Two-level Adiabatic Transition Probability for Small Avoided Crossings Generated by Tangential Intersections

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Abstract

In this paper, the asymptotic behaviors of the transition probability for two-level avoided crossings are studied under the limit where two parameters (adiabatic parameter and energy gap parameter) tend to zero. This is a continuation of our previous works where avoided crossings are generated by tangential intersections and obey a non-adiabatic regime. The main results elucidate not only the asymptotic expansion of transition probability but also a quantum interference caused by several avoided crossings and a coexistence of two-parameter regimes arising from different vanishing orders.

1 Introduction

In quantum mechanics, especially in the quantum chemistry, the adiabatic approximation and the Born-Oppenheimer approximation are widely used. The adiabatic theorem, the motivation of these approximations, asserts that in the slowly varying Hamiltonian the quantum effect like the transition between the energy-levels hardly occurs. From this point of view, it is important to accurately describe how much slowing down the variation shrinks the transition probability.

In this paper, we study a mathematical model such that the transition probability is not always small even in case of the adiabatic approximation. Since the transition probability intuitively depends on the size of the smallest gap between energy-levels, the approaching (resp. receding) speed to (resp. from) the smallest gap, and the quantum interference, we consider asymptotic behavior in a two-parameter singular limit $h, \varepsilon \to +0$ of solutions to the time-dependent Schrödinger equation

$$ih\frac{d}{dt}\psi(t) = H(t;\varepsilon)\psi(t), \quad t \in \mathbb{R}. \tag{1.1}$$

Here, the Hamiltonian $H(t;\varepsilon)$ is given as a 2×2 matrix-valued function

$$H(t;\varepsilon) := \begin{pmatrix} V(t) & \varepsilon \\ \varepsilon & -V(t) \end{pmatrix}, \tag{1.2}$$

where $V(t)$ is a real-valued smooth function and h, ε are small positive parameters. Its two eigenvalues are

$$E_\pm(t;\varepsilon) = \pm\sqrt{V(t)^2 + \varepsilon^2}.$$

In this model, the ratio t/h is interpreted as the time variable, $E_\pm(t;\varepsilon)$ are the two energy-levels of $H(t;\varepsilon)$, and the adiabatic limit $h \to 0$ corresponds to the slow variation of the Hamiltonian $H(t;\varepsilon)$ compared with the time.

According to the adiabatic theorem, one expects that for a solution $\psi(t)$ to (1.1), the projection $\Pi_-(t;\varepsilon)\psi(t)$ onto the eigenspace associated with $E_-(t;\varepsilon)$ is "small" for every $t \in \mathbb{R}$ if $\psi(t_0)$ belongs to the eigenspace associated with $E_+(t_0;\varepsilon)$ at some $t_0 \in \mathbb{R}$. More simply, we can say that the adiabatic theorem asserts the smallness of the transition probability. Here, we call

$$P(\varepsilon, h) := \lim_{t\to+\infty} \left\| \Pi_-(t;\varepsilon) J_\ell^+(t) \right\|_{\mathbb{C}^2}^2 \tag{1.3}$$

the transition probability, where J_ℓ^+ is the normalized solution to (1.1) such that $\lim_{t\to-\infty} \left\|\Pi_-(t;\varepsilon)J_\ell^+(t)\right\|_{\mathbb{C}^2} = 0$ with $\left\|J_\ell^+(t)\right\|_{\mathbb{C}^2} = 1$ (this solution will be introduced in Appendix A.1). These limits exist under suitable conditions on V near infinity (Condition A in this paper). Note that $\|\psi(t)\|_{\mathbb{C}^2}^2 = \sum_\pm \|\Pi_\pm(t;\varepsilon)\psi(t)\|_{\mathbb{C}^2}^2$ is constant in t for any solution ψ.

As long as $\varepsilon > 0$, the two energy-levels $E_\pm(t;\varepsilon)$ are smooth functions of t, and never intersect with each other:

$$\inf_{t\in\mathbb{R}} |E_+(t;\varepsilon) - E_-(t;\varepsilon)| = \inf_{t\in\mathbb{R}} 2\sqrt{V(t)^2 + \varepsilon^2} \geq 2\varepsilon > 0. \tag{1.4}$$

This quantity called the energy-gap is bounded from below by 2ε even if V vanishes at some point. This phenomenon occurring near each zero of V is called an avoided crossing. The simplest case $V(t) = vt$ with a positive constant v is investigated individually by L.D. Landau and C. Zener in 30's [16, 23]. The transition probability

$$P(\varepsilon, h) = \exp\left(-\frac{\pi\varepsilon^2}{vh}\right) \tag{1.5}$$

for this case is known as the Landau-Zener formula. This is exact and true for any positive ε, h. For fixed $\varepsilon > 0$, this formula implies that the transition probability is exponentially small with respect to $h > 0$. There are many results generalizing the Landau-Zener formula. Under some analyticity condition, such an exponential decay estimate is obtained even in case of more general Hamiltonian, for example operator-valued unbounded Hamiltonians as in [4, 5, 12, 13, 17], while a smoothness condition without an analyticity yields nothing but a polynomial decay with respect to h as in [15]. Note that in the general setting, the condition of the energy-gap is replaced with the gap condition, which mandates that the spectrum is decomposed into a disjoint union of two subsets and that the distance between them is positive. The history of these generalizations can be consulted in the survey [9] and in the books [8, 19].

The transition probability may become larger when the energy-gap is also small. In our model, this situation occurs if $V(t)$ vanishes at some t and if ε (see (1.4)) is sufficiently small compared with h. One observes from Landau-Zener formula (1.5) that the transition probability is small and the adiabatic approximation is reasonable if $\varepsilon \gg h^{1/2}$. However, one also observe that it is almost one if $\varepsilon \ll h^{1/2}$. The former situation is called the adiabatic regime, and the latter the non-adiabatic regime, which was discussed in [22] and also in [3, 18].

The leading term of the transition probability is given by the same formula by replacing v with $|V'(0)|$ when $V(t)$ vanishes only at $t = 0$ and $V'(0) \neq 0$, namely, the situation that $V(t)$ and $-V(t)$ intersect transversely at $t = 0$ (see [12] and also its microlocal version [3]). From the viewpoint of the energy-levels, the approaching/receding $|E_+(t;\varepsilon) - E_-(t;\varepsilon)| - 2\varepsilon = 2(\sqrt{V(t)^2 + \varepsilon^2} - \varepsilon)$ near a transversal crossing of $\pm V$ is of order $|t|$.

In the tangential case $V'(0) = 0$, the transition probability is studied by one of the authors under the condition $\varepsilon \gg h^{m/(m+1)}$ corresponding to the adiabatic regime, where m stands for the vanishing order of V at $t = 0$ as in [20, 21] (equivalently, $|E_+(t;\varepsilon) - E_-(t;\varepsilon)| - 2\varepsilon$ is of order $|t|^m$). In this case, transition probability is exponentially small as $h\varepsilon^{-(m+1)/m}$ tends to 0. The analyticity of V and the adiabatic regime condition are necessary for applying the exact WKB method. In fact, the "complex crossing points" of the energy-levels, which are the zeros of $E_+(t;\varepsilon) - E_-(t;\varepsilon)$ on the complex plane and are called turning points in the WKB method, are essential for this case. The adiabatic regime condition implies that these complex crossing points are not too close to each other.

On the other hand, the situation corresponding to the non-adiabatic regime $\varepsilon \ll h^{m/(m+1)}$ is studied by the other author [10]. He applied other classical method (also used in [2]) to a little bit more general setting. The transition probability is almost one as in the Landau-Zener formula only when m is odd, and that it is still small of order $\varepsilon h^{-m/(m+1)}$ when m is even.

One of other generalizations is the existence of several avoided crossings. Following the classical probability theory, one may think that the transition probability is obtained by multiplying and summing the non-negative "local transition probability" around each avoided crossing. However, as well as other quantum situations, only a complex-valued probability amplitude is associated with each avoided crossing. Then the "total" probability amplitude is given by multiplying and summing them, and the transition probability is its absolute square. This phenomenon is treated in [14, 21, 22].

This paper is a continuation of the authors' previous works in the viewpoint of dealing with several avoided crossings generated by tangential intersections with different vanishing orders in the non-adiabatic regime. Our first result, Theorem 1, concerns several tangential avoided crossings in the non-adiabatic regime, that is, $\varepsilon \ll h^{m/(m+1)}$ with m the maximum among the avoided crossings. It shows that the transition probability is almost one when the number of odd avoided crossings is odd and that it is small of order $\varepsilon h^{-m/(m+1)}$ when the number is even. The effect of the quantum interference appears in the coefficient of the term of order $\varepsilon h^{-m/(m+1)}$. In Formula (2.8), the second term describes the quantum interference while the first term is given by the sum of absolute square of the local transition probability amplitudes. In particular, this coefficient vanishes in some cases. We also show some concrete models (see Remark 2.4 and Examples 2.5 and 2.7).

One notices that the border of the parameter regimes for each avoided crossing depends on the vanishing order m. Consequently, there are parameter regimes which is adiabatic for some avoided crossings and non-adiabatic for the others when there are several tangential intersections of $V(t)$ and $-V(t)$. Our second result , Theorem 2, concerns this situation, and shows that the leading term of the transition probability depends on the parity of the number of odd avoided crossings in the non-adiabatic regime. Since the local probability amplitude around an avoided crossing in the non-adiabatic and adiabatic regime has already been computed in Theorem 1 and in [21], Theorem 2 is obtained by combining them. The novelties of this paper are to examine precisely the transition probability in the intermediate regime, where the non-adiabatic regime and the adiabatic one coexist, and to elucidate a possibility of "switching of the transition probability" by varying two parameters ε, h continuously without changing $V(t)$ as in Example 2.11. Note that the situation neither adiabatic nor non-adiabatic regime, namely, $\varepsilon \sim h^{m/(m+1)}$ for some $m \geq 2$, has not been treated yet, although the case for $m = 1$ was treated by [7].

Our proof is based on the classical method. We first introduce the Jost solutions $J_\ell^\pm = J_\ell^\pm(t; \varepsilon, h)$ and $J_r^\pm = J_r^\pm(t; \varepsilon, h)$ admitting the asymptotic behavior (2.1) at infinity, and in particular, J_ℓ^+ satisfies (1.3) (see Appendix A.1 for the construction). Then the total transition probability amplitude and the transition probability are $s_{21}(\varepsilon, h)$ and the square of its modulus, where $s_{21}(\varepsilon, h)$ stands for the $(2,1)$-entry of the scattering matrix $S(\varepsilon, h)$ defined by

$$(J_\ell^+(t; \varepsilon, h), J_\ell^-(t; \varepsilon, h)) = (J_r^+(t; \varepsilon, h), J_r^-(t; \varepsilon, h))S(\varepsilon, h).$$

Note that one has

$$\Pi_-(t; \varepsilon)J_\ell^+(t; \varepsilon, h) - s_{21}(\varepsilon, h)J_r^-(t; \varepsilon, h) \to 0 \quad \text{as } t \to +\infty.$$

To study the entries of $S(\varepsilon, h)$, we continue the solutions $J_\ell^\pm$ from $-\infty$ to $+\infty$. More precisely, we construct solutions which approximately belong to the eigenspece associated with $E_\pm(t; \varepsilon)$ away from any avoided crossings, and compute the transfer matrices between the bases consisting of such solutions. The transfer matrix is almost diagonal when there is no avoided crossing between two points. Thus, the transfer matrix T_k across each avoided crossing near t_k is crucial to obtain the transition probability. The four entries of T_k are the probability amplitudes of the local transition at the vanishing point t_k.

The asymptotic behavior of T_k around each avoided crossing near t_k is given in Theorem 3. As we mentioned above, the exact WKB solutions used in the previous work [21] concerning avoided crossings generated by tangential intersection are no longer valid in the non-adiabatic regime. The solutions are constructed in Section 3 by the method of successive approximations (MSA for short) due to [2, 10]. For example, the $(1,2)$ and $(2,1)$-entries of $T(\varepsilon,h)$ correspond to the local transition probability amplitude from E_+ to E_- and from E_- to E_+ when the vanishing order m is odd and $V(t)(t-t_k)\geq 0$ near t_k. The leading term of them is given by applying the degenerate stationary phase method (Lemma 3.2) to the oscillatory integral (4.7), where the derivative of the phase function $\mp 2\int_0^t V(r)dr$ off-course has a zero of the same order as V.

This paper is organized as follows. In Section 2, we make precise the definitions and settings, and state our main results Theorems 1 and 2. We construct the solutions by the method of successive approximations (MSA) in Section 3, and prove the connection formulas Theorem 3 and Proposition 4.2 by using these solutions in Section 4. Finally, we will complete the proofs in Section 5. To obtain the product of $2n+1$ matrices of SU(2), we employ an algebraic formula shown in Appendix A.2.

2 Results

2.1 Assumptions and main result

As mentioned in the introduction, we focus on the non-adiabatic regime and work under the C^∞-category without any assumption on the analyticity. We notice that the assumption on $V(t)$ and the setting of the problem are sightly different from the previous work [22] but the definitions of the transition probability in the series of our works are the same. We first assume the following:

Condition A. *The function $V(t)\in C^\infty(\mathbb{R};\mathbb{R})$ has a limit $V_r\in\mathbb{R}\setminus\{0\}$ (resp. $V_\ell\in\mathbb{R}\setminus\{0\}$) as $t\to+\infty$ (resp. $-\infty$), and satisfies*

$$V-V_r\in L^1([0,+\infty)),\quad V-V_\ell\in L^1((-\infty,0]),\quad V'\in L^1(\mathbb{R}).$$

For simplicity, we assume $V_r>0$. Based on the argument in Appendix A.1 under Condition A, one sees the unique existence of Jost solutions $J_\bullet^\pm(t)$ ($\bullet\in\{\ell,r\}$) which satisfy the asymptotic conditions:

$$\begin{aligned}
J_r^+(t)&\sim\exp\left[-\frac{it}{h}\sqrt{V_r^2+\varepsilon^2}\right]\begin{pmatrix}\cos\theta_r\\ \sin\theta_r\end{pmatrix}\quad &&\text{as } t\to+\infty,\\
J_r^-(t)&\sim\exp\left[+\frac{it}{h}\sqrt{V_r^2+\varepsilon^2}\right]\begin{pmatrix}-\sin\theta_r\\ \cos\theta_r\end{pmatrix}\quad &&\text{as } t\to+\infty,\\
J_\ell^+(t)&\sim\exp\left[-\frac{it}{h}\sqrt{V_\ell^2+\varepsilon^2}\right]\begin{pmatrix}\cos\theta_\ell\\ \sin\theta_\ell\end{pmatrix}\quad &&\text{as } t\to-\infty,\\
J_\ell^-(t)&\sim\exp\left[+\frac{it}{h}\sqrt{V_\ell^2+\varepsilon^2}\right]\begin{pmatrix}-\sin\theta_\ell\\ \cos\theta_\ell\end{pmatrix}\quad &&\text{as } t\to-\infty,
\end{aligned}\tag{2.1}$$

where $\tan 2\theta_\bullet=\varepsilon/V_\bullet$ with $0<\theta_\bullet<\pi/2$ (equivalently determined by $\theta_\bullet=\arctan(\varepsilon^{-1}(\sqrt{V_\bullet^2+\varepsilon^2}-V_\bullet))$). The pairs (J_r^+,J_r^-) and (J_ℓ^+,J_ℓ^-) form bases of the solution space. Each of them corresponds to one of the eigenvalues $\pm\sqrt{V_r^2+\varepsilon^2}$ and $\pm\sqrt{V_\ell^2+\varepsilon^2}$ of $H(t,\varepsilon)$ at the infinity. Note that a function $\psi = {}^t(\psi_1,\psi_2)$ is a solution to (1.1) if and only if ${}^t(-\overline{\psi_2},\overline{\psi_1})$ is so. This implies that $(J_r^+(t), J_r^-(t))$ and $(J_\ell^+(t), J_\ell^-(t))$ are orthonormal bases on $\mathbb{C}^2$ at each $t\in\mathbb{R}$. Then we can introduce the scattering matrix $S(\varepsilon,h)$ as the change of basis between the pairs of Jost solutions:

$$\left(J_\ell^+, J_\ell^-\right) = \left(J_r^+, J_r^-\right) S(\varepsilon,h), \quad S(\varepsilon,h) = \begin{pmatrix} s_{11}(\varepsilon,h) & s_{12}(\varepsilon,h) \\ s_{21}(\varepsilon,h) & s_{22}(\varepsilon,h) \end{pmatrix}. \tag{2.2}$$

This matrix is unitary. In particular, one has $|s_{11}| = |s_{22}|$, $|s_{12}| = |s_{21}|$, and $|s_{11}|^2 + |s_{21}|^2 = 1$.

Definition 2.1. *The transition probability $P(\varepsilon,h)$ is defined by*

$$P(\varepsilon,h) := |s_{21}(\varepsilon,h)|^2.$$

Remark 2.2. *The above definition of the transition probability is equivalent to* (1.3)*. In fact, one has $\|J_\bullet^\pm(t)\|_{\mathbb{C}^2} = 1$ for any t, and*

$$\begin{aligned} &\lim_{t\to-\infty}\left\|\Pi_\pm J_\ell^\pm(t)\right\|_{\mathbb{C}^2} = 1, &&\lim_{t\to-\infty}\left\|\Pi_\mp J_\ell^\pm(t)\right\|_{\mathbb{C}^2} = 0,\\ &\lim_{t\to+\infty}\left\|\Pi_\pm J_r^\pm(t)\right\|_{\mathbb{C}^2} = 1, &&\lim_{t\to+\infty}\left\|\Pi_\mp J_r^\pm(t)\right\|_{\mathbb{C}^2} = 0. \end{aligned}$$

Condition B. *The function $V(t)$ has a finite number of zeros $t_1 > \cdots > t_n$ on $\mathbb{R}$, where each zero t_k for $k = 1,\dots n$ is of finite order denoted by m_k.*

This assumption implies that for $k = 1,\dots,n$,

$$V^{(l)}(t_k) = 0 \quad (1 \le l < m_k), \quad v_k := V^{(m_k)}(t_k) \neq 0. \tag{2.3}$$

Let m_* denote the maximal order of the zeros:

$$m_* = \max_{j\in\{1,2,\dots,n\}} m_j \tag{2.4}$$

and let Λ_* denote the index set of $k \in \{1,2,\dots,n\}$ which attains m_* (i.e., $m_k = m_* \iff k \in \Lambda_*$). Put

$$\sigma_k := \sum_{j=1}^{k} m_j \tag{2.5}$$

for $k = 1,2,\dots,n$. Then $V_r = \lim_{t\to+\infty} V(t) > 0$ implies that σ_k determines the sign of $V(t)$ on each interval (t_{k+1}, t_k), and in particular σ_n determines the sign of V_ℓ, namely $(-1)^{\sigma_k} V(t) > 0$ for $t_{k+1} < t < t_k$ and $(-1)^{\sigma_n} V_\ell > 0$.

As we mentioned in the introduction, the ratio of ε and (a specific power of) h is crucial. We set

$$\mu_* := \mu_{m_*}, \tag{2.6}$$

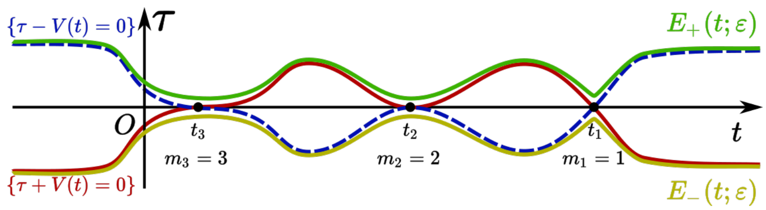

Figure 1: An example of $V(t)$ and energies $E_\pm(\varepsilon, h)$

where

$$\mu_m = \mu_m(\varepsilon, h) := \varepsilon h^{-\frac{m}{m+1}} \tag{2.7}$$

for each $m \in \mathbb{N}$. We focus on the regime $\mu_* \ll 1$. In the case where there exists at least one avoided crossing generated by a tangential intersection, that is $m_* \geq 2$, we obtain the following result.

Theorem 1. *Assume Conditions A, B and $m_* \geq 2$. Then there exist $\mu_0 > 0$ and $h_0 > 0$ such that, for any ε and h with $\mu_*(\varepsilon, h) \in (0, \mu_0]$ and $h \in (0, h_0]$, the transition probability $P(\varepsilon, h)$ has the asymptotic expansions:*

$$P(\varepsilon, h) = \begin{cases} 1 - C_*(h)\,\mu_*^2 \quad + \quad \mathcal{O}\left(\mu_*^2\left(\mu_* + h^{\frac{1}{m_*(m_*+1)}}\right)\right) & \text{if } \sigma_n \text{ is odd,} \\ \quad C_*(h)\,\mu_*^2 \quad + \quad \mathcal{O}\left(\mu_*^2\left(\mu_* + h^{\frac{1}{m_*(m_*+1)}}\right)\right) & \text{if } \sigma_n \text{ is even,} \end{cases}$$

where the coefficient $C_*(h)$ consists of the product of two factors γ_* and $\delta_*(h)$, that is $C_*(h) = \gamma_*\delta_*(h)$, which are given by

$$\gamma_* = 4\left(\frac{(m_*+1)!}{2}\right)^{\frac{2}{m_*+1}} \Gamma\left(\frac{m_*+2}{m_*+1}\right)^2 \left(1 - \frac{1+(-1)^{m_*}}{2}\sin^2\left(\frac{\pi}{2(m_*+1)}\right)\right),$$

$$\delta_*(h) = \sum_{j \in \Lambda_*} |v_j|^{-\frac{2}{m_*+1}} + 2\sum_{\substack{j,k \in \Lambda_* \\ j<k}} |v_j v_k|^{-\frac{1}{m_*+1}} \cos\left(\frac{2}{h}\int_{t_k}^{t_j} V(t)dt + \theta_{m_*}^{j,k}\right), \tag{2.8}$$

with

$$\theta_{m_*}^{j,k} = \begin{cases} (\operatorname{sgn} v_j)\dfrac{\pi}{m_*+1} & \text{if } m_* \text{ is odd and } \operatorname{sgn} v_j = -\operatorname{sgn} v_k, \\ 0 & \text{otherwise.} \end{cases}$$

Here, Γ stands for the standard Gamma function $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$.

Remark 2.3. *When every avoided crossing is generated by a transversal intersection, that is, $m_* = 1$, Theorem 1 is proven in [22] under an additional assumption that V is analytic near the real line. Our method also deduces the same asymptotic formula under Conditions A, B and the additional condition that $\tilde{\mu}_1 := (\log(1/h))^{1/2}\varepsilon h^{-1/2}$, replaced with μ_1, is sufficiently small (see also [10, Remark 1.2]).*

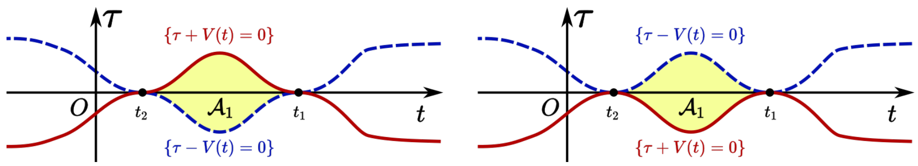

Figure 2: Cases (b) (left) and (c) (right) in Example 2.5

Remark 2.4. *The factor γ_* depends only on the highest order m_* of the zeros and never vanishes while the factor $\delta_*(h)$ depends also on the behavior of V not only the local property at zeros and may vanish. This vanishing phenomenon corresponds to the destructive quantum interference. Suppose, for example, that $|v_j|$ among $j \in \Lambda_*$ are the same. Put $N_* := \#\Lambda_*$ and $n_* := \min \Lambda_*$. Then the condition for $\delta_*(h)$ to vanish is given by*

$$N_* + 2 \left(\sum_{j \in \Lambda_* \setminus \{n_*\}} \cos \mathcal{V}_j + \sum_{\substack{j,k \in \Lambda_* \setminus \{n_*\} \\ j<k}} \cos(\mathcal{V}_j - \mathcal{V}_k) \right) = 0, \tag{2.9}$$

where

$$\mathcal{V}_j := \frac{2}{h} \int_{t_{n_*}}^{t_j} V(t) dt + \frac{1 - (-1)^{m_*}}{2} (\operatorname{sgn} v_j) \frac{\pi}{2(m_* + 1)}. \tag{2.10}$$

The algebraic curve (2.9) in $(N_* - 1)$-variables $\{\mathcal{V}_j\}_{j \in \Lambda_* \setminus \{n_*\}}$ appears as so-called Fermi surface in the context of the discrete Laplacian on the $(N_* - 1)$-dimensional diamond lattice, which is a generalization of the hexagonal lattice (see [1]).

The rest of this subsection is devoted to the concrete expression of the transition probability in Theorem 1 for typical models by means of the following geometric quantity on the (time-energy) phase space. For each $k = 1, 2, \ldots, n-1$, we denote the area enclosed by $V(t)$ and $-V(t)$ between t_{k+1} and t_k by

$$\mathcal{A}_k := 2 \int_{t_{k+1}}^{t_k} |V(t)| \, dt. \tag{2.11}$$

Example 2.5 (Two avoided crossings)**.** *Let the number n of avoided crossings be two. Then the transition probability $P(\varepsilon, h)$ is 1 (resp. 0) modulo $\mathcal{O}(\mu_*^2)$ if the sum $\sigma_2 = m_1 + m_2$ of the order of zeros is odd (resp. even). In particular, when the two zeros have the same order, one sees that $P(\varepsilon, h) = \mathcal{O}(\mu_*^2)$ independent of the parity of the order. We give the coefficient $C_*(h)$ attached to μ_*^2 in each situation:*

(a). *$m_1 > m_2$;*

$$C_*(h) = \gamma_{m_1} |v_1|^{-\frac{2}{m_1+1}}. \tag{2.12}$$

(b). $m_1 = m_2 \in 2\mathbb{Z} - 1$ *and* $|v_1| = |v_2|$*;*

$$C_*(h) = 4\gamma_{m_1} |v_1|^{-\frac{2}{m_1+1}} \cos^2 \left(\frac{\mathcal{A}_1}{2h} - \frac{\pi}{2(m_1+1)} \right). \tag{2.13}$$

(c). $m_1 = m_2 \in 2\mathbb{Z}$ *and* $|v_1| = |v_2|$*;*

$$C_*(h) = 4\gamma_{m_1} |v_1|^{-\frac{2}{m_1+1}} \cos^2 \frac{\mathcal{A}_1}{2h}. \tag{2.14}$$

Remark 2.6. *In Cases* (b) *and* (c) *of Example 2.5, we see that* $C_*(h)$ *may vanish and the order of the transition probability varies due to the destructive quantum interference under the Bohr-Sommerfeld type quantization rule*

$$\begin{cases} \dfrac{\mathcal{A}_1}{h} + \pi \in 2\pi\mathbb{Z} & \textit{Case } (\mathrm{b}), \\ \dfrac{\mathcal{A}_1}{h} + \dfrac{m_1\pi}{m_1+1} \in 2\pi\mathbb{Z} & \textit{Case } (\mathrm{c}). \end{cases} \tag{2.15}$$

This condition is a generalization of that shown in [22] (for $m_1 = 1$*).*

Example 2.7 (Three avoided crossings)**.** *Let* $n = 3$*. The transition probability is determined modulo* $\mathcal{O}(\mu_*^2)$ *by the sum* $(m_1 + m_2 + m_3)$ *whereas the coefficient* $C_*(h)$ *attached to* μ_*^2 *is determined by zeros* t_j *only for* $j \in \Lambda_*$ *and by integrals of* V *between them. In particular, when* $\#\Lambda_* \leq 2$ *and* $\Lambda_* \neq \{1, 3\}$*, the coefficient* $C_*(h)$ *is given by the same formula as a model with two avoided crossings.*

(a). $\Lambda_* = \{1, 3\}$ *and* $|v_1| = |v_2|$*;*

$$C_*(h) = 4\gamma_{m_1} |v_1|^{-\frac{2}{m_1+1}} \cos^2 \left(\frac{\mathcal{A}_1 + (-1)^{m_2}\mathcal{A}_2}{h} \right). \tag{2.16}$$

(b). $m_1 = m_2 = m_3 \in 2\mathbb{Z} - 1$ *and* $|v_1| = |v_2| = |v_3|$*;*

$$\begin{aligned} C_*(h) = \gamma_{m_1} |v_1|^{-\frac{2}{m_1+1}} \Big[3 + 2 \Big(&\cos\Big(\frac{\mathcal{A}_1}{h} - \frac{\pi}{m_1+1} \Big) \\ &+ \cos\Big(\frac{\mathcal{A}_2}{h} - \frac{\pi}{m_1+1} \Big) + \cos\Big(\frac{\mathcal{A}_1 - \mathcal{A}_2}{h} \Big) \Big) \Big]. \end{aligned} \tag{2.17}$$

(c). $m_1 = m_2 = m_3 \in 2\mathbb{Z}$ *and* $|v_1| = |v_2| = |v_3|$*;*

$$C_*(h) = \gamma_{m_1} |v_1|^{-\frac{2}{m_1+1}} \left[3 + 2 \left(\cos \frac{\mathcal{A}_1}{h} + \cos \frac{\mathcal{A}_2}{h} + \cos \left(\frac{\mathcal{A}_1 + \mathcal{A}_2}{h} \right) \right) \right].$$

Remark 2.8. *While the destructive quantum interference condition in the case* $n = 2$ *is that the area on the phase space is quantized (i.e. discretized) as in* (2.15)*, that condition in* $n = 3$ *is that two areas lie along the Fermi curve.*

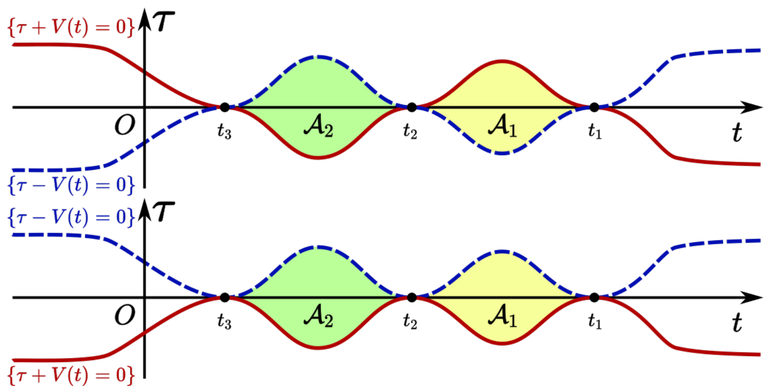

Figure 3: Cases b (above) and c (below) in Example 2.7

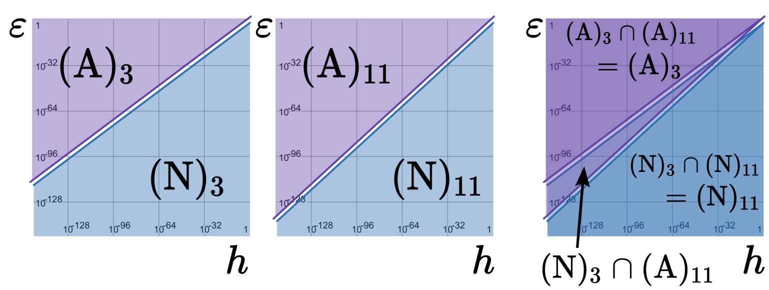

Figure 4: Adiabatic and non-adiabatic regimes $(A)_m$ and $(N)_m$ for $m = 3, 11$ (logarithmic scale, $10^{-150} \leq \varepsilon, h \leq 1$, $(A)_m = \{(\varepsilon, h);\, \mu_m \geq 100\}$, $(N)_m = \{(\varepsilon, h);\, 0 < \mu_m \leq 0.01\}$).

2.2 Coexistence of the two parameter regimes

Recall that the quantum dynamics around each avoided crossing near $t = t_k$ depends principally on the magnitude of the parameter μ_{m_k}. More precisely, $\mu_{m_k} \ll 1$ and $\mu_{m_k} \gg 1$ correspond to the non-adiabatic and adiabatic regimes (note that the regime $\mu_{m_k} \sim 1$ is studied only for the transversal case $m_k = 1$ [3]). This parameter is different for two zeros of $V(t)$ with different order, thus the transition problem with several avoided crossings generated by tangential intersections admits various regimes.

Note that μ_m obeys the algebraic order relation:

$$m < m' \iff \mu_m < \mu_{m'}. \tag{2.18}$$

The regime $\mu_{m_*} \ll 1$ considered in Theorem 1 corresponds to non-adiabatic regime $\mu_{m_k} \ll 1$ for every $k \in \{1, \ldots, n\}$. Conversely, the regime $\mu_{m_\circledast} \gg 1$ (with $m_\circledast$ standing for the minimum order $\min_{k\in\{1,\ldots,n\}} m_k$) considered in [20] corresponds to adiabatic regime $\mu_{m_k} \gg 1$ for every k.

Here, we consider the case that the two different regimes coexist, that is, the set of indices is decomposed into a disjoint union of two parts

$$\{1, 2, \ldots, n\} = \overline{\Lambda_\sharp} \sqcup \overline{\Lambda_\flat}$$

such that

$$\mu_{m_k} \gg 1 \quad (\forall k \in \overline{\Lambda_\sharp}), \qquad \mu_{m_k} \ll 1 \quad (\forall k \in \overline{\Lambda_\flat}).$$

Again by (2.18), this corresponds to

$$\mu_\sharp := \mu_{m_\sharp} \gg 1, \quad \mu_\flat := \mu_{m_\flat} \ll 1,$$

where we put $m_\flat := \max_{k\in\overline{\Lambda_\flat}} m_k$ and $m_\sharp := \min_{k\in\overline{\Lambda_\sharp}} m_k$. We also put

$$\Lambda_\flat := \{k;\, m_k = m_\flat\}, \quad \Lambda_\sharp := \{k;\, m_k = m_\sharp\}.$$

Figure 4 illustrates the regimes for $m = 3, 11$. When each zero of V is either of order 3 or 11, we here study the regime $(N)_3 \cap (A)_{11}$ while Theorem 1 and [20] concern the regime $(N)_{11}$ and $(A)_3$, respectively. In Figure 5, the problem here corresponds to $(N)_1 \cap (A)_2$ or $(N)_2 \cap (A)_3$. Note also that these figures are displayed with a logarithmic scale. Hence the borders between regimes are straight lines. Indeed, the border $\mu_m = c$ for some $c > 0$ is rewritten as $\log \varepsilon = \log c + \frac{m}{m+1} \log h$.

In the study of adiabatic regime, one of the authors employed the exact-WKB method [20] which requires the function V to be analytic. Hence we also suppose the additional condition.

Condition C. *$V(t)$ is real-analytic on an interval containing $[t_n, t_1]$.*

Under this condition, when ε is small enough, there exist $2m_k$ zeros of $V(t)^2 + \varepsilon^2$ near each $t = t_k$ like the power roots. We call these zeros turning points and

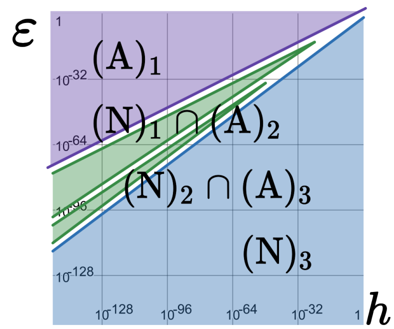

Figure 5: Adiabatic and non-adiabatic regimes $(A)_m$ and $(N)_m$ for $m = 1, 2, 3$.

denote the nearest two turning points to the real axis on the upper half-plane by $\zeta_{k,1}(\varepsilon), \zeta_{k,m_k}(\varepsilon)$, which behave like

$$\zeta_{k,j}(\varepsilon) \sim t_k + \left(\frac{m_k!}{v_k}\varepsilon\right)^{1/m_k} \exp\left[\frac{2j-1}{2m_k}\pi i\right]$$

as $\varepsilon \to 0$. As in [22], the action integral $A_{k,j}(\varepsilon)$ for $j = 1, m_k$ is given by

$$A_{k,j} := 2\int_{t_k}^{\zeta_{k,j}(\varepsilon)} \sqrt{V(t)^2 + \varepsilon^2}\, dt,$$

where the path is the segment from t_k to $\zeta_{k,j}(\varepsilon)$ and the branch of the square root of the integrand is ε at $t = t_k$. Note that $\operatorname{Im} A_{k,j} > 0$ on this branch and there exist $a_{k,j} > 0$ such that

$$\operatorname{Im} A_{k,j} = a_{k,j}\varepsilon^{(m_k+1)/m_k} + \mathcal{O}\left(\varepsilon^{(m_k+2)/m_k}\right)$$

as $\varepsilon \to 0$.

Roughly speaking, the absolute value of the "probability amplitude of the transition around an avoided crossing near t_k" is small in the limit $\mu_{m_k} \to +\infty$. Contrary to the non-adiabatic case $\Lambda_\flat$, this fact is independent of the parity of m_k. The probability amplitude has the same order as

$$\exp\left[-a_k\mu_{m_k}^{(m_k+1)/m_k}\right] \ll 1, \quad a_k := \min\{\operatorname{Im} a_{k,1}, \operatorname{Im} a_{k,m_k}\}.$$

From the sake of distinguishing this difference, we introduce

$$\overline{\Lambda_\sharp^{\mathrm{odd}}} = \{k \in \overline{\Lambda_\sharp}\,;\, m_k : \text{ odd}\} = \{k(1), k(2), \ldots, k(N)\},$$

where $N = \#\overline{\Lambda_\sharp^{\mathrm{odd}}}$ stands for the number of the elements of $\overline{\Lambda_\sharp^{\mathrm{odd}}}$, and the elements are labeled in ascending order $k(1) < k(2) < \cdots < k(N)$.

We also introduce the effective energy $\tilde{V}(t) = \tilde{V}(t; m_\flat, m_\sharp)$ in this regime by

$$\tilde{V}(t) = \begin{cases} -V(t) & \text{on } (t_{k(2l-1)}, t_{k(2l)}), \\ V(t) & \text{otherwise}, \end{cases} \tag{2.19}$$

where $t_{k(2l)}$ is taken as $-\infty$ when $t_{k(2l-1)} = t_{k(N)}$ (see also (2.21)).

Putting $a := \min_{k\in\Lambda_\sharp} a_k$ and introducing two functions

$$\epsilon_1 = \epsilon_1(m_\flat, m_\sharp, a) = \mu_\flat + \exp\left[-a\mu_\sharp^{(m_\sharp+1)/m_\sharp}\right],$$
$$\epsilon_2 = \epsilon_2(m_\flat, m_\sharp, a) = \mu_\flat\left(\mu_\flat + h^{\frac{1}{m_\flat(m_\flat+1)}}\right) + \mu_\sharp^{-(m_\sharp+1)/m_\sharp}\exp\left[-a\mu_\sharp^{(m_\sharp+1)/m_\sharp}\right],$$

we state the asymptotic expansion of the transition probability in this intermediate regime:

Theorem 2. *Assume Conditions A, B and C. Then there exist $0 < \mu_0 < 1$ and $h_0 > 0$ such that, for any ε and h satisfying $\mu_\flat < \mu_0 < \mu_0^{-1} < \mu_\sharp$, and $h \in (0, h_0]$, the transition probability $P(\varepsilon, h)$ has the asymptotic expansions:*

$$P(\varepsilon,h) = \begin{cases} 1 - \mathcal{L}(\varepsilon,h) \quad + \quad \mathcal{E}(\varepsilon,h) & \text{if } (\sigma_n + N) \text{ is odd,} \\[1ex] \quad \mathcal{L}(\varepsilon,h) \quad + \quad \mathcal{E}(\varepsilon,h) & \text{if } (\sigma_n + N) \text{ is even,} \end{cases}$$

where the leading term $\mathcal{L}(\varepsilon,h) = \mathcal{O}(\epsilon_1^2)$ and the error therm $\mathcal{E}(\varepsilon,h) = \mathcal{O}(\epsilon_1\epsilon_2)$.

Remark 2.9. *The parity which characterizes the transition probability depends not only on σ_n determined by V but also on N determined by the regime. This implies that the switch of $P(\varepsilon,h)$ occurs with changing the regime without doing the energy V (see Figure 6).*

As we mentioned in Section 2.2, Theorem 2 covers the range of the pair of the parameters (ε, h) included in the parameter regime determined by $m_\flat$ and $m_\sharp$. Regarding ε as a function of h like a one-parameter problem, we find the typical cases, which realize the intermediate regime $\mu_\sharp \to \infty$ and $\mu_\flat \to 0$.

Polynomial case: If $\varepsilon \sim h^\alpha$ with

$$\frac{m_\flat}{m_\flat+1} < \alpha < \frac{m_\sharp}{m_\sharp+1},$$

the contribution coming from $\Lambda_\sharp$ is exponentially small.

Logarithmic case: If $\varepsilon = (h\log(1/h^\rho))^{m_\sharp/(m_\sharp+1)}$ with some positive constant ρ, the contribution coming from $\Lambda_\sharp$ must be taken into account, since $\exp[-a\mu_\sharp^{(m_\sharp+1)/m_\sharp}] = h^{a\rho}$.

In the former case, the leading term is similar to that in Theorem 1 and is given by

$$\mathcal{L}(\varepsilon,h) = \mu_\flat^2\left(\sum_{j\in\Lambda_\flat}\gamma_\flat|v_{j+1}|^{-\frac{2}{m_\flat+1}} + 2\sum_{\substack{j,k\in\Lambda_\flat\\ j<k}}\mathrm{Re}\, C_{j,k}^{\flat\flat}(\varepsilon,h)\cos\left[\frac{1}{h}\int_{t_k}^{t_j}\tilde{V}(t)dt\right]\right), \tag{2.20}$$

where the factor $C^{\flat\flat}_{j,k}(\varepsilon,h)$ is of $\mathcal{O}(1)$ and consulted in (5.16). In other cases including the latter case, the leading term is more complicated than (2.20). In fact, the leading term $\mathcal{L}(\varepsilon,h)$ is of the form:

$$\mu_\flat^2\left(\sum_{j\in\Lambda_\flat}\gamma_\flat|v_{j+1}|^{-\frac{2}{m_\flat+1}}+2\sum_{\substack{j,k\in\Lambda_\flat\\ j<k}}\operatorname{Re}C^{\flat\flat}_{j,k}(\varepsilon,h)\cos\left[\frac{1}{h}\int_{t_k}^{t_j}\tilde{V}(t)dt\right]\right)$$
$$+\sum_{k\in\Lambda_\sharp}\exp\left[-2a_k\mu_\sharp^{(m_\sharp+1)/m_\sharp}\right]$$
$$+2\sum_{\substack{j\in\Lambda_\flat,k\in\Lambda_\sharp\\ j<k}}\operatorname{Re}C^{\flat\sharp}_{j,k}(\varepsilon,h)\mu_\flat\exp\left[-a_k\mu_\sharp^{(m_\sharp+1)/m_\sharp}\right]\cos\left[\frac{1}{h}\int_{t_k}^{t_j}\tilde{V}(t)dt\right]$$
$$+2\sum_{\substack{j,k\in\Lambda_\sharp\\ j<k}}\operatorname{Re}C^{\sharp\sharp}_{j,k}(\varepsilon,h)\exp\left[-(a_j+a_k)\mu_\sharp^{(m_\sharp+1)/m_\sharp}\right]\cos\left[\frac{1}{h}\int_{t_k}^{t_j}\tilde{V}(t)dt\right],$$

where $C^{\flat\sharp}_{j,k}(\varepsilon,h)$ and $C^{\sharp\sharp}_{j,k}(\varepsilon,h)$ are of $\mathcal{O}(1)$ and referred in (5.17) and (5.18) respectively.

Remark 2.10. *The mixed terms coming from ϵ_1^2 correspond to quantum interference terms referred in Remark 2.4. The phase shift caused by the integral of the energy V changes into the phase shift done by that of the effective energy $\tilde{V}$ as in Figure 6.*

In our method, we represent the Jost solution J_ℓ^+ by several bases. As we mentioned in the introduction, each basis is consists of solutions corresponding to an eigenvector associated with $E_\pm(t,\varepsilon)$ in each region between two avoided crossings. Consequently, the absolute value of the coefficients gives $\|\Pi_\pm(t;\varepsilon)J_\ell^+\|_{\mathbb{C}^2}$. Moreover, one observes from our proof that

$$1\sim\|\Pi_{\sigma(t)}(t;\varepsilon)J_\ell^+(t)\|_{\mathbb{C}^2}>\|\Pi_{-\sigma(t)}(t;\varepsilon)J_\ell^+(t)\|_{\mathbb{C}^2}\sim 0, \tag{2.21}$$

for any $t\in\mathbb{R}$, where $\sigma(t):=(-1)^{\sigma_n+N}\operatorname{sgn}\tilde{V}(t)$. In this sense, (the square of) the modulus of the probability amplitude that the energy follows the curve $\tau=E_{\sigma(t)}(t;\varepsilon)$ is almost 1. In Figure 6, we draw this curve in green.

Example 2.11. *Figure 6 shows an example case with $m_k=2k-1$ ($k=1,2,3$).*

3 Construction of exact solutions

In this section, we construct exact solutions which form a local basis near each vanishing point $t=t_k$ by means of a method of successive approximations due to [6]. While the equation treated there is a second order 2×2 system of time-independent Schrödinger equations, our equation in this paper is a first order 2×2 system.

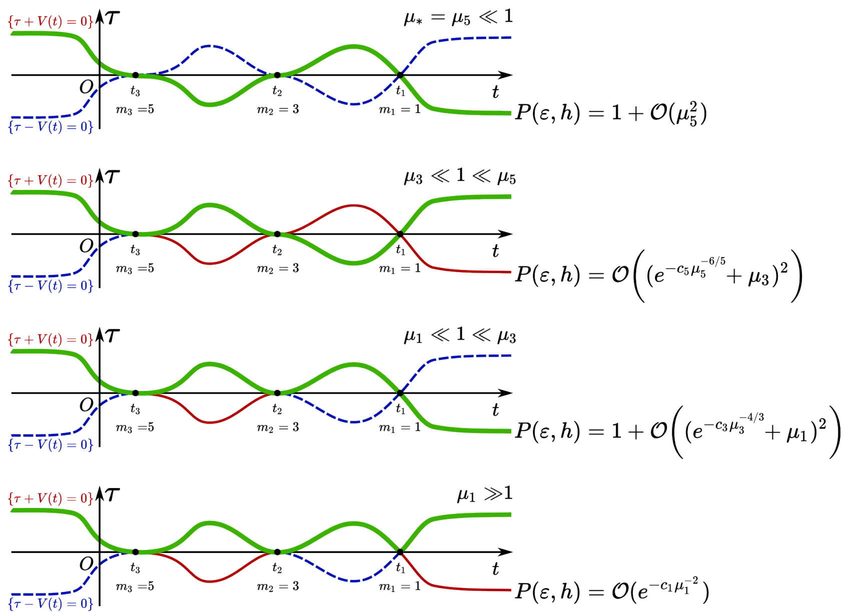

Figure 6: Change of the effective energy of Example 2.11 for each regime

3.1 Estimates of fundamental solutions

For simplicity, we assume $t_k = 0$, and let m denote m_k. Let I be a small interval containing the vanishing point 0 in its interior. We fix

$$u^{\pm}(t) = \exp\left(\mp \frac{i}{h} \int_0^t V(s) ds \right)$$

as a particular solution to

$$(hD_t \pm V(t))\, u = 0 \quad \text{on} \quad I \tag{3.1}$$

respectively, where D_t stands for $-id/dt$. For any point $a \in I$, we define an integral operator $K_a^{\pm}$ by

$$K_a^{\pm} f(t) := \frac{i}{h} u^{\pm}(t) \int_a^t \frac{f(s)}{u^{\pm}(s)} ds \quad \text{for} \quad f \in C(I), \tag{3.2}$$

where $C(I)$ is the Banach space of continuous functions on I equipped with the norm $\|f\|_{C(I)} := \sup_{x \in I} |f(x)|$. Since

$$(hD_t \pm V(t))\, K_a^{\pm} f = f \quad \text{for} \quad f \in C(I), \tag{3.3}$$

for any $a \in I$, the integral operator $K_a^{\pm} : C(I) \to C(I)$ is well-defined as a fundamental solution of $hD_t \pm V(t)$ respectively for the signs $\pm$.

Using these fundamental solutions $K^{\pm}_{a^{\pm}}$ with base points $a^{\pm} \in I$ respectively, our equation (1.1) turns into the integral system with arbitrary constants $c^+, c^- \in \mathbb{C}$:

$$\begin{cases} \psi_1(t) = -\varepsilon K^{+}_{a^+}\psi_2(t) + c^+ u^+(t), \\ \psi_2(t) = -\varepsilon K^{-}_{a^-}\psi_1(t) + c^- u^-(t). \end{cases} \tag{3.4}$$

Depending on the choice of the base points a^+ and a^-, the initial value for ψ_1 and ψ_2 at these points are determined:

$$\psi_1(a^+) = c^+ u^+(a^+), \quad \psi_2(a^-) = c^- u^-(a^-).$$

In the next subsection, we show a construction of the unique solution to the system by an iteration. For this purpose, we give the following estimates for the fundamental solutions. Note that they are independent of ε, and this estimate gives the critical rate $\mu_m = \varepsilon h^{-m/(m+1)}$.

Let $\|\cdot\|_q$ for $q \in \mathbb{R}$ be a norm on the space of continuously differentiable functions $C^1(I)$ defined by

$$\|f\|_q := \sup_I |f| + h^q \sup_I |f'| \quad f \in C^1(I). \tag{3.5}$$

Proposition 3.1. *For any $a^{\pm} \in I$, there exists $C > 0$ such that*

$$\left\|(u^{\pm})^{-1} K^{\pm}_{a^{\pm}}(u^{\mp} f)\right\|_{\frac{1}{m+1}} \le C h^{-\frac{m}{m+1}} \|f\|_{\frac{1}{m+1}} \tag{3.6}$$

for $h > 0$ small enough.

For the sake of the proof of Proposition 3.1, we introduce the the following lemma which plays an important role in this paper.

Lemma 3.2. *On a compact interval $I \subset \mathbb{R}$, consider the integral*

$$\mathcal{I}_I(h) := \int_I f(t) \exp\left(\frac{2i}{h}\int_{t_0}^t V(s)ds\right) dt, \tag{3.7}$$

with a continuously differentiable function $f \in C^1(I)$ possibly depending on h. Then there exists a constant $C > 0$ independent of f (but depending on V) such that

$$|\mathcal{I}_I(h)| \le Ch \sup_I (|f| + |f'|), \tag{3.8}$$

for $h > 0$ small enough when V does not vanish on I. If t_0 is the unique zero in I of V, one has

$$|\mathcal{I}_I(h)| \le C\left(h^{\frac{1}{m+1}} \sup_I |f| + h^{\frac{2}{m+1}} \sup_I |f'|\right) = Ch^{\frac{1}{m+1}} \|f\|_{\frac{1}{m+1}}, \tag{3.9}$$

where m denotes the order of vanishing at t_0. Moreover if f is independent of h, we have

$$\mathcal{I}_I(h) = f(t_0)\omega_m h^{\frac{1}{m+1}} + \mathcal{O}(h^{\frac{2}{m+1}}). \tag{3.10}$$

Here, the constant ω_m is given by

$$\omega_m = 2\left(\frac{(m+1)!}{2|V^{(m)}(t_0)|}\right)^{\frac{1}{m+1}} \Gamma\left(\frac{m+2}{m+1}\right)\eta_m, \tag{3.11}$$

with

$$\eta_m := \begin{cases} \cos\left(\dfrac{\pi}{2(m+1)}\right) & m: \textit{even}, \\ \exp\left(\dfrac{\operatorname{sgn}(V^{(m)}(t_0))i\pi}{2(m+1)}\right) & m: \textit{odd}. \end{cases} \tag{3.12}$$

Proof of Lemma 3.2. Suppose that V does not vanish on I, that is, a non-stationary case. Then we have for $t \in I$

$$\frac{h}{2iV(t)}\frac{d}{dt}\exp\left(\frac{2i}{h}\int_{t_0}^{t} V(s)ds\right) = \exp\left(\frac{2i}{h}\int_{t_0}^{t} V(s)ds\right). \tag{3.13}$$

This with an integration by parts and the compactness of I implies the estimate from above of $|\mathcal{I}_I(h)|$ by $Ch\sup_I(|f|+|f'|)$.

Suppose next that t_0 is the unique zero of V in I. Take a smooth cut-off function χ such that $\chi(t) = 1$ for $|t-t_0| < Ch^{1/(m+1)}$ and $\chi(t) = 0$ for $|t-t_0| > 2Ch^{1/(m+1)}$. On the support of $1-\chi$, one has the estimate

$$\left|\frac{d}{dt}\left(\frac{f(t)}{2iV(t)}\right)\right| \le \frac{C}{t^{m+1}}\left(\sup_I|f| + t\sup_I|f'|\right).$$

This with a similar argument as above implies that contribution coming from the support of $1-\chi$ to the integral is estimated by $Ch^{1/(m+1)}\|f\|_{\frac{1}{m+1}}$. The other part is estimated by $h^{1/(m+1)}\sup_I|f|$ since the support of χ is $\mathcal{O}(h^{1/(m+1)})$, and the estimate (3.9) follows.

We then suppose also that f is independent of h. Let $g = g(t)$ be the smooth function defined near t_0 such that

$$2\int_{t_0}^{t} V(s)ds = (t-t_0)^{m+1}g(t), \quad g(t) = \frac{2V^{(m)}(t_0)}{(m+1)!} + \mathcal{O}(t-t_0).$$

Take a smooth cut-off function χ whose value is 1 near t_0 and supported only on a small neighborhood where the change of the variable $\tau = (t-t_0)|g(t)|^{1/(m+1)}$ is valid. Then one has

$$\int_I \chi(t)f(t)\exp\left(\frac{2i}{h}\int_{t_0}^{t} V(s)\right)ds = |g(t_0)|^{-\frac{1}{m+1}}\int_{\mathbb{R}}\chi(t(\tau))\tilde{f}(\tau)e^{\sigma i\tau^{m+1}/h}d\tau$$

with $\sigma = \operatorname{sgn} g(t_0)$ and a smooth function $\tilde{f} = \tilde{f}(\tau)$ satisfying $\tilde{f}(0) = f(t_0)$. The resulting asymptotic formula is obtained from this integral by applying the method of degenerate stationary phase (see e.g. [11]). The non-stationary estimate (3.8) is applicable on the support of $1-\chi$.

□

Based on this lemma, let us prove Proposition 3.1.

Proof of Proposition 3.1. For $f \in C^1(I)$, we have by definition

$$(u^{\pm})^{-1} K^{\pm}_{a^{\pm}} (u^{\mp} f)(t) = \frac{i}{h} \int_{a^{\pm}}^{t} \exp\left(\mp \frac{2i}{h} \int_0^s V(r)dr\right) f(s)ds.$$

According to (3.9) of Lemma 3.2, this integral is estimated by $Ch^{\frac{1}{m+1}} \|f\|_{\frac{1}{m+1}}$. For the derivative, we have

$$\frac{d}{dt}\left[(u^{\pm})^{-1} K^{\pm}_{a^{\pm}} (u^{\mp} f)(t)\right] = \frac{i}{h} \exp\left(\mp \frac{2i}{h} \int_0^t V(r)dr\right) f(t).$$

This is clearly bounded by $h^{-1} \sup_I |f|$. $\square$

Remark 3.3. *The argument in the proof of Lemma 3.2 shows that the estimate* (3.6) *becomes better as $\mathcal{O}(h)$ if the integral interval does not contain $t = 0$.*

3.2 Method of successive approximations (MSA)

From Proposition 3.1, it follows that for each ${}^t(c^+, c^-) \in \mathbb{C}^2$, there uniquely exists a solution to the integral system (3.4). By a linearity of the system, the solution is given by the linear combination $c_1 w_1(t) + c_2 w_2(t)$ of the solutions $w_1(t)$ and $w_2(t)$ corresponding to the choices $\boldsymbol{e}_1 = {}^t(1, 0)$ and $\boldsymbol{e}_2 = {}^t(0, 1)$ for ${}^t(c^+, c^-)$. Moreover, the solution can be constructed by MSA:

$$w_1(t) = w_1(t; a^-, a^+) := \begin{pmatrix} \sum_{k\geq 0} (\varepsilon^2 K^+_{a^+} K^-_{a^-})^k u^+ \\ -\varepsilon K^-_{a^-} \sum_{k\geq 0} (\varepsilon^2 K^+_{a^+} K^-_{a^-})^k u^+ \end{pmatrix} \tag{3.14}$$

$$\left(\text{resp. } w_2(t) = w_2(t; a^+, a^-) := \begin{pmatrix} -\varepsilon K^+_{a^+} \sum_{k\geq 0} (\varepsilon^2 K^-_{a^-} K^+_{a^+})^k u^- \\ \sum_{k\geq 0} (\varepsilon^2 K^-_{a^-} K^+_{a^+})^k u^- \end{pmatrix}\right) \tag{3.15}$$

in I for a fixed small $\varepsilon h^{-\frac{m}{m+1}}$. These iteration formulas imply that the solutions admit the asymptotic expansions when $\mu_m := \varepsilon h^{-\frac{m}{m+1}} \to 0$ as follows:

$$\begin{aligned} w_1(t; a^-, a^+) &= \begin{pmatrix} u^+(t) + \varepsilon^2 K^+_{a^+} K^-_{a^-} u^+(t) \\ -\varepsilon K^-_{a^-} u^+(t) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\mu_m^4) \\ \mathcal{O}(\mu_m^3) \end{pmatrix}, \\ w_2(t; a^+, a^-) &= \begin{pmatrix} -\varepsilon K^+_{a^+} u^-(t) \\ u^-(t) + \varepsilon^2 K^-_{a^-} K^+_{a^+} u^-(t) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\mu_m^3) \\ \mathcal{O}(\mu_m^4) \end{pmatrix}. \end{aligned} \tag{3.16}$$

Notice that Proposition 3.1 allows us to choose arbitrarily the base points $a^{\pm}$ of the fundamental solutions in this construction. As we mentioned in Remark 3.3, we obtain better asymptotic formulas

$$w_1(t;a^-,a^+) = \begin{pmatrix} u^+(t) + \mathcal{O}(\varepsilon^2/h) \\ \mathcal{O}(\varepsilon) \end{pmatrix} \quad \text{on} \quad I \cap \{\pm t > 0\} \tag{3.17}$$

if $\pm a^- > 0$, that is, the integral interval $[a^-, t]$ of $K^-_{a^-}$ does not contain the zero, and likewise

$$w_2(t;a^+,a^-) = \begin{pmatrix} \mathcal{O}(\varepsilon) \\ u^-(t) + \mathcal{O}(\varepsilon^2/h) \end{pmatrix} \quad \text{on} \quad I \cap \{\pm t > 0\} \tag{3.18}$$

if $\pm a^+ > 0$, that is, the integral interval $[a^+, t]$ of $K^+_{a^+}$ does not.

Take (ε, h)-independent constants r and ℓ such that $\ell < 0 < r$ and $[\ell, r] \subset I$. We define the four MSA solutions $w_{1.r}, w_{2,r}, w_{1,\ell}$, and $w_{2,\ell}$ in I as

$$\begin{aligned} w_{1,r}(t) &:= w_1(t;r,r), \quad w_{2,r}(t) := w_2(t;r,r), \\ w_{1,\ell}(t) &:= w_1(t;\ell,\ell), \quad w_{2,\ell}(t) := w_2(t;\ell,\ell). \end{aligned} \tag{3.19}$$

According to Proposition 3.1 and the asymptotic formula (3.16), one sees that the asymptotic behaviors of these MSA solutions as $\mu_m \to 0$ are clear on $[\ell, r]$, and also that $w_{j,r}(t)$ (resp. $w_{j,\ell}(t)$) $(j = 1, 2)$ behave like the initial data $u_j(t)$ near $t = r$ (resp. $t = \ell$) for a fixed small μ_m. Moreover the pairs $(w_{1,r}(t), w_{2,r}(t))$ and $(w_{1,\ell}(t), w_{2,\ell}(t))$ form bases of the space of solutions.

4 Connection formulas

In this section, we also treat a zero of $V(t)$ as $t_k = 0$ for simplicity. The purpose of this section is to establish the two connection formulas. One of them connects across the vanishing point and the other does between the consecutive vanishing points.

4.1 Across the vanishing point

The crucial point of this proof is the connection formula between the two bases $(w_{1,r}(t), w_{2,r}(t))$ and $(w_{1,\ell}(t), w_{2,\ell}(t))$ introduced by (3.19) in the previous section. The claim of this subsection is the asymptotic behavior of the transfer matrix $T(\varepsilon, h)$ as follows:

Theorem 3. *For the solutions defined by* (3.19)*, we have*

$$\begin{pmatrix} w_{1,\ell}(t) & w_{2,\ell}(t) \end{pmatrix} = \begin{pmatrix} w_{1,r}(t) & w_{2,r}(t) \end{pmatrix} T(\varepsilon, h), \tag{4.1}$$

where the 2×2*-matrix* $T = T(\varepsilon, h)$ *admits the following asymptotic formula:*

$$T(\varepsilon, h) = I_2 - i\mu_m T_{\mathrm{sub}} + \mathcal{O}(\mu_m^2 + \mu_m h^{\frac{1}{m+1}}) \tag{4.2}$$

as $(\varepsilon, h) \to (0,0)$ *with* $\mu_m = \varepsilon h^{-\frac{m}{m+1}} \to 0$ *and*

$$T_{\mathrm{sub}} = \begin{pmatrix} 0 & \omega_m \\ \overline{\omega_m} & 0 \end{pmatrix}, \tag{4.3}$$

where ω_m *is given by* (3.11) *with* $t_0 = 0$.

Remark 4.1. *By construction and the symmetry* $\overline{K_a^{\pm} f} = -K_a^{\mp}\overline{f}$, *we have*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{w_{2,\bullet}} = w_{1,\bullet} \quad (\bullet = r, \ell). \tag{4.4}$$

This makes T *symmetrical in the following sense:*

$$T = \begin{pmatrix} \tau_1 & -\tau_2 \\ \overline{\tau_2} & \overline{\tau_1} \end{pmatrix} \in \mathrm{SU}(2), \tag{4.5}$$

where $\mathrm{SU}(2)$ *is the special unitary group of degree* 2 *(see §A.2). Namely,* T *is unitary* ($|\tau_1|^2 + |\tau_2|^2 = 1$). *This is a consequence that the time evolution by* H *is unitary and that for* $\bullet = \ell, r$, *the basis*

$$(w_{1,\bullet}(\bullet), w_{2,\bullet}(\bullet)) = \left(\begin{pmatrix} u^+(\bullet) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u^-(\bullet) \end{pmatrix} \right)$$

of $\mathbb{C}^2$ *is orthonormal.*

Proof of Theorem 3. Since the pair $(w_{1,r}, w_{2,r})$ forms a basis of the space of solutions, the matrix $(w_{1,r}(t)\, w_{2,r}(t))$ is invertible for any $t \in I$. Then (4.1) is rewritten as

$$T(\varepsilon, h) = (w_{1,r}(t)\, w_{2,r}(t))^{-1} (w_{1,\ell}(t)\, w_{2,\ell}(t)),$$

where the right-hand side is also independent of t. Substituting r for t, we have

$$\begin{aligned} T(\varepsilon, h) &= \begin{pmatrix} u^-(r) & 0 \\ 0 & u^+(r) \end{pmatrix} (w_{1,\ell}(r)\, w_{2,\ell}(r)) \\ &= \mathrm{Id} - \begin{pmatrix} 0 & u^-(r)\varepsilon K_\ell^+ u^-(r) \\ u^+(r)\varepsilon K_\ell^- u^+(r) & 0 \end{pmatrix} + \mathcal{O}(\mu_m^2). \end{aligned} \tag{4.6}$$

Here, we have used $u^+ u^- = 1$ and (3.16).

In order to prove Theorem 3, it is enough to compute the asymptotic behavior of the quantity $u^-(r)\varepsilon K_\ell^+ u^-(r)$ as $(\varepsilon, h) \to (0,0)$ with $\mu_m \to 0$. The computation of

$$u^-(r)\varepsilon K_\ell^+ u^-(r) = \frac{i\varepsilon}{h} \int_\ell^r \exp\left(\frac{2i}{h} \int_0^t V(s) ds \right) dt \tag{4.7}$$

is carried out based on Lemma 3.2. In fact, since the integral interval $[\ell, r]$ of (4.7) contains the zero of $V(t)$, we have

$$\begin{aligned} u^-(r)\varepsilon K_\ell^+ u^-(r) &= -\frac{i\varepsilon}{h} \left(\omega_m h^{\frac{1}{m+1}} + \mathcal{O}(h^{\frac{2}{m+1}}) \right) \\ &= -i\mu_m \omega_m + \mathcal{O}(\mu_m h^{\frac{1}{m+1}}). \end{aligned} \tag{4.8}$$

Combining (4.6) and (4.8), we obtain Theorem 3.

□

Hence this theorem implies that the transfer matrix T_k in (5.1) is given by

$$T_k(\varepsilon, h) = \begin{pmatrix} 1 & -i\omega_{m_k}\mu_{m_k} \\ -i\overline{\omega}_{m_k}\mu_{m_k} & 1 \end{pmatrix} + \mathcal{O}(\mu_{m_k}^2 + \mu_{m_k} h^{\frac{1}{m_k+1}}) \tag{4.9}$$

as $(\varepsilon, h) \to (0, 0)$ with $\mu_{m_k} = \varepsilon h^{-\frac{m_k}{m_k+1}} \to 0$.

4.2 Between the vanishing points

In addition to Theorem 3, which gives the connection formula around the vanishing point of $V(t)$, a similar argument yields the following proposition, which gives the connection formula between two consecutive zeros of $V(t)$.

Let $t_{k+1} < t_k$ be two consecutive zeros of $V(t)$ (i.e. $V \neq 0$ on $]t_{k+1}, t_k[$) with multiplicities m_k, m_{k+1}, and let ℓ_j, r_j $(j = k, k+1)$ be base points such that $\ell_{k+1} < t_{k+1} < r_{k+1} < \ell_k < t_k < r_k$. The configuration of these points implies that each interval $I_j := [\ell_j, r_j]$ includes t_j and does not intersect with each other. We set $m_* = \max\{m_k, m_{k+1}\}$. We can consider two bases $(w_{1,\ell_k}, w_{2,\ell_k})$ and $(w_{1,r_{k+1}}, w_{2,r_{k+1}})$, which are similarly given by the formulas (3.19) with $t_0 = t_k$ and $t_0 = t_{k+1}$ respectively.

Proposition 4.2. *The change of basis $T_{k,k+1}(\varepsilon, h)$ given by*

$$(w_{1,r_{k+1}}, w_{2,r_{k+1}}) = (w_{1,\ell_k}, w_{2,\ell_k}) T_{k.k+1}(\varepsilon, h) \tag{4.10}$$

admits the following asymptotic behavior as $(\varepsilon, h) \to (0, 0)$ with $\mu_{m_*} \to 0$:

$$T_{k,k+1}(\varepsilon, h) = \begin{pmatrix} \exp\left(-\frac{i}{h}\int_{t_{k+1}}^{t_k} V(t)dt\right) & 0 \\ 0 & \exp\left(\frac{i}{h}\int_{t_{k+1}}^{t_k} V(t)dt\right) \end{pmatrix} + \mathcal{O}\left(\frac{\varepsilon^2}{h}\right). \tag{4.11}$$

Remark 4.3. *The error term in (4.11) is rewritten as $\mathcal{O}(\varepsilon^2/h) = \mathcal{O}(\mu_1^2)$. From the order relation (2.18) of μ_m with respect to m, this error is smaller than that in (4.2) when $m > 1$.*

Proof of Proposition 4.2. We can derive from (4.10) the expression of $T_{k,k+1}$ by a similar way to the proof of Theorem 3. We set $u_k^{\pm} = \exp\left(\mp i \int_{t_k}^t V(s)ds/h\right)$, and compute the matrix $T_{k,k+1}$ by using the value at $t = \ell_k$:

$$\begin{aligned} &T_{k,k+1}(\varepsilon, h) \\ &= (w_{1,\ell_k}\ w_{2,\ell_k})^{-1}(w_{1,r_{k+1}}\ w_{2,r_{k+1}})\big|_{t=\ell_k} \\ &= \begin{pmatrix} u_k^-(\ell_k) & 0 \\ 0 & u_k^+(\ell_k) \end{pmatrix} \left(\begin{pmatrix} u_{k+1}^+(\ell_k) & 0 \\ 0 & u_{k+1}^-(\ell_k) \end{pmatrix} + \mathcal{O}\left(\frac{\varepsilon^2}{h}\right) \right). \end{aligned}$$

Here, we have used the asymptotic formulas (3.17) and (3.18). Note that the integral interval $[r_{k+1}, \ell_k]$ of the fundamental solutions for the construction does not contain any zero of $V(t)$. We deduce (4.11) from this with the identity

$$u_k^{\pm}(\ell_k)u_{k+1}^{\mp}(\ell_k) = \exp\left(\pm\frac{i}{h}\int_{t_{k+1}}^{t_k} V(t)dt\right). \tag{4.12}$$

□

5 End of the proofs

By using the transfer matrices T_k $(k = 1, 2, \ldots, n)$, $T_{k,k+1}$ $(k = 1, 2, \ldots, n-1)$ introduced in the previous section and T_r, T_ℓ in Appendix A.1, the scattering matrix $S = S(\varepsilon, h)$ is represented as

$$S = T_r^{-1}T_1T_{1,2}T_2T_{2,3}T_3\cdots T_{n-1,n}T_nT_\ell. \tag{5.1}$$

Here, the matrix T_r (resp. T_ℓ) has a similar form to $T_{k,k+1}$ which connects Jost solutions $J_r^{\pm}$ (resp. $J_\ell^{\pm}$) and local solutions near t_1 (resp. t_n). The previous section shows the asymptotic behaviors of T_k and $T_{k,k+1}$ (see (4.9), (4.11)), and the appendix does those of T_r and T_ℓ (see (A.17), (A.18), (A.20)). The formula of T_ℓ depending on the sign of V_ℓ can be rewritten as

$$T_\ell = \begin{cases} T_{n,n+1} & (\sigma_n \text{ is even}) \\ T_{n,n+1}J & (\sigma_n \text{ is odd}), \end{cases} \tag{5.2}$$

by means of $\sigma_n = \sum_{k=1}^n m_k$ and

$$T_{n,n+1} := \begin{pmatrix} \exp\left(-\frac{i}{h}R_\ell\right) & 0 \\ 0 & \exp\left(+\frac{i}{h}R_\ell\right)\end{pmatrix}, \qquad J := \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \tag{5.3}$$

where J is a complex structure on $\mathbb{C}^2$, that is, $J^2 = -\mathrm{Id}$. Note that all of these transfer matrices are elements of SU(2), which is mentioned in §A.2. By using the notation (A.23), the scattering matrix S is expressed as $S = T_r^{-1}\mathcal{T}_n$ (resp. $S = T_r^{-1}\mathcal{T}_nJ$) if σ_n is even (resp. odd). This implies that, when σ_n is even (resp. odd), the off-daiagnoal entry s_{21} equals $e^{-iR_r/h}\tau_{21}^n$ (resp. $e^{-iR_r/h}\tau_{22}^n$). From Lemma A.4 alone, it is complicated to examine the asymptotics of τ_{22}^n up to the coefficient of $\mathcal{O}(\mu^2)$. However, thanks to the unitarity of the scattering matrix S, that is, $|s_{11}|^2 + |s_{21}|^2 = 1$, the computation of the asymptotic behavior of $|s_{21}|^2$ can be reduced to that of $|\tau_{21}^n|^2$ even if σ_n is odd.

5.1 Proof of Theorem 1

Let us demonstrate the proof of Theorem 1. Taking the complex numbers α_k, $\beta_k(\mu)$ and ν_k in Lemma A.4 as

$$\alpha_k = 1, \qquad \beta_k(\mu) = -i\overline{\omega_{m_k}}\mu_{m_k}, \qquad \nu_k = \exp\left(-\frac{i}{h}\int_{t_{k+1}}^{t_k} V(t)dt\right),$$

and noting $\mu_{m_k} \ll \mu_*$ for any k, we have from the algebraic formula (A.28) the asymptotic behavior of $|\tau_{21}^n|^2$ as follows:

$$\begin{aligned}|\tau_{21}^n|^2 = \mu_*^2 \sum_{j\in\Lambda_*} |\omega_{m_j}|^2 + 2\mu_*^2 \,\mathrm{Re} \sum_{\substack{j,k\in\Lambda_* \\ j<k}} \overline{\omega_{m_j}} \omega_{m_k} e^{-\frac{2i}{h}\int_{t_k}^{t_j} V(t)dt} \\ + \mathcal{O}(\mu_*^3) + \mathcal{O}(\mu_*\mu_{*-1}),\end{aligned} \tag{5.4}$$

where μ_{*-1} stands for μ_{m_*-1}. Concerning the error terms in (5.4), the former one, i.e. $\mathcal{O}(\mu_*^3)$, is a higher order error term coming from Λ_* and the latter is a cross term between the largest vanishing order and the second largest one. This error coming from a cross term is at most $\mathcal{O}(\mu_*\mu_{*-1})$. If m_* is odd, ω_{m_j} is not real for $j \in \Lambda_*$, and the following phase shift term may arise from the product $\overline{\omega_{m_j}}\omega_{m_k}$ depending on the sign of v_j and v_k:

$$\arg(\overline{\omega_{m_j}}\omega_{m_k}) = \frac{((\operatorname{sgn} v_k) - (\operatorname{sgn} v_j))\pi}{2(m_*+1)} = \begin{cases} -\dfrac{(\operatorname{sgn} v_j)\pi}{2(m_*+1)} & \text{if } \operatorname{sgn} v_j = -\operatorname{sgn} v_k, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Therefore, when m_* is odd, the quantity $|\tau_{21}|^2$ behaves like

$$\begin{aligned}\mu_*^2(\omega_*^o)^2 \left(\sum_{j\in\Lambda_*} |v_j|^{-\frac{2}{m_*+1}} + 2\mathrm{Re} \sum_{\substack{j,k\in\Lambda_* \\ j<k}} |v_j v_k|^{-\frac{1}{m_*+1}} \cos\left(\frac{2}{h}\int_{t_k}^{t_j} V(t)dt + \theta^{j,k} \right) \right) \\ + \mathcal{O}(\mu_*^3) + \mathcal{O}(\mu_*\mu_{*-1}),\end{aligned} \tag{5.6}$$

where

$$\omega_*^o = 2\left(\frac{(m_*+1)!}{2} \right)^{\frac{1}{m_*+1}} \Gamma\left(\frac{m_*+2}{m_*+1} \right),$$

$$\theta^{j,k} = \begin{cases} (\operatorname{sgn} v_j)\dfrac{\pi}{2(m_*+1)} & \text{if } \operatorname{sgn} v_j = -\operatorname{sgn} v_k, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, ω_{m_j} is real when m_* is even. In this case, the quantity $|\tau_{21}|^2$ can be computed similarly as the formula (5.6) by replacing ω_*^o with $\omega_*^o \cos(\pi/2(m_*+1))$ and $\theta^{j,k}$ with 0. Therefore we have completed the proof of Theorem 1.

□

5.2 Proof of Theorem 2

The intermediate regime where $\mu_\flat \to 0$ and $\mu_\sharp \to \infty$ requires making use of the transfer matrix based on the exact WKB method in [21] under Condition

C at the vanishing points governed by an adiabatic regime. Namely, in the intermediate regime, the transfer matrix T_k is replaced as

$$T_k(\varepsilon, h) = \begin{cases} T_k^{(N)}(\varepsilon, h) & k \in \overline{\Lambda_\flat}, \\ T_k^{(A)}(\varepsilon, h) & k \in \overline{\Lambda_\sharp}, \end{cases} \tag{5.7}$$

where $T_k^{(N)}$ is equal to (4.9) and $T_k^{(A)}$ will be given below. The key of the proof of Theorem 2 is to reduce the computation of the product including the different kinds of the transfer matrices in (5.7) to the same computation as in the proof of Theorem 1.

According to the exact WKB method in [21], we obtained the existences of the exact WKB solutions $\psi_\bullet^\pm$ ($\bullet = \ell, r$) near a vanishing point and their asymptotic behaviors away from the vanishing point under an adiabatic regime. Moreover we got the change of basis between $(\psi_\ell^+, \psi_\ell^-)$ and (ψ_r^+, ψ_r^-). By matching the asymptotic behaviors of the exact WKB solutions $\psi_\bullet^\pm$ and the MSA solutions $w_{j,\bullet}$ ($j \in \{1,2\}, \bullet \in \{\ell, r\}$) on their semiclassical wave front sets referred in [10], we have the connection formula between them. Consequently, in the case where both m_k and σ_{k-1} are even, the transfer matrix $T_k^{(A)}$ is of the form:

$$T_k^{(A)} := T_k^{\mathrm{w}} = \begin{pmatrix} \alpha_k & -\bar{\beta}_k \\ \beta_k & \bar{\alpha}_k \end{pmatrix} \in \mathrm{SU}(2),$$

where α_k, β_k have the asymptotic expansions as $\mu_{m_k} \to \infty$:

$$\begin{aligned} \alpha_k = {} & \exp\left[-\frac{i}{2h}(A_{k,1} - A_{k,m_k})\right] \\ & + (-1)^{m_k} \exp\left[-\frac{i}{2h}(A_{k,1} - 2\overline{A_{k,1}} + A_{k,m_k})\right] + \mathcal{O}\left(\mu_{m_k}^{-\frac{m_k+1}{m_k}}\right), \end{aligned} \tag{5.8}$$

$$\begin{aligned} \beta_k = {} & (-1)^{\sigma_k} \Big\{ \exp\left[-\frac{i}{2h}(A_{k,1} - \overline{A_{k,m_k}})\right] \\ & \qquad - (-1)^{m_k} \exp\left[-\frac{i}{2h}(A_{k,1} - 2\overline{A_{k,1}} + \overline{A_{k,m_k}})\right] \Big\} \\ & + \mathcal{O}\left(\mu_{m_k}^{-\frac{m_k+1}{m_k}} \exp\left[-a\mu_{m_k}^{(m_k+1)/m_k}\right]\right), \end{aligned} \tag{5.9}$$

with the notations given in §2.2. Actually α_k is of $\mathcal{O}(1)$ and β_k is exponentially small of $\mathcal{O}(\exp[-a\mu_{m_k}^{(m_k+1)/m_k}])$ as $\mu_{m_k} \to \infty$. Notice that the above case where both m_k and σ_{k-1} are even implies that the eigenvalue of $H(t, \varepsilon)$ (i.e. the energy of the system: $\sqrt{V(t)^2 + \varepsilon^2}$) and the energy without an interaction (i.e. $V(t)$) have the same sign before and behind the vanishing point $t = t_k$. Conversely, in the case where the sign of $\sqrt{V(t)^2 + \varepsilon^2}$ does not coincide with that of $V(t)$ near the vanishing point, the correspondence of the exact WKB solutions to the MSA solutions varies. In fact, $T_k^{(A)}$ for $k \in \overline{\Lambda_\sharp}$ depends on m_k and σ_{k-1} as follows:

$$T_k^{(A)} = \begin{cases} T_k^{\mathrm{w}} & \text{if } (m_k, \sigma_{k-1}) = (\mathrm{even}, \mathrm{even}), \\ \mathcal{C}^{\sigma_{k-1}} T_k^{\mathrm{w}} & \text{if } (m_k, \sigma_{k-1}) = (\mathrm{even}, \mathrm{odd}), \\ iQ \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} T_k^{\mathrm{w}} & \text{if } (m_k, \sigma_{k-1}) = (\mathrm{odd}, \mathrm{even}), \\ iQ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mathcal{C}^{\sigma_{k-1}} T_k^{\mathrm{w}} & \text{if } (m_k, \sigma_{k-1}) = (\mathrm{odd}, \mathrm{odd}), \end{cases} \tag{5.10}$$

where the matrix $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ equipped with useful properties:

$$Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} Q, \qquad Q^2 = \mathrm{Id}, \tag{5.11}$$

and $\mathcal{C}$ is an operator of taking a complex conjugate, that is $\mathcal{C} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$. Notice that each matrix in (5.10) from which are removed the factor iQ if it exists belongs to $\mathrm{SU}(2)$. Introducing the notation of the transfer matrix T_k' belonging to $\mathrm{SU}(2)$ as follows:

$$T_k' = \begin{cases} T_k^{(N)} & k \in \overline{\Lambda_\flat} \\ T_k^{\mathrm{w}} & k \in \overline{\Lambda_\sharp} \quad (m_k, \sigma_{k-1}) = (\mathrm{even}, \mathrm{even}), \\ \mathcal{C}^{\sigma_{k-1}} T_k^{\mathrm{w}} & k \in \overline{\Lambda_\sharp} \quad (m_k, \sigma_{k-1}) = (\mathrm{even}, \mathrm{odd}), \\ \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} T_k^{\mathrm{w}} & k \in \overline{\Lambda_\sharp} \quad (m_k, \sigma_{k-1}) = (\mathrm{odd}, \mathrm{even}), \\ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mathcal{C}^{\sigma_{k-1}} T_k^{\mathrm{w}} & k \in \overline{\Lambda_\sharp} \quad (m_k, \sigma_{k-1}) = (\mathrm{odd}, \mathrm{odd}), \end{cases} \tag{5.12}$$

and recalling the commutative property (5.11), we can express the scattering matrix in the intermediate regime by

$$S = T_r^{-1} \prod_{k=1}^{n} T_k T_{k,k+1} = T_r^{-1} \left(\prod_{k=1}^{n} \widetilde{T_k'} \widetilde{T_{k,k+1}} \right) (iQ)^N , \tag{5.13}$$

where the notation of $\widetilde{A_j}$ for the 2×2 matrix A_j depending on j stands for

$$\widetilde{A_j} = \begin{cases} QA_jQ & (k(2l-1) \le j < k(2l)), \\ A_j & \text{otherwise}, \end{cases} \tag{5.14}$$

with the same notations as in §2.2. Remark that if $A_j \in \mathrm{SU}(2)$ then $QA_jQ \in \mathrm{SU}(2)$. Hence the expression (5.13) implies that the algebraic lemma (A.27) can be applied directly to the computation of the off-diagonal entry of the product, and that the transition probability depends also on the parity of the number $N = \#\overline{\Lambda_\sharp^{\rm odd}}$.

As a sequel to this computation of the product in (5.13), we can derive the dependence on $\mu_\flat, \mu_\sharp$ more precisely. Denoting the $(1,1)$-entry of $\widetilde{T'_k}$ by $\widetilde{\alpha'_k}$, we see that $\widetilde{\alpha'_k}$ is of $\mathcal{O}(1)$ and, in particular $\widetilde{\alpha'_k} = 1$ for $k \in \Lambda_\sharp$. Setting, similarly, the $(2,1)$-entry of $\widetilde{T'_k}$ by $\widetilde{\beta'_k}$, we can rewrite $\widetilde{\beta'_k}$ as

$$\widetilde{\beta'_k} = \begin{cases} p_k \mu_\flat & (k \in \Lambda_\flat), \\ q_k(m_k, \sigma_{k-1}) \exp[-a_k \mu_\sharp^{(m_\sharp+1)/m_\sharp}] & (k \in \Lambda_\sharp), \end{cases} \tag{5.15}$$

where p_k and $q_k(m_k, \sigma_{k-1})$ are uniquely determined by (5.9), (5.12) and (5.14). Notice that p_k and $q_k(m_k, \sigma_{k-1})$ are of $\mathcal{O}(1)$ in each regime. On the other hand, $\widetilde{T_{k,k+1}}$ can be regarded as the matrix $T_{k,k+1}$ by replacing $\tilde{V}(t)$ (see (2.19)) with $V(t)$. From this fact, it is deduced that the asymptotic of the transition probability in the intermediate regime is determined by the effective energy $\tilde{V}$. Hence, we can obtain the asymptotic behavior of $|\tau_{21}|^2$ as follows:

$$\begin{aligned} &\mu_\flat^2 \left(\sum_{j \in \Lambda_\flat} \gamma_\flat |v_{j+1}|^{-\frac{2}{m_\flat+1}} + 2 \sum_{\substack{j,k \in \Lambda_\flat \\ j<k}} \mathrm{Re}\, C^{\flat\flat}_{j,k}(\varepsilon, h) \cos\left[\frac{1}{h} \int_{t_k}^{t_j} \tilde{V}(t) dt \right] \right) \\ &\quad + \sum_{k \in \Lambda_\sharp} \exp\left[-2 a_k \mu_\sharp^{(m_\sharp+1)/m_\sharp} \right] \\ &\quad + 2 \sum_{\substack{j \in \Lambda_\flat, k \in \Lambda_\sharp \\ j<k}} \mathrm{Re}\, C^{\flat\sharp}_{j,k}(\varepsilon, h) \mu_\flat \exp\left[-a_k \mu_\sharp^{(m_\sharp+1)/m_\sharp} \right] \cos\left[\frac{1}{h} \int_{t_k}^{t_j} \tilde{V}(t) dt \right] \\ &\quad + 2 \sum_{\substack{j,k \in \Lambda_\sharp \\ j<k}} \mathrm{Re}\, C^{\sharp\sharp}_{j,k}(\varepsilon, h) \exp\left[-(a_j + a_k) \mu_\sharp^{(m_\sharp+1)/m_\sharp} \right] \cos\left[\frac{1}{h} \int_{t_k}^{t_j} \tilde{V}(t) dt \right] \\ &\quad + \mathcal{O}(\epsilon_1 \epsilon_2), \end{aligned}$$

where ϵ_1, ϵ_2 are given in §2.2 and

$$C^{\flat\flat}_{j,k}(\varepsilon, h) = p_j\, \overline{p_k}, \tag{5.16}$$

$$C^{\flat\sharp}_{j,k}(\varepsilon, h) = \left(\prod_{\kappa=j+1}^{k-1} \widetilde{\alpha'_\kappa}^2 \right) \widetilde{\alpha'_k}\, p_j\, \overline{q_k(m_k, \sigma_{k-1})}, \tag{5.17}$$

$$C^{\sharp\sharp}_{j,k}(\varepsilon, h) = \widetilde{\alpha'_k} \left(\prod_{\kappa=j+1}^{k-1} \widetilde{\alpha'_\kappa}^2 \right) \widetilde{\alpha'_k}\, q_j(m_j, \sigma_{j-1})\, \overline{q_k(m_k, \sigma_{k-1})}. \tag{5.18}$$

The proof of Theorem2 have been completed.

A Appendix

A.1 Jost solutions

In this subsection A.1, we give the existence of the Jost solutions for the definition of the scattering matrix. We remark that the smallness of h is not required for the argument here.

We first consider the Jost solutions $J_r^{\pm}$ near $+\infty$. A discussion for $J_l^{\pm}$ is done similarly but the difference is that we are assuming that V_r is positive (Condition A). Let H_r denote the limiting Hamiltonian at $+\infty$:

$$H_r := \begin{pmatrix} V_r & \varepsilon \\ \varepsilon & -V_r \end{pmatrix}. \tag{A.1}$$

The functions defined by

$$\varphi_r^+(t) = e^{-i\lambda_r t/h} \begin{pmatrix} \cos\theta_r \\ \sin\theta_r \end{pmatrix}, \quad \varphi_r^-(t) = e^{+i\lambda_r t/h} \begin{pmatrix} -\sin\theta_r \\ \cos\theta_r \end{pmatrix}, \tag{A.2}$$

where $\lambda_r = \sqrt{V_r^2 + \varepsilon^2}$ and $\tan 2\theta_r = \varepsilon/V_r$ $(0 < \theta_r < \pi/4)$, are particular solutions to $hD_t\psi + H_r\psi = 0$ and form a basis of $\mathbb{C}^2$ for each $t \in \mathbb{R}$.

Proposition A.1. *There uniquely exists a pair of solutions (ϕ_r^+, ϕ_r^-) to the system* (1.1) *such that*

$$\lim_{t\to+\infty} \left(\phi_r^{\pm}(t) - \varphi_r^{\pm}(t)\right) = 0. \tag{A.3}$$

Proof. Let $U(t)$ be a 2×2-matrix valued C^1-function. We have

$$\frac{d}{dt}(\Phi_r(t)U(t)) = \Phi_r'(t)U(t) + \Phi_r(t)U'(t) = \frac{1}{ih}H_r\Phi_r(t)U(t) + \Phi_r(t)U'(t)$$

with $\Phi_r := (\varphi_r^+, \varphi_r^-)$. Thus, if $U(t)$ satisfies

$$U' = \frac{1}{ih}\Phi_r^{-1}(H - H_r)\Phi_r U, \tag{A.4}$$

each column of the matrix-valued function $\Phi_r U$ is a solution to the equation (1.1). Put $A_r(t) := \Phi_r^{-1}(H - H_r)\Phi_r$. From the identity

$$H - H_r = (V(t) - V_r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5}$$

and Condition A, the matrix-valued function $A_r(t)$:

$$A_r(t) = (V(t) - V_r)\begin{pmatrix} \cos 2\theta_r & -e^{+2i\lambda_r t/h}\sin 2\theta_r \\ -e^{-2i\lambda_r t/h}\sin 2\theta_r & -\cos 2\theta_r \end{pmatrix}, \tag{A.6}$$

is integrable on the half-line $[0, \infty[$. Then the function

$$U_r(t) := \exp\left(-\frac{i}{h}\int_{+\infty}^t A_r(s)ds\right), \tag{A.7}$$

is well-defined and solves the equation (A.4) with the boundary condition

$$\lim_{t\to+\infty} U_r(t) = \mathrm{Id}.$$

Here we recall that Id stands for the 2×2 unit matrix. We finally obtain the solutions $\phi_r^{\pm}(t)$ with the asymptotic behavior (A.3):

$$(\phi_r^+(t), \phi_r^-(t)) := \Phi_r(t) U_r(t). \tag{A.8}$$

□

From Proposition A.1 and the trace-free property of $H(t;\varepsilon)$, the pair of (ϕ_r^+, ϕ_r^-) forms a basis. Similarly, this fact implies that ϕ_r^+ (resp. ϕ_r^-) coincides with the Jost solution J_r^+ (resp. J_r^-).

Next, we give the asymptotic behaviors of $\phi_r^{\pm}$ as $\varepsilon\to 0$ near some fixed point t_r. Take $t_r > t_1$ (recall that $t_1 = \max\{t\in\mathbb{R}\,;\, V=0\}$ is the first zero of V) satisfying

$$\int_{+\infty}^{t_r} (V(t) - V_r)\, dt \neq 0. \tag{A.9}$$

We introduce some kind of the action integral taking into account of contributions from the infinity as

$$R_r = V_r t_r + \int_{+\infty}^{t_r} (V(s) - V_r)\, ds,$$

and put

$$u_r^{\pm} = \exp\left(\mp \frac{i}{h} \int_{t_r}^{t} V(s) ds \right).$$

Proposition A.2. *We have*

$$\phi_r^+(t) = e^{-iR_r/h} \begin{pmatrix} u_r^+ + \mathcal{O}(\varepsilon^2/h) \\ \mathcal{O}(\varepsilon) \end{pmatrix}, \quad \phi_r^-(t) = e^{+iR_r/h} \begin{pmatrix} \mathcal{O}(\varepsilon) \\ u_r^- + \mathcal{O}(\varepsilon^2/h) \end{pmatrix}$$

as $(\varepsilon^2/h, \varepsilon) \to (0,0)$ *uniformly for* t *in a small neighborhood of* t_r*.*

Before proving Proposition A.2, we prepare the following.

Lemma A.3. *Let* A *be a matrix of the form:*

$$A = \frac{i}{h} \begin{pmatrix} -a & b \\ \bar{b} & a \end{pmatrix}$$

with $a\in\mathbb{R}\setminus\{0\}$ *and* $b\in\mathbb{C}$*. For* $|b/a| \ll 1$*, one has*

$$e^A = \begin{pmatrix} e^{-ia/h} + \mathcal{O}((b/a)^2) & \mathcal{O}(b/a) \\ \mathcal{O}(b/a) & e^{+ia/h} + \mathcal{O}((b/a)^2) \end{pmatrix}.$$

Proof. Since $A^2 = -h^{-2}(a^2 + |b|^2)\mathrm{Id}$, an algebraic computation gives

$$e^A = \left(\cos \frac{z}{h}\right) \mathrm{Id} + \frac{i}{z} \left(\sin \frac{z}{h}\right) \begin{pmatrix} -a & b \\ \bar{b} & a \end{pmatrix} \tag{A.10}$$

where $z = \sqrt{a^2 + |b|^2}$. We have $z = \mathrm{sgn}(a)a(1 + \mathcal{O}(b^2/a^2))$ under $|b/a| \ll 1$ and $a \neq 0$. This gives the following asymptotic formula:

$$e^A = \begin{pmatrix} e^{-ia/h} + \mathcal{O}((b/a)^2) & 0 \\ 0 & e^{+ia/h} + \mathcal{O}((b/a)^2) \end{pmatrix} + \frac{i}{z} \left(\sin \frac{z}{h}\right) \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}.$$

The lemma follows from $b/z = \mathcal{O}(b/a)$. □

Proof of Proposition A.2. From the expression (A.6), U_r defined by (A.7) is written as

$$U_r(t; \varepsilon, h) = \exp\left(\frac{i}{h} \begin{pmatrix} -\mathcal{I}_r(t) \cos 2\theta_r(\varepsilon) & \mathcal{J}_r(t;h) \sin 2\theta_r(\varepsilon) \\ \overline{\mathcal{J}_r(t;h)} \sin 2\theta_r(\varepsilon) & \mathcal{I}_r(t) \cos 2\theta_r(\varepsilon) \end{pmatrix} \right), \tag{A.11}$$

where

$$\mathcal{I}_r(t) = \int_{+\infty}^{t} (V(s) - V_r) ds, \qquad \mathcal{J}_r(t;h) = \int_{+\infty}^{t} (V(s) - V_r) e^{+2is\lambda_r/h} ds.$$

Apply Lemma A.3 with

$$a = a(t, \varepsilon) = \mathcal{I}_r(t) \cos 2\theta_r(\varepsilon), \quad b = b(t, \varepsilon, h) = \mathcal{J}_r(t;h) \sin 2\theta_r(\varepsilon). \tag{A.12}$$

By the choice of t_r with the condition (A.9), $a = \mathcal{I}_r(t) \cos 2\theta_r(\varepsilon)$ never vanishes for t near t_r. By definition, we have $\theta_r(\varepsilon) = \mathcal{O}(\varepsilon)$, and consequently $|b/a| = \mathcal{O}(\varepsilon)$. Then Lemma A.3 shows

$$U_r(t; \varepsilon, h) = \begin{pmatrix} e^{-i\mathcal{I}_r(t)/h} + \mathcal{O}(\varepsilon^2/h) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & e^{+i\mathcal{I}_r(t)/h} + \mathcal{O}(\varepsilon^2/h) \end{pmatrix} \tag{A.13}$$

Note that we have the following decomposition of $\mathcal{I}_r(t)$:

$$\mathcal{I}_r(t) = \mathcal{I}_r(t_r) + \int_{t_r}^{t} (V(s) - V_r) ds = R_r + \int_{t_r}^{t} V(s) ds - V_r t. \tag{A.14}$$

Since $\Phi_r(t)$ admits the asymptotic formula

$$\Phi_r(t) = \begin{pmatrix} \varphi_r^+ & \varphi_r^- \end{pmatrix} = \left(1 + \mathcal{O}\left(\frac{\varepsilon^2}{h}\right)\right) \begin{pmatrix} e^{-iV_r t/h} & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & e^{+iV_r t/h} \end{pmatrix}, \tag{A.15}$$

as $(\varepsilon^2/h, \varepsilon) \to (0, 0)$, Proposition A.2 follows from (A.13) and (A.14). □

The asymptotic formula

$$\begin{pmatrix} J_r^+ & J_r^- \end{pmatrix} = \begin{pmatrix} u_r^+ + \mathcal{O}(\varepsilon^2/h) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & u_r^- + \mathcal{O}(\varepsilon^2/h) \end{pmatrix} \begin{pmatrix} e^{-iR_r/h} & 0 \\ 0 & e^{+iR_r/h} \end{pmatrix} \tag{A.16}$$

is directly deduced from Propositions A.1 and A.2. Therefore, we obtain

$$T_r = \begin{pmatrix} e^{-iR_r/h} + \mathcal{O}(\varepsilon^2/h) & \mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon^2) & e^{+iR_r/h} + \mathcal{O}(\varepsilon^2/h) \end{pmatrix}. \tag{A.17}$$

For the Jost solutions $J_\ell^\pm(t)$, the same argument as above works when $V_\ell > 0$ and a similar one induces the existences and the asymptotic behaviors when $V_\ell < 0$.

In the case where $V_\ell > 0$, by exchanging the sub-index r for ℓ, one sees

$$T_\ell = \begin{pmatrix} e^{-iR_\ell/h} + \mathcal{O}(\varepsilon^2/h) & \mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon^2) & e^{+iR_\ell/h} + \mathcal{O}(\varepsilon^2/h) \end{pmatrix}. \tag{A.18}$$

Here

$$R_\ell = V_\ell t_\ell + \int_{-\infty}^{t_\ell} (V(s) - V_\ell)\, ds$$

with $t_\ell < t_n = \min\{t \in \mathbb{R}\,;\, V = 0\}$ satisfying that the second integral term in the right-hand side does not vanish.

In the case where $V_\ell < 0$, we choose instead of (A.2) particular solutions $\varphi_\ell^\pm(t)$ to $hD_t\psi + H_\ell\psi = 0$ as

$$\varphi_\ell^+(t) = e^{-i\lambda_\ell t/h} \begin{pmatrix} \sin\eta_\ell \\ \cos\eta_\ell \end{pmatrix}, \quad \varphi_\ell^-(t) = e^{+i\lambda_\ell t/h} \begin{pmatrix} -\cos\eta_\ell \\ \sin\eta_\ell \end{pmatrix}, \tag{A.19}$$

where $\lambda_\ell = \sqrt{V_\ell^2 + \varepsilon^2}$ and $\tan 2\eta_\ell = \varepsilon/(-V_\ell)$ $(0 < \eta_\ell < \pi/4)$. They coincide with the leading terms of Jost solutions $J_\ell^\pm(t)$ when $V_\ell < 0$ and satisfy the asymptotic formulas:

$$\varphi_\ell^+ \sim e^{+iV_\ell t/h} \begin{pmatrix} \mathcal{O}(\varepsilon) \\ 1 + \mathcal{O}(\varepsilon^2/h) \end{pmatrix}, \quad \varphi_\ell^- \sim e^{-iV_\ell t/h} \begin{pmatrix} -1 + \mathcal{O}(\varepsilon^2/h) \\ \mathcal{O}(\varepsilon) \end{pmatrix}$$

as $(\varepsilon^2/h, \varepsilon) \to (0, 0)$ for each t. One sees that, with (A.19), Proposition A.1 also holds. One also have similar asymptotic formulas to those of Proposition A.2:

$$\phi_\ell^+(t) = e^{+iR_\ell/h} \begin{pmatrix} \mathcal{O}(\varepsilon) \\ u_\ell^+ + \mathcal{O}(\varepsilon^2/h) \end{pmatrix}, \quad \phi_\ell^-(t) = e^{-iR_\ell/h} \begin{pmatrix} -u_\ell^- + \mathcal{O}(\varepsilon^2/h) \\ \mathcal{O}(\varepsilon) \end{pmatrix}$$

as $(\varepsilon^2/h, \varepsilon) \to (0, 0)$ uniformly in a small neighborhood of $t = t_\ell$, where $u_\ell^\pm = \exp(\mp i \int_{t_\ell}^t V(s) ds/h)$. We obtain

$$T_\ell = \begin{pmatrix} \mathcal{O}(\varepsilon) & -e^{-iR_\ell/h} + \mathcal{O}(\varepsilon^2/h) \\ e^{+iR_\ell/h} + \mathcal{O}(\varepsilon^2/h) & \mathcal{O}(\varepsilon) \end{pmatrix}. \tag{A.20}$$

A.2 Algebraic lemma

In order to know the asymptotic behavior of the scattering matrix (5.1), it suffices to compute the products of the matrices of the following forms:

$$T_k(\mu) = \begin{pmatrix} \alpha_k & -\overline{\beta_k(\mu)} \\ \beta_k(\mu) & \overline{\alpha_k} \end{pmatrix}, \qquad T_{k,k+1} = \begin{pmatrix} \nu_k & 0 \\ 0 & \overline{\nu_k} \end{pmatrix},$$

where α_k, β_k, and ν_k are complex numbers such that $\det T_k = \det T_{k,k+1} = 1$, namely $|\alpha_k|^2 + |\beta_k|^2 = |\nu_k|^2 = 1$, and $\beta_k(\mu) = \mathcal{O}(\mu)$ as $\mu \to 0$. Notice that, in our context (see (4.2)), T_k and $T_{k,k+1}$ have this form with the numbers given modulo $\mathcal{O}(\mu^2)$ by

$$\alpha_k \equiv 1, \quad \beta_k \equiv -i\overline{\omega_k}\mu_{m_k} = \mathcal{O}(\mu_{m_k}), \quad \nu_k \equiv \exp\left(-\frac{i}{h}\int_{t_{k+1}}^{t_k} V(t)dt\right).$$

In this subsection we give an algebraic formula by means of these notations α_k, $\beta_k(\mu)$ and ν_k for simplicity. We know that the product of them is of the form

$$T_k T_{k,k+1} = \begin{pmatrix} \alpha_k\nu_k & -\overline{\beta_k\nu_k} \\ \beta_k\nu_k & \overline{\alpha_k\nu_k} \end{pmatrix}. \tag{A.21}$$

Let SU(2) be the special unitary group of degree 2 given by

$$\mathrm{SU}(2) = \left\{ T \in M_2(\mathbb{C})\,;\, T = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, a, b \in \mathbb{C},\ \det T = 1 \right\}. \tag{A.22}$$

One sees that all of the above matrices belong to SU(2). Denoting the products of these matrices by

$$\mathcal{T}_n = T_1 T_{1,2} T_2 \cdots T_n T_{n,n+1} = \begin{pmatrix} \tau_{11}^n & \tau_{12}^n \\ \tau_{21}^n & \tau_{22}^n \end{pmatrix} \in \mathrm{SU}(2), \tag{A.23}$$

we get the following lemma:

Lemma A.4. *As $\mu \to 0$, the following asymptotic formulas hold.*

$$\tau_{11}^n(\mu) = \prod_{j=1}^n \alpha_j\nu_j + \mathcal{O}(\mu^2), \tag{A.24}$$

$$\tau_{21}^n(\mu) = \sum_{j=1}^n \left(\prod_{\kappa=1}^{j-1} \overline{\alpha_\kappa}\right) \beta_j(\mu) \left(\prod_{k=j+1}^n \alpha_k\right) \left(\prod_{\kappa=1}^{j-1} \overline{\nu_\kappa}\right) \left(\prod_{k=j}^n \nu_k\right) + \mathcal{O}(\mu^2), \tag{A.25}$$

$$\tau_{12}^n(\mu) = -\overline{\tau_{21}^n}(\mu), \qquad \tau_{22}^n(\mu) = \overline{\tau_{11}^n}(\mu), \tag{A.26}$$

with the convention that $\prod_{\kappa=1}^0 \overline{\alpha_\kappa} = \prod_{\kappa=1}^0 \overline{\nu_\kappa} = 1$.

The proof of this lemma is based on the mathematical induction for the product of the matrices (A.21). A simple computation of $|\tau_{21}^n|^2$, which corresponds to the transition probability, gives

$$|\tau_{21}^n|^2 = \sum_{j=1}^{n} |\beta_j(\mu)|^2 + 2\mathrm{Re}\left[\sum_{1\le j<k\le n} \beta_j(\mu)\alpha_j \left(\prod_{\kappa=j+1}^{k-1} \alpha_\kappa^2\right) \alpha_k \overline{\beta_k}(\mu) \left(\prod_{\kappa=j}^{k-1} \nu_\kappa^2\right)\right] + \mathcal{O}(\mu^3), \tag{A.27}$$

with the convention that $\prod_{\kappa=j+1}^{j} \alpha_\kappa^2 = \prod_{\kappa=j}^{j-1} \nu_\kappa = 1$. Note that we used $|\alpha_k| = 1 + \mathcal{O}(\mu^2)$ and $|\nu_k| = 1$ in the above computation. In particular, when $\alpha_k = 1 + \mathcal{O}(\mu^2)$, we have

$$|\tau_{21}^n|^2 = \sum_{j=1}^{n} |\beta_j(\mu)|^2 + 2\mathrm{Re}\left[\sum_{1\le j<k\le n} \beta_j(\mu)\overline{\beta_k(\mu)} \left(\prod_{\kappa=j}^{k-1} \nu_\kappa^2\right)\right] + \mathcal{O}(\mu^3). \tag{A.28}$$

Remark that the factor γ_n does not appear explicitly in the leading term in the formulas (A.27) and (A.28).

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