S . The stabilizer generators contain essential information about the error-correcting capabilities of the code, eliminating the need to construct the code space V_S explicitly. Stabilizer codes are also known as additive quantum codes due to their one-to-one correspondence with trace hermitian self-orthogonal additive classical codes over GF(4) [21].

Pauli group elements can be represented as binary vectors, allowing for an efficient representation of the stabilizer group. Namely, up to a phase, any Pauli group element can be expressed as a product of X and Z operators, each raised to a power of either 0 or 1,

$$g \propto (X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}) \cdot (Z_1^{\beta_1} Z_2^{\beta_2} \dots Z_n^{\beta_n}), \quad g \in \mathcal{P}_n \quad (2.1)$$

Here α_i and β_i are binary variables. Combining them together, we can represent a Pauli element using a binary vector $\mathbf{c} = (\boldsymbol{\alpha}|\boldsymbol{\beta})^2$. Multiplying two Paulis then corresponds to adding their respective binary vectors mod 2.

In this representation, the stabilizer group generators can be specified by a binary $(n-k) \times 2n$ matrix, where each row is vector representation of one of the $(n-k)$ generators. This matrix H is called the stabilizer matrix (also known as the parity check matrix). We can generate binary representations of all elements of the stabilizer group as linear combinations (mod 2) of the generators of the group,

$$\mathbf{c} = H^T \mathbf{x} \pmod{2}, \quad \mathbf{x} \in \mathbb{Z}_2^{n-k}, \quad \mathbf{c} \in \mathbb{Z}_2^{2n}, \quad \mathbb{Z}_2 \equiv \{0, 1\}, \quad (2.2)$$

where \mathbf{c} is the binary representation of a given stabilizer group element, sometimes called a ‘‘codeword.’’

The normalizer of S , denoted $N_{\mathcal{P}_n}(S)$, consists of all elements of Pauli group that map the stabilizer group to itself under conjugation, $N_{\mathcal{P}_n}(S) = \{g \in \mathcal{P}_n \mid gSg^{-1} = S\}$. It is clear that $S \subseteq N_{\mathcal{P}_n}(S)$. The operators present in the normalizer but not in stabilizer represent logical Pauli operators on the k encoded logical qubits. These operators map states from the code subspace into each other, which makes them undetectable errors.

¹ The n qubit Pauli group, \mathcal{P}_n , consists of all n -fold tensor products of the Pauli operators I, X, Y, Z , with an overall phase i^k with an integer k .

² We use bold symbols to represent vectors.

The generators of the normalizer group, $N_{\mathcal{P}_n}(S)$, can be represented as columns of the normalizer matrix G (also known as the generator matrix), which is a $(2n) \times (n+k)$ matrix of full rank that satisfies

$$H \Lambda G = 0 \pmod{2}, \quad \Lambda := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad (2.3)$$

with I_n being the $n \times n$ identity matrix. The full rank of G ensures that all generators of the normalizer are linearly independent, while the condition (2.3) ensures that all normalizer elements commute with the stabilizer group. Matrix G also satisfies $\dim \ker(G^T \Lambda G) = n - k$.

B. Minimum distance

The weight of a Pauli operator – an element from the Pauli group – is the number of qubits on which it acts non-trivially. The weight of a Pauli operator can be calculated using the binary representation $\mathbf{c} = (\boldsymbol{\alpha}|\boldsymbol{\beta})$ as follows,

$$w(\mathbf{c}) = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{\beta} - \boldsymbol{\alpha} \cdot \boldsymbol{\beta}. \quad (2.4)$$

The minimum distance of a ($k \neq 0$) stabilizer code is the smallest weight of an operator in the set of all logical Pauli operators, i.e., the set of elements in the normalizer $N_{\mathcal{P}_n}(S)$ but not in the stabilizer S . The minimum code distance is also called the minimal Hamming distance.

The code minimum distance d can then be expressed in terms of the normalizer matrix, G ,

$$d = \min_{\mathbf{c}} w(\mathbf{c}), \quad \mathbf{c} = G \mathbf{x} \pmod{2}, \quad \mathbf{x} \in \mathbb{Z}_2^{n+k}, \quad \mathbf{c} \notin \ker(G^T \Lambda). \quad (2.5)$$

Here the constraint $\mathbf{c} \notin \ker(G^T \Lambda)$ is equivalent to $\mathbf{c} \notin \text{Im}(H^T)$ which ensures that we minimize only over the set of logical Pauli operations. A stabilizer code with minimum distance d can correct all errors of weight $\lfloor \frac{d-1}{2} \rfloor$ or less, and can also correct all erasure errors³ of weight $d - 1$ or less [2].

C. Self-dual stabilizer codes

Self-dual codes represent a special class of stabilizer codes with $k = 0$. In this case the code space V_S is one-dimensional, often referred to as a stabilizer state. These codes cannot store or be used to manipulate quantum information, instead, they are protocols for error detection. By initializing a quantum processor in a stabilizer state and evolving in time such that it should return to the original state, we can detect and identify any errors that may have accumulated during this evolution. For self-dual codes, the minimum distance of the code is the smallest weight among the weights of all non-trivial elements of the stabilizer group,

$$d = \min_{\mathbf{c}} w(\mathbf{c}), \quad \mathbf{c} = G \mathbf{x} \pmod{2}, \quad \mathbf{x} \in \mathbb{Z}_2^n, \quad \mathbf{c} \neq 0^{2n}. \quad (2.6)$$

For self-dual codes, we can chose $G = H^T$. An $[[n, 0, d]]$ code can detect all errors affecting up to $d - 1$ qubits, and can identify all errors of weight $\lfloor \frac{d-1}{2} \rfloor$ or less [2].

³ Erasure errors are errors in which the qubits affected by the error are known.

III. ALGORITHM

In this section, we reformulate the problem of finding the minimum distance of a stabilizer code as a QUBO (quadratic unconstrained binary optimization) problem.

A. Quadratic binary cost function

Any generator matrix can be written as $G^T = (A \ B)$, where A and B are $n \times (n + k)$ binary matrices. Using the definition of the Pauli operator weight (2.4), the code minimum distance (2.5) can be written as

$$d = \min_{\mathbf{x} \notin \mathcal{C}} \sum_{i=1}^n (\alpha_i^2 + \beta_i^2 - \alpha_i \beta_i), \quad \mathbf{x} \in \mathbb{Z}_2^{n+k},$$

$$\mathcal{C} = \begin{cases} \{0^n\}, & k = 0, \\ \{\mathbf{x} \mid G^T \Lambda G \mathbf{x} = \mathbf{0}\}, & k \neq 0. \end{cases} \quad (3.1)$$

Here, $\boldsymbol{\alpha} = (A \mathbf{x} \bmod 2)$ and $\boldsymbol{\beta} = (B \mathbf{x} \bmod 2)$. The minimization problem above is a constrained optimization problem with a quadratic cost function in terms of binary variables $\{x_i\}$. It closely resembles a QUBO problem [9], except for the presence of constraints and mod 2 sums in the definition of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The subsequent sections explain how to deal with each of these differences.

B. Reformulation of the mod 2 summation

Here, we describe a simple method to trade modular sums into conventional ones by introducing auxiliary integer variables, which subsequently can be parameterized by several binary variables. This leads to a quadratic binary optimization problem that involves only conventional sums.

To eliminate modular sums, we introduce two vectors,

$$\mathbf{a} := A \mathbf{x}, \quad \mathbf{b} := B \mathbf{x}. \quad (3.2)$$

Here matrix multiplication assumes conventional summation. Next, we introduce parity \mathbb{P} ,

$$\mathbb{P}(m) = m \bmod 2. \quad (3.3)$$

With this definition, we readily recognize $\alpha_i = \mathbb{P}(a_i)$ and $\beta_i = \mathbb{P}(b_i)$. It can be checked straightforwardly that

$$\alpha_i^2 + \beta_i^2 - \alpha_i \beta_i = \frac{1}{2} (\mathbb{P}(a_i) + \mathbb{P}(b_i) + \mathbb{P}(a_i + b_i)). \quad (3.4)$$

To implement (3.3) we introduce minimization over an auxiliary integer variable t ,

$$\mathbb{P}(m) = \min_t (m - 2t)^2, \quad 0 \leq t \leq \lfloor \frac{\max m}{2} \rfloor. \quad (3.5)$$

The resulting cost function becomes

$$d = \min_{\mathbf{t}, \mathbf{u}, \mathbf{v}} \frac{1}{2} \left(\sum_{i=1}^n (a_i - 2t_i)^2 + (b_i - 2u_i)^2 + (a_i + b_i - 2v_i)^2 \right), \quad (3.6)$$

where $\mathbf{t}, \mathbf{u}, \mathbf{v}$ are vectors of auxiliary integer variables. To express integer variables in terms of binary variables, we use their binary representation, i.e.

$$t_i = \sum_{j=1}^s 2^{j-1} t_{ij}, \quad t_{ij} \in \mathbb{Z}_2. \quad (3.7)$$

Here $s = \lceil \ln \left(\frac{\max a_i}{2} \right) \rceil$ is the number of auxiliary binary variables. This number is bounded by $\ln \left(\frac{n+k}{2} \right)$. Putting it all together, we arrive at

$$d = \min_{\mathbf{x} \notin \mathcal{C}} \min_{\mathbf{t}, \mathbf{u}, \mathbf{v}} \left(\begin{array}{c} \mathbf{x}^T (A^T A + B^T B + B^T A) \mathbf{x} \\ - \sum_{i=1}^n \sum_{j=1}^{n+k} \frac{x_j}{2} \left(\sum_{l=1}^s 2^l (A_{ij} t_{il} + B_{ij} u_{il}) + \sum_{l=1}^{2s} (A_{ij} + B_{ij}) 2^l v_{il} \right) \\ + \sum_{i=1}^n \sum_{j,l=1}^s 2^{j+l-1} (t_{ij} t_{il} + u_{ij} u_{il}) + \sum_{i=1}^n \sum_{j,l=1}^{2s} 2^{j+l-1} v_{ij} v_{il} \end{array} \right), \quad (3.8)$$

where $\mathbf{t}, \mathbf{u}, \mathbf{v}$ now represent matrices of binary variables t_{ij}, u_{ij}, v_{ij} . The cost function above can be expressed in terms of a binary symmetric matrix

$$\mathbf{d} = \mathbf{z}^T Q \mathbf{z}, \quad (3.9)$$

where \mathbf{z} is a vector of the primary and auxiliary binary variables

$$\mathbf{z} = (x_1, x_2, \dots, x_{n+k}, t_{1,1}, t_{2,1}, \dots, t_{n,1}, \dots, t_{1,s}, \dots, t_{n,s}, u_{1,1}, \dots, u_{n,s}, v_{1,1}, \dots, v_{n,2s}), \quad (3.10)$$

and

$$Q := \begin{pmatrix} A^T A + B^T B + B^T A & -(\mathbf{w}_s \otimes A)^T & -(\mathbf{w}_s \otimes B)^T & -(\mathbf{w}_{2s} \otimes (A + B))^T \\ -\mathbf{w}_s \otimes A & 8\mathbf{w}_s \mathbf{w}_s^T \otimes I_n & 0 & 0 \\ -\mathbf{w}_s \otimes B & 0 & 8\mathbf{w}_s \mathbf{w}_s^T \otimes I_n & 0 \\ -\mathbf{w}_{2s} \otimes (A + B) & 0 & 0 & 8\mathbf{w}_{2s} \mathbf{w}_{2s}^T \otimes I_n \end{pmatrix}. \quad (3.11)$$

Here we introduced the column vector $\mathbf{w}_s := \frac{1}{2}(1, 2, 4, \dots, 2^{s-1})$. The total number of auxiliary binary variables is $4ns$, with $s \leq \ln \frac{n+k}{2} < \ln n$. Thus, the total number of binary variables, both primary and auxiliary, grows asymptotically as $O(n \ln n/n_0)$, for some constant n_0 . Notably, the multiplicative overhead in the number of variables increases only logarithmically with n . Finding the minimum distance of the corresponding stabilizer code is equivalent to finding the minimum of (3.9) subject to constraints.

1. Self dual codes

For self-dual codes, we can leverage code equivalence (discussed in the next section) to reduce the number of auxiliary variables required. It has been shown that every self-dual code is equivalent to a graph code [22]. Graph codes are self-dual codes associated with a simple undirected graph, their generator matrix can be written as $G^T = (I_n \ B)$, where B is the adjacency matrix of the corresponding graph.

Using this representation of the generator matrix, the i -th term in the optimization problem (3.1) becomes $x_i^2 + \beta_i(\beta_i - x_i)$. We can replace β with b , as defined in (3.2), using the following identity

$$x_i^2 - \beta_i(\beta_i - x_i) = x_i^2 + \min_{u_i}(b_i - 2u_i)(b_i - 2u_i - x_i), \quad (3.12)$$

where u_i is an integer variable and can be represented in terms of binary variables as before. The corresponding Q matrix then becomes:

$$Q = \begin{pmatrix} I_n + B^2 - B & 2(\mathbf{w}_s \otimes (I_n - 2B))^T \\ 2\mathbf{w}_s \otimes (I_n - 2B) & 8\mathbf{w}_s \mathbf{w}_s^T \otimes I_n \end{pmatrix}. \quad (3.13)$$

C. Reformulating constraints

1. Trade constraints for penalty term

Different stabilizer codes are called equivalent if their stabilizer generators can be mapped to each other using conjugation by local Clifford operations⁴, and qubit permutations. The definition of code equivalence ensures that equivalent codes have the same minimum distance. Code equivalences, together with the freedom to select generators of the normalizer, can be used to bring the normalizer matrix G into various convenient forms [2, 23].

A particular convenient form to which the normalizer matrix can be efficiently brought by a quick classical algorithm is $G = (L \ H^T)$, where L and H^T are $2n \times 2k$ and $2n \times (n - k)$ binary matrices, respectively. Here, L represents $2k$ linearly independent logical Pauli operators, while H^T is the parity check matrix containing generators of the stabilizer group. For non-self-dual ($k \neq 0$) codes, the constraint $(G^T \Lambda G)\mathbf{x} \neq \mathbf{0}$ can be imposed by penalizing \mathbf{x} with vanishing first $2k$ components x_i , that correspond to logical operations. This is achieved by adding the following quadratic term to the utility function

$$\min_m n \left(\sum_{i=1}^{2k} x_i - m \right)^2, \quad 1 \leq m \leq 2k. \quad (3.14)$$

Here, m is an auxiliary integer variable. Multiplying by n ensures that the penalty is greater than the minimum distance (since $d < n$). To express the cost function in terms of binary variables, we trade the integer variable m for r auxiliary binary variables using its binary representation: $m = 1 + \sum_{j=1}^r 2^{j-1} e_j$, $e_j \in \{0, 1\}$. It is enough to introduce $r = \lceil \ln 2k \rceil \leq \lceil \ln n \rceil$ binary variables to cover the required range for m .

⁴ Local Clifford operations are tensor products of single-qubit unitaries that map Pauli operators to themselves under conjugation.

For Self-Dual codes, the constraint $\mathbf{x} \neq 0^n$ can be satisfied by a term similar to (3.14) with the sum going from $i = 1$ to n instead. m in this case can be represented in terms of $r = \lceil \ln n \rceil$ binary variables. The number of auxiliary variables introduced in both cases is bounded by $\lceil \ln n \rceil$.

To incorporate these penalty terms into our optimization problem, we extend the vector of variable to

$$\mathbf{z} = (x_1, x_2, \dots, x_{n+k}, t_{1,1}, t_{2,1}, \dots, t_{n,1}, \dots, t_{1,s}, \dots, t_{n,s}, u_{1,1}, \dots, u_{n,s}, v_{1,1}, \dots, v_{n,2s}, e_1, \dots, e_r). \quad (3.15)$$

The minimization problem becomes

$$d = n + \min_{\mathbf{z}} \mathbf{z}^T Q \mathbf{z}, \quad (3.16)$$

with the following rows and columns added to Q :

$$Q := \begin{matrix} n+k & & r \\ n+k \{ & \left(\begin{array}{c|c} \overbrace{n(\mathbf{v}\mathbf{v}^T - 2\mathbf{v} \odot I_{n+k})} & \overbrace{-2n\mathbf{v}\mathbf{w}_r^T} \\ 0 & 0 \end{array} \right) & \\ r \{ & \left(\begin{array}{c|c} \hline -2n\mathbf{w}_r\mathbf{v}^T & 0 \\ \hline 0 & 4n(\mathbf{w}_r\mathbf{w}_r^T + \mathbf{w}_r \odot I_r) \end{array} \right) & \\ & + & r \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array} \right). \end{matrix} \quad (3.17)$$

The column vector \mathbf{v} is defined by $v_i = 1$ for $i \leq 2k$ and $v_i = 0$ otherwise. By \odot , we denote the Hadamard (element-wise) product, $(A \odot B)_{ij} := A_{ij}B_{ij}$. Introducing the penalty term which implements the constraint on \mathbf{x} increases the number of binary variables only by $\ln n$, preserving the asymptotic growth of the total number of variables of order $O(n \ln n/n_0)$.

2. Trade constraints for multiple cost functions

An alternative approach to incorporate the constraints is to go through all values $i = 1, \dots, 2k$, and for each value of i set $x_1, \dots, x_{i-1} = 0$ and $x_i = 1$, and solve the resulting QUBO for the remaining variables. The best (minimal) value among the $2k$ QUBOs would be the global minimum of the original constrained problem. This approach requires solving $O(n)$ distinct QUBOs of sizes $O(n \ln n/n_0)$, which is normally less favorable than the approach presented in the previous section. However, it proves to be highly efficient for specific classes of code, particularly for circulant self-dual codes.

Circulant self-dual codes is a family of codes defined by a normalizer matrix admitting the form $G^T = (I_n|B)$, with B an $n \times n$ binary circulant matrix [24].⁵ These codes are of interest since many optimal self-dual codes discovered to date belong to this family [25].⁶ To meet the constraint $\mathbf{x} \neq 0^n$, it is sufficient to set $x_1 = 1$ and solve QUBO for the remaining variables.

IV. ADIABATIC QUANTUM COMPUTING AND QUANTUM ANNEALING

Adiabatic Quantum Computing (AQC) is a universal model for quantum computation based on the adiabatic theorem. AQC starts by initializing the quantum processor in an

⁵ Circulant matrix is a matrix with each row being a cyclic shift of the row above.

⁶ Codes that meet the linear programming upper bound on minimum distance are called extremal, while those that achieve the highest possible minimum distance but do not satisfy the linear programming upper bound are referred to as optimal.

easy-to-prepare ground state of some chosen initial Hamiltonian, $H(0) = H_I$. Subsequently, the system's Hamiltonian is slowly evolved over an extended time, t_a , transitioning towards a problem Hamiltonian, $H(t_a) = H_p$, whose ground state encodes the solution to a problem of interest. The adiabatic theorem guarantees that the evolved state $|\Psi(t_a)\rangle$ will have a large overlap with the ground state of the problem Hamiltonian provided that the evolution was done slow enough [26]. In fact, this overlap, often termed the probability of success, $P_s := |\langle 0, t_a | \Psi(t_a) \rangle|^2$ satisfies

$$\lim_{t_a \rightarrow \infty} P_s = 1. \quad (4.1)$$

Here, the state $|m, t\rangle$ satisfies $H(t) |m, t\rangle = E_m(t) |m, t\rangle$ with $E_m(t)$ denoting the m -th lowest eigenvalue of $H(t)$.

For Hamiltonians that can be written as $H(t) = \tilde{H}(\gamma)$, where $\gamma := t/t_a \in [0, 1]$, and \tilde{H} has no explicit dependence on t_a , the adiabatic condition can be expressed as

$$t_a \gg \frac{\hbar \mathcal{E}}{g_{\min}^2}, \quad \mathcal{E} := \max_{\gamma} \left| \langle 1, \gamma | \partial_{\gamma} \tilde{H} | 0, \gamma \rangle \right|, \quad g_{\min} := \min_{\gamma} |E_1(\gamma) - E_0(\gamma)|. \quad (4.2)$$

This condition ensures that P_s is finite for finite values of t_a [26]. The performance of AQC depends on the chosen adiabatic path, specifically the choice of the initial Hamiltonian and the trajectory of evolution towards the problem Hamiltonian. For Hamiltonians exhibiting final ground state degeneracy, g_{\min} is calculated between the instantaneous ground state and the first excited state that does not evolve eventually into the final ground state subspace [27]. In this scenario, we define the probability of success, P_s , as the overlap between the final evolved state and the subspace of final degenerate ground-states. Notably, AQC has been shown to be polynomially equivalent, in terms of the resource overhead, to universal gate-based quantum computation [28].

Quantum Annealing (QA) is a heuristic approach that follows the same process as AQC but relaxes the condition of adiabaticity. The departure from adiabatic evolution introduces excitations in the system's state, voiding the guarantee of convergence to the ground state of the problem Hamiltonian. To mitigate the limitations of nonadiabaticity, the algorithm repeats the entire evolution multiple times, generating various candidate final states. The algorithm then reports the best overall result. The computational power of QA and whether it offers a speedup over classical computation is currently an open area of research [16], with studies comparing the performance of QA against classical algorithms like Simulated Annealing (SA) and Simulated Quantum Annealing (SQA) [29].

Quantum annealers are devices designed to run the QA algorithm. D-Wave systems have introduced multiple commercially available quantum annealers [30]. In this paper, we focus on the D-wave Advantage 4.1 quantum annealer, which operates by evolving a mixed-field quantum Ising Hamiltonian of the form

$$H(\gamma) = \underbrace{-\frac{\mathcal{A}(\gamma)}{2} \sum_i X_{(i)}}_{\text{initial Hamiltonian}} + \underbrace{\frac{\mathcal{B}(\gamma)}{2} \left(\sum_i h_i Z_{(i)} + \sum_{i>j} J_{ij} Z_{(i)} Z_{(j)} \right)}_{\text{problem Hamiltonian}}. \quad (4.3)$$

Here, $\mathcal{A}(\gamma)$ and $\mathcal{B}(\gamma)$ are functions that control the transition between the initial and problem Hamiltonians, satisfying $\mathcal{A}(0) \gg \mathcal{B}(0)$ and $\mathcal{A}(1) \ll \mathcal{B}(1)$. The parameters γ and t_a are referred to as the annealing parameter and the annealing time, respectively. The existence

of a map between any QUBO cost function and the problem Hamiltonian of the quantum Ising model [31] makes the D-Wave Advantage 4.1 quantum annealer a natural tool to solve the QUBOs constructed in the previous section.

A. Necessary runtime analysis

The runtime necessary to evaluate the optimum for the worst-case instance of an NP-complete optimization problem is believed to be exponentially large in problem size for both classical and quantum algorithms. Hypothetically however, it could be that quantum algorithms offer a modest (sub-exponential) speedup. It is also conceivable that quantum algorithms may offer systematically better performance on average, i.e. for typical instances [32]. In practice there are many factors affecting AQC runtime necessary to solve a QUBO problem e.g. associated with the circulant self-dual optimal stabilizer codes. We discuss the most important of them below, assuming for simplicity that the quantum device possesses sufficient connectivity to enable direct mapping of QUBO variables onto the physical qubits. The effects of limited connectivity, which are by far the most important, will be discussed in later sections.

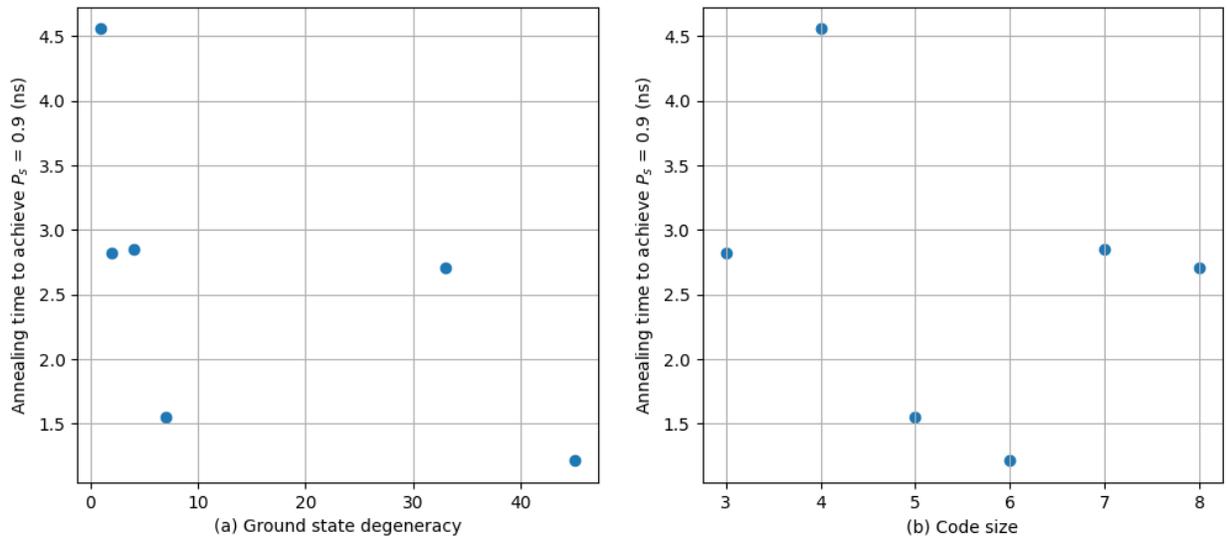


FIG. 1. a) Graph of annealing time required to achieve $P_s = 0.9$, calculated using the form of $\mathcal{A}(s)$ and $\mathcal{B}(s)$ utilized by the D-Wave Advantage 4.1 quantum annealer. For larger code sizes, the dimensions of the corresponding Ising Hamiltonian exceeded computational feasibility. b) Graph depicting the relationship between annealing time and ground-state degeneracy of the problem Hamiltonian.

It is well-known that the degeneracy of the ground state, even in absence of quantum noise and decoherence, may significantly influence necessary annealing time t_a . Given that circulant codes are cyclic, our problem Hamiltonian $H(\gamma = 1)$ always exhibits multiple ground states. Ground state degeneracy can both increase and decrease t_a , depending on the degeneracy of the excited states [27, 33, 34]. To address this question in our case we numerically simulated the time-dependent Schrodinger equation describing the annealing process for codes of sizes $n = 3 - 8$. In these simulations annealing time was varied until the success rate $P_s = 0.9$ was achieved. The results, shown in the panel a) of Fig. 1 are

consistent with the inverse correlation between the ground state degeneracy and minimal necessary t_a .

This conclusion is very preliminary, as the degeneracy of final Hamiltonian is merely one of the factors. Another important factor is the smallest value of gap of $H(\gamma)$, which may change significantly for large n (we checked that for codes of sizes $n = 3 - 7$ it does not). For the Hamiltonian employed, it can be demonstrated that \mathcal{E} scales polynomially with the problem size, as follows from the observation that the eigenvalues of H_I and H_P are also at most polynomial. Yet, understanding the scaling of g_{min} with n is usually a very difficult problem, see [32].

Our simulations shown in Fig. 1, panel b) suggest that theoretically required annealing times for the small code sizes $n \sim 3 - 8$ fall within the nanosecond range. In practice, as we discussed in the next section, reasonable necessary annealing time on the D-Wave Advantage 4.1 platform for these problems is of the order of microseconds. This suggests theoretically quantum annealer is operating firmly within the adiabatic regime, but in practice its quality of performance is dominated by other factors related to quantum noise. The factors discussed above are much less important, at least for moderate problem sizes n . This instructs our approach in the next section to optimize the annealing time t_a empirically.

V. RESULTS OF SIMULATIONS ON THE D-WAVE QA PLATFORM

A. Quantum-only implementation

We tested our QUBO-based algorithm by running implementations on the D-Wave Advantage 4.1 quantum annealer. We ran instances of optimal circulant self-dual codes of small length, taken from the database composed by M. Grassl [25]. The constraint $\mathbf{x} \neq 0^n$ was imposed by setting $x_1 = 1$ as described in section III C 2. To quantify how close the quantum annealer got to the optimal value, we calculated the so-called approximation ratio for various code sizes. The approximation ratio is the ratio between the smallest value generated by the quantum annealer and the actual minimum distance. As shown in Figure 2, the quantum annealer was able to successfully find exact minimum distance for codes of sizes up to 12. For larger code sizes, despite numerous re-runs, QA could only find conservative upper bounds. Our results strongly suggest that QA alone can not currently compete with purely classical algorithms/implementations, but there could be a room to use QA as a tool to obtain rigorous upper bounds. Efficiency of this method should be addressed separately by comparing the upper bounds with the theoretically known ones [21].

To systematically analyze the performance of quantum annealer, we study how the quality of outcome depends on the annealing time t_a , a free parameter of implementation of the quantum algorithm. At least naively, longer annealing time would make time evolution more adiabatic. But since the platform is noisy, a longer annealing time can lead to an accumulation of errors. Thus, we can expect that certain values of t_a would yield optimal performance.

To study t_a dependence, we plot the rate of success for the QUBO associated with the given code of length $n = 9$ using various annealing times. The rate of success is defined as the proportion of runs where the optimal result was obtained out of the total number of runs conducted. As shown in Figure 3, the rate of success initially increases as we increase the annealing time, reaching its peak at approximately $4.5 \mu s$. However, increasing the annealing time beyond this point resulted in deteriorating performance. Considering the results from

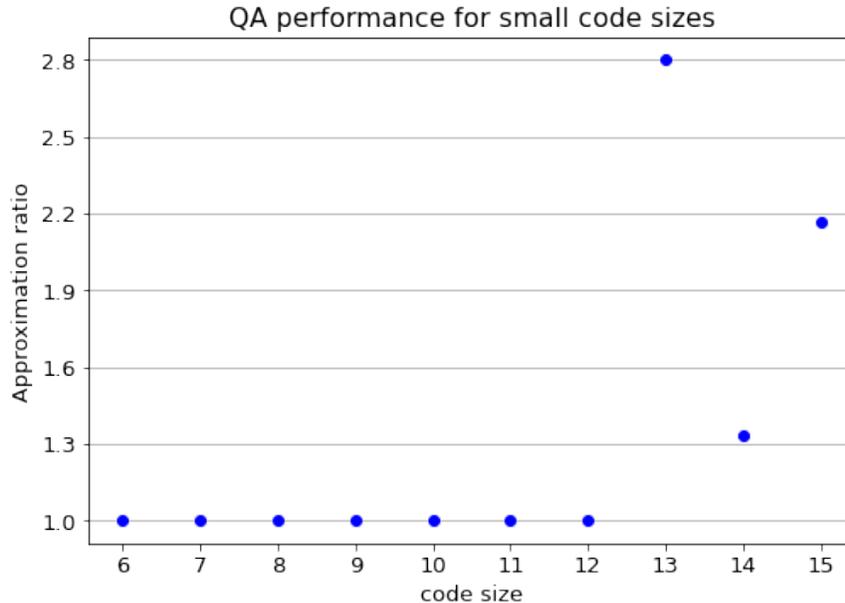


FIG. 2. Approximation Ratio (AR) of best output generated by runs of quantum annealer solving the QUBO associated with optimal circulant codes of length $6 \leq n \leq 15$. AR equals one means the algorithm found the global minimum (exact value of the minimum code distance). Larger AR means that despite numerous re-runs, the algorithm failed to find the global minimum and only produced a conservative bound. We performed 40 runs (one run is a 100 anneals) for each code size.

the previous section, which suggested that, at least theoretically, we are operating within the adiabatic regime, this deterioration in performance is likely due to accumulated noise. Empirically, we find that an annealing time of $t_a = 4.5 \mu s$ is close to optimum for most code sizes, and therefore, we use this value in all of our simulations.

Suboptimal performance of quantum annealer for relatively small code sizes $n \geq 13$ can likely be attributed to limited connectivity between physical qubits of the platform. If two physical qubits are connected, the platform can directly implement two-qubit gates. Thus, for best performance, any two binary variables in the cost function that share a cross-term should be mapped to two physically connected qubits. However, in many practical scenarios, each QUBO variable can share cross-terms with numerous other variables, exceeding the connectivity limitations of individual device qubits in the contemporary available platforms. A common workaround is to map a single QUBO variable to a chain of physical qubits, a process referred to as embedding [35]. This leads to an overhead in the number of qubits employed by the platform. Figure 4 shows that the number of necessary physical qubits increases approximately *quadratically* with the size of the original QUBO problem, associated with the code. The increase in the number of necessary qubits significantly complicates successful implementation of the QA algorithm. As the number of physical qubits grows, the corresponding search space also expands. Furthermore, since current quantum devices are noisy, each additional qubit introduces additional sources of noise that can affect the final result of the algorithm.

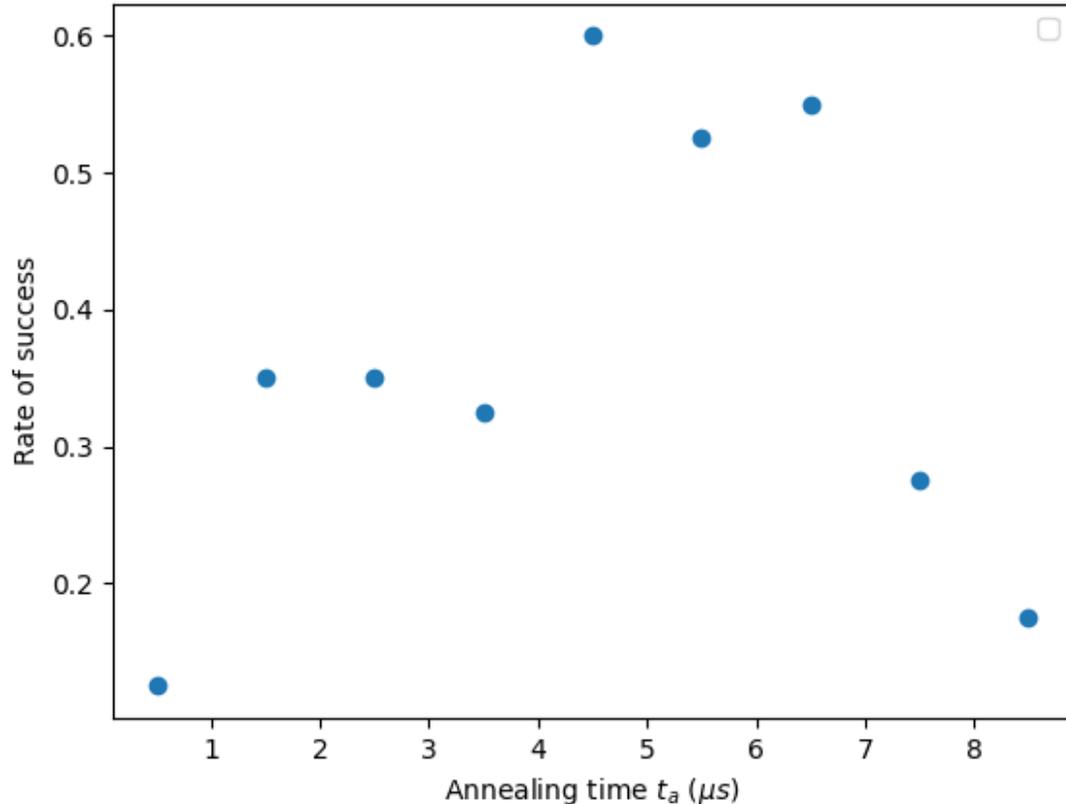


FIG. 3. The success rate of the QUBO-based algorithm run on D-Wave Advantage 4.1 as a function of different annealing times t_a . Each data point represents the success rate, calculated as the number of times the optimal value was achieved out of 40 runs (each comprising a 100 anneals). All runs were optimizing the same QUBO associated with an optimal circulant code of length $n = 9$. The annealing times ranged from 0.5 to $8.5\mu s$ in increments of $1\mu s$.

B. Hybrid implementation

To mitigate challenges of limited connectivity, we shift our focus to hybrid quantum-classical implementation. Specifically, we use Qbsolv algorithm, a hybrid algorithm developed by the D-Wave Systems. Qbsolv employs a classical algorithm to partition QUBO problems into smaller and more manageable pieces. These “subproblems” can be solved using QA, or any other suitable QUBO solving algorithm, before stitching them together [36]. This approach offers two significant benefits. First, it allows solving problems of bigger size, exceeding current limits of quantum platforms. Second, the subproblems have fewer connectivity constraints, which reduces the number of necessary physical qubits, and in this way decreases the source of errors.

To compare performance of hybrid approach against a fully classical one, we conducted runs solving subproblems using purely classical Simulated Annealing (SA) algorithm as well as QA using the D-Wave platform. In what follows we refer to these runs as “SA-assisted” and “QA-assisted.” To ensure a fair comparison, for each run we imposed the same overall

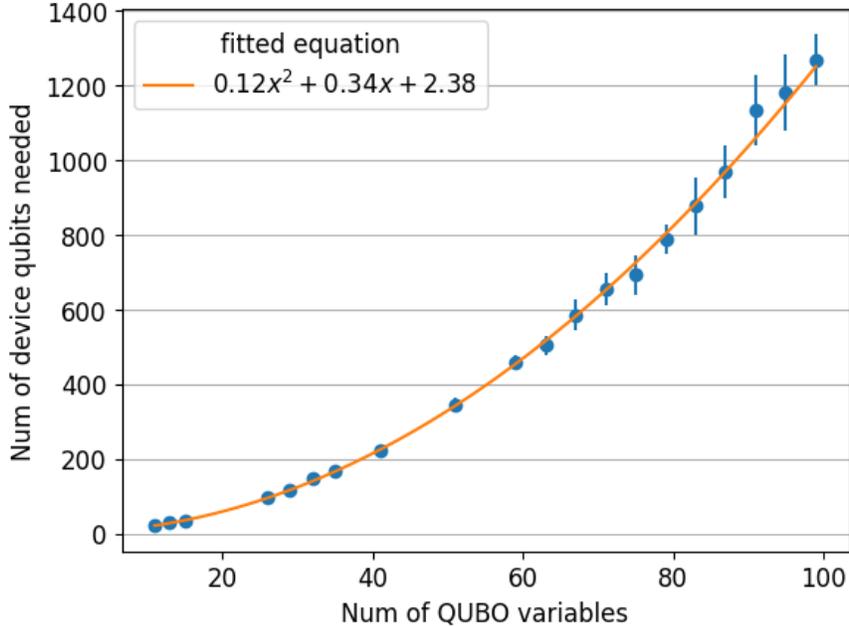


FIG. 4. Mean number of physical qubits employed by the QA platform (i.e. number of qubits after the so-called embedding) vs. the number of binary variables of the underlying QUBO problem. Standard deviation, due to heuristic nature of the embedding algorithm, is shown as the error bars. It is usually small. To generate the plot, we used optimal circulant codes of different sizes, reformulated the minimum distance problem as QUBO and then embed it onto the QA platform 10 times. We note, that QUBO associated with different codes can have different connectivity requirements, which can affect the number of qubits after embedding. The data points (blue) conform well to quadratic fit (orange), which suggests that on average the number of qubits after embedding grows quadratically.

time limit, which reflects the time-cost from the end-user point of view. However, it should be noted that the QA-assisted runs experience hidden latency of unknown duration, due to remote execution of QA on the D-Wave platform. In contrast, SA-assisted runs were executed locally and did not experience this issue. Therefore, although end-user time was the same, the QA implementation was effectively subject to a more stringent time constraint.

We performed a different number of runs for different codes, but each QUBO problem, associated with a given code, was minimized the same number of times using QA and then SA-assisted implementations. Since the subproblems are of the similar size to the problems discussed in the previous section, we use an annealing time of $t_a = 4.5 \mu s$ also for the hybrid quantum-assisted simulations.

Our results, shown in Figure 5, demonstrate a comparable performance between the two approaches. For codes of length up to $n = 20$, both approaches managed to find exact minimum distance d . For longer codes of length $n \lesssim 30$, both algorithms, with some exceptions, typically find the exact d , but as n increased, there were codes for which either implementation failed despite many repetitive runs. Of the 17 codes for which either of the implementations did not find the exact result, in approximately 58% of cases both algorithms yielded only conservative bounds. This may suggest that for certain codes, the resulting QUBO problems are inherently hard, regardless of which implementation is used. In certain cases, for codes of length $n = 21, 26, 40$ and 49 (24% of all cases when either

approach failed), the hybrid approach succeeded in finding the QUBO global minimum while the classical approach failed. On the contrary, the reverse scenario occurred only three times (18% of all cases), for codes of length $n = 29, 31$ and 38 .

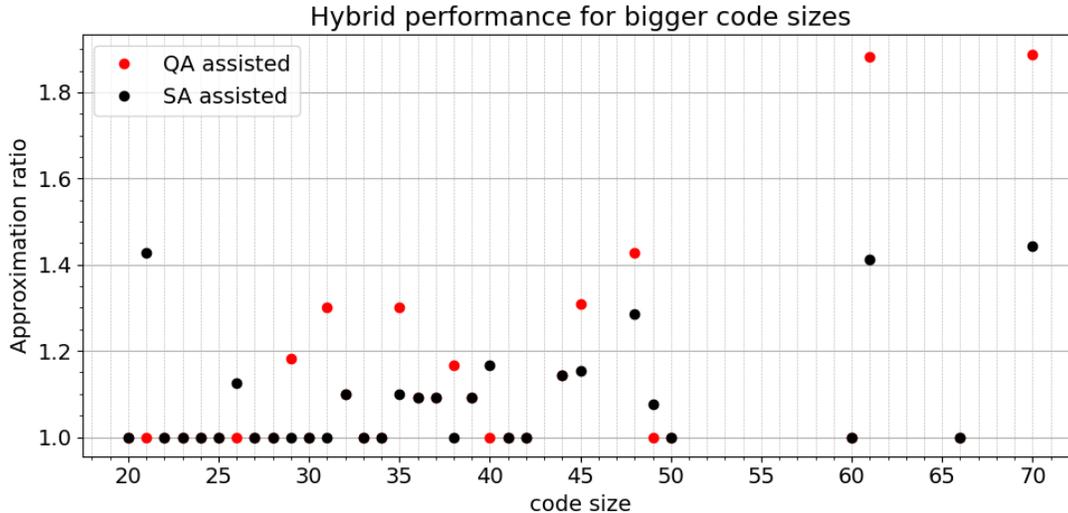


FIG. 5. Approximation Ratio (AR) for minimum distance d for codes of varying length, obtained with help of QUBO formulation, optimized using hybrid QA-assisted or purely classical SA-assisted approach. For each code length n , the number of QA and SA-assisted runs were the same (but different for different n). Due to limited computational resources, We bypassed certain code lengths in favor of larger values.

VI. DISCUSSION

In this work we introduced an approach to reformulate the problem of finding the minimum distance of a binary quantum stabilizer code as a QUBO (quadratic unconstrained binary optimization) problem. The underlying motivation behind this was to ultimately harness the capabilities of contemporary quantum platforms to help design better quantum error-correcting protocols. As a result, we reformulated a well-known NP-hard problem of coding theory in a way suitable for solving using quantum processors. We then investigated the practical advantages of this approach by running problem instances on the D-Wave Advantage 4.1 platform. Our QUBO reformulation requires $O(n \ln n)$ optimization variables, representing only a logarithmic multiplicative overhead.

The QUBO formulation of the minimum distance problem offers a conceptual and possibly practical advantage. First, QUBO is an active area of research. Therefore practical implementation of our QUBO-based approach can benefit from many existing heuristics and speedups. Second, the QUBO-based formulation can be solved using quantum, classical, or hybrid methods. While our study was focused on investigating practical advantage of solving QUBO using quantum annealing, one can envision in the future using coherent gate-based quantum devices and algorithms like the Quantum Approximate Optimization Algorithm (QAOA) [37] to tackle the minimum distance and similar problems.

In our simulations, we compared purely quantum, classical, and hybrid approaches to solving the QUBO associated with the minimal distance problem. Our focus was on optimal

self-dual codes of length n , i.e. self-dual codes with the largest possible minimal distance d for given n . Minimum distances for these codes with small or moderate n are known, providing a good testing ground. We found that the purely quantum approach fails to produce the optimum (only providing conservative values for d), already for $n \gtrsim 10$. Purely classical and quantum-classical hybrid approaches have shown much better and mutually comparable performance, yielding correct values of d for codes up to $n \sim 60$, see Figure 5. In the cases where the algorithms failed to produce the correct distance, they provided rigorous upper bounds. This suggests that for even larger n , when deterministic approaches become completely impractical, heuristics can be employed to solve these QUBOs to find rigorous upper bounds on the minimum distance of specific code instances. Whether these upper bounds can be made tighter than theoretical upper bounds established by linear programming techniques [21] remains an open question.

Overall, our simulations have demonstrated potential of hybrid approach, that combines quantum annealing with classical algorithm used to partition the problem into smaller pieces. Our study highlights that the limited connectivity of quantum devices, along with the associated embedding problem, represents a primary obstacle on the path to practical advantage of QA. As is shown in Figure 4, embedding the QUBOs onto the quantum platform introduces roughly a quadratic overhead in the number of variables. Due to intrinsic noise, the overhead in the number of qubits not only enlarges the search space but also leads to significantly more errors, which results in the suboptimal minimum.

Running simulations on the D-Wave Advantage 4.1 platform, we concluded that the QUBO-based approach currently does not offer a practical advantage over the classical deterministic minimum distance algorithm introduced by G. White and M. Grassl [4]. However, as the fault-tolerant quantum devices continue to develop, this situation might change. While classical deterministic algorithm would require the runtime exponentially growing with n , AQC-based approach may show better scaling. Understanding this scaling theoretically, and thus certifying possible quantum advantage of the QUBO-based approach, would be an important task for the future.

To conclude, we proposed a reformulation of a theoretically and practically important NP-hard problem as a QUBO problem, contributing to an ongoing effort to reformulate various problems in a form suitable for quantum platforms. Looking ahead, an important step would be to extend our approach to other problems of classical and quantum information theory and discrete mathematics. In particular, the shortest lattice vector problem, a known NP-hard problem, which is conceptually similar to code minimum distance problem discussed in this work, would be a natural next candidate.

VII. ACKNOWLEDGMENTS

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