SHARP $\ell^q(L^p)$ DECOUPLING FOR PARABOLOIDS

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ABSTRACT. In this short expository note, we prove the following result, which is a special case of the main theorem in [GOZZK23]. For each $n \geq 2$ and $p, q \in [2, \infty]$, we prove upper bounds of $\ell^q(L^p)$ decoupling constants for paraboloids in $\mathbb{R}^n$, as well as presenting extremisers for each case. Both are sharp up to ε-losses.

1. INTRODUCTION

Let $d \geq 2$ and $v_i \in \{-1, 1\}$ for $i = 1, \ldots, d-1$. Write $\xi = (\xi_1, \ldots, \xi_{d-1})$, $x' = (x_1, \ldots, x_{d-1})$ and $x = (x', x_d)$. Define the paraboloid

$$\phi(\xi) = \sum_{i=1}^{d-1} v_i \xi_i^2, \tag{1}$$

and its associated extension operator

$$Eg(x_1, \ldots, x_d) = \int_{[0,1]^{d-1}} g(\xi) e(x' \cdot \xi + x_d \phi(\xi)) d\xi. \tag{2}$$

For $\delta \in \mathbb{N}^{-1/2}$, let $\mathcal{P}(\delta)$ be a tiling of $[0,1]^{d-1}$ by cubes of side length $\delta^{1/2}$.

For $p, q > 0$ and each nonzero $g \in L^1([0,1]^{d-1})$, denote

$$R(g) = R_{p,q}(g) := \sup_{B:d\text{-cube},\ l(B)=\delta^{-1}} \frac{\|Eg\|_{L^p(B)}}{\|\|E(g1_Q)\|_{L^p(w_B)}\|_{l^q(Q \in \mathcal{P}(\delta))}}, \tag{3}$$

where w_B is a weight function adapted to B, such as (see [BD17b])

$$w_B(x) = \left(1 + \frac{|x - c_B|}{\delta^{-1}}\right)^{-100d},$$

where c_B is the centre of B. Let

$$D_{p,q}(\delta) = \sup\{R(g) : 0 \neq g \in L^1([0,1]^{d-1})\}. \tag{4}$$

Denote

$$p_d = \frac{2(d+1)}{d-1}, \quad N = \delta^{-\frac{d-1}{2}}. \tag{5}$$

In the case of an elliptic paraboloid, Bourgain and Demeter [BD15] proved

Theorem 1. *Let $v_i = 1$ for $i = 1, \dots, d-1$. Then for every $p, q \in [2, \infty]$ we have*

$$D_{p,2}(\delta) \lessapprox \max\{1, N^{\frac{1}{2}-\frac{p_d}{2p}}\}. \tag{6}$$

Here and throughout this article, $A(\delta) \lessapprox B(\delta)$ will always mean that for every $\varepsilon > 0$, there exists some C_ε such that $A(\delta) \leq C_\varepsilon \delta^{-\varepsilon} B(\delta)$ for every $\delta \in (0,1)$. We define $\gtrapprox$ and $\approx$ in a similar way.

For general $v_i = \pm 1$, Bourgain and Demeter [BD17a] also proved

Theorem 2. *Let $p \in [2, \infty]$. Then*

$$D_{p,p}(\delta) \lessapprox \max\{N^{\frac{1}{2}-\frac{1}{p}}, N^{1-\frac{1}{p}-\frac{p_d}{2p}}\}. \tag{7}$$

They also proved a version of $\ell^2(L^p)$ decoupling in the case of hyperbolic paraboloids:

Theorem 3. *Let $p \in [2, \infty]$. Let $d(v)$ denote the minimum number of positive and negative entries of v_i, $1 \leq i \leq d-1$, and denote*

$$p_v = \frac{2(d+1-d(v))}{d-1-d(v)}. \tag{8}$$

Then

$$D_{p,2}(\delta) \lessapprox \begin{cases} N^{\frac{d(v)}{d-1}\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{if} \quad 2 \leq p \leq p_v, \\ N^{\frac{1}{2}-\frac{p_d}{2p}}, & \text{if} \quad p_v \leq p \leq \infty. \end{cases} \tag{9}$$

Note that $p_d \leq p_v$, with equality if and only if $d(v) = 0$, that is, we are in the case of elliptic paraboloids.

1.1. Main results.

In the case of an elliptic paraboloid, we prove that

Theorem 4. *Let $v_i = 1$ for $i = 1, \dots, d-1$. Then for all $p, q \in [2, \infty]$, we have*

$$D_{p,q}(\delta) \approx \max\{N^{\frac{1}{2}-\frac{1}{q}}, N^{1-\frac{p_d}{2p}-\frac{1}{q}}\}. \tag{10}$$

More generally, for hyperbolic paraboloids, we also have

Theorem 5. *In the $(1/p, 1/q)$ interpolation diagram (see Figure 1), let*

$$A_1 = (0,0), \quad A_2 = \left(0, \frac{1}{2}\right), \quad A_3 = \left(\frac{1}{p_v}, \frac{1}{2}\right), \quad A_4 = \left(\frac{1}{p_d}, \frac{1}{p_d}\right)$$

$$A_5 = \left(\frac{1}{p_d}, 0\right), \quad A_6 = \left(\frac{1}{2}, 0\right), \quad A_7 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

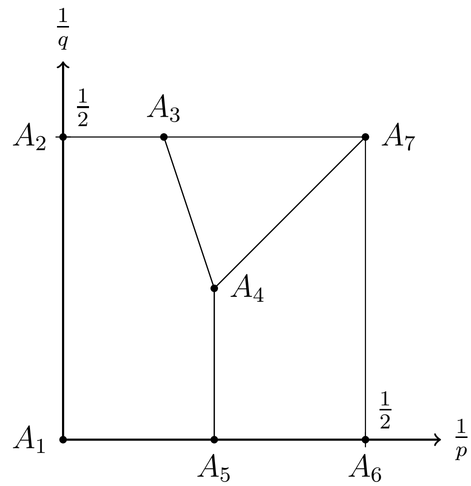

FIGURE 1. The interpolation diagram

Then the following holds:

(11)
$$D_{p,q}(\delta) \approx \begin{cases} N^{\frac{1}{2}-\frac{1}{q}}, & \text{if } (1/p,1/q) \text{ lies in trapezoid } A_4A_5A_6A_7, \\ N^{\frac{1}{2}-\frac{1}{q}+\frac{d(v)}{d-1}\left(\frac{1}{q}-\frac{1}{p}\right)}, & \text{if } (1/p,1/q) \text{ lies in triangle } A_3A_4A_7, \\ N^{1-\frac{p_d}{2p}-\frac{1}{q}}, & \text{if } (1/p,1/q) \text{ lies in pentagon } A_1A_2A_3A_4A_5. \end{cases}$$

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2. EXTREMISERS FOR ELLIPTIC PARABOLOIDS

2.1. **The constant test function.**

Theorem 6. *Let $g \equiv 1$. For every $p \in [2,\infty]$, $q \in (0,\infty]$ we have*

$$R(g) \gtrsim N^{1-\frac{p_d}{2p}-\frac{1}{q}}. \tag{12}$$

In particular, if $p \geq p_d$, then the constant function is an extremizer for $D_{p,q}(\delta)$ for elliptic paraboloids.

Proof. If $p < \infty$, we use the equivalent neighbourhood version of decoupling. For each Q, let f_Q be a Schwartz function such that $\widehat{f_Q}$ is essentially the normalised indicator function of the δ-neighbourhood of the graph of ϕ over Q. Let $f = \sum_Q f_Q$. Let B be centred at the origin and have side length δ^{-1}.

For $|x| \leq c \ll 1$, we have $|e(x' \cdot \xi + x_d\phi(\xi))| > 1/2$. Thus $|f(x)| \sim N$ over $[-c,c]^{d-1}$, and hence

$$\|f\|_{L^p(B)} \geq \|f\|_{L^p([-c,c]^{d-1})} \sim N. \tag{13}$$

On the other hand, we have $f_Q \sim 1_{T_Q}$ where T_Q is a rectangle with dimensions $(\delta^{-1/2})^{d-1} \times \delta^{-1}$. Thus

$$\|f_Q\|_{L^p(w_B)} \sim \delta^{-\frac{d-1}{2p}-\frac{1}{p}}, \tag{14}$$

which holds for every Q. Thus we have

$$R(g) \gtrsim \frac{N}{N^{\frac{1}{q}}\delta^{-\frac{d-1}{2p}-\frac{1}{p}}} = N^{1-\frac{1}{q}-\frac{1}{p}}\delta^{\frac{1}{p}} = N^{1-\frac{pd}{2p}-\frac{1}{q}}.$$

For $p = \infty$, it is easy to see that $\|E1\|_{L^\infty(B)} = |E1(0)| \sim 1$, and $\|E1_Q\|_{L^\infty(w_B)} = |E1_Q(0)| \sim |Q| = N^{-1}$. Thus we also have

$$R(g) \gtrsim \frac{1}{N^{\frac{1}{q}-1}} = N^{1-\frac{1}{q}}. \tag{15}$$

□

2.2. The exponential sum test function.

Lemma 7. *For $\delta \in \mathbb{N}^{-2}$ we define the function on $\mathbb{T}^2$ as*

$$f(x,y) = \sum_{j=1}^{\delta^{-1/2}} e\left(jx + j^2 y\right).$$

Then for all $2 \le p \le \infty$, we have

$$\|f\|_{L^p(\mathbb{T}^2)} \approx \begin{cases} \delta^{-\frac{1}{4}} & \text{if } 2 \le p \le 6 \\ \delta^{-\frac{1}{2}+\frac{3}{2p}} & \text{if } 6 \le p \le \infty \end{cases}.$$

Proof. We first prove the upper bound by testing

$$g(s) = \sum_{j=1}^{\delta^{-1/2}} \Delta_{j\delta^{1/2}}(s),$$

where Δ_a is the delta-mass at a. (A more rigorous argument is to take an approximation to the identity at each of the delta masses.) Then we have

$$Eg(x,y) = \sum_{j=1}^{\delta^{-1/2}} e\left(j\delta^{1/2}x + j^2\delta y\right) = f(\delta^{1/2}x, \delta y).$$

Also, for each j we have

$$Eg_j(x,y) = e\left(j\delta^{1/2}x + j^2\delta y\right).$$

Thus $\|Eg_j\|_{L^p(w_B)} \sim \delta^{-\frac{2}{p}}$, and so

$$\|\|Eg_j\|_{L^p(w_B)}\|_{l^2(j)} \sim \delta^{-\frac{1}{4}}\delta^{-\frac{2}{p}}.$$

But by periodicity, we have

$$\|Eg\|_{L^p(B)} = \delta^{-\frac{2}{p}}\|f\|_{L^p(\mathbb{T}^2)}.$$

The desired upper bound then follows from Theorem 1 with $d = 2$.

The lower bound follows from [BB10]. We provide a little more detail for clarity. The case $p = \infty$ is trivial, taking $x = y = 0$. The case $6 \le p < \infty$ follows by considering (x, y) near the origin; the detail can be found in Theorem 2.2 of [Yip20] (with $s = p/2$, and the proof there is easily seen to work for all real numbers $s > 0$.) The case $p = 2$ and $p = 4$ follows from the first (trivial) bound of Theorem 2.3 of [Yip20].

Thus, the only case remaining is the case when $2 < p < 6$ and $p \ne 4$. Assume $2 < p < 4$ first. Then $4 \in (p, 6)$ and using the log-convexity of L^p-norms, we have

$$\|f\|_{L^4(\mathbb{T}^2)} \le \|f\|^{1-\theta}_{L^p(\mathbb{T}^2)}\|f\|^{\theta}_{L^6(\mathbb{T}^2)},$$

where $\frac{1-\theta}{p} + \frac{\theta}{6} = \frac{1}{4}$. But since $\|f\|_{L^4(\mathbb{T}^2)} \sim \delta^{-1/4}$ and $\|f\|_{L^6(\mathbb{T}^2)} \sim \delta^{-1/4}$, we also have $\|f\|_{L^p(\mathbb{T}^2)} \gtrsim \delta^{-1/4}$. The case $4 < p < 6$ is similar. $\square$

Theorem 8. *Let $g = \sum_Q \Delta(c_Q, \phi(c_Q))$ be the sum of delta-masses at $(c_Q, \phi(c_Q))$. Then for every $p \in [2, \infty]$, $q \in (0, \infty]$ we have*

$$R(g) \gtrsim \max\{N^{\frac{1}{2}-\frac{1}{q}}, N^{1-\frac{3}{p}-\frac{1}{q}}\}. \tag{16}$$

As a result, if $2 \le p \le p_d$, then g is an extremizer for $D_{p,q}(\delta)$ for elliptic paraboloids.

Remark. For $d \ge 3$ and $p > p_d$, the bound $N^{1-\frac{3}{p}-\frac{1}{q}}$ does not agree with the sharp bound $N^{1-\frac{p_d}{2p}-\frac{1}{q}}$. Thus to get sharp decoupling, we will need the constant test function. Thus, only the first lower bound $N^{\frac{1}{2}-\frac{1}{q}}$ will be useful in our proof later. Nevertheless, if $d = 2$, $p_d = 6$, then g is an extremizer for every $p \ge 2$.

Proof. Let B be centred at the origin and have side length δ^{-1}. We have $|E(g1_Q)(x)| \equiv 1$, and thus

$$\|\|E(g1_Q)\|_{L^p(w_B)}\|_{l^q(Q\in\mathcal{P}(\delta))} \sim N^{\frac{1}{q}}|B|^{\frac{1}{p}} = N^{\frac{1}{q}+\frac{2d}{p(d-1)}}. \tag{17}$$

For the left hand side, we have

$$\begin{aligned} Eg(x) &= \sum_Q e(x'c_Q + x_d\phi(c_Q)) \\ &= \prod_{i=1}^{d-1}\sum_{j_i=1}^{\delta^{-1/2}} e(j_i\delta^{1/2}x_i + v_i j_i^2\delta x_d). \end{aligned}$$

For $p<\infty$,

$$\begin{aligned}
&\int_B |Eg(x)|^p dx\\
&=\int_{-\delta^{-1}}^{\delta^{-1}}\left(\prod_{i=1}^{d-1}\int_{-\delta^{-1}}^{\delta^{-1}}\left|\sum_{j_i=1}^{\delta^{-1/2}} e(j_i\delta^{1/2}x_i+v_ij_i^2\delta x_d)\right|^p dx_i\right)dx_d\\
&=\delta^{1+\frac{d-1}{2}}\int_{-1}^{1}\left(\prod_{i=1}^{d-1}\int_{-\delta^{-1/2}}^{\delta^{-1/2}}\left|\sum_{j_i=1}^{\delta^{-1/2}} e(j_ix_i+v_ij_i^2x_d)\right|^p dx_i\right)dx_d\\
&=\delta^{-d}\int_{-1}^{1}\left(\prod_{i=1}^{d-1}\int_{-1}^{1}\left|\sum_{j_i=1}^{\delta^{-1/2}} e(j_ix_i+v_ij_i^2x_d)\right|^p dx_i\right)dx_d, \qquad (18)
\end{aligned}$$

where the last line follows from periodicity. By taking complex conjugation and symmetry, it is easy to see that for each i we have

$$\int_{-1}^{1}\left|\sum_{j_i=1}^{\delta^{-1/2}} e(j_ix_i+v_ij_i^2x_d)\right|^p dx_i=\int_{-1}^{1}\left|\sum_{j=1}^{\delta^{-1/2}} e(jt+j^2x_d)\right|^p dt:=S_p(x_d).$$

Thus

$$\int_B |Eg(x)|^p dx=\delta^{-d}\int_{-1}^{1}S_p(x_d)^{d-1}dx_d. \qquad (19)$$

In particular, this computation shows that the test function g disregards the signs of v_i, and thus it should only extremize the case of elliptic paraboloids.

We then use Jensen's inequality:

$$\begin{aligned}
&\int_{-1}^{1}S_p(x_d)^{d-1}dx_d\\
&\gtrsim\left(\int_{-1}^{1}S_p(x_d)dx_d\right)^{d-1}\\
&\sim\max\{(\delta^{-\frac12})^{\frac{p}{2}(d-1)},(\delta^{-\frac12})^{(p-3)(d-1)}\}\\
&=\max\{N^{\frac{p}{2}},N^{p-3}\},
\end{aligned}$$

where the $\sim$ follows from Lemma 7. This gives

$$\|Eg\|_{L^p(B)}\gtrsim\delta^{-\frac{d}{p}}\max\{N^{\frac12},N^{1-\frac{3}{p}}\}. \qquad (20)$$

If $p=\infty$, it is also easy to see that $\|Eg\|_{L^\infty(B)}\ge|Eg(0)|\sim N$, which agrees with the expression above.

As a result, we have

$$R(g) \gtrsim \frac{\delta^{-\frac{d}{p}} \max\{N^{\frac{1}{2}}, N^{1-\frac{3}{p}}\}}{N^{\frac{1}{q}+\frac{2d}{p(d-1)}}} = \max\{N^{\frac{1}{2}-\frac{1}{q}}, N^{1-\frac{3}{p}-\frac{1}{q}}\}. \tag{21}$$

□

Remark. When $d = 2$, $p = 6$, a more refined number-theoretic argument (see [BB10] and Theorem 3.5 of [Yip20]) proves $\|Eg\|_{L^p(B)} \gtrsim N^{7/6}(\log N)^{1/6}$, implying $D_{6,2}(\delta) \gtrsim |\log\delta|^{1/6}$. So in general, we must have at least logarithmic losses in the upper bound. In fact, in [GMW24], the authors proved that when $d = 2$, $p = 6$, we can have $D_{6,2}(\delta) \lesssim |\log\delta|^{O(1)}$.

2.3. Proof of Theorem 4.

Proof. The upper bound follows from Theorem 1 and Hölder's inequality. The lower bound follows from Theorems 6 and 8. □

3. EXTREMISERS FOR HYPERPOLIC PARABOLOIDS

We are now going to prove Theorem 5. By renaming the variables, we may let

$$\phi(\xi) = \xi_1^2 + \cdots + \xi_{d(v)}^2 - \xi_{d(v)+1}^2 - \cdots - \xi_{d-1}^2, \tag{22}$$

where we recall $d(v) \leq (d-1)/2$ is the minimum number of positive and negative entries of v_i, $1 \leq i \leq d-1$.

First, if $p \geq p_v$, then since $p_v \geq p_d$, by Theorem 3 with Hölder's inequality and Theorem 6, our claim follows. Thus it suffices to consider $2 \leq p \leq p_v$.

3.1. The hyperplane test function.

Theorem 9. *Let $p, q \in [2,\infty]$ and h be an extremizer for the decoupling for the elliptic paraboloid in dimension $d-1-2d(v)$ (set $h \equiv 1$ if $d-1-2d(v) = 0$). Define $g(\xi) = h(\xi_{2d(v)+1}, \cdots, \xi_{d-1}) \prod_{j=1}^{d(v)} \Delta(\xi_j - \xi_{d(v)+j})$. Then*

$$R(g) \gtrsim N^{\frac{1}{2}-\frac{1}{q}+\frac{d(v)}{d-1}(\frac{1}{p}-\frac{1}{q})}. \tag{23}$$

Proof. Let B be centred at the origin and have side length δ^{-1}. By the separability of Eg, we can easily compute that

(24)

$$Eg(x) = \tilde{E}h(x_{2d(v)+1}, \ldots, x_{d-1}, x_d) \prod_{k=1}^{d(v)} \left(\sum_{j_k=1}^{\delta^{-1/2}} e(\delta^{\frac{1}{2}} j_k (x_k + x_{k+d(v)})) \right),$$

where $\tilde{E}$ is the extension operator for the elliptic paraboloid over the unit cube in $d-1-2d(v)$ dimensions. We first compute

$$\begin{aligned}
&\int_{[-\delta^{-1},\delta^{-1}]^{2d(v)}} \prod_{k=1}^{d(v)} \left|\sum_{j_k=1}^{\delta^{-1/2}} e(\delta^{\frac12} j_k(x_k+x_{k+d(v)}))\right|^p dx_1\cdots dx_{2d(v)}\\
&=\left(\int_{[-\delta^{-1},\delta^{-1}]^2} \left|\sum_{j=1}^{\delta^{-1/2}} e(\delta^{\frac12} j(x+y))\right|^p dxdy\right)^{d(v)}\\
&\sim \left(\int_{-\delta^{-1}}^{\delta^{-1}} \delta^{-1}\left|\sum_{j=1}^{\delta^{-1/2}} e(\delta^{\frac12} jz)\right|^p dz\right)^{d(v)}\\
&\sim \delta^{-\frac{(p+3)d(v)}{2}}.
\end{aligned}$$

Thus

$$\|Eg\|_{L^p(B)} \sim \delta^{-\frac{(p+3)d(v)}{2p}} \|\tilde{E}h\|_{L^p(\tilde{B})}, \tag{25}$$

where $\tilde{B}\subseteq \mathbb{R}^{d-2d(v)}$ is the cube centred at 0 and has side length δ^{-1}.

Similarly, given each $Q:=\prod_{l=1}^{d-1} I_l \subseteq [0,1]^{d-1}$ of side length $\delta^{1/2}$. Let $\tilde{Q}=\prod_{l=2d(v)+1}^{d-1} I_l$, we see $E(g1_Q)(x)$ is nonzero only if and for each $1\le l\le d(v)$ we have $I_l=I_{l+d(v)}$. Thus

$$|E(g1_Q)(x)| = |\tilde{E}_{\tilde{Q}} h(x_{2d(v)+1},\dots,x_{d-1},x_d)| \prod_{l=1}^{d(v)} 1_{I_l=I_{l+d(v)}}. \tag{26}$$

For such Q we have

$$\|E(g1_Q)\|_{L^p(B)} \sim \|\tilde{E}_{\tilde{Q}}h\|_{L^p(\tilde{B})}\delta^{-\frac{2d(v)}{p}}. \tag{27}$$

Thus

$$\|\|E(g1_Q)\|_{L^p(B)}\|_{l^q(Q)} \sim \|\|\tilde{E}_{\tilde{Q}}h\|_{L^p(\tilde{B})}\|_{l^q(\tilde{Q})}\delta^{-\frac{2d(v)}{p}}\delta^{-\frac{d(v)}{2q}}. \tag{28}$$

For $\tilde{E}h$, the critical exponent this time becomes $p_{d-2d(v)}$ instead of p_d. Since $p\le p_v\le p_{d-2d(v)}$, we use Theorem 8. Since h is defined to be an extremizer, we have

$$\|\tilde{E}h\|_{L^p(\tilde{B})} \sim (\delta^{-\frac{d-1-2d(v)}{2}})^{\frac12-\frac1q}\|\|\tilde{E}_{\tilde{Q}}h\|_{L^p(\tilde{B})}\|_{l^q(\tilde{Q})} \tag{29}$$

This implies that

$$\begin{aligned}
R(g) &\gtrsim \delta^{-\frac{(p+3)d(v)}{2p}}\delta^{\frac{2d(v)}{p}}\delta^{\frac{d(v)}{2q}}(\delta^{-\frac{d-1-2d(v)}{2}})^{\frac12-\frac1q}\\
&= N^{\frac12-\frac1q}N^{\frac{d(v)}{d-1}(\frac1q-\frac1p)}.
\end{aligned}$$

$\square$

3.2. **The case $p_d \le p \le p_v$.** Combining the sharp lower bounds in Theorems 6 and 9, we get

$$D_{p,q}(\delta) \gtrsim \max\{N^{1-\frac{p_d}{2p}-\frac{1}{q}}, N^{\frac{1}{2}-\frac{1}{q}} N^{\frac{d(v)}{d-1}(\frac{1}{q}-\frac{1}{p})}\}. \tag{30}$$

We need to determine which one is larger. Setting

$$1 - \frac{p_d}{2p} - \frac{1}{q} = \frac{1}{2} - \frac{1}{q} + \frac{d(v)}{d-1}\left(\frac{1}{q} - \frac{1}{p}\right), \tag{31}$$

that is,

$$\frac{1}{2} - \frac{p_d}{2p} = \frac{d(v)}{d-1}\left(\frac{1}{q} - \frac{1}{p}\right), \tag{32}$$

we see this corresponds to the critical line l_1 that passes through the points $(1/p_v, 1/2)$ and $(1/p_d, 1/p_d)$ in the $(1/p, 1/q)$ interpolation diagram.

- If $(1/p, 1/q)$ lies above l_1, then we have

$$D_{p,q}(\delta) \gtrsim N^{\frac{1}{2}-\frac{1}{q}+\frac{d(v)}{d-1}(\frac{1}{p}-\frac{1}{q})}. \tag{33}$$

  We claim that $\lessapprox$ also holds in this case. By interpolation, it suffices to prove upper bounds at the following three points:

$$\left(\frac{1}{p_v}, \frac{1}{2}\right), \quad \left(\frac{1}{p_d}, \frac{1}{p_d}\right), \quad \left(\frac{1}{p_d}, \frac{1}{2}\right). \tag{34}$$

  The point $(\frac{1}{p_v}, \frac{1}{2})$ has been settled by the case $p \ge p_v$, since we can check by direct computation that

$$\frac{d(v)}{d-1}\left(\frac{1}{2} - \frac{1}{p_v}\right) = \frac{1}{2} - \frac{p_d}{p_v}. \tag{35}$$

  The upper bound at $(\frac{1}{p_d}, \frac{1}{p_d})$ follows from Theorem 2 when $p = p_d$. The upper bound at $(\frac{1}{p_d}, \frac{1}{2})$ follows from Theorem 3 when $p = p_d$.
- If $(1/p, 1/q)$ lies below l_1, then we have

$$D_{p,q}(\delta) \gtrsim N^{1-\frac{p_d}{2p}-\frac{1}{q}}. \tag{36}$$

  We have $\lessapprox$ also holds in this case, by the upper bounds at $(\frac{1}{p_v}, \frac{1}{2})$ and $(\frac{1}{p_d}, \frac{1}{p_d})$, together with Hölder's inequality.

3.3. **The case $2 \le p \le p_d$.** Combining the sharp lower bounds in Theorems 8 and 9, we get

$$D_{p,q}(\delta) \gtrapprox \max\{N^{\frac{1}{2}-\frac{1}{q}}, N^{\frac{1}{2}-\frac{1}{q}} N^{\frac{d(v)}{d-1}(\frac{1}{q}-\frac{1}{p})}\}. \tag{37}$$

It is easy to see that the critical line l_2 in this case is given by $1/p = 1/q$.

- If $1/p \le 1/q$, then we have

$$D_{p,q}(\delta) \gtrapprox N^{\frac{1}{2}-\frac{1}{q}+\frac{d(v)}{d-1}(\frac{1}{p}-\frac{1}{q})}. \tag{38}$$

  We have $\lessapprox$ also holds in this case, which follows immediately by interpolating the upper bounds we already obtained at $(\frac{1}{p_d}, \frac{1}{p_d})$ and $(\frac{1}{p_d}, \frac{1}{2})$ with $(\frac{1}{2}, \frac{1}{2})$ given by Plancherel.
- If $1/p \ge 1/q$, then we have

$$D_{p,q}(\delta) \gtrapprox N^{\frac{1}{2}-\frac{1}{q}}. \tag{39}$$

  We have $\lessapprox$ also holds in this case, by the upper bound at $(\frac{1}{p_d}, \frac{1}{p_d})$ and Plancherel at $(\frac{1}{2}, \frac{1}{2})$, together with Hölder's inequality.

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