Torsion and Chern-Simons gravity in 4D space-times from a Geometrodynamical four-form

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Abstract

In Hermann Minkowski's pioneering formulation of special relativity, the space-time geometry in any inertial frame is described by the line-element $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. It is interesting to note that not only the Minkowski metric $\eta_{\mu\nu}$ is invariant under proper Lorentz transformations, the totally antisymmetric Levi-Civita tensor $e_{\mu\nu\alpha\beta}$ too is.

In Einstein's general relativity (GR), $\eta_{\mu\nu}$ of the flat space-time gets generalized to a dynamical, space-time dependent metric tensor $g_{\mu\nu}$ that characterizes a curved space-time geometry. In the present study, it is put forward that the flat space-time Levi-Civita tensor gets elevated to a dynamical four-form field $\tilde{w}$ in curved space-time manifolds, i.e. $e_{\mu\nu\alpha\beta} \to w_{\mu\nu\alpha\beta}(x) = \phi(x)\ e_{\mu\nu\alpha\beta}$ so that $\tilde{w} = \frac{1}{4!}\ w_{\mu\nu\rho\sigma}\ \tilde{d}x^\mu \wedge \tilde{d}x^\nu \wedge \tilde{d}x^\rho \wedge \tilde{d}x^\sigma$. It is shown that this geometrodynamical four-form field, extends GR by leading naturally to a torsion in the theory as well as to a Chern-Simons gravity. Furthermore, it is demonstrated that the scalar-density $\phi(x)$ associated with the geometrodynamical four form $\tilde{w}$ may be used to construct a generalized exterior derivative that converts a p-form density to a (p+1)-form density of identical weight.

In order to subject the hypothesized four-form field $\tilde{w}$ to observational evidence, we first argue that the associated scalar-density $\phi(x)$ corresponds to an axion-like pseudo-scalar field in the Minkowski space-time, and that it can also masquerade as dark matter. Thereafter, we provide a simple semi-classical analysis in which a self-gravitating Bose-Einstein condensate of such ultra-light pseudo-scalars leads to the formation of a supermassive black hole. A brief analysis of propagation of weak gravitational waves in the presence of $\tilde{w}$ is also considered in this article.

I. INTRODUCTION

By envisaging a four dimensional manifold's space-time geometry be described by a line-element, $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, Hermann Minkowski had overturned the idea of treating space and time disjointly. Furthermore, the cause and effect relation, in the realm of classical physics, attained a significant elucidation in the Minkowskian framework - infinitesimally separated events (assuming a signature (+ - - -)) for which $ds^2 < 0$ ($ds^2 > 0$) are causally disconnected (causally connected). Of course, the underlying physics involved in it is that an effect essentially follows a cause due to transfer of energy from the latter to the former, and that the rate of flow of energy is limited by the speed c. What is usually not stressed is that, because of this causality criteria, the imaginary numbers entered physics in a fundamental manner even before the advent of quantum mechanics, i.e. the proper distance ds between any two causally disconnected events is always an imaginary number.

Although space and time coordinates get mixed up while transiting from one inertial frame to another, the Minkowski metric $\eta_{\mu\nu}$ itself remains invariant. In fact, this is utilized in the Minkowskian framework to define an arbitrary Lorentz transformation $x^\mu \rightarrow x^{'\mu} = \Lambda^\mu{}_\nu x^\nu$ by the condition,

$$\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \; \eta_{\alpha\beta} \tag{1}$$

It must be pointed out that there is another tensor in the flat space-time that is invariant under proper Lorentz transformations: the totally antisymmetric Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. Since, under $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$,

$$\epsilon_{\mu\nu\alpha\beta} \quad \rightarrow \quad \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \Lambda^\gamma{}_\alpha \Lambda^\kappa{}_\beta \; \epsilon_{\sigma\tau\gamma\kappa} \tag{2}$$

$$= det(\Lambda) \; \epsilon_{\mu\nu\alpha\beta} \quad , \tag{3}$$

and $det(\Lambda) = 1$ for proper Lorentz Transformations, it is clear that the flat space-time Levi-Civita tensor does not change under arbitrary boosts and rotations of inertial frames.

Given this special status that the Minkowski metric and the Levi-Civita tensor enjoy in the Minkowskian space-time, as well as the fact that $\eta_{\mu\nu}$ metamorphoses into a dynamical space-time metric $g_{\mu\nu}(x)$ in GR, raises a pertinent point of of elevating $\epsilon_{\mu\nu\alpha\beta}$ into a dynamical field. In the present article, we explore such a possibility by positing a geometrodynamical field, $w_{\mu\nu\rho\sigma}(x)$, totally antisymmetric in its indices, that is independent of the metric tensor and that influences space-time physics. Then, the interesting feature that $\eta_{\mu\nu}$ and $\epsilon_{\mu\nu\alpha\beta}$ both share in the flat space-time becomes relevant also in arbitrary curved space-times [1].

We propose in this article that this geometrodynamical field represents a new physical degree of freedom that couples universally to all matter. In the following section, we discuss some of the differential geometric aspects of $w_{\mu\nu\rho\sigma}(x)$ that may be significant in the context of space-time physics.

II. GEOMETRODYNAMICAL FOUR-FORM FIELD AND TORSION

Now, in any 4D space-time manifold, because of its complete antisymmetry, $w_{\mu\nu\rho\sigma}(x)$ has only one algebraically independent component and, hence, renders itself to be expressed as,

$$w_{\mu\nu\rho\sigma} = \phi(x)\epsilon_{\mu\nu\rho\sigma} \quad , \tag{4}$$

where $\phi(x)$ is a scalar-density of weight +1, with $\phi(x) \to \phi'(x') = \phi(x)/J(x,x')$ under a general coordinate transformation $x^\mu \to x'^\mu$, $J(x,x')$ being the corresponding Jacobian.

In a coordinate basis, this four-form field may be expressed as,

$$\tilde{w} = \frac{1}{4!}\ w_{\mu\nu\rho\sigma}(x)\ \tilde{d}x^\mu \wedge \tilde{d}x^\nu \wedge \tilde{d}x^\rho \wedge \tilde{d}x^\sigma \tag{5}$$

$$= \frac{1}{4!}\ \phi(x)\ \epsilon_{\mu\nu\rho\sigma}\ \tilde{d}x^\mu \wedge \tilde{d}x^\nu \wedge \tilde{d}x^\rho \wedge \tilde{d}x^\sigma \tag{6}$$

Eq.(6) reflects the equivalence between four-forms and 0-forms in the case of four dimensional manifolds.

Since we demand that this new dynamical field be independent of the metric tensor, instead of raising the indices of $w_{\mu\nu\rho\sigma}$ using the metric tensor, the totally antisymmetric components $w^{\mu\nu\rho\sigma}$ corresponding to the four-form $\tilde{w}$ are obtained from the associated p-vector $\mathbf{w}$, so that one obtains the relation, $w^{\mu\nu\rho\sigma}\ w_{\mu\nu\rho\sigma} = -4!$ [2]. This implies,

$$w^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}/\phi(x) \quad . \tag{7}$$

We assume throughout that $\phi = w_{0123}$ has no physical dimension. As the canonical volume-form in GR is given by,

$$\tilde{V} = \frac{1}{4!}\sqrt{-g}\ \ \epsilon_{\mu\nu\rho\sigma}\ \tilde{d}x^\mu \wedge \tilde{d}x^\nu \wedge \tilde{d}x^\rho \wedge \tilde{d}x^\sigma \ , \tag{8}$$

our geometrodynamical four-form field may be expressed in terms of a scalar field $\chi(x) \equiv \frac{\phi(x)}{\sqrt{-g}}$ entailing,

$$\tilde{w} = \chi(x)\tilde{V} \tag{9}$$

A fundamental four-form field that varies with space and time may be utilized to extend GR by having a dynamical torsion field in the theory. Introducing an asymmetric affine connection (e.g. see [3, 4]),

$$\bar{\Gamma}^{\mu}_{\alpha\beta} \equiv \Gamma^{\mu}_{\alpha\beta} - a_T\ g^{\mu\nu} g^{\gamma\lambda} \frac{\partial \chi}{\partial x^{\lambda}}\ w_{\nu\gamma\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} - a_T\ g^{\mu\nu} \chi^{\ ;\gamma}\ w_{\nu\gamma\alpha\beta}\ , \tag{10}$$

where $\Gamma^{\mu}_{\alpha\beta}$ is the standard Christoffel-Levi Civita connection while a_T is a dimensionless constant that is a measure of the coupling between the metric tensor and the four-form, we define a metric compatible torsion field,

$$S_{\alpha\beta}{}^{\mu} = \frac{1}{2}\left(\bar{\Gamma}^{\mu}_{\alpha\beta} - \bar{\Gamma}^{\mu}_{\beta\alpha}\right) = -a_T\ g^{\mu\nu} \chi^{\ ;\gamma}\ w_{\nu\gamma\alpha\beta} = -a_T \sqrt{-g}\ g^{\mu\nu} g^{\gamma\sigma}\ \chi\ \frac{\partial \chi}{\partial x^{\sigma}}\ \epsilon_{\nu\gamma\alpha\beta} \tag{11}$$

In the language of differential geometry, since torsion is a vector valued 2-form, the above expression for it can be seen as arising from an inner contraction (e.g. see [2, 5]) of $\tilde{w}$ with the vector field $\chi^{;\gamma} \frac{\partial}{\partial x^{\gamma}}$ followed by inner contractions with four vector fields $-g^{\mu\nu} \frac{\partial}{\partial x^{\nu}}$, $\mu = 0, 1, ..3$.

Covariant derivatives of tensor fields are obtained in the usual way,

$$D_\nu A^\mu = \partial_\nu A^\mu + \bar{\Gamma}^{\mu}_{\nu\beta} A^\beta = A^{\mu}{}_{;\nu} - a_T\ g^{\mu\alpha} \chi^{\ ;\gamma} w_{\alpha\gamma\nu\beta} A^\beta\ \ , \tag{12}$$

$$D_\nu B_\mu = \partial_\nu B_\mu - \bar{\Gamma}^{\beta}_{\mu\nu} B_\beta = B_{\mu;\ \nu} + a_T g^{\beta\sigma} \chi^{;\gamma} w_{\sigma\gamma\mu\nu} B_\beta\ \ , \tag{13}$$

and,

$$D_\nu C_{\mu\tau} = \partial_\nu C_{\mu\tau} - \bar{\Gamma}^{\beta}_{\mu\nu} C_{\beta\tau} - \bar{\Gamma}^{\beta}_{\tau\nu} C_{\mu\beta} = C_{\mu\tau;\ \nu} + a_T g^{\beta\sigma} \chi^{;\gamma} w_{\sigma\gamma\mu\nu} C_{\beta\tau} + a_T g^{\beta\sigma} \chi^{;\gamma} w_{\sigma\gamma\tau\nu} C_{\mu\beta}\ , \tag{14}$$

with $C_{\mu\tau;\ \nu}$ being the covariant derivative of $C_{\mu\tau}$ in standard GR when there is no torsion (i.e. $a_T = 0$ case). From eq.(14), the metric compatibility can be easily proved by showing that $D_\alpha g_{\mu\nu} = 0$. Similarly, eq.(10) entails that,

$$\bar{\Gamma}^{\mu}_{\alpha\beta} A^\alpha A^\beta = \Gamma^{\mu}_{\alpha\beta} A^\alpha A^\beta$$

implying that the torsion in our case is autoparallel compatible. In our case, the contortion tensor $K_{\alpha\beta}{}^{\mu}$ and $S_{\alpha\beta\gamma}$ have the expressions,

$$K_{\alpha\beta}{}^{\mu} \equiv -S_{\alpha\beta}{}^{\mu} + S_{\beta}{}^{\mu}{}_{\alpha} - S^{\mu}{}_{\alpha\beta} = -S_{\alpha\beta}{}^{\mu} \tag{15}$$

and,

$$S_{\alpha\beta\gamma} = g_{\gamma\mu} S_{\alpha\beta}{}^{\mu} = a_T \ \chi^{\ ;\sigma} w_{\sigma\alpha\beta\gamma} = a_T \sqrt{-g} \ g^{\gamma\sigma} \chi \ \frac{\partial \chi}{\partial x^{\gamma}} \epsilon_{\sigma\alpha\beta\gamma} \tag{16}$$

which is simply an inner contraction of $\tilde{w}$ with the vector field $\chi^{\ ;\sigma} \frac{\partial}{\partial x^{\sigma}}$.

The important point to note is that the torsion vanishes if either $a_T = 0$ or if the scalar field $\chi(x)$ that corresponds to the geometrodynamical four form is independent of both space as well as time. **For the sake of simplicity, in the remaining portions of the article we will concentrate on the case $a_T = 0$ while assuming $\frac{\partial \chi}{\partial x^{\alpha}}$ to be non-vanishing**. It is straightforward to extend the analysis to the $a_T \neq 0$ case in order to study the effects of torsion. For the rest of the sections, we have chosen a unit system in which $\hbar = c = 1$ and the metric signature: (+ - - -).

III. ACTION, FIELD EQUATIONS AND CHERN-SIMONS EXTENSIONS

(i) Dynamical equations

As ϕ is a scalar-density of weight +1,

$$\phi_{\ ;\beta} = \phi_{\ ,\beta} - \Gamma^{\alpha}_{\alpha\beta} \ \phi \quad ,$$

where we follow the standard notation,

$$f_{\ ,\beta} \equiv \partial_{\beta} f \quad .$$

Using the above equation along with the relation $w^{\mu\nu\rho\sigma} \ w_{\mu\nu\rho\sigma} = - 4!$, the covariant derivatives of the four-form field and its corresponding p-vector are easily shown to be,

$$w_{\mu\nu\alpha\beta \ ;\lambda} = [(\ln \phi)_{,\lambda} - \Gamma^{\sigma}_{\sigma\lambda}] w_{\mu\nu\alpha\beta} \quad , \tag{17}$$

and,

$$w^{\mu\nu\alpha\beta}{}_{;\lambda} = -[(\ln \phi)_{,\lambda} - \Gamma^{\sigma}_{\sigma\lambda}] w^{\mu\nu\alpha\beta} \quad , \tag{18}$$

respectively.

The action $\mathcal{A}$ for the standard matter, geometry as well as the dynamical four-form that is invariant under general coordinate transformations may be expressed as,

$$\mathcal{A} = -\frac{m_{Pl}^2}{16\pi} \int R \sqrt{-g} d^4 x + \int L \sqrt{-g} d^4 x +$$

$$+\frac{a_1}{4!}\int \phi \; w^{\mu\nu\alpha\beta}{}_{;\lambda} w_{\mu\nu\alpha\beta}{}^{;\lambda} \; d^4x + a_2\int \phi(x) d^4x, \tag{19}$$

where $m_{Pl} \equiv \sqrt{\hbar c/G}$ and L are the Planck mass and the Lagrangian density of the matter fields, respectively, with a_1 and a_2 being real valued constants of the theory with dimensions (mass)^2 and (mass)^4, respectively. The part of the action in eq.(19) that pertains to the four-form is by no means unique. For instance, a term $\propto \int R\; \phi d^4x$ could be added to the above action, but we limit ourselves to gravitational minimal coupling, at present. Later, we shall discuss supplementing the above action with Chern-Simons terms induced by the geometrodynamical four-form.

By extremizing $\mathcal{A}$ with respect to $g_{\mu\nu}$ and ϕ, respectively, we obtain the following equations of motion,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{m_{Pl}^2}[T_{\mu\nu} + \Theta_{\mu\nu}], \tag{20}$$

$$\chi\;{}^{;\alpha}_{;\alpha} \equiv \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\alpha\beta}\chi_{,\alpha})_{,\beta} = \frac{1}{2}\Big[g^{\mu\nu}\frac{\chi_{,\mu}\chi_{,\nu}}{\chi} + \frac{a_2}{a_1}\chi\Big], \tag{21}$$

where the scalar field $\chi(x)$ is defined, as earlier, to be $\chi \equiv \frac{\phi}{\sqrt{-g}}$, while $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are the energy-momentum tensors for the standard matter and the geometrodynamical four-form, respectively. The expression for the latter is given by,

$$\Theta_{\mu\nu} = 2a_1[\frac{\chi_{,\mu}\chi_{,\nu}}{\chi} - g_{\mu\nu}\chi\;{}^{;\alpha}_{;\alpha}]. \tag{22}$$

When χ satisfies the equation of motion given by eq.(21), the above energy-momentum tensor gets simplified to,

$$\Theta_{\mu\nu} = 2a_1\Big[\frac{\chi_{,\mu}\chi_{,\nu}}{\psi} - \frac{g_{\mu\nu}}{2}\Big(g^{\alpha\beta}\frac{\chi_{,\alpha}\chi_{,\beta}}{\chi} + \frac{a_2}{a_1}\chi\Big)\Big]. \tag{23}$$

(ii) Chern-Simons extensions

In this sub-section, we construct (3+1)-dimensional Chern-Simons (CS) terms. Our CS extensions of GR follows closely the seminal paper of Jackiw and Pi [6], except for a crucial difference: they had introduced an external fixed four-vector v_μ instead of a dynamical $\phi_{;\mu}$ in their formulation. Because of this, their model had a manifest violation of Lorentz invariance. In our case, both general as well as Lorentz covariances are maintained throughout [1].

First, we consider a Chern-Simons (CS) term that couples electromagnetic field to the four-form $\tilde{w}$,

$$\mathcal{A}_{CS} = \mathrm{J} \int w^{\mu\nu\alpha\beta} F_{\mu\nu} A_\alpha \phi_{;\beta} \; d^4x$$

$$= \mathrm{J} \int \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} A_\alpha (\ln \chi)_{,\beta} \; d^4x \tag{24}$$

where J is a dimensionless constant and $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. The action expressed in eq.(24) is invariant under diffeomorphism as well as gauge transformations, $A_\mu \to A_\mu + \partial_\mu \xi(x)$.

The upshot of adding $\mathcal{A}_{CS}$ (eq.(24)) to the standard action for an electromagnetic field interacting with charge particles in the presence of gravitation $\mathcal{A}_{EM}$ is that, upon extremizing the full action $\mathcal{A}_{EM} + \mathcal{A}_{CS}$ with respect to A_μ, one obtains the following modified Einstein-Maxwell equation,

$$F^{\alpha\beta}_{\;\;\;;\beta} = -4\pi j^\alpha + 8\pi \mathrm{J} \; w^{\mu\nu\alpha\beta} F_{\mu\nu} \chi_{,\beta} \tag{25}$$

with j^α being the 4-current density associated with charge particles.

Use of the covariant derivative to the above equation with respect to x^α leads to the charge density continuity equation that entails conservation of electric charge Q,

$$(\sqrt{-g} j^\alpha)_{,\alpha} = 0 \quad \Rightarrow \quad Q = \int j^0 \sqrt{-g} \; d^3x \quad . \tag{26}$$

Application of the above CS formulation to study its effects on magnetohydrodynamics as well as the $\vec{E}.\vec{B} \neq 0$ regions of a pulsar magnetosphere as a source of propagating $\tilde{w}$ have been studied earlier [7].

We continue adopting the procedure laid out in the paper by Jackiw and Pi to construct a Chern-Simons action for gravity in the 3+1-dimensional space-time. Instead of their fixed Lorentz vector v_μ, we use the covariant derivative of $\phi(x)$ so that,

$$\mathcal{A}_{GCS} = \mathrm{H} \int w^{\mu\nu\alpha\beta} [\Gamma^\sigma_{\nu\tau} \partial_\alpha \Gamma^\tau_{\beta\sigma} + \frac{2}{3} \Gamma^\sigma_{\nu\tau} \Gamma^\tau_{\alpha\eta} \Gamma^\eta_{\beta\sigma}] \phi_{;\mu} \; d^4x$$

$$= \mathrm{H} \int \epsilon^{\mu\nu\alpha\beta} [\Gamma^\sigma_{\nu\tau} \partial_\alpha \Gamma^\tau_{\beta\sigma} + \frac{2}{3} \Gamma^\sigma_{\nu\tau} \Gamma^\tau_{\alpha\eta} \Gamma^\eta_{\beta\sigma}] (\ln \chi)_{,\mu} \; d^4x \quad , \tag{27}$$

where H is a dimensionless, real constant. After integrating by parts once, eq.(27) can be expressed in terms of the Riemann tensor as,

$$\mathcal{A}_{GCS} = -\frac{\mathrm{H}}{2}\int \ln\chi \; {}^{*}RRd^4x \quad , \tag{28}$$

with,

$${}^{*}RR \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R^{\tau}{}_{\sigma\alpha\beta}R^{\sigma}{}_{\tau\mu\nu} = 8\Big[R^{\tau\sigma}{}_{01}R_{\sigma\tau 23} + R^{\tau\sigma}{}_{12}R_{\sigma\tau 03} + R^{\tau\sigma}{}_{13}R_{\sigma\tau 20}\Big] \tag{29}$$

and,

$${}^{*}R^{\tau\rho\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R^{\tau\rho}{}_{\alpha\beta} \tag{30}$$

We have used the following conventions pertaining to the Riemann curvature and Ricci tensors,

$$R^{\tau}{}_{\sigma\alpha\beta} = \partial_\alpha\Gamma^{\tau}_{\sigma\beta} - \partial_\beta\Gamma^{\tau}_{\sigma\alpha} + \Gamma^{\tau}_{\alpha\eta}\Gamma^{\eta}_{\beta\sigma} - \Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\alpha\sigma}$$

and $R_{\alpha\beta} = R^{\tau}{}_{\alpha\tau\beta}$. From eq.(27) we note that $\ln\chi$ acts like the parameter θ in Jackiw and Pi 's paper in which the external vector v_μ is set as $\theta_{,\mu}$. Adding $\mathcal{A}_{CS}+\mathcal{A}_{GCS}$ to $\mathcal{A}$ of eq.(19) and then extremizing the full action with respect to ϕ and $g_{\mu\nu}$ leads to,

$$\chi^{\;;\alpha}_{\;;\alpha} = \frac{1}{2}\Big[g^{\mu\nu}\frac{\chi_{,\mu}\chi_{,\nu}}{\chi} + \frac{a_2}{a_1}\chi + \frac{\mathrm{J}}{2a_1}\chi\; w^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} - \frac{\mathrm{H}}{4a_1}\chi\; w^{\mu\nu\alpha\beta}R^{\tau}{}_{\sigma\alpha\beta}R^{\sigma}{}_{\tau\mu\nu}\Big], \tag{31}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{m_{Pl}^2}[T_{\mu\nu} + \Theta_{\mu\nu} + C_{\mu\nu}], \tag{32}$$

where the modified Cotton tensor $C^{\mu\nu}$ is defined as,

$$C^{\mu\nu} \equiv -2\frac{\mathrm{H}}{\sqrt{-g}}\Big[\frac{1}{4}{}^{*}RRg^{\mu\nu} - (\ln\chi)_{;\alpha;\beta}\Big({}^{*}R^{\beta\mu\alpha\nu} + {}^{*}R^{\beta\nu\alpha\mu}\Big) + (\ln\chi)_{,\alpha}\Big(\epsilon^{\alpha\mu\sigma\tau}R^{\nu}{}_{\sigma;\tau} + \epsilon^{\alpha\nu\sigma\tau}R^{\mu}{}_{\sigma;\tau}\Big)\Big]. \tag{33}$$

The first term in the RHS of Eq.(33) is new and is not present in the expression for the Cotton tensor as delineated in [6]. It appears in our work because under an infinitesmal variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, the change in $(\ln\chi)_{,\mu}$ occurring in eq.(28) is given by,

$$\delta(\ln\chi)_{,\mu} = \delta(\phi_{;\mu}/\phi) = -\delta\Gamma^{\alpha}_{\alpha\mu} = -\frac{1}{2}\delta(g^{\alpha\beta}g_{\alpha\beta,\mu}) \quad . \tag{34}$$

However, when the equation of motion for the dynamical four-form given by eq.(31) is substituted in eq.(22), its energy-momentum tensor in the presence of CS-term takes the form,

$$\Theta_{\mu\nu} = 2a_1\Big[\frac{\chi_{,\mu}\chi_{,\nu}}{\chi} - \frac{g_{\mu\nu}}{2}\Big(g^{\alpha\beta}\frac{\chi_{,\alpha}\chi_{,\beta}}{\chi} + \frac{a_2}{a_1}\chi\Big)\Big] + \frac{H}{2\sqrt{-g}}{}^{*}RRg_{\mu\nu} \tag{35}$$

so that when eq.(35) is substituted in eq.(32) the first term in the RHS of eq.(33) cancels with the last term in the RHS of eq.(35). In other words, the term ${}^*RRg_{\mu\nu}$ does not appear in or contribute to the Einstein equations given by eq.(32).

Assuming that the geometrodynamical four-form is a dark energy candidate, the above dynamical equations were studied in an earlier work to address the observed late time acceleration of the expansion rate of the universe ([1] and the references therein). Our objective, in this article, is somewhat different since we propose here that the particle associated with $\tilde{w}$ is a pseudo-scalar which acts as a cold dark matter (CDM) candidate aiding in the formation of supermassive black holes when the universe was young. This is the subject of Section IV.

(iii) Generalized Exterior Derivative

Apart from the possibility of torsion and Chern-Simons extensions ensuing from the four-form field $\tilde{w}$, there may also be another differential geometric significance of this field. The scalar-density ϕ can be used to have an antiderivation acting on antisymmetric tensor-densities of arbitrary weights.

Suppose $\tilde{\alpha}$ is a p-form density with weight w such that its components $\alpha_{\nu_1\nu_2..\nu_p}$ transform to $J^{-w}(x,x')\ \alpha_{\nu_1\nu_2..\nu_p}$, under a general coordinate transformation $x \to x'$, $J(x,x')$ being the corresponding Jacobian. We define a generalized exterior derivative $\tilde{d}_w$ acting on $\tilde{\alpha}$ in the following manner,

$$\begin{aligned}\tilde{d}_w\tilde{\alpha} \equiv \frac{1}{p!}\ \partial_\mu\alpha_{\nu_1\nu_2..\nu_p}\ \tilde{d}x^\mu \wedge \tilde{d}x^{\nu_1} \wedge ... \wedge \tilde{d}x^{\nu_p} - \\ -w\partial_\mu(\ln\phi)\ \tilde{d}x^\mu \wedge \tilde{\alpha} = \tilde{d}\tilde{\alpha} - w\tilde{d}\ (\ln\phi)\ \wedge \tilde{\alpha}.\end{aligned}$$

It is easy to see that $\tilde{d}_w\tilde{\alpha}$ is a (p+1)-form density of weight w. If α_1 and α_2 are scalar-densities of weights w_1 and w_2, respectively, then the above equation leads to,

$$\tilde{d}_w\alpha_i = \partial_\mu\alpha_i\ \tilde{d}x^\mu - w_i\alpha_i\ \partial_\mu(\ln\phi)\ \tilde{d}x^\mu, \quad i = 1,2 \tag{36}$$

which are one-form densities and, furthermore, one can show that,

$$\tilde{d}_w(\alpha_1\tilde{d}_w\alpha_2) = \tilde{d}_w\alpha_1 \wedge \tilde{d}_w\alpha_2. \tag{37}$$

This generalized exterior derivative also satisfies (a) $\tilde{d}_w \tilde{d}_w = 0$ and (b) $\tilde{d}_w(\tilde{\alpha} \wedge \tilde{\beta}) = \tilde{d}_w \tilde{\alpha} \wedge \tilde{\beta} + (-1)^p \tilde{\alpha} \wedge \tilde{d}_w \tilde{\beta}$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are p- and q-form densities of weights w_1 and w_2, respectively.

These properties are sufficient to qualify $\tilde{d}_w$ to be a well defined antiderivation on differential form-densities [5].

From eq.(36) it follows that,

$$\tilde{d}_w \sqrt{-g} = -\sqrt{-g}\ \tilde{d} \ln(\frac{\phi}{\sqrt{-g}}), \tag{40}$$

since $\sqrt{-g}$ is a scalar-density of weight +1. There are other physically meaningful tensor densities e.g. the scalar density Dirac delta function and antisymmetric tensor-densities, e.g. dual of $F^{\mu\nu}$, on which $\tilde{d}_w$ can act.

In passing, we note that $\tilde{d}_w \phi = 0$, which is analogous to $g_{\mu\nu;\lambda} = 0$.

IV. DARK BOSONS, BOSE-EINSTEIN CONDENSATES AND FORMATION OF SUPERMASSIVE BLACK HOLES

The wave-particle duality of quantum mechanics transcends to field-particle duality in quantum field theory. Particles associated with quantum fields are intimately linked with their Poincare' algebra and special relativistic covariance in the standard framework of quantum field theory (QFT). In the context of the four-form field $\tilde{w}$, or equivalently the scalar-density $\phi(x)$, we may ask what kind of particle is associated with it?

Now, $\phi(x)$ in the geometrically flat Minkowski space-time transforms to,

$$\phi \to \phi' = J(t, \vec{r}; t, -\vec{r})\ \phi = -\phi \quad , \tag{41}$$

under a space reversal $\vec{r} \to -\vec{r}$ as the Jacobian $J(t, \vec{r}; t, -\vec{r}) = -1$. Since ϕ changes its sign under a parity transformation, the quantum particle associated with the field $\tilde{w}$ is a pseudo scalar, and thus is a boson.

Because such a pseudo scalar is likely to interact with matter with a strength at the most comparable with that of gravity, we propose here that $\tilde{w}$ is the candidate for ultra-light cold dark matter (CDM), although such ultralight CDM has been linked with axions in the literature (e.g. see [8], and the references therein). Can this help us in solving the intriguing problem of supermassive black holes (SMBHs)?

So far, around a million quasars have been observed, each powered by accretion of matter onto SMBHs weighing $\gtrsim 10^7 M_\odot$ [9–12]. Quasar J0100+2802 found at a redshift, $z = 6.33$, is associated with a SMBH of mass $\approx 10^{10}\ M_\odot$, while another (J2157–3602) at a redshift, $z = 4.7$, has a black hole weighing $\cong 2.4 \times 10^{10}\ M_\odot$. Similarly, quasars J1120+0641 and J1342+0928 located at redshifts $z = 7.09$ and $z = 7.54$, respectively, are associated with SMBHs with mass $\gtrsim 10^8\ M_\odot$. These findings pose a severe challenge to the existing models that invoke seed black holes (BHs) growing via accretion, since one needs extremely large seed BHs with mass $\gtrsim 10^3\ M_\odot$ at $z \gtrsim 40$ [12, 13].

Recently, using the JWST observations, a very distant SMBH with mass $\sim 10^6\ M_\odot$ has been discovered at a redshift of 10.6 [14]. Of course, even heavier SMBHs at lower redshifts, like with mass $\cong 4 \times 10^{10}\ M_\odot$ at the centre of Holm 15A galaxy belonging to the galaxy cluster Abell 85 have been detected [15]. At a redshift of 3.96, a SMBH weighing $\cong 1.7 \times 10^{10}\ M_\odot$, accreting matter at a rate one solar mass per day has been seen [16].

In order to address the formation of SMBHs when the universe was $\lesssim 10^9$ yrs old, we had earlier put forward a scenario using the framework of Gross-Pitaevskii equation in which ultra heavy BHs are created because of gravitational contraction of Bose-Einstein condensates (BECs) of ultra-light bosonic CDM [17, 18]. In this section, we provide a simple semi-classical analysis that captures the essential physics of the problem.

Considering a galactic DM halo of size R_h that is constituted of these gravitationally bound dark bosons of mass m, a significant fraction of the bosons that have sufficiently low momenta p would form a BEC if their de Broglie wavelength,

$$\lambda_{DB} \sim \frac{h}{p} \gtrsim \left(\frac{3N}{4\pi R_h^3}\right)^{-1/3} = R_h\left(\frac{3N}{4\pi}\right)^{-1/3} = R_h\left(\frac{3M}{4\pi m}\right)^{-1/3} \qquad (42) \quad ,$$

where M and N are the total mass of dark bosons and their number, respectively, making up the BEC. Eq.(42) represents the condition for BEC formation since it entails that the de Broglie wavelength be larger than the mean separation between nearby bosons.

In the context of the CDM scenario and assuming a positive cosmological constant, the energy E_b of a typical dark boson is given by [19],

$$E_b \sim \frac{p^2}{2m} - \frac{GMm}{R_h} + \frac{\Lambda\ r^2}{6} mc^2 < 0 \qquad (43) \quad .$$

The condition $E_b < 0$ follows from the fact that the DM halo is a gravitationally bound structure. Hence,

$$p^2 < \frac{2GMm^2}{R_h} - \frac{\Lambda\ r^2}{3} m^2 c^2 \qquad (44)\ \ .$$

Since a very weakly interacting dark boson can be anywhere within the DM halo, application of the Heisenberg's uncertainty principle implies that,

$$\Delta p \sim p \gtrsim \frac{\hbar}{2R_h} \qquad (45)\ .$$

From eqs.(42) and (45), we have an inequality,

$$\frac{h}{4\pi p} \lesssim R_h \lesssim \frac{h}{p}\left(\frac{3N}{4\pi}\right)^{1/3} \qquad (46)\ .$$

that is self-consistent since $N \gg 1$.

Using the result $p \sim \frac{\hbar}{2R_h}$ from eq.(45) in eq.(43), we obtain,

$$E_b(R_h) \sim \frac{\hbar^2}{8mR_h^2} - \frac{GMm}{R_h} + \frac{\Lambda\ R_h^2}{6} mc^2 < 0 \qquad (47)\ \ ,$$

that leads to a condition,

$$R_h \gtrsim \frac{\hbar^2}{8GMm^2}\left(1 - \frac{R_h^3 \Lambda c^2}{6GM}\right)^{-1} \qquad (48)\ \ .$$

In the context of the ΛCDM model, the numerical value of the cosmological constant that ensues from the cosmological constant density parameter, $\Omega_{\Lambda,0} \cong 0.7$ and the Hubble parameter, $\mathrm{H}_0 \cong 68$ km/s/Mpc, is $\Lambda \cong 10^{-56}\ cm^{-2}$.

Even with a low dark matter halo mass $M \cong 10^6\ M_\odot$, one finds $\Lambda c^2/6GM \cong 10^{-68}\ cm^{-3}$. Hence, for galactic scales $R_h \lesssim 10^{22}\ cm$, $R_h^3 \Lambda c^2/6GM \ll 1$ so that eq.(48) can be approximated to,

$$R_h \gtrsim \frac{\hbar^2}{8GMm^2}\left(1 + \frac{R_h^3 \Lambda c^2}{6GM}\right) \approx \frac{\hbar^2}{8GMm^2} \qquad (49)\ \ .$$

By minimizing E_b with respect to R_h, one can obtain an estimate of the BEC size,

$$\frac{\partial E_b}{\partial R_h} = \frac{\Lambda\ R_h}{3} mc^2 + \frac{GMm}{R_h^2} - \frac{\hbar^2}{4mR_h^3} = 0 \qquad (50)$$

which leads to a quartic equation to be solved,

$$\Lambda R_h^4 + \frac{3GM}{c^2} R_h - \frac{3\hbar^2}{4m^2c^2} = 0 \qquad (51)\ \ .$$

The above equation can be solved exactly and, out of the four roots, only one is real and positive. However, given the smallness of the quantity $\lambda_M \equiv \Lambda c^2/6GM \cong 10^{-68}\ cm^{-3}$ in our case, even up to the cubic expansion of this root, R_{bec}, in terms of λ_M, the solution to eq.(51) corresponding to the minimum energy configuration of the BEC is (upto λ_M^3 orders),

$$R_{bec} \cong \frac{\hbar^2}{4GMm^2} \cong 22\Big(\frac{10^9 M_\odot}{M}\Big)\Big(\frac{10^{-22}\ \mathrm{eV}}{m}\Big)^2\ \mathrm{pc} \qquad (52)\ ,$$

corresponding to a single boson energy,

$$E_{min} = E_b(R_{bec}) \cong -\frac{GMm}{2R_{bec}} = -2\ mc^2\Big(\frac{m_{Pl}^2}{m\ M}\Big)^{-2} \qquad (53)\ .$$

Eqs.(52) and (53) are exact when $\Lambda = 0$, ensuing trivially from eq.(51). These equations entail that for larger BEC mass M not only its size is smaller, it is also more tightly bound since its energy is more negative. The physical repercussion of these features is that for a sufficiently large mass M, R_{bec} can be very close to the corresponding Schwarzschild radius $R_s \equiv 2GM/c^2$ so that even a slight perturbation would cause an irreversible gravitational collapse leading to the formation of a BH.

The BEC would form a BH if its size R_{bec} goes below the Schwarzschild radius R_S,

$$R_S = \frac{2GM}{c^2} = \frac{2M}{m_{Pl}^2} \qquad (54)\ ,$$

where m_{Pl} is the Planck mass.

Using the criteria $R_{bec} \lesssim R_S$ along with eqs.(52) to (54), the condition for the BEC to implode into a BH is given by the inequality,

$$\Big(\frac{m_{Pl}^2}{4M\ m}\Big)^2 \lesssim 1 \qquad (55)\ ,$$

$$\Rightarrow\ m\ M \gtrsim\ 0.25\ m_{Pl}^2 = 3\times 10^9\ \Big(\frac{m}{10^{-20}\ \mathrm{eV}}\Big)^{-1} M_\odot \qquad (56)\ .$$

A more rigorous analysis carried out earlier by employing a Gross-Pitaevskii equation framework had shown that the dynamical evolution of ultra-light dark bosons in the BEC phase leads to the formation of a BH on time scales of 10^8 years with a similar constraint given by eq.(56) [17, 18].

The Hawking temperature T_{BH} of a BH of mass M is given by [20],

$$k_B T_{BH} = \frac{m_{Pl}^2}{8\pi M} \qquad (57)\ .$$

It is interesting to note that when eq.(56) is substituted in eq.(57), one gets,

$$k_B T_{BH} \lesssim \frac{m}{2\pi} \qquad (59)\ ,$$

implying that a SMBH created from the collapse of a BEC of dark bosons would be predominantly radiating massless particles as well as these dark bosons through Hawking evaporation!

V. PROPAGATION OF GRAVITATIONAL WAVES IN THE PRESENCE OF THE GEOMETRODYNAMICAL FOUR-FORM

We consider in this section the effect of $\tilde{w}$ on the propagation of gravitational waves (GWs) by assuming that, except for a weak GW and the geometrodynamical four form, there is no other matter present. Considering a traceless-transverse (TT)-gauge for a plane GW travelling along the x-axis,

$$h_{\oplus} \equiv h_{22}(t,x) = -h_{33}(t,x) \qquad h_{\otimes} \equiv h_{23}(t,x) = h_{32}(t,x)$$

we express eqs.(19) and (28), by making the weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and retaining only upto quadratic terms in $h_{\oplus}$ and $h_{\otimes}$ for the gravitational part of the action [21],

$$\mathcal{A} \approx -\frac{m_{Pl}^2}{8\pi}\int [h_{\oplus}(\ddot{h}_{\oplus} - h''_{\oplus}) + h_{\otimes}(\ddot{h}_{\otimes} - h''_{\otimes})]d^4x + \mathcal{A}_{GCS} + \mathcal{A}_{\phi} \tag{60}$$

with,

$$\begin{aligned}\mathcal{A}_{GCS} \approx -H\int \eta^{\tau\lambda}\Big(\frac{\partial^2 \ln\chi}{\partial x\partial x^{\tau}}(h_{\oplus}h_{\otimes,0\lambda} - h_{\otimes}h_{\oplus,0\lambda}) + \frac{\partial^2 \ln\chi}{\partial t\partial x^{\tau}}(h_{\otimes}h_{\oplus,1\lambda} - h_{\oplus}h_{\otimes,1\lambda})\Big)d^4x\\ -H\int \frac{\partial \ln\chi}{\partial x}\Big(h_{\oplus}(\ddot{h}_{\otimes,0} - h''_{\otimes,0}) - h_{\otimes}(\ddot{h}_{\oplus,0} - h''_{\oplus,0})\Big)d^4x\\ +H\int \frac{\partial \ln\chi}{\partial t}\Big(h_{\oplus}(\ddot{h}_{\otimes,1} - h''_{\otimes,1}) - h_{\otimes}(\ddot{h}_{\oplus,1} - h''_{\oplus,1})\Big)d^4x\end{aligned}$$

and,

$$\mathcal{A}_{\phi} = \int \Big[\frac{a_1}{\chi}\eta^{\mu\nu}\chi_{,\mu}\chi_{,\nu} + a_2\chi - \frac{a_1}{\chi}\Big(h_{\oplus}(\chi_{,2}\chi_{,2} - \chi_{,3}\chi_{,3}) + 2h_{\otimes}\chi_{,2}\chi_{,3}\Big)\Big]d^4x$$

After defining $\psi(x^{\mu}) \equiv \ln(\chi(x^{\mu}))$, the dynamical equations of motion that ensue from variations of $\mathcal{A}$ (eq.(60)) with respect to $h_{\oplus}$, $h_{\otimes}$ and ψ are given by,

$$\begin{aligned}\ddot{h}_{\oplus} - h''_{\oplus} = -\frac{8\pi}{m_{Pl}^2}\Big[H\Big\{\frac{\partial^2\psi}{\partial t\partial x}(\ddot{h}_{\otimes} + h''_{\otimes}) - h_{\otimes,01}\Big(\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x^2}\Big) + \frac{\partial\psi}{\partial x}(\ddot{h}_{\otimes,0} - h''_{\otimes,0}) - \frac{\partial\psi}{\partial t}(\ddot{h}_{\otimes,1} - h''_{\otimes,1})\Big\} +\\ + a_1\exp(\psi)\Big\{\Big(\frac{\partial\psi}{\partial y}\Big)^2 - \Big(\frac{\partial\psi}{\partial z}\Big)^2\Big\}\Big]\end{aligned} \tag{61}$$

$$\ddot{h}_\otimes - h''_\otimes = \frac{8\pi}{m_{Pl}^2}\Big[H\Big\{\frac{\partial^2\psi}{\partial t\partial x}(\ddot{h}_\oplus + h''_\oplus) - h_{\oplus,01}\Big(\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x^2}\Big) + \frac{\partial\psi}{\partial x}(\ddot{h}_{\oplus,0} - h''_{\oplus,0}) - \frac{\partial\psi}{\partial t}(\ddot{h}_{\oplus,1} - h''_{\oplus,1})\Big\} +$$

$$+2a_1 \exp(\psi)\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial z}\Big] \tag{62}$$

and

$$\partial^\mu\partial_\mu\psi + \frac{1}{2}\eta^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu} - \frac{a_2}{4a_1} = h_\oplus\Big[\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} + \frac{1}{2}\Big(\frac{\partial\psi}{\partial y}\Big)^2 - \frac{1}{2}\Big(\frac{\partial\psi}{\partial z}\Big)^2\Big] + 2h_\otimes\Big[\frac{\partial^2\psi}{\partial y\partial z} + \frac{1}{2}\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial z}\Big]\,. \tag{63}$$

(The derivatives with respect to time t and x-coordinate are denoted by dot and prime, respectively.)

A simplification occurs if we assume ψ to depend only on the x-coordinate and time, like the GW amplitude itself. With $\psi = \psi(t,x)$, the last terms in eqs.(61) and (62) drop out. Introducing the complex circularly polarized GW amplitudes [21],

$$h_R = \frac{1}{\sqrt{2}}(h_\oplus + ih_\otimes) \qquad h_L = \frac{1}{\sqrt{2}}(h_\oplus - ih_\otimes)$$

one finds from eqs.(61) and (62) that,

$$\ddot{h}_R - h''_R = -\frac{8\pi Hi}{m_{Pl}^2}\Big[\dot{\psi}\frac{\partial}{\partial x}(\ddot{h}_R - h''_R) - \psi'\frac{\partial}{\partial t}(\ddot{h}_R - h''_R) + \dot{h}'_R(\ddot{\psi} + \psi'') - \dot{\psi}'(\ddot{h}_R + h''_R)\Big] \tag{64}$$

$$\ddot{h}_L - h''_L = \frac{8\pi Hi}{m_{Pl}^2}\Big[\dot{\psi}\frac{\partial}{\partial x}(\ddot{h}_L - h''_L) - \psi'\frac{\partial}{\partial t}(\ddot{h}_L - h''_L) + \dot{h}'_L(\ddot{\psi} + \psi'') - \dot{\psi}'(\ddot{h}_L + h''_L)\Big] \tag{65}$$

Since $\psi = \psi(t,x)$, the equation of motion for the four-form given by eq.(63) simplifies to,

$$\ddot{\psi} - \psi'' + \frac{1}{2}(\dot{\psi}^2 - \psi'^{\,2}) + 2\mu^2 = 0 \tag{66}$$

where $\mu \equiv \sqrt{-\frac{a_2}{4a_1}}$ acts as the mass of the four-form field. Seeking exact solutions, involving monochromatic and circularly polarized GWs, we substitute,

$$h_R(t,x) = h\exp(i(\omega t - kx))$$

in eq.(64), leading to a relation,

$$\omega^2 - k^2 = -\beta[k(\omega^2 - k^2)\dot{\psi} + \omega(\omega^2 - k^2)\psi' - i\omega k(\ddot{\psi} + \psi'') - i(\omega^2 + k^2)\dot{\psi}'] \tag{67}$$

where $\beta \equiv \frac{8\pi H}{m_{Pl}^2}$.

If $w = \pm k$, eq.(67) implies,

$$2\dot{\psi}' \pm (\ddot{\psi} + \psi'') = 0 \quad . \tag{68}$$

The only self-consistent solution of eqs.(66) and (68) is,

$$\psi(t,x) = (b_0 - \frac{\mu^2}{b_0})t + (b_0 + \frac{\mu^2}{b_0})x \tag{69}$$

where b_0 is an integration constant. But the above solution is unphysical, as it implies an exponentially growing ψ. On the other hand, when $w^2 > k^2$,

$$k\dot{\psi} + \omega\psi' + \frac{1}{\beta} = 0 \Rightarrow \psi'' = \frac{k^2}{w^2}\,\ddot{\psi} \tag{70}$$

and therefore, $\psi = \psi(x - \frac{\omega}{k}t)$, leading to an exact solution for $b_1 \geq \lambda$,

$$\psi = \ln\left(\sqrt{\frac{b_1}{\lambda}}\cos^2\left(\pm\sqrt{\frac{\lambda}{2}}(t - \frac{k}{\omega}x) + b_2\right)\right) \tag{71}$$

where $\lambda \equiv \frac{2\omega^2\mu^2}{\omega^2 - k^2}$, while b_1 and b_2 are integration constants.

Hence, $\chi(t - \frac{k}{\omega}x) = \exp(\psi) \propto \cos^2\left(\pm\sqrt{\frac{\lambda}{2}}(t - \frac{k}{\omega}x) + b_2\right)$, with $|\omega| > |k|$, is physically meaningful and is an acceptable solution. Indeed, we find that it is possible to have circularly polarized, monochromatic GWs with phase velocity exceeding the speed of light. The caveat, however, is that eq.(66) being a nonlinear differential equation, deriving exact solutions corresponding to GWs with superposed wave modes is non-trivial.

VI. CONCLUSIONS

Based on the invariances of the Minkowski tensor and the flat space-time Levi-Civita tensor under proper Lorentz transformations, the present study explores the possibility of extending Einstein's geometrical theory of gravitation by including another geometrical degree of freedom $\tilde{w}$, which is simply a generalisation of the Levi-Civita symbol, in the theory. This geometrodynamical four-form $\tilde{w}$ leads not only to a dynamical torsion it also generates Chern-Simon (CS) extensions in the 3+1-dimensional space-times. Torsion, among several other interesting implications, is also important in teleparallel gravity [22].

Pursuing closely the seminal work of Jackiw and Pi [6] but avoiding the use of an unphysical Lorentz vector, CS coupling between electromagnetic fields and $\tilde{w}$ comes about naturally, bringing about a modification of Einstein-Maxwell equations. Adopting once again the formulation of [6], a gravitational CS term is constructed that leads to a modified Cotton tensor. However, when the dynamical equation of the four-form is used in the ensuing Einstein equation, it is shown that it is the standard Cotton tensor that affects the space-time dynamics.

The scalar-density ϕ associated with $\tilde{w}$ leads to a well-defined exterior derivative that turns a differential p-form density into a (p+1)-form density of same weight. Since the notion of an n-form and exterior derivative in a differential manifold does not require either an affine connection or a metric, further studies are required to investigate the role of $\tilde{w}$ in situations where metric is degenerate as well as its impact on the manifold-orientability.

The particle aspects of the geometrodynamical four-form field is considered and is shown to be a pseudo-scalar. It is shown that the Bose-Einstein condensates of such pseudo-scalars can give rise to formation of supermassive black holes through an interplay of self-gravity and quantum mechanics. Therefore, self-gravitating BECs of pseudo-scalar particles associated with the geometrodynamical four-form field may solve the long standing problem concerning frequent discovery of tens of billion solar mass SMBHs at epochs when the universe was barely a billion years old.

It is also demonstrated that in the presence of the dynamical four-form, gravitational waves can be decoupled using circularly polarized waveforms. Moreover, exact, self-consistent monochromatic gravitational wave solutions of the wave equation can be obtained. However, much more work is needed to obtain realistic solutions of gravitational waves, with group velocities $\leq c$, propagating in the background consisting of $\tilde{w}$.

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