

Geometric Characteristics in Phaseless Operator and Structured Matrix Recovery*

Gao Huang[†] and Song Li[‡]

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Abstract

In this paper, we first propose a unified approach for analyzing the stability of phaseless operator for both amplitude and intensity measurement on an arbitrary geometric set, thus characterizing the robust performance of phase retrieval via the empirical minimization method. The unified analysis entails randomly embedding a concave lifting operator in tangent space. Similarly, we investigate the structured matrix recovery problem through the robust injectivity of a linear rank-one measurement operator on an arbitrary matrix set, where the core of our analysis lies in bounding the empirical chaos process. We introduce Talagrand's γ_α -functionals to characterize the relationship between the required number of measurements and the geometric constraints. We also generate adversarial noise to demonstrate the sharpness of the recovery bounds in these two scenarios.

Keywords: Phaseless Operator; Low Rank Plus Sparse; Talagrand's Functionals; Empirical Chaos Process; Adversarial Noise.

1 Introduction

Phase retrieval refers to the problem of reconstructing an unknown signal $\mathbf{x}_0 \in \mathbb{F}^n$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ from m phaseless measurements of the form

$$b_k = \mathcal{A}_k^\ell(\mathbf{x}_0), \quad k = 1, \dots, m. \quad (1)$$

Here, the sample $\mathcal{A}_k^\ell(\mathbf{x}_0)$ may be in two forms: For $\ell = 1$, the amplitude measurement is $\mathcal{A}_k^1(\mathbf{x}_0) = |\langle \phi_k, \mathbf{x}_0 \rangle|$, and for $\ell = 2$, the intensity measurement is $\mathcal{A}_k^2(\mathbf{x}_0) = |\langle \phi_k, \mathbf{x}_0 \rangle|^2$. The

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[†]School of Mathematical Science, Zhejiang University, Hangzhou 310027, P. R. China, E-mail address: hgmath@zju.edu.cn

[‡]Corresponding Author, School of Mathematical Science, Zhejiang University, Hangzhou 310027, P. R. China, E-mail address: songli@zju.edu.cn

collection of measurement vectors $\Phi = \{\phi_k\}_{k=1}^m$ in $V = \mathbb{F}^n$ is known. Phase retrieval has gained significant attention in the past few decades in various fields due to its wide range of applications, including X-ray crystallography, astronomy, optics, and diffraction imaging [7, 27].

As far as applications are concerned, the robust performance of reconstruction is perhaps the most important consideration. We investigate the phaseless measurements with bounded noise \mathbf{z} :

$$\mathbf{b} = \mathcal{A}_\Phi^\ell(\mathbf{x}_0) + \mathbf{z}, \quad (2)$$

where $\mathcal{A}_\Phi^\ell(\mathbf{x})$ is phaseless operator $\mathcal{A}_\Phi^\ell : \mathbb{F}^n \rightarrow \mathbb{R}^m$ defined by

$$\mathcal{A}_\Phi^\ell(\mathbf{x}) = \begin{pmatrix} |\langle \phi_1, \mathbf{x} \rangle|^\ell \\ \vdots \\ |\langle \phi_m, \mathbf{x} \rangle|^\ell \end{pmatrix}. \quad (3)$$

One of the main goals of this paper is to establish a unified framework for amplitude and intensity measurement that obtains stable recovery conditions while disregarding specific recovery approaches and establishing recovery assurances within the framework of the empirical minimization method. These two motivations point to our focus on the stability of the phaseless operator \mathcal{A}_Φ^ℓ .

The stability of \mathcal{A}_Φ^ℓ can infer stable uniqueness, signifying the identification of conditions that lead to the determination of a unique solution. It has demonstrated that if there is no additional information about \mathbf{x}_0 , a unique recovery requires at least $m = \mathcal{O}(n)$ measurements [1, 2, 19]. However, prior assumptions on \mathbf{x}_0 , such as sparse, can greatly reduce the number of measurements [19, 44]. Thus, we assume that $\mathbf{x}_0 \in \mathcal{K}$ to capture the geometric structure of the signal. We are concerned about the relationship between the number of measurements m that ensure the unique or stable recovery of \mathbf{x}_0 and the intrinsic geometric properties of the set \mathcal{K} . In many cases of interest, \mathcal{K} behaves as if it is a low-dimensional set; thus, m is significantly smaller than the dimension n . We now define the stability of the phaseless operator \mathcal{A}_Φ^ℓ on the set \mathcal{K} .

Definition 1 (Stability of \mathcal{A}_Φ^ℓ). For $q \geq 1, \ell = 1, 2$, $\mathcal{A}_\Phi^\ell : \mathbb{F}^n \rightarrow \mathbb{R}^m$ is C -stable on a set \mathcal{K} , if for every $\mathbf{u}, \mathbf{v} \in \mathcal{K}$,

$$\|\mathcal{A}_\Phi^\ell(\mathbf{u}) - \mathcal{A}_\Phi^\ell(\mathbf{v})\|_q \geq C d_\ell(\mathbf{u}, \mathbf{v}). \quad (4)$$

Here are the different metrics we use: For $\ell = 1$, if $\mathbb{F} = \mathbb{C}$, $d_1(\mathbf{u}, \mathbf{v}) = \min_{\theta \in [0, 2\pi)} \|\mathbf{u} - e^{i\theta}\mathbf{v}\|_2$ and if $\mathbb{F} = \mathbb{R}$, $d_1(\mathbf{u}, \mathbf{v}) = \min\{\|\mathbf{u} - \mathbf{v}\|, \|\mathbf{u} + \mathbf{v}\|\}$. For $\ell = 2$, $d_2(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^*\|_F$. It can be seen that the definition of \mathcal{A}_Φ^ℓ can eliminate issues where $\mathcal{A}_\Phi^\ell(\mathbf{u}) = \mathcal{A}_\Phi^\ell(\mathbf{v})$ whenever $\mathbf{u} = c\mathbf{v}$ for some scalar c of unit modulus.

As mentioned above, the most natural way to estimate \mathbf{x}_0 is via an empirical ℓ_q ($q \geq 1$) risk minimization such that

$$\begin{aligned} & \text{minimize} && \|\mathcal{A}_\Phi^\ell(\mathbf{x}) - \mathbf{b}\|_q \\ & \text{subject to} && \mathbf{x} \in \mathcal{K}. \end{aligned} \quad (5)$$

We assume that $\mathbf{x}_0 \in \mathcal{K}$ and set the solutions of (5) as \mathbf{x}_*^ℓ . The robust performance of model (5) then refers to estimating the error between \mathbf{x}_*^ℓ and \mathbf{x}_0 .

By lifting intensity measurement, phase retrieval can be cast as a structured matrix recovery problem [7]. The latter also has a wide range of applications, including face recognition, recommender systems, linear system identification and control; see e.g., [38].

We focus on the structured matrix recovery problem from rank-one measurements, given the linear operator $\mathbf{A}_\Phi : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^m$ and a corrupted vector of measurements as

$$\mathbf{b} = \mathbf{A}_\Phi(\mathbf{X}_0) + \mathbf{z}, \quad (6)$$

with $\mathbf{A}_\Phi(\mathbf{X}_0)(k) = \langle \phi_{a,k} \phi_{b,k}^*, \mathbf{X}_0 \rangle$. We wish to estimate $\mathbf{X}_0 \in \mathbb{R}^{n_1 \times n_2}$ with a specific structure, for instance, low rank or low rank plus sparse. Here, \mathbf{z} is assumed to be bounded and we can also choose symmetric rank-one measurement matrices $\Phi = \{\phi_k \phi_k^*\}_{k=1}^m$. The advantage of rank-one measurement systems over so-called Gaussian measurement systems is that the former requires much less storage space than the latter [6, 13].

Estimating structured matrix \mathbf{X}_0 from measurements (6) via empirical ℓ_q ($q \geq 1$) risk minimization favored by a lot of algorithms:

$$\begin{aligned} & \text{minimize} && \|\mathbf{A}_\Phi(\mathbf{X}) - \mathbf{b}\|_q \\ & \text{subject to} && \mathbf{X} \in \mathcal{M}, \end{aligned} \quad (7)$$

where we assume $\mathbf{X}_0 \in \mathcal{M}$ to capture the geometric structure of the matrix and we assume that \mathcal{M} is symmetric. Motivation is similar to deducing the stability of \mathcal{A}_Φ^ℓ , the robust performance of model (7) requires investigating the robust injectivity of \mathbf{A}_Φ , which is defined as follows.

Definition 2 (Robust Injectivity of \mathbf{A}_Φ). For $q \geq 1$, we say $\mathbf{A}_\Phi : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^m$ is C -robust injective on a set \mathcal{M} if for all $\mathbf{X} \in \mathcal{M}$,

$$\|\mathbf{A}_\Phi(\mathbf{X})\|_q \geq C \|\mathbf{X}\|_F. \quad (8)$$

Then it is also interesting to ask for the measurement number that can ensure the unique or stable recovery of \mathbf{X}_0 based on the intrinsic geometric properties of \mathcal{M} . In summary, our goal is to address the following two specific issues:

Question I: What is the required number of measurements to ensure the stability of the phaseless operator \mathcal{A}_Φ^ℓ or the robust injectivity of \mathbf{A}_Φ related to the intrinsic geometric properties of \mathcal{K} or \mathcal{M} ?

Question II: How to evaluate the robustness performance of models (5) and (7) using the empirical minimization approach, regardless of the algorithms used?

Roadmap and Contributions: This paper is based on the assumption that the measurement vectors ϕ, ϕ_a, ϕ_b are subgaussian random vectors that satisfy certain conditions; see Section 2.3. To prove the stability of \mathcal{A}_Φ^ℓ , we first investigate the random embedding of the concave lifting operator \mathcal{B}_Φ^p on a specific tangent space, where $\mathcal{B}_\Phi^p : \mathbb{F}^{n \times n} \rightarrow \mathbb{R}^m$ is defined by:

$$\mathcal{B}_\Phi^p(\mathbf{X}) = \frac{1}{m} \begin{pmatrix} |\langle \phi_1 \phi_1^*, \mathbf{X} \rangle|^p \\ \vdots \\ |\langle \phi_m \phi_m^*, \mathbf{X} \rangle|^p \end{pmatrix}, \quad 0 < p \leq 1. \quad (9)$$

The specific tangent space refers to the fact that we set $\mathbf{X} = \mathbf{X}_{\mathbf{h},\mathbf{g}} := \mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*$ (the tangent space of rank-1 matrix) in (9) and focus on the estimate of

$$C_1 \|\mathbf{X}_{\mathbf{h},\mathbf{g}}\|_F^p \leq \|\mathcal{B}_\Phi^p(\mathbf{X}_{\mathbf{h},\mathbf{g}})\|_1 \leq C_2 \|\mathbf{X}_{\mathbf{h},\mathbf{g}}\|_F^p \quad (10)$$

where \mathbf{h} and \mathbf{g} belong to set \mathcal{K} . The stability of \mathcal{A}_Φ^1 can then be derived from the random embedding of the concave lifting operator $\mathcal{B}_\Phi^{1/2}$, while the stability of \mathcal{A}_Φ^2 is linked to the operator \mathcal{B}_Φ^1 .

The robust injectivity of \mathbf{A}_Φ can be cast as a nonnegative empirical process. Mendelson's small ball method [29, 34] can provide a lower bound for it. The connection between the geometric properties of \mathcal{M} and the measurement number m is determined by the empirical process in the small ball method (see Theorem 1). This is done by finding the upper bound of the suprema of empirical chaos process

$$\overline{S(\mathcal{M})} = \mathbb{E} \sup_{\mathbf{X} \in \mathcal{M}} \left\langle \sum_{k=1}^m \phi_{a,k} \phi_{b,k}^*, \mathbf{X} \right\rangle \quad (11)$$

for an arbitrary set \mathcal{M} . This illuminates our utilization of the generic chaining method [40].

Talagrand's γ_α -functionals are employed to quantify the geometric characteristic for the random embedding of \mathcal{B}_Φ^p and the suprema of empirical chaos process $\overline{S(\mathcal{M})}$. These mean that Talagrand's γ_α -functionals associated with \mathcal{K} or \mathcal{M} can determine the necessary measurement number m to guarantee the stability of \mathcal{A}_Φ^ℓ or robust injectivity of \mathbf{A}_Φ .

The main contributions of this paper can be attributed to two framework aspects. On the one hand, our work is a unified framework applicable to arbitrary geometric sets \mathcal{K} and \mathcal{M} . Talagrand's γ_α -functionals can quantify the required number of measurements to guarantee the stability of \mathcal{A}_Φ^ℓ and the robust injectivity of \mathbf{A}_Φ . Some examples that can be investigated include \mathcal{K} being a sparse set and \mathcal{M} being a low-rank plus sparse set with a near-optimal measurement number. On the other hand, the robust performance of models (5) and (7) can be achieved by ensuring the stability of \mathcal{A}_Φ^ℓ and robust injectivity of \mathbf{A}_Φ . By constructing adaptive adversarial noise \mathbf{z} , the recovery bounds $\frac{\|\mathbf{z}\|_q}{m^{1/q}}$ of the models are theoretically sharp, indicating the existence of noise \mathbf{z} such that the solutions to the models satisfy

$$d_\ell(\mathbf{x}_\star^\ell, \mathbf{x}_0) \gtrsim \frac{\|\mathbf{z}\|_q}{m^{1/q}} \quad \text{and} \quad \|\mathbf{X}_\star - \mathbf{X}_0\|_F \gtrsim \frac{\|\mathbf{z}\|_q}{m^{1/q}}.$$

As far as we know, our work is the first to show a connection between rank-one measurements (such as phase retrieval and structured matrix recovery) and Talagrand's γ_α -functionals, as well as chaos process.

Our contributions also include two specific aspects. We first introduce the stability of the phaseless operator \mathcal{A}_Φ^1 with random measurements, which was widely discussed in classical stable phase retrieval [2, 3, 23]. Thus, we extend the random stability of \mathcal{A}_Φ^2 that first considered in [19]. The author in [37] characterized the robust injectivity of \mathbf{A}_Φ with rank-one measurement by restricting the set \mathcal{M} to at most rank- R matrices; however, the required number of measurements they obtained is dependent on the additional factor R , making it

unusable if \mathcal{M} exceeds the constraint of at most rank- R . We use the empirical chaos process to demonstrate that the factor R appears to be an artifact of the proof if \mathcal{M} is symmetric.

Related Work: In [19], Eldar and Mendelson considered the stability of \mathcal{A}_Φ^2 in the real case in the sense of ℓ_1 -norm such that

$$\|\mathcal{A}_\Phi^2(\mathbf{u}) - \mathcal{A}_\Phi^2(\mathbf{v})\|_1 \geq C \|\mathbf{u} - \mathbf{v}\|_2 \cdot \|\mathbf{u} + \mathbf{v}\|_2$$

and then derived the robust performance of model (5) for the intensity case via empirical ℓ_q ($q \geq 1$) risk minimization with random noise. The Gaussian width was employed to quantify the number of measurements needed for the geometric set of \mathbf{x}_0 . [30] provided an incoherence-based analysis of the stability of \mathcal{A}_Φ^2 under random Bernoulli measurements, following the approach and set taken by [19]. The metric above is evidently unsuitable for complex case. Besides, Bandeira et al. [2] defined the stability of $\mathcal{A}_\Phi^1(\mathbf{u})$ in the real case in the presence of adversarial noise under a specific estimator with metric $d_1(\mathbf{u}, \mathbf{v})$. However, they stated that for such a metric, $\mathcal{A}_\Phi^2(\mathbf{u})$ was no longer stable in their definition. These prompt us to establish a unified form for the stability of the phaseless operator \mathcal{A}_Φ^ℓ ; see (4).

In addition, [8, 28, 32] provided the theoretical guarantee for the lifting model based on intensity measurement with positive semidefinite (PSD) cone restriction. [25, 45] concluded the robust performance of the amplitude model with \mathcal{K} being specific sets, entire space, and sparse sets, and [12, 26] concluded the intensity model. We provide a unified perspective on the robustness performance of (5) by the stability of phaseless operator \mathcal{A}_Φ^ℓ on geometric set \mathcal{K} . Consequently, specific cases, for instance, sparse sets can be derived from our consistent results, as demonstrated in Corollary 1 below.

[11, 37, 41] used Gordon's "escape through a mesh" theorem and the Gaussian width to describe geometrically the robust injectivity of \mathbf{A}_Φ for gaussian measurement. However, the coupling of random variables in rank-one measurement operator \mathbf{A}_Φ prevents the Gaussian width from reflecting the number of measurements required related to the geometric set. Talagrand's γ_α -functionals then illuminate our hope of using the chaos process for the characterization of geometric relationships. The chaos process is a powerful tool for processing structured measurements in compressed sensing [31], blind deconvolution [35], low-rank tensor recovery [24], and other related applications. The rank-one measurement has gained significant attention in recent years [6, 13, 18, 22, 33, 37], due to its applicability.

Notation: We review all notation used in this paper in order to ease readability. A variety of norms are used throughout this paper: Let $\{\sigma_k\}_{k=1}^r$ be a singular value sequence of rank- r matrix \mathbf{X} in descending order. $\|\mathbf{X}\|_* = \sum_{k=1}^r \sigma_k$ is the nuclear norm; $\|\mathbf{X}\|_F = (\sum_{k=1}^r \sigma_k^2)^{1/2}$ is the Frobenius norm; $\|\mathbf{X}\|_{op} = \sigma_1$ is the operator norm. In addition, $\Gamma(x)$ denotes the Gamma function, cone $(\mathcal{T}) := \{tx : t \geq 0, x \in \mathcal{T}\}$ is the conification of set \mathcal{T} , \mathbb{S}_{ℓ_2} denotes the ℓ_2 unit ball of \mathbb{F}^n , and \mathbb{S}_F denotes the Frobenius unit ball of $\mathbb{F}^{n_1 \times n_2}$.

Outline: The organization of the remainder of this paper is as follows: Some preliminaries are placed in Section 2. The main results are presented in Section 3. In Section 4, we give proof of the stability of the phaseless operator \mathcal{A}_Φ^ℓ and the robust performance of model (5). In Section 5, we provide proof of the robust injectivity of \mathbf{A}_Φ and the robust performance of model (7). In Section 6, we show the recovery bounds are theoretically sharp. The Appendix presents proofs of some auxiliary conclusions.

2 Preliminaries

2.1 Talagrand's Functionals

The following definition is Talagrand's γ_α -functionals [40, Definition 2.7.3] and forms the core of the geometric characteristics of this paper.

Definition 3. For a metric space (\mathcal{T}, d) , an admissible sequence of \mathcal{T} is an increasing sequence $(\mathcal{A}_n)_{n \geq 0}$ of partitions of \mathcal{T} such that for every $n \geq 1$, $|\mathcal{A}_n| \leq 2^{2^n}$ and $|\mathcal{A}_0| = 1$. We denote by $A_n(t)$ the unique element of \mathcal{A}_n that contains t . For $\alpha \geq 1$, define the γ_α -functional by

$$\gamma_\alpha(\mathcal{T}, d) = \inf_{\mathcal{T}} \sup_{t \in \mathcal{T}} \sum_{n=0}^{\infty} 2^{n/\alpha} \Delta(A_n(t)),$$

where the infimum is taken with respect to all admissible sequences of \mathcal{T} and $\Delta(A_n(t))$ denotes the diameter of $A_n(t)$ for d .

We require some properties of γ_α -functionals. The first is that they can be bounded in terms of covering numbers $\mathcal{N}(\mathcal{T}, d, u)$ by the well-known Dudley integral [40],

$$\gamma_\alpha(\mathcal{T}, d) \leq C \int_0^{\text{diam}(\mathcal{T})} (\log \mathcal{N}(\mathcal{T}, d, u))^{1/\alpha} du. \quad (12)$$

This type of entropy integral was introduced by Dudley [17] to bound the suprema of gaussian process. In addition, Sudakov's minoration inequality [40, Exercise 2.7.8] provides a lower bound for $\gamma_2(\mathcal{T}, d)$,

$$\gamma_2(\mathcal{T}, d) \geq cu \sqrt{\log \mathcal{N}(\mathcal{T}, d, u)}. \quad (13)$$

The following proposition presents the subadditivity of γ_α -functionals in vector space. Its proof can be found in [35, Lemma 2.1].

Proposition 1 (Subadditivity of γ_α -Functionals). Let (\mathcal{T}, d) be an arbitrary vector space. Suppose $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T}$. Then

$$\gamma_\alpha(\mathcal{T}_1 + \mathcal{T}_2, d) \lesssim \gamma_\alpha(\mathcal{T}_1, d) + \gamma_\alpha(\mathcal{T}_2, d).$$

2.2 Properties of ψ_s -Norm

Recall that for $s \geq 1$, the ψ_s -norm of a random variable X is defined as

$$\|X\|_{\psi_s} := \inf\{t > 0 : \mathbb{E} \exp(|X|^s / t^s) \leq 2\}. \quad (14)$$

In particular, random variable X is called subgaussian if $\|X\|_{\psi_2} < \infty$ and sub-exponential if $\|X\|_{\psi_1} < \infty$. We present several useful properties of ψ_s -norm.

Proposition 2 (Properties of ψ_s -Norm). Let X be a random variable and $s \geq 1$.

- (a) $\|X^q\|_{\psi_s} = \|X\|_{\psi_{qs}}^q$ for all $q \geq 1$. In particular, $\|X^2\|_{\psi_1} = \|X\|_{\psi_2}^2$;
- (b) $\|X\|_{\psi_s} \lesssim \|X\|_{\psi_l}$ for all $l \geq s$;
- (c) $(\mathbb{E}|X|^q)^{1/q} \lesssim q^{\frac{1}{s}} \|X\|_{\psi_s}$ for all $q \geq 1$.

Proof. See Appendix A. □

In particular, property (a) provides us with the relationship of the ψ_s -norm between random variables X and X^q . Property (b) implies that for $s \geq 1$, all random variables with finite ψ_s -norm are also sub-exponential random variables. Property (c) tells us for $q \geq 1$, all q -th moments of X exist whenever $\|X\|_{\psi_s}$ is finite.

2.3 Subgaussian Measurement

Throughout this paper, we will focus on measurement systems $\Phi = \{\phi_k\}_{k=1}^m$ in phase retrieval and $\Phi = \{\phi_{a,k}, \phi_{b,k}^*\}_{k=1}^m$ in structured matrix recovery. Here $\phi_k, \phi_{a,k}$ and $\phi_{b,k}$ are given as independent copies of a random vector ϕ , whose entries are assumed to be i.i.d. subgaussian random variables ϕ with ψ_2 -norm K , expectation $\mathbb{E}[\phi] = 0$ and variance $\mathbb{E}|\phi|^2 = 1$. It should be noted that by Proposition 2.(c) and $\mathbb{E}|\phi|^2 = 1$, we have $K \gtrsim 1$. To avoid some ambiguities, we may need an additional assumption in phase retrieval that $\mathbb{E}|\phi|^4 > 1$. For instance, in the real case of i.i.d. Rademacher random variables, it is impossible to distinguish between vector $\mathbf{x}_0 = \mathbf{e}_1$ and vector $\tilde{\mathbf{x}}_0 = \mathbf{e}_2$. Note that in this scenario we have $\mathbb{E}|\phi|^2 = \mathbb{E}|\phi|^4 = 1$. To simplify the description, we provide the following definition.

Definition 4. We call a random vector ϕ suitable K -subgaussian if its entries ϕ are i.i.d. subgaussian with ψ_2 -norm K , $\mathbb{E}[\phi] = 0$ and $\mathbb{E}|\phi|^2 = 1$. Besides, ϕ is called suitable (K, β) -subgaussian if its entries ϕ are i.i.d. subgaussian with ψ_2 -norm K , $\mathbb{E}[\phi] = 0$, $\mathbb{E}|\phi|^2 = 1$ and $\mathbb{E}|\phi|^4 \geq 1 + \beta$ for some $\beta > 0$.

Furthermore, the following two lemmas provide upper bounds for high-order moments.

Lemma 1. Let $q \geq 1$ and $\phi \in \mathbb{F}^n$ be an suitable K -subgaussian random vector. Then for any $\mathbf{X} \in \mathbb{F}^{n \times n}$,

$$(\mathbb{E}|\phi^* \mathbf{X} \phi|^q)^{1/q} \lesssim (qK^2) \cdot \|\mathbf{X}\|_F + \|\mathbf{X}\|_*. \quad (15)$$

Proof. See Appendix B.1. □

Lemma 2. Let $q \geq 1$ and $\phi_a \in \mathbb{F}^{n_1}, \phi_b \in \mathbb{F}^{n_2}$ be independent suitable K -subgaussian random vectors. Then for any $\mathbf{X} \in \mathbb{F}^{n_1 \times n_2}$,

$$(\mathbb{E}|\phi_a^* \mathbf{X} \phi_b|^q)^{1/q} \lesssim (qK^2) \cdot \|\mathbf{X}\|_F. \quad (16)$$

Proof. See Appendix B.2. □

2.4 Small Ball Method

Mendelson's small ball method is a powerful method in signal processing and machine learning as it can provide a lower bound for nonnegative empirical processes. We present the following theorem without providing a proof here, as we found other versions comparable to ours in [15, 41].

Theorem 1. Let \mathcal{F} be a class of functions from \mathbb{R}^n into \mathbb{R} and $\{\phi_k\}_{k=1}^m$ be independent copies of a random vector ϕ in \mathbb{F}^n . Consider the marginal tail function

$$\mathcal{Q}_\xi(\mathcal{F}; \phi) = \inf_{f \in \mathcal{F}} \mathbb{P}(|f(\phi)| \geq \xi) \quad (17)$$

and suprema of empirical process

$$\mathcal{R}_m(\mathcal{F}; \phi) = \mathbb{E} \sup_{f \in \mathcal{F}} \left(\frac{1}{m} \sum_{k=1}^m \varepsilon_k f(\phi_k) \right), \quad (18)$$

where $\{\varepsilon_k\}_{k=1}^m$ is a Rademacher sequence independent of everything else.

Then for any $q \geq 1, \xi > 0$ and $t > 0$, with probability exceeding $1 - \exp(-2t^2)$,

$$\inf_{f \in \mathcal{F}} \left(\frac{1}{m} \sum_{k=1}^m |f(\phi_k)|^q \right)^{1/q} \geq \xi \mathcal{Q}_{2\xi}(\mathcal{F}; \phi) - 2\mathcal{R}_m(\mathcal{F}; \phi) - \frac{\xi t}{\sqrt{m}}. \quad (19)$$

3 Main Results

3.1 Stability of Phaseless Operator

We first present the stability results of the phaseless operator \mathcal{A}_Φ^ℓ for both amplitude and intensity cases on an arbitrary geometric set \mathcal{K} . The outcome is the robust performance of phase retrieval within the framework of empirical risk minimization.

Theorem 2. Let $\ell = 1, 2$ and $q \geq 1$. Suppose $\Phi = \{\phi_k\}_{k=1}^m$ are i.i.d. suitable (K, β) -subgaussian random vectors and measurement number

$$m \geq L_\ell \cdot \gamma_2^2(\text{cone}(\mathcal{K}) \cap \mathbb{S}_{\ell_2}). \quad (20)$$

Then with probability exceeding $1 - \mathcal{O}(e^{-c_\ell m})$: For all $\mathbf{u}, \mathbf{v} \in \mathcal{K}$,

$$\|\mathcal{A}_\Phi^\ell(\mathbf{u}) - \mathcal{A}_\Phi^\ell(\mathbf{v})\|_q \geq C_\ell m^{1/q} d_\ell(\mathbf{u}, \mathbf{v}). \quad (21)$$

Besides, the model solutions \mathbf{x}_*^ℓ to (5) satisfy

$$d_\ell(\mathbf{x}_*^\ell, \mathbf{x}_0) \leq \frac{2 \|\mathbf{z}\|_q}{C_\ell m^{1/q}}. \quad (22)$$

Here, L_1, L_2, C_1, C_2, c_1 and c_2 are positive constants dependent only on K and β .

Remark 1. [19] demonstrated the stability of \mathcal{A}_Φ^2 in the real case and in the sense of ℓ_1 -norm via an empirical process method. Our result is unified for both \mathcal{A}_Φ^1 and \mathcal{A}_Φ^2 , both real and complex cases, in the sense of ℓ_q -norm ($q \geq 1$) on an arbitrary set \mathcal{K} . The proof only involves the random embedding of concave lifting operator \mathcal{B}_Φ^p .

Remark 2. The geometric characterization in the aforementioned theorem involves Talagrand's γ_2 -functional, which is distinct from Gaussian width in [19]. Nevertheless, these two are equivalent in \mathbb{R}^n [42, Theorem 8.6.1], but the former is more convenient to represent in the complex case.

Subsequently, we investigate some special cases of \mathcal{K} in Theorem 2 and determine the required measurement number m . The first case involves the entire space $\mathcal{K} = \mathbb{F}^n$. [9, 43, 46] focused on employing ℓ_2 -minimization to solve model (5), whereas [8, 16] utilized ℓ_1 -minimization. The second case is that \mathcal{K} is a sparse set, and then model (5) can be attributed to the sparse phase retrieval. When considering algorithms, the required measurement number is quadratic scaling in sparsity [4, 5]. The following corollary indicates that $m = \mathcal{O}(s \log n)$ when only considering the robust performance of model (5); it is also consistent with the results in [19, 25, 26, 45]. The final case is that \mathcal{K} is a finite set in \mathbb{F}^n .

Corollary 1. The required number of measurements m in Theorem 2 is:

- I. $\mathcal{K} = \mathbb{F}^n$, then $m \underset{K, \beta}{\gtrsim} n$.
- II. $\mathcal{K} = \mathcal{S}_{n,s} := \{\mathbf{x} \in \mathbb{F}^n : \|\mathbf{x}\|_0 \leq s\}$, then $m \underset{K, \beta}{\gtrsim} s \log(en/s)$.
- III. \mathcal{K} is a finite set that $|\mathcal{K}| < \infty$, then $m \underset{K, \beta}{\gtrsim} \log |\mathcal{K}|$.

3.2 Robust Injectivity of Structured Matrix Recovery

This subsection presents the robust injectivity results of \mathbf{A}_Φ on an arbitrary matrix set \mathcal{M} and deduces the robust performance of structured matrix recovery within the framework of empirical risk minimization.

Theorem 3. Let $q \geq 1$. Suppose $\Phi = \{\phi_{a,k} \phi_{b,k}^*\}_{k=1}^m$ where $\phi_{a,k}, \phi_{b,k}$ are independent copies of a suitable K -subgaussian random vector and measurement number

$$m \gtrsim K^{12} \cdot \gamma_2^2(\text{cone}(\mathcal{M}) \cap \mathbb{S}_F, \|\cdot\|_F) + K^{10} \cdot \gamma_1(\text{cone}(\mathcal{M}) \cap \mathbb{S}_F, \|\cdot\|_{op}). \quad (23)$$

Then with probability exceeding $1 - e^{-\mathcal{O}(\frac{m}{K^8})}$: For all $\mathbf{X} \in \mathcal{M}$,

$$\|\mathbf{A}_\Phi(\mathbf{X})\|_q \gtrsim \frac{m^{1/q}}{K^8} \cdot \|\mathbf{X}\|_F. \quad (24)$$

In addition, the model solution \mathbf{X}_\star to (7) satisfies

$$\|\mathbf{X}_\star - \mathbf{X}_0\|_F \lesssim K^8 \cdot \frac{\|\mathbf{z}\|_q}{m^{1/q}}. \quad (25)$$

Remark 3. The above theorem actually applies to symmetric rank-one measurements as well ($\phi_{a,k} = \phi_{b,k}$ are i.i.d. suitable (K, β) -subgaussian). Therefore, we can also derive the intensity case ($\ell = 2$) in Theorem 2 using a slightly different measurement number, though it involves γ_1 -functional. The statement is not reiterated here.

Remark 4. As shown in [37, Lemma 2.12], the author characterizes the robust injectivity of A_Φ by restricting the set \mathcal{M} to at most rank- R matrices, provided

$$m \gtrsim_K R \cdot \gamma_2^2 \left(\text{cone}(\mathcal{M}) \cap \mathbb{S}_F, \|\cdot\|_F \right) + \gamma_1 \left(\text{cone}(\mathcal{M}) \cap \mathbb{S}_F, \|\cdot\|_{op} \right). \quad (26)$$

The estimate above is far from optimal and unusable, as the additional factor R appears to be an artifact of the proof. The empirical chaos process can aid in resolving this issue and remove this artifact, and in many cases we can acquire the near-optimal measurement number.

We investigate some special cases of \mathcal{M} in Theorem 3. The first case is all at most rank- R matrices in $\mathbb{F}^{n_1 \times n_2}$:

$$\mathcal{S}^R = \{ \mathbf{X} \in \mathbb{F}^{n_1 \times n_2} : \text{rank}(\mathbf{X}) \leq R \}. \quad (27)$$

Furthermore, the low rank plus sparse case

$$\mathcal{S}_{s_1, s_2}^r = \left\{ \mathbf{X} \in \mathbb{F}^{n_1 \times n_2} : \text{rank}(\mathbf{X}) \leq r, \|\mathbf{X}\|_{2,0} \leq s_1, \|\mathbf{X}\|_{0,2} \leq s_2 \right\}, \quad (28)$$

where $\|\cdot\|_{2,0}$ and $\|\cdot\|_{0,2}$ count the number of non-zero rows and columns, has been extensively studied in the past few years [18, 21, 36, 37]. We offer the following corollary.

Corollary 2. The measurement number in Theorem 3 we require are:

I. $\mathcal{M} = \mathcal{S}^R$, then $m \gtrsim_K R(n_1 + n_2)$.

II. $\mathcal{M} = \mathcal{S}_{s_1, s_2}^r$, then $m \gtrsim_K r(s_1 + s_2) \max \left\{ \log \left(\frac{en_1}{s_1} \right), \log \left(\frac{en_2}{s_2} \right) \right\}$.

3.3 Sharp Recovery Bound

We show that the recovery bounds $\frac{\|\mathbf{z}\|_q}{m^{1/q}}$ in Theorem 2 and Theorem 3 are both sharp. We present the following theorems.

Theorem 4. Let $\ell = 1, 2$ and $1 \leq q < \infty$ and suppose $\Phi = \{\phi_k\}_{k=1}^m$ are independent copies of a suitable K -subgaussian random vector. For any fixed $\mathbf{x}_0 \in \mathbb{F}^n$, there exists a class of adversarial noise \mathbf{z} such that with probability exceeding $1 - 1/m$, the solutions \mathbf{x}_*^ℓ to (5) satisfy

$$d_\ell(\mathbf{x}_*^\ell, \mathbf{x}_0) \gtrsim \frac{\|\mathbf{z}\|_q}{(\sqrt{q}K)^\ell \cdot m^{1/q}}. \quad (29)$$

Remark 5. [26, 45] provided the sharp recovery bound $\frac{\|z\|_2}{\sqrt{m}}$ for ℓ_2 -minimization, but extra assumptions were needed for the measurement number and signal \mathbf{x}_0 . Furthermore, their method is not applicable for ℓ_q -minimization except for $p = 2$, due to the use of gradient descent methods. Our strategy successfully tackles ℓ_q -minimization by constructing adversarial noise \mathbf{z} .

Theorem 5. Let $1 \leq q < \infty$ and suppose $\Phi = \{\phi_{a,k} \phi_{b,k}^*\}_{k=1}^m$ where $\phi_{a,k}, \phi_{b,k}$ are independent copies of a suitable K -subgaussian random vector. For any fixed $\mathbf{X}_0 \in \mathbb{R}^{n_1 \times n_2}$, there exists a class of adversarial noise \mathbf{z} such that with probability exceeding $1 - 1/m$, the solution \mathbf{X}_* to (7) satisfies

$$\|\mathbf{X}_* - \mathbf{X}_0\|_F \gtrsim \frac{\|\mathbf{z}\|_q}{(qK^2) \cdot m^{1/q}}. \quad (30)$$

Remark 6. The aforementioned theorem can be expanded to include symmetric rank-one measurements, demonstrating that the sharpness of the recovery bounds stated in [28, Theorem 1.4, Corollary 2.1].

4 Proofs of Stability Results

We first demonstrate the random embedding of the concave lifting operator \mathcal{B}_Φ^p with $0 < p \leq 1$. The stability of phaseless operator \mathcal{A}_Φ^1 then is based on operator $\mathcal{B}_\Phi^{1/2}$, and we establish a relationship between \mathcal{A}_Φ^2 and \mathcal{B}_Φ^1 . The robust performance of model (5) can be derived from the stability of \mathcal{A}_Φ^ℓ . Our unified results allow us to identify several instances of \mathcal{K} .

4.1 Random Embedding of Concave Lifting Operator \mathcal{B}_Φ^p

Theorem 6. Let $0 < p \leq 1$ and $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{F}^n$. Suppose $\Phi = \{\phi_k\}_{k=1}^m$ are i.i.d. suitable (K, β) -subgaussian random vectors and the measurement number satisfies

$$m \geq C_1 \max\{\gamma_2^2(\text{cone}(\mathcal{K}_1) \cap \mathbb{S}_{\ell_2}, \ell_2), \gamma_2^2(\text{cone}(\mathcal{K}_2) \cap \mathbb{S}_{\ell_2}, \ell_2)\}.$$

Then the following holds with probability exceeding $1 - \mathcal{O}(e^{-C_2 m})$: Let $\mathbf{X}_{\mathbf{u}, \mathbf{v}} := \mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^*$, for all $\mathbf{u} \in \mathcal{K}_1, \mathbf{v} \in \mathcal{K}_2$, operator \mathcal{B}_Φ^p satisfies

$$C_3 \|\mathbf{X}_{\mathbf{u}, \mathbf{v}}\|_F^p \leq \|\mathcal{B}_\Phi^p(\mathbf{X}_{\mathbf{u}, \mathbf{v}})\|_1 \leq C_4 \|\mathbf{X}_{\mathbf{u}, \mathbf{v}}\|_F^p. \quad (31)$$

Here, C_1, C_2, C_3 and C_4 are positive absolute constants dependent only on K, β and p .

Proof. Step 1: Moment Argument. For fixed $\mathbf{u} \in \mathcal{K}_1, \mathbf{v} \in \mathcal{K}_2$, let $\Psi_{\mathbf{u}, \mathbf{v}} := \frac{\mathbf{X}_{\mathbf{u}, \mathbf{v}}}{d_2(\mathbf{u}, \mathbf{v})} = \frac{\mathbf{X}_{\mathbf{u}, \mathbf{v}}}{\|\mathbf{X}_{\mathbf{u}, \mathbf{v}}\|_F}$ and set random variable

$$X_{\mathbf{u}, \mathbf{v}} = |\phi^* \Psi_{\mathbf{u}, \mathbf{v}} \phi|.$$

Since $\text{rank}(\Psi_{\mathbf{u},\mathbf{v}}) \leq 2$ and $\|\Psi_{\mathbf{u},\mathbf{v}}\|_F = 1$, using the eigenvalue decomposition of $\Psi_{\mathbf{u},\mathbf{v}}$, we can assume that

$$\Psi_{\mathbf{u},\mathbf{v}} = \lambda_1 \mathbf{x}\mathbf{x}^* + \lambda_2 \mathbf{y}\mathbf{y}^*,$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ satisfy $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy $\lambda_1^2 + \lambda_2^2 = 1$. On the one hand, we can obtain that

$$\begin{aligned} \mathbb{E}X_{\mathbf{u},\mathbf{v}}^p &= \mathbb{E}|\lambda_1 |\phi^* \mathbf{x}|^2 + \lambda_2 |\phi^* \mathbf{y}|^2|^p \\ &\leq (|\lambda_1| \cdot \mathbb{E}|\phi^* \mathbf{x}|^2 + |\lambda_2| \cdot \mathbb{E}|\phi^* \mathbf{y}|^2)^p \\ &= (|\lambda_1| + |\lambda_2|)^p \leq 2, \end{aligned} \quad (32)$$

where in the second line we use Jensen's inequality. On the other hand, by Hölder's inequality,

$$\mathbb{E}X_{\mathbf{u},\mathbf{v}}^2 \leq (\mathbb{E}X_{\mathbf{u},\mathbf{v}}^p)^{\frac{2}{4-p}} \cdot (\mathbb{E}X_{\mathbf{u},\mathbf{v}}^4)^{\frac{2-p}{4-p}}.$$

Thus we have

$$\mathbb{E}X_{\mathbf{u},\mathbf{v}}^p \geq \frac{(\mathbb{E}X_{\mathbf{u},\mathbf{v}}^2)^{\frac{4-p}{2}}}{(\mathbb{E}X_{\mathbf{u},\mathbf{v}}^4)^{\frac{2-p}{2}}}. \quad (33)$$

By direct calculation, we can get that

$$\begin{aligned} \mathbb{E}X_{\mathbf{u},\mathbf{v}}^2 &= \mathbb{E}|\phi^* \Psi_{\mathbf{u},\mathbf{v}} \phi|^2 = \mathbb{E}(\lambda_1 |\phi^* \mathbf{x}|^2 + \lambda_2 |\phi^* \mathbf{y}|^2)^2 \\ &= \lambda_1^2 \mathbb{E}|\phi^* \mathbf{x}|^4 + \lambda_2^2 \mathbb{E}|\phi^* \mathbf{y}|^4 + 2\lambda_1 \lambda_2 \mathbb{E}|\phi^* \mathbf{x}|^2 \mathbb{E}|\phi^* \mathbf{y}|^2 \\ &= (\beta + 1)(\lambda_1^2 + \lambda_2^2) + 2\lambda_1 \lambda_2 \\ &\geq \beta(\lambda_1^2 + \lambda_2^2) = \beta > 0. \end{aligned} \quad (34)$$

By Lemma 1, we also have

$$\mathbb{E}X_{\mathbf{u},\mathbf{v}}^4 \lesssim (K^2 + \|\Psi_{\mathbf{u},\mathbf{v}}\|_*)^4 \lesssim K^8, \quad (35)$$

where we use the fact that $\text{rank}(\Psi_{\mathbf{u},\mathbf{v}}) \leq 2$ and $K \gtrsim 1$. Therefore, from (34) and (35), it can be concluded that

$$\mathbb{E}X_{\mathbf{u},\mathbf{v}}^p \gtrsim \frac{\beta^{\frac{4-p}{2}}}{K^{4(2-p)}} > 0. \quad (36)$$

Step 2: Fixed Point Argument. We claim that $X_{\mathbf{u},\mathbf{v}}^p$ have finite ψ_1 -norm. We have that

$$\begin{aligned} \|X_{\mathbf{u},\mathbf{v}}\|_{\psi_1} &= \|\lambda_1 |\phi^* \mathbf{x}|^2 + \lambda_2 |\phi^* \mathbf{y}|^2\|_{\psi_1} \\ &\leq |\lambda_1| \cdot \|\phi^* \mathbf{x}\|_{\psi_1}^2 + |\lambda_2| \cdot \|\phi^* \mathbf{y}\|_{\psi_1}^2 \\ &= |\lambda_1| \cdot \|\phi^* \mathbf{x}\|_{\psi_2} + |\lambda_2| \cdot \|\phi^* \mathbf{y}\|_{\psi_2} \lesssim K. \end{aligned}$$

Proposition 2.(a) implies that $\|X_{\mathbf{u},\mathbf{v}}^p\|_{\psi_{1/p}} = \|X_{\mathbf{u},\mathbf{v}}\|_{\psi_1}^p$. Therefore, by Proposition 2.(b), $X_{\mathbf{u},\mathbf{v}}^p$ is sub-exponential with ψ_1 -norm:

$$\|X_{\mathbf{u},\mathbf{v}}^p\|_{\psi_1} \lesssim \|X_{\mathbf{u},\mathbf{v}}^p\|_{\psi_{1/p}} \lesssim K^p \lesssim K.$$

Bernstein-type inequality in [42] then yields that for a fixed pair (\mathbf{u}, \mathbf{v}) and any $\varepsilon_1 > 0$,

$$\frac{\beta^{\frac{4-p}{2}}}{K^{4(2-p)}} - \varepsilon_1 \lesssim \frac{1}{m} \sum_{k=1}^m X_{\mathbf{u},\mathbf{v}}^p \leq 2 + \varepsilon_1 \quad (37)$$

with probability exceeding $1 - 4 \exp\left(-cm \min\left\{\frac{\varepsilon_1^2}{K^2}, \frac{\varepsilon_1}{K}\right\}\right)$.

Step 3: Tangent Space Conversion. We set $\mathbf{h} = \mathbf{u} + \mathbf{v} \in \mathcal{K}_1 + \mathcal{K}_2$ and $\mathbf{g} = \mathbf{u} - \mathbf{v} \in \mathcal{K}_1 - \mathcal{K}_2$. A simple calculation yields

$$\Psi_{\mathbf{u},\mathbf{v}} = \frac{\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^*}{d_2(\mathbf{u}, \mathbf{v})} = \frac{\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*}{\|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F} := \tilde{\Psi}_{\mathbf{h},\mathbf{g}}. \quad (38)$$

By homogeneity, we can assume that $\mathbf{h}, \mathbf{g} \in \mathbb{S}_{\ell_2}$. Thus we have

$$\mathbf{h} \in \text{cone}(\mathcal{K}_1 + \mathcal{K}_2) \cap \mathbb{S}_{\ell_2} := \mathcal{K}^+ \text{ and } \mathbf{g} \in \text{cone}(\mathcal{K}_1 - \mathcal{K}_2) \cap \mathbb{S}_{\ell_2} := \mathcal{K}^-.$$

Let \mathcal{K}_ϵ^+ and \mathcal{K}_ϵ^- be the ϵ -net of \mathcal{K}^+ and \mathcal{K}^- . Then for all $\mathbf{h} \in \mathcal{K}^+, \mathbf{g} \in \mathcal{K}^-$, there exist $\mathbf{h}_0 \in \mathcal{K}_\epsilon^+$ and $\mathbf{g}_0 \in \mathcal{K}_\epsilon^-$ such that $\|\mathbf{h} - \mathbf{h}_0\|_2 \leq \epsilon, \|\mathbf{g} - \mathbf{g}_0\|_2 \leq \epsilon$. We then claim that

$$\left\| \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\|_F \leq 16\epsilon. \quad (39)$$

Firstly,

$$\|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F^2 = 2 + \mathbf{h}^* \mathbf{g} \mathbf{h}^* \mathbf{g} + \mathbf{g}^* \mathbf{h} \mathbf{g}^* \mathbf{h} = 2 + 2|\mathbf{h}^* \mathbf{g}|^2 \in [2, 4].$$

Then

$$\begin{aligned} & \|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^* - \mathbf{h}_0\mathbf{g}_0^* - \mathbf{g}_0\mathbf{h}_0^*\|_F \\ &= \|\mathbf{h}(\mathbf{g} - \mathbf{g}_0)^* + \mathbf{g}(\mathbf{h} - \mathbf{h}_0)^* + (\mathbf{h} - \mathbf{h}_0)\mathbf{g}_0^* + (\mathbf{g} - \mathbf{g}_0)\mathbf{h}_0^*\|_F \\ &\leq 4\epsilon. \end{aligned}$$

Finally, we have that

$$\begin{aligned} & \left\| \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\|_F \\ &= \frac{\|(\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*) \cdot \|\mathbf{h}_0\mathbf{g}_0^* + \mathbf{g}_0\mathbf{h}_0^*\|_F - (\mathbf{h}_0\mathbf{g}_0^* + \mathbf{g}_0\mathbf{h}_0^*) \cdot \|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F\|_F}{\|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F \cdot \|\mathbf{h}_0\mathbf{g}_0^* + \mathbf{g}_0\mathbf{h}_0^*\|_F} \\ &\leq \left| \|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F - \|\mathbf{h}_0\mathbf{g}_0^* + \mathbf{g}_0\mathbf{h}_0^*\|_F \right| \cdot \|\mathbf{h}_0\mathbf{g}_0^* + \mathbf{g}_0\mathbf{h}_0^*\|_F \\ &\quad + \|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^* - \mathbf{h}_0\mathbf{g}_0^* - \mathbf{g}_0\mathbf{h}_0^*\|_F \cdot \|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F \\ &\leq \|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^* - \mathbf{h}_0\mathbf{g}_0^* - \mathbf{g}_0\mathbf{h}_0^*\|_F \cdot (\|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F + \|\mathbf{h}\mathbf{g}^* + \mathbf{g}\mathbf{h}^*\|_F) \\ &\leq 4\epsilon \cdot 4 \leq 16\epsilon. \end{aligned}$$

Subsequently, as $\tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0}$ is at most rank-4, similar to the decomposition in **Step 1**, we can assume

$$\frac{\tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0}}{\left\| \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\|_F} = \sum_{k=1}^4 \lambda_k \mathbf{x}_k \mathbf{x}_k^*,$$

where $\mathbf{x}_k \in \mathbb{F}^n$ satisfy $\|\mathbf{x}_k\|_2 = 1$, $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{i,j}$ and $\lambda_k \in \mathbb{R}$ satisfy $\sum_{k=1}^4 \lambda_k^2 = 1$. Then similar to the argument in **Step 2**,

$$\mathbb{E} \left| \left\langle \phi\phi^*, \frac{\tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0}}{\left\| \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\|_F} \right\rangle \right|^p \leq \left| \sum_{k=1}^4 |\lambda_k| \cdot \mathbb{E} |\phi^* \mathbf{x}_k|^2 \right|^p = \left| \sum_{k=1}^4 |\lambda_k| \right|^p \leq 4. \quad (40)$$

Then by Bernstein-type inequality,

$$\frac{1}{m} \sum_{k=1}^m \left| \left\langle \phi\phi^*, \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\rangle \right|^p \leq (4 + \epsilon_2) \cdot \left\| \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\|_F^p \leq (4 + \epsilon_2) \cdot 16\epsilon \quad (41)$$

with probability exceeding $1 - 2 \exp \left(-cm \min \left\{ \frac{\epsilon_2^2}{K^2}, \frac{\epsilon_2}{K} \right\} \right)$.

Step 4: Uniform Argument. To achieve consistent result, we choose ϵ_1, ϵ_2 small enough ($\epsilon_1, \epsilon_2 \leq K$) and $\epsilon_1 = \epsilon = \mathcal{O} \left(\frac{\beta^{\frac{4-p}{2}}}{2K^{4(2-p)}}. By (37) and (41), for any $\mathbf{u} \in \mathcal{K}_1, \mathbf{v} \in \mathcal{K}_2$:$

$$\begin{aligned} \|\mathcal{B}_{\Phi}^p(\Psi_{\mathbf{u},\mathbf{v}})\|_1 &= \frac{1}{m} \sum_{k=1}^m \left| \left\langle \phi\phi^*, \tilde{\Psi}_{\mathbf{h},\mathbf{g}} \right\rangle \right|^p \\ &\geq \frac{1}{m} \sum_{k=1}^m \left| \left\langle \phi\phi^*, \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\rangle \right|^p - \frac{1}{m} \sum_{k=1}^m \left| \left\langle \phi\phi^*, \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\rangle \right|^p \\ &\gtrsim \left(\frac{\beta^{\frac{4-p}{2}}}{K^{4(2-p)}} - \epsilon_1 \right) - (4 + \epsilon_2) \cdot 10\epsilon \gtrsim \frac{\beta^{\frac{4-p}{2}}}{K^{4(2-p)}} \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{B}_{\Phi}^p(\Psi_{\mathbf{u},\mathbf{v}})\|_1 &\leq \frac{1}{m} \sum_{k=1}^m \left| \left\langle \phi\phi^*, \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\rangle \right|^p + \frac{1}{m} \sum_{k=1}^m \left| \left\langle \phi\phi^*, \tilde{\Psi}_{\mathbf{h},\mathbf{g}} - \tilde{\Psi}_{\mathbf{h}_0,\mathbf{g}_0} \right\rangle \right|^p \\ &\leq (2 - \epsilon_1) + (4 + \epsilon_2) \cdot 10\epsilon \leq 4. \end{aligned}$$

By Sudakov's minoration inequality (13) and the subadditivity of γ_2 -functional in Proposition 1,

$$\begin{aligned} \log \mathcal{N}(\mathcal{K}^+, \ell_2, \epsilon) &\lesssim \epsilon^{-2} \gamma_2^2(\mathcal{K}^+, \ell_2) \\ &\lesssim \epsilon^{-2} \max \left\{ \gamma_2^2(\text{cone}(\mathcal{K}_1) \cap \mathbb{S}_{\ell_2}, \ell_2), \gamma_2^2(\text{cone}(\mathcal{K}_2) \cap \mathbb{S}_{\ell_2}, \ell_2) \right\}. \end{aligned} \quad (42)$$

Similarly, we have

$$\log \mathcal{N}(\mathcal{K}^-, \ell_2, \epsilon) \lesssim \epsilon^{-2} \max \{ \gamma_2^2(\text{cone}(\mathcal{K}_1) \cap \mathbb{S}_{\ell_2}, \ell_2), \gamma_2^2(\text{cone}(\mathcal{K}_2) \cap \mathbb{S}_{\ell_2}, \ell_2) \}.$$

Thus, provided m obeys condition of the theorem, the successful probability exceeding

$$\begin{aligned} & 1 - [4 \exp(-cm\epsilon_1^2/K^2) + 2 \exp(-cm\epsilon_2^2/K^2)] \cdot \mathcal{N}(\mathcal{K}^+, \ell_2, \epsilon) \cdot \mathcal{N}(\mathcal{K}^-, \ell_2, \epsilon) \\ & \geq 1 - 6 \exp(-cm\epsilon_1^2/K^2 + \log \mathcal{N}(\mathcal{K}^+, \ell_2, \epsilon) + \log \mathcal{N}(\mathcal{K}^-, \ell_2, \epsilon)) \\ & \geq 1 - \mathcal{O}\left(e^{-cm\epsilon_1^2/K^2}\right). \end{aligned}$$

□

4.2 Proof of Theorem 2

The distance between \mathbf{x}_\star^ℓ and \mathbf{x}_0 in model (5) can be bounded through the following proposition, which indicates that we only need to determine the stability condition (4) for \mathcal{A}_Φ^ℓ .

Proposition 3. If \mathcal{A}_Φ^ℓ is C -stable with respect to ℓ_q -norm, then we have

$$d_\ell(\mathbf{x}_\star^\ell, \mathbf{x}_0) \leq 2 \|\mathbf{z}\|_q / C. \quad (43)$$

Proof. The optimality of \mathbf{x}_\star^ℓ yields

$$\begin{aligned} 0 & \geq \|\mathcal{A}_\Phi^\ell(\mathbf{x}_\star^\ell) - \mathbf{b}\|_q - \|\mathcal{A}_\Phi^\ell(\mathbf{x}_0) - \mathbf{b}\|_q \\ & = \|\mathcal{A}_\Phi^\ell(\mathbf{x}_\star^\ell) - \mathcal{A}_\Phi^\ell(\mathbf{x}_0) - \mathbf{z}\|_q - \|\mathbf{z}\|_q \\ & \geq \|\mathcal{A}_\Phi^\ell(\mathbf{x}_\star^\ell) - \mathcal{A}_\Phi^\ell(\mathbf{x}_0)\|_q - 2\|\mathbf{z}\|_q \\ & \geq Cd_\ell(\mathbf{x}_\star^\ell, \mathbf{x}_0) - 2\|\mathbf{z}\|_q. \end{aligned}$$

□

Case I: Intensity Measurement. We set $p = 1$ and $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$ in Theorem 6, then with certain probability and provided $m \gtrsim_{K, \beta} \gamma_2^2(\text{cone}(\mathcal{K}) \cap \mathbb{S}_{\ell_2}, \ell_2)$, we have

$$\begin{aligned} \frac{\|\mathcal{A}_\Phi^2(\mathbf{u}) - \mathcal{A}_\Phi^2(\mathbf{v})\|_q}{m^{1/q}} & \geq \frac{\|\mathcal{A}_\Phi^2(\mathbf{u}) - \mathcal{A}_\Phi^2(\mathbf{v})\|_1}{m} \\ & = \frac{1}{m} \sum_{k=1}^m |\langle \phi \phi^*, \mathbf{u} \mathbf{u}^* - \mathbf{v} \mathbf{v}^* \rangle| \\ & = \|\mathcal{B}_\Phi^1(\mathbf{X}_{\mathbf{u}, \mathbf{v}})\|_1 \gtrsim d_2(\mathbf{u}, \mathbf{v}). \end{aligned}$$

Case II: Amplitude Measurement. We first declare the following three facts.

Fact 1. The first one establishes the correlation between $d_2(\mathbf{u}, \mathbf{v})$ and $d_1(\mathbf{u}, \mathbf{v})$:

$$2d_2(\mathbf{u}, \mathbf{v}) \geq (\|\mathbf{u}\|_2 + \|\mathbf{v}\|_2) \cdot d_1(\mathbf{u}, \mathbf{v}). \quad (44)$$

Indeed, we choose $\theta := \text{Phase}(\mathbf{u}^* \mathbf{v})$ and set $\bar{\mathbf{v}} = \exp(i\theta) \mathbf{v}$, then $\langle \mathbf{u}, \bar{\mathbf{v}} \rangle \geq 0$ and

$$d_1^2(\mathbf{u}, \mathbf{v}) = d_1^2(\mathbf{u}, \bar{\mathbf{v}}) = \|\mathbf{u}\|_2^2 + \|\bar{\mathbf{v}}\|_2^2 - 2\langle \mathbf{u}, \bar{\mathbf{v}} \rangle.$$

We have that

$$\begin{aligned} d_2^2(\mathbf{u}, \mathbf{v}) &= d_2^2(\mathbf{u}, \bar{\mathbf{v}}) \\ &= \|\mathbf{u}\mathbf{u}^* - \bar{\mathbf{v}}\bar{\mathbf{v}}^*\|_F^2 = \|\mathbf{u}\|_2^4 + \|\bar{\mathbf{v}}\|_2^4 - 2|\langle \mathbf{u}, \bar{\mathbf{v}} \rangle|^2 \\ &= \left(\sqrt{\|\mathbf{u}\|_2^4 + \|\bar{\mathbf{v}}\|_2^4} - \sqrt{2}\langle \mathbf{u}, \bar{\mathbf{v}} \rangle \right) \cdot \left(\sqrt{\|\mathbf{u}\|_2^4 + \|\bar{\mathbf{v}}\|_2^4} + \sqrt{2}\langle \mathbf{u}, \bar{\mathbf{v}} \rangle \right) \\ &\geq \frac{1}{2} (\|\mathbf{u}\|_2^2 + \|\bar{\mathbf{v}}\|_2^2 - 2\langle \mathbf{u}, \bar{\mathbf{v}} \rangle) \cdot (\|\mathbf{u}\|_2^2 + \|\bar{\mathbf{v}}\|_2^2 + 2\langle \mathbf{u}, \bar{\mathbf{v}} \rangle) \\ &\geq \frac{1}{4} d_1^2(\mathbf{u}, \mathbf{v}) \cdot (\|\mathbf{u}\|_2 + \|\mathbf{v}\|_2)^2. \end{aligned}$$

Fact 2. Furthermore, we set $p = 1/2$, $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{K}_2 = \emptyset$ in Theorem 6, then with certain probability and if $m \underset{K, \beta}{\gtrsim} (\text{cone}(\mathcal{K}) \cap \mathbb{S}_{\ell_2}, \ell_2)$, we have

$$\frac{1}{m} \sum_{k=1}^m |\langle \phi, \mathbf{u} \rangle| = \left\| \mathcal{B}_{\Phi}^{1/2}(\mathbf{u}\mathbf{u}^*) \right\|_1 \lesssim \|\mathbf{u}\|_2, \text{ for all } \mathbf{u} \in \mathcal{K}. \quad (45)$$

Fact 3. We then set $p = 1/2$, $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$ in Theorem 6, then with certain probability and if $m \underset{K, \beta}{\gtrsim} \gamma_2^2(\text{cone}(\mathcal{K}) \cap \mathbb{S}_{\ell_2}, \ell_2)$,

$$\frac{1}{m} \sum_{k=1}^m |\langle \phi \phi^*, \mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^* \rangle|^{1/2} = \left\| \mathcal{B}_{\Phi}^{1/2}(\mathbf{X}_{\mathbf{u}, \mathbf{v}}) \right\|_1 \gtrsim d_2^{1/2}(\mathbf{u}, \mathbf{v}), \text{ for all } \mathbf{u}, \mathbf{v} \in \mathcal{K}. \quad (46)$$

Finally, provided $m \underset{K, \beta}{\gtrsim} \gamma_2^2(\text{cone}(\mathcal{K}) \cap \mathbb{S}_{\ell_2}, \ell_2)$, we have that

$$\begin{aligned} \frac{\|\mathcal{A}_{\Phi}^1(\mathbf{u}) - \mathcal{A}_{\Phi}^1(\mathbf{v})\|_q}{m^{1/q}} &\geq \frac{\|\mathcal{A}_{\Phi}^1(\mathbf{u}) - \mathcal{A}_{\Phi}^1(\mathbf{v})\|_1}{m} = \frac{1}{m} \sum_{k=1}^m ||\langle \phi_k, \mathbf{u} \rangle| - |\langle \phi_k, \mathbf{v} \rangle|| \\ &\geq \frac{\left(\frac{1}{m} \sum_{k=1}^m \sqrt{||\langle \phi_k, \mathbf{u} \rangle|^2 - |\langle \phi_k, \mathbf{v} \rangle|^2|} \right)^2}{\frac{1}{m} \sum_{k=1}^m (|\langle \phi_k, \mathbf{u} \rangle| + |\langle \phi_k, \mathbf{v} \rangle|)} \\ &= \frac{\left\| \mathcal{B}_{\Phi}^{1/2}(\mathbf{X}_{\mathbf{u}, \mathbf{v}}) \right\|_1^2}{\left\| \mathcal{B}_{\Phi}^{1/2}(\mathbf{u}\mathbf{u}^*) \right\|_1 + \left\| \mathcal{B}_{\Phi}^{1/2}(\mathbf{v}\mathbf{v}^*) \right\|_1} \\ &\gtrsim \frac{d_2(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_2 + \|\mathbf{v}\|_2} \geq \frac{1}{2} d_1(\mathbf{u}, \mathbf{v}). \end{aligned}$$

We used the Cauchy-Schwarz inequality for the first two inequalities, **Fact 2** and **Fact 3** for the third inequality, and **Fact 1** for the final inequality.

4.3 Proof of Corollary 1

We only need to determine γ_2 -functional for various \mathcal{K} .

Case I. If $\mathcal{K} = \mathbb{F}^n$, then $\text{cone}(\mathbb{F}^n) \cap \mathbb{S}_{\ell_2} = \mathbb{S}_{\ell_2}$. By $\mathcal{N}(\mathbb{S}_{\ell_2}, \ell_2, u) \leq (1 + \frac{2}{u})^n$ [42, Corollary 4.2.13] and Dudley integral in (12), we have

$$\begin{aligned}\gamma_2(\mathbb{S}_{\ell_2}, \ell_2) &\leq C \int_0^1 \sqrt{\log \mathcal{N}(\mathbb{S}_{\ell_2}, \ell_2, u)} du \\ &= C\sqrt{n} \int_0^1 \sqrt{\log(1 + 2/u)} du \leq \tilde{C}\sqrt{n}.\end{aligned}$$

Case II. If $\mathcal{K} = \mathcal{S}_{n,s}$, let $\mathcal{D}_{n,s} = \{\mathbf{x} \in \mathbb{F}^n : \|\mathbf{x}\|_2 = 1, \|\mathbf{x}\|_0 \leq s\}$. The volumetric argument yields

$$\mathcal{N}(\mathcal{D}_{n,s}, \ell_2, u) \leq \sum_{k=1}^s \binom{n}{k} \cdot \left(1 + \frac{2}{u}\right)^s \leq \left(\frac{en}{s}\right)^s \cdot \left(1 + \frac{2}{u}\right)^s,$$

so that

$$\begin{aligned}\gamma_2(\text{cone}(\mathcal{K}_{n,s}) \cap \mathbb{S}_{\ell_2}, \ell_2) &= \gamma_2(\mathcal{D}_{n,s}, \ell_2) \\ &\leq C\sqrt{s} \left(\sqrt{\log(en/s)} + \int_0^1 \sqrt{\log(1 + 2/u)} du \right) = \tilde{C}\sqrt{s \log(en/s)}.\end{aligned}$$

Case III. If $|\mathcal{K}| < \infty$, we can restrict n to satisfy that $|\mathcal{A}_n| \leq 2^{2^n} \leq |\mathcal{K}|$. Thus $2^n \leq \log |\mathcal{K}|$. By the definition of γ_2 -functional,

$$\begin{aligned}\gamma_2(\text{cone}(\mathcal{K}) \cap \mathbb{S}_{\ell_2}, \ell_2) &\leq \text{diam}(\text{cone}(\mathcal{K}) \cap \mathbb{S}_{\ell_2}) \cdot \sum_{n=0}^{\log \log |\mathcal{K}|} 2^{n/2} \\ &\leq \sum_{n=0}^{\log \log |\mathcal{K}|} 2^{n/2} \lesssim \sqrt{\log |\mathcal{K}|}.\end{aligned}$$

5 Proofs of Robust Injectivity Results

We first investigate the suprema of chaos process and its empirical form with subgaussian random vectors. Combining this with the small ball method, we characterize the robust injectivity of \mathbf{A}_Φ and the robust performance of model (7) on an arbitrary matrix set \mathcal{M} . Some special cases, such as \mathcal{M} being a low rank plus sparse matrix set can be investigated.

5.1 Suprema of Chaos Process

Let ϕ_a, ϕ_b be independent suitable K -subgaussian random vectors and \mathcal{M} be a matrix set, we will first found the upper bound for the quantity

$$S(\mathcal{M}) = \mathbb{E} \sup_{\mathbf{X} \in \mathcal{M}} \langle \phi_a \phi_b^*, \mathbf{X} \rangle.$$

Besides, let $\phi_{a,k}, \phi_{b,k} (k = 1, \dots, m)$ be independent suitable K -subgaussian random vectors, then we consider the empirical form for $S(\mathcal{M})$:

$$\overline{S(\mathcal{M})} = \mathbb{E} \sup_{\mathbf{X} \in \mathcal{M}} \left\langle \sum_{k=1}^m \phi_{a,k} \phi_{b,k}^*, \mathbf{X} \right\rangle.$$

We would like in fact to understand the values of $S(\mathcal{M})$ and $\overline{S(\mathcal{M})}$ as function of the geometry of \mathcal{M} .

Theorem 7. For a symmetric matrix set \mathcal{M} , we have

$$S(\mathcal{M}) \lesssim K^2 \cdot \gamma_2(\mathcal{M}, \|\cdot\|_F) + K^2 \cdot \gamma_1(\mathcal{M}, \|\cdot\|_{op}) \quad (47)$$

and

$$\overline{S(\mathcal{M})} \lesssim \sqrt{m} K^2 \cdot \gamma_2(\mathcal{M}, \|\cdot\|_F) + K^2 \cdot \gamma_1(\mathcal{M}, \|\cdot\|_{op}). \quad (48)$$

Remark 7. Theorem 7 is also suitable for symmetric measurement such that $\phi_a = \phi_b$.

Proof. Our proof is based on [40, Theorem 15.1.4].

Step1: Preliminary. We denote by $\Delta_1(A)$ and $\Delta_2(A)$ the diameter of the set A for norm $\|\cdot\|_{op}$ and $\|\cdot\|_F$, respectively. We then consider an admissible sequence $(\mathcal{B}_n)_{n \geq 0}$ such that

$$\sum_{n \geq 0} 2^n \Delta_1(B_n(\mathbf{X})) \leq 2\gamma_1(\mathcal{M}, \|\cdot\|_{op}), \quad \forall \mathbf{X} \in \mathcal{M} \quad (49)$$

and an admissible sequence $(\mathcal{C}_n)_{n \geq 0}$ such that

$$\sum_{n \geq 0} 2^{n/2} \Delta_2(C_n(\mathbf{X})) \leq 2\gamma_2(\mathcal{M}, \|\cdot\|_F), \quad \forall \mathbf{X} \in \mathcal{M}.$$

Here $B_n(\mathbf{X})$ is the unique element of (\mathcal{B}_n) that contains \mathbf{X} (etc.). The definition of partitions \mathcal{A}_n of \mathcal{M} is as follows: we set $\mathcal{A}_0 = \{\mathcal{M}\}$, and for $n \geq 1$, we define \mathcal{A}_n as the partition generated by \mathcal{B}_{n-1} and \mathcal{C}_{n-1} , i.e., the partition that consists of the sets $B \cap C$ for $B \in \mathcal{B}_{n-1}$ and $C \in \mathcal{C}_{n-1}$. Thus $|\mathcal{A}_n| \leq |\mathcal{B}_{n-1}| \cdot |\mathcal{C}_{n-1}| \leq 2^{2^n}$ and the sequence $(\mathcal{A}_n)_{n \geq 0}$ is admissible.

Step2: Chaining Method. We construct a subset \mathcal{M}_n of \mathcal{M} by taking exactly one point in each set A of \mathcal{A}_n and thus $|\mathcal{M}_n| \leq 2^{2^n}$. For any $\mathbf{X} \in \mathcal{M}$, we consider $\pi_n(\mathbf{X}) \in \mathcal{M}_n$ such that $\pi_n(\mathbf{X})$ are successive approximations of \mathbf{X} . We assume that \mathcal{M}_0 consists of a single element \mathbf{X}_0 , which means that $\pi_0(\mathbf{X}) = \mathbf{X}_0$. Let random variable $Y_{\mathbf{X}} = \langle \phi_a \phi_b^*, \mathbf{X} \rangle$, then we have that

$$Y_{\mathbf{X}} - Y_{\mathbf{X}_0} = \sum_{n \geq 1} (Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}).$$

By Hanson-Wright inequality, there exists numerical constant $c > 0$ and for $v > 0$

$$\begin{aligned} & \mathbb{P}(|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}| \geq v) \\ & \leq 2 \exp \left(-c \min \left\{ \frac{v^2}{K^4 \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_F^2}, \frac{v}{K^2 \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_{op}} \right\} \right). \end{aligned} \quad (50)$$

We reformulate the above inequality as follows: when $u \geq 0$, we have

$$\begin{aligned} & \mathbb{P}(|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}| \geq u \cdot K^2 2^{n/2} \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_F / \sqrt{c} + u^2 \cdot K^2 2^n \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_{op} / c) \\ & \leq 2 \exp(-u^2 2^{n-1}). \end{aligned}$$

We define the event $\Omega_{u,n}$ by

$$\begin{aligned} |Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}| & \leq u \cdot K^2 2^{n/2} \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_F / \sqrt{c} \\ & \quad + u^2 \cdot K^2 2^n \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_{op} / c, \quad \forall \mathbf{X} \in \mathcal{M}. \end{aligned}$$

The number of possible pairs $(\pi_n(\mathbf{X}), \pi_{n-1}(\mathbf{X}))$ is bounded by

$$|\mathcal{M}_n| \cdot |\mathcal{M}_{n-1}| \leq 2^{2^{n+1}}.$$

Thus, we have

$$\mathbb{P}(\Omega_{u,n}^c) \leq 2 \cdot 2^{2^{n+1}} \exp(-u^2 \cdot 2^{n-1}).$$

We define $\Omega_u = \cap_{n \geq 1} \Omega_{u,n}$. Then

$$\begin{aligned} \mathbb{P}(u) = \mathbb{P}(\Omega_u^c) & \leq \sum_{n \geq 1} \mathbb{P}(\Omega_{u,n}^c) \leq \sum_{n \geq 1} 2 \cdot 2^{2^{n+1}} \exp(-u^2 \cdot 2^{n-1}) \\ & \leq \sum_{n \geq 1} 2 \cdot 2^{2^{n+1}} \exp(-u^2/2 - 2^{n+1}) \\ & \leq \sum_{n \geq 1} 2 \cdot \left(\frac{2}{e}\right)^{2^{n+1}} \exp(-u^2/2) \leq 4 \exp(-u^2/2), \end{aligned}$$

where in the second line we use $u^2 \cdot 2^{n-1} \geq u^2/2 + 2^{n+1}$ for any $n \geq 1, u > 0$. Thus when Ω_u occurs, we have

$$\begin{aligned} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| & \leq u \cdot K^2 \sum_{n \geq 1} 2^{n/2} \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_F / \sqrt{c} \\ & \quad + u^2 \cdot K^2 \sum_{n \geq 1} 2^n \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_{op} / c \\ & := u \cdot K^2 S_1 + u^2 \cdot K^2 S_2, \end{aligned}$$

so that

$$\begin{aligned}
& \mathbb{P} \left(\sup_{\mathbf{X} \in \mathcal{M}} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| \geq 2 \max \{u \cdot K^2 S_1, u^2 \cdot K^2 S_2\} \right) \\
& \leq \mathbb{P} \left(\sup_{\mathbf{X} \in \mathcal{M}} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| \geq u \cdot K^2 S_1 + u^2 \cdot K^2 S_2 \right) \\
& \leq \mathbb{P}(u) \leq 4 \exp(-u^2/2).
\end{aligned} \tag{51}$$

There exists constant c such that $c \min \left\{ \frac{t}{K^2 S_2}, \frac{t^2}{K^4 S_1^2} \right\} \leq u^2/2$, thus we can rewrite (51) as

$$\mathbb{P} \left(\sup_{\mathbf{X} \in \mathcal{M}} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| \geq t \right) \leq 4 \exp \left(c \min \left\{ \frac{t}{K^2 S_2}, \frac{t^2}{K^4 S_1^2} \right\} \right). \tag{52}$$

Step3: Expectation Form. In particular, we have that

$$\begin{aligned}
S(\mathcal{M}) &= \mathbb{E} \sup_{\mathbf{X} \in \mathcal{M}} Y_{\mathbf{X}} \\
&= \mathbb{E} \sup_{\mathbf{X} \in \mathcal{M}} (Y_{\mathbf{X}} - Y_{\mathbf{X}_0}) \leq \mathbb{E} \sup_{\mathbf{X} \in \mathcal{M}} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| \\
&\leq \int_0^\infty \mathbb{P} \left(\sup_{\mathbf{X} \in \mathcal{M}} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| \geq t \right) dt \\
&\leq \int_0^\infty 4 \exp \left(c \min \left\{ \frac{t}{K^2 S_2}, \frac{t^2}{K^4 S_1^2} \right\} \right) dt \\
&\lesssim (K^2 S_1) \int_0^\infty \mathbb{P} \left(\sup_{\mathbf{X} \in \mathcal{M}} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| \geq u \cdot K^2 S_1 \right) du \\
&\quad + (K^2 S_2) \int_0^\infty \mathbb{P} \left(\sup_{\mathbf{X} \in \mathcal{M}} |Y_{\mathbf{X}} - Y_{\mathbf{X}_0}| \geq u^2 \cdot K^2 S_2 \right) du \\
&\lesssim K^2 S_1 + K^2 S_2.
\end{aligned}$$

Besides, we have that

$$\begin{aligned}
S_1 &= \sum_{n \geq 1} 2^{n/2} \|Y_{\pi_n(\mathbf{X})} - Y_{\pi_{n-1}(\mathbf{X})}\|_F / \sqrt{c} \\
&\leq \sum_{n \geq 1} 2^{n/2} \|Y_{\mathbf{X}} - Y_{\pi_{n-1}(\mathbf{X})}\|_F / \sqrt{c} + \sum_{n \geq 1} 2^{n/2} \|Y_{\mathbf{X}} - Y_{\pi_n(\mathbf{X})}\|_F / \sqrt{c} \\
&\lesssim \sum_{n \geq 0} 2^{n/2} \Delta_2(C_n(\mathbf{X})) \\
&\lesssim \gamma_2(\mathcal{M}, \|\cdot\|_F).
\end{aligned}$$

Similarly, we have $S_2 \lesssim \gamma_1(\mathcal{M}, \|\cdot\|_{op})$. These lead to (47).

Step4: Empirical Form. We then use (47) to prove (48). Let random variable $\overline{Y_X} = \langle \sum_{k=1}^m \phi_{a,k} \phi_{b,k}^*, \mathbf{X} \rangle$. We can rewrite it as

$$\overline{Y_X} = (\phi_{b,1}^* \quad \cdots \quad \phi_{b,m}^*) \begin{pmatrix} \mathbf{X} & & \\ & \ddots & \\ & & \mathbf{X} \end{pmatrix} \begin{pmatrix} \phi_{a,1} \\ \vdots \\ \phi_{a,m} \end{pmatrix} := \tilde{\phi}_b^* \mathbf{Z} \tilde{\phi}_a,$$

where $\tilde{\phi}_a, \tilde{\phi}_b \in \mathbb{R}^{mn}$, $\mathbf{Z} \in \mathbb{R}^{mn \times mn}$. As $\|\mathbf{Z}\|_F = \sqrt{m} \|\mathbf{X}\|_F$ and $\|\mathbf{Z}\|_{op} = \|\mathbf{X}\|_{op}$, we can rewrite (50)

$$\begin{aligned} & \mathbb{P}(|\overline{Y_{\pi_n(\mathbf{X})}} - \overline{Y_{\pi_{n-1}(\mathbf{X})}}| \geq v) \\ & \leq 2 \exp \left(-c \min \left\{ \frac{v^2}{mK^4 \|\overline{Y_{\pi_n(\mathbf{X})}} - \overline{Y_{\pi_{n-1}(\mathbf{X})}\|_F^2}, \frac{v}{K^2 \|\overline{Y_{\pi_n(\mathbf{X})}} - \overline{Y_{\pi_{n-1}(\mathbf{X})}\|_{op}} \right\} \right). \end{aligned}$$

We can then obtain (48) by following the above steps. \square

5.2 Proof of Theorem 3

We set

$$\mathcal{F} = \{\langle \phi_a \phi_b^*, \mathbf{X} \rangle : \mathbf{X} \in \text{cone}(\mathcal{M}) \cap \mathbb{S}_F\}.$$

Since ϕ_a and ϕ_b are independent and suitable random vectors, we have

$$\mathbb{E} |\phi_a^* \mathbf{X} \phi_b|^2 = \|\mathbf{X}\|_F = 1.$$

Note that by Paley-Zygmund inequality [14],

$$\begin{aligned} \mathbb{P}(|\phi_a^* \mathbf{X} \phi_b|^2 \geq \xi \|\mathbf{X}\|_F) &= \mathbb{P}(|\phi_a^* \mathbf{X} \phi_b|^2 \geq \xi \mathbb{E} |\phi_a^* \mathbf{X} \phi_b|^2) \\ &\geq (1 - \xi)^2 \frac{(\mathbb{E} |\phi_a^* \mathbf{X} \phi_b|^2)^2}{\mathbb{E} |\phi_a^* \mathbf{X} \phi_b|^4}. \end{aligned}$$

Besides, by Lemma 2,

$$\mathbb{E} |\phi_a^* \mathbf{X} \phi_b|^4 \lesssim K^8.$$

Thus we can determine that the marginal tail function

$$\mathcal{Q}_\xi(\mathcal{F}; \phi_a \phi_b^*) \gtrsim (1 - \xi)^2 / K^8. \quad (53)$$

Giné–Zinn symmetrization principle [42, Lemma 6.4.2] and empirical chaos process in Theorem 7 imply the suprema of empirical process satisfies

$$\begin{aligned} \mathcal{R}_m(\mathcal{F}; \phi_a \phi_b^*) &\leq \frac{2}{m} \overline{S(\mathcal{M})} \\ &\lesssim K^2 \cdot \frac{\gamma_2(\text{cone}(\mathcal{M}) \cap \mathbb{S}_F, \|\cdot\|_F)}{\sqrt{m}} + K^2 \cdot \frac{\gamma_1(\text{cone}(\mathcal{M}) \cap \mathbb{S}_F, \|\cdot\|_{op})}{m}. \end{aligned} \quad (54)$$

Finally, we use the small ball method. We set $\xi = \sqrt{2}/4, t = C\frac{\sqrt{m}}{K^8}$ in Theorem 1, then with probability exceeding $1 - e^{-C\frac{m}{K^8}}$, we have

$$\frac{\|\mathbf{A}_\Phi(\mathbf{X})\|_q}{m^{1/q}} = \left(\frac{1}{m} \sum_{k=1}^m |\langle \phi_a \phi_b^*, \mathbf{X} \rangle|^q \right)^{1/q} \gtrsim \frac{1}{K^8}, \quad (55)$$

provided m obeys the condition of the theorem.

The proof of (26) then follows from similar argument to Proposition 3.

5.3 Proofs of Corollary 2

We determine the upper bounds of the γ_1 -functional and γ_2 -functional in Theorem 3.

Case I : $\mathcal{M} = \mathcal{S}^R$. [10, Lemma 3.1] provided the entropy number of $\text{cone}(\mathcal{S}^R) \cap \mathbb{S}_F$ with respect to $\|\cdot\|_F$:

$$\log \mathcal{N}(\text{cone}(\mathcal{S}^R) \cap \mathbb{S}_F, \|\cdot\|_F, \epsilon) \leq R(n_1 + n_2 + 1) \cdot \log\left(\frac{9}{\epsilon}\right). \quad (56)$$

Thus, by Dudley integral we have

$$\begin{aligned} \gamma_2(\text{cone}(\mathcal{S}^R) \cap \mathbb{S}_F, \|\cdot\|_F) &\lesssim \int_0^1 \sqrt{\log \mathcal{N}(\text{cone}(\mathcal{S}^R) \cap \mathbb{S}_F, \|\cdot\|_F, \epsilon)} d\epsilon \\ &\leq \int_0^1 \sqrt{R(n_1 + n_2 + 1) \cdot \log\left(\frac{9}{\epsilon}\right)} d\epsilon \\ &\lesssim \sqrt{R(n_1 + n_2)} \end{aligned} \quad (57)$$

and due to $\|\cdot\|_{op} \leq \|\cdot\|_F$

$$\begin{aligned} \gamma_1(\text{cone}(\mathcal{S}^R) \cap \mathbb{S}_F, \|\cdot\|_{op}) &\leq \gamma_1(\text{cone}(\mathcal{S}^R) \cap \mathbb{S}_F, \|\cdot\|_F) \\ &\lesssim \int_0^1 R(n_1 + n_2 + 1) \cdot \log\left(\frac{9}{\epsilon}\right) d\epsilon \\ &\lesssim R(n_1 + n_2). \end{aligned} \quad (58)$$

Combining (57) and (58) we can get $m \gtrsim_K R(n_1 + n_2)$.

Case II : $\mathcal{M} = \mathcal{S}_{s_1, s_2}^r$. In this case, we first provide the following lemma, whose proof can be found in [20, Lemma 7.2].

Lemma 3 (Entropy Number of Cone $(\mathcal{S}_{s_1, s_2}^r) \cap \mathbb{S}_F$). For all $\epsilon > 0$, the covering number of $\text{cone}(\mathcal{S}_{n_1, n_2}^r) \cap \mathbb{S}_F$ with respect to $\|\cdot\|_F$ satisfies

$$\begin{aligned} \log \mathcal{N}(\text{Cone}(\mathcal{S}_{s_1, s_2}^r) \cap \mathbb{S}_F, \|\cdot\|_F, \epsilon) \\ \leq r(s_1 + s_2 + 1) \log\left(\frac{9}{\epsilon}\right) + r s_1 \log\left(\frac{e n_1}{s_1}\right) + r s_2 \log\left(\frac{e n_2}{s_2}\right). \end{aligned} \quad (59)$$

Then by the above lemma and similar to **Case I**, we have

$$\begin{aligned}
& \gamma_2 \left(\text{cone} \left(\mathcal{S}_{s_1, s_2}^r \right) \cap \mathbb{S}_F, \|\cdot\|_F \right) \\
& \lesssim \sqrt{r(s_1 + s_2 + 1)} + \sqrt{rs_1 \log \left(\frac{en_1}{s_1} \right)} + \sqrt{rs_2 \log \left(\frac{en_2}{s_2} \right)} \\
& \lesssim r(s_1 + s_2) \max \left\{ \log \left(\frac{en_1}{s_1} \right), \log \left(\frac{en_2}{s_2} \right) \right\}
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
& \gamma_1 \left(\text{cone} \left(\mathcal{S}_{s_1, s_2}^r \right) \cap \mathbb{S}_F, \|\cdot\|_{op} \right) \leq \gamma_1 \left(\text{cone} \left(\mathcal{S}_{s_1, s_2}^r \right) \cap \mathbb{S}_F, \|\cdot\|_F \right) \\
& \lesssim r(s_1 + s_2) \max \left\{ \log \left(\frac{en_1}{s_1} \right), \log \left(\frac{en_2}{s_2} \right) \right\}.
\end{aligned} \tag{61}$$

Combining the above estimation we can get $m \gtrsim_K r(s_1 + s_2) \max \left\{ \log \left(\frac{en_1}{s_1} \right), \log \left(\frac{en_2}{s_2} \right) \right\}$.

6 Adversarial Noise to Sharp Recovery Bound

This section will showcase the potent application of adversarial noise. We construct adaptive adversarial noise to show that the recovery bounds $\frac{\|\mathbf{z}\|_q}{m^{1/q}}$ are theoretically sharp in both phase retrieval and structured matrix recovery.

6.1 Proof of Theorem 4

Prior to demonstrating Theorem 4, we present the subsequent lemma.

Lemma 4. Let $\ell = 1, 2$ and $1 \leq q < \infty$. Suppose ϕ is a suitable K -subgaussian random vector. Then

$$\mathcal{M}_\ell(K, q) := \sup_{\mathbf{u}, \mathbf{v} \in \mathbb{F}^n} \mathbb{E} \left| \frac{|\langle \phi, \mathbf{u} \rangle|^\ell - |\langle \phi, \mathbf{v} \rangle|^\ell}{d_\ell(\mathbf{u}, \mathbf{v})} \right|^q \lesssim (\sqrt{q}K)^{\ell q}. \tag{62}$$

Proof. See Appendix C. □

Now we choose any $\mathbf{x}_\star \in \mathbb{F}^n$ such that \mathbf{x}_\star and \mathbf{x}_0 are not in the same equivalence class, i.e., $\mathbf{x}_\star = c\mathbf{x}_0$ for some $|c| = 1$. Then we set

$$\mathbf{z} = \mathcal{A}_\Phi^\ell(\mathbf{x}_\star) - \mathcal{A}_\Phi^\ell(\mathbf{x}_0). \tag{63}$$

Let loss function $\mathcal{L}_q(\mathbf{x}) = \min \|\mathcal{A}_\Phi^\ell(\mathbf{x}) - \mathbf{b}\|_q$ with $\mathbf{b} = \mathcal{A}_\Phi^\ell(\mathbf{x}_0) + \mathbf{z}$. Thus $\mathcal{L}_q(\mathbf{x}_\star) = 0$ and $\mathcal{L}_q(\mathbf{x}_0) = \|\mathcal{A}_\Phi^\ell(\mathbf{x}_\star) - \mathcal{A}_\Phi^\ell(\mathbf{x}_0)\|_q = \|\mathbf{z}\|_q > 0$.

By Chebyshev's inequality, with probability exceeding $1 - 1/t^2$, we have

$$\begin{aligned} \frac{\|\mathbf{z}\|_q^q}{d_\ell^q(\mathbf{x}_*, \mathbf{x}_0)} &= \frac{\|\mathcal{A}_\Phi^\ell(\mathbf{x}_*) - \mathcal{A}_\Phi^\ell(\mathbf{x}_0)\|_q^q}{d_\ell^q(\mathbf{x}_*, \mathbf{x}_0)} = \sum_{k=1}^m \left| \frac{|\langle \phi_k, \mathbf{x}_* \rangle|^\ell - |\langle \phi_k, \mathbf{x}_0 \rangle|^\ell}{d_\ell(\mathbf{x}_*, \mathbf{x}_0)} \right|^q \\ &\leq m \left(\mathbb{E} \left| \frac{|\langle \phi, \mathbf{x}_* \rangle|^\ell - |\langle \phi, \mathbf{x}_0 \rangle|^\ell}{d_\ell(\mathbf{x}_*, \mathbf{x}_0)} \right|^q + t \cdot \sqrt{\mathbb{E} \left| \frac{|\langle \phi, \mathbf{x}_* \rangle|^\ell - |\langle \phi, \mathbf{x}_0 \rangle|^\ell}{d_\ell(\mathbf{x}_*, \mathbf{x}_0)} \right|^{2q} / m} \right) \\ &\leq m \left(\mathcal{M}_\ell(K, q) + t \sqrt{\mathcal{M}_\ell(K, 2q) / m} \right). \end{aligned}$$

We choose $t = \sqrt{m}$ and by Lemma 4, we can get

$$d_\ell(\mathbf{x}_*, \mathbf{x}_0) \geq \frac{\|\mathbf{z}\|_q}{\left(\mathcal{M}_\ell(K, q) + \sqrt{\mathcal{M}_\ell(K, 2q)} \right)^{1/q} \cdot m^{1/q}} \gtrsim \frac{\|\mathbf{z}\|_q}{(\sqrt{q}K)^\ell \cdot m^{1/q}}, \quad (64)$$

with probability exceeding $1 - 1/m$.

6.2 Proof of Theorem 5

The proof for Theorem 5 is similar to Theorem 4. For any $\mathbf{X}_* \in \mathbb{R}^{n_1 \times n_2}$, we set

$$\mathbf{z} = \mathbf{A}_\Phi(\mathbf{X}_*) - \mathbf{A}_\Phi(\mathbf{X}_0) = \mathbf{A}_\Phi(\mathbf{X}_* - \mathbf{X}_0). \quad (65)$$

Then $\mathbf{b} = \mathbf{A}_\Phi(\mathbf{X}_0) + \mathbf{z} = \mathbf{A}_\Phi(\mathbf{X}_*)$ and the solution to model (7) is \mathbf{X}_* . By Lemma 2 and Chebyshev's inequality, we have

$$\begin{aligned} \frac{\|\mathbf{z}\|_q^q}{\|\mathbf{X}_* - \mathbf{X}_0\|_F^q} &= \frac{\|\mathbf{A}_\Phi(\mathbf{X}_* - \mathbf{X}_0)\|_q^q}{\|\mathbf{X}_* - \mathbf{X}_0\|_F^q} \\ &\leq m \left(\mathbb{E} \left| \frac{\langle \phi_a \phi_b^*, \mathbf{X}_* - \mathbf{X}_0 \rangle}{\|\mathbf{X}_* - \mathbf{X}_0\|_F} \right|^q + \sqrt{\mathbb{E} \left| \frac{\langle \phi_a \phi_b^*, \mathbf{X}_* - \mathbf{X}_0 \rangle}{\|\mathbf{X}_* - \mathbf{X}_0\|_F} \right|^{2q}} \right) \\ &\lesssim m (qK^2)^q, \end{aligned}$$

with probability exceeding $1 - 1/m$. Thus we can get

$$\|\mathbf{X}_* - \mathbf{X}_0\|_F \gtrsim \frac{\|\mathbf{z}\|_q}{(qK^2) \cdot m^{1/q}}. \quad (66)$$

A Properties of ψ_s -Norm

(a) By definition of ψ_s norm and variable substitution,

$$\begin{aligned} \|X^q\|_{\psi_s} &= \inf\{t > 0 : \mathbb{E} \exp(|X|^{qs}/t^s) \leq 2\} \\ &= \inf\{u^p : u > 0 \text{ and } \mathbb{E} \exp(|X|^{qs}/u^{qs}) \leq 2\} \\ &= (\inf\{u > 0 : \mathbb{E} \exp(|X|^{qs}/u^{qs}) \leq 2\})^q \\ &= \|X\|_{\psi_{qs}}^q. \end{aligned}$$

(b) We first claim that if $\|X\|_{\psi_s} \leq K < \infty$, then $\mathbb{P}(|X| \geq t) \leq 2 \exp(-t^s/K^s)$ for all $t \geq 0$. This follows from the definition of ψ_s -norm and Markov's inequality:

$$\mathbb{P}(|X| \geq t) = \mathbb{P}(e^{X^s/K^s} \geq e^{t^s/K^s}) \leq \frac{\mathbb{E}e^{X^s/K^s}}{e^{t^s/K^s}} \leq 2 \exp(-t^s/K^s). \quad (67)$$

Without loss of generality, we can assume $\|X\|_{\psi_l} = 1$. By (67), we have

$$\mathbb{P}(|X| \geq t) \leq 2 \exp(-t^l) = 2 \exp(-t^s \cdot t^{l-s}).$$

When $t > 1$, we have $t^{l-s} \geq 1 > \log 2$. When $t \in [0, 1]$, we have $2 \exp(-t^s \log 2) = 2 \cdot 2^{-t^s} \geq 1$. These lead to that

$$\mathbb{P}(|X| \geq t) \leq 2 \exp(-t^s \log 2). \quad (68)$$

With a change of variable $x = e^{\frac{u}{3/\log 2}}$ on interval $(1, \infty)$ we have

$$\begin{aligned} \mathbb{E} \exp\left(\frac{|X|^s}{3/\log 2}\right) &= \int_0^\infty \mathbb{P}\left(e^{\frac{|X|^s}{3/\log 2}} \geq x\right) dx \\ &\leq \int_0^1 1 dx + \frac{1}{3/\log 2} \int_0^\infty \mathbb{P}(|X|^s \geq u) e^{\frac{u}{3/\log 2}} du. \end{aligned}$$

By (68), $\mathbb{P}(|X|^s \geq u) = \mathbb{P}(|X| \geq u^{1/s}) \leq 2 \exp(-u \log 2)$, then we get

$$\begin{aligned} \mathbb{E} \exp\left(\frac{|X|^s}{3/\log 2}\right) &\leq 1 + \frac{2 \log 2}{3} \int_0^\infty \exp\left(-u \log 2 + \frac{u}{3/\log 2}\right) du \\ &\leq 1 + \frac{2 \log 2}{3} \int_0^\infty \exp\left(-\frac{2 \log 2}{3} u\right) du = 2. \end{aligned}$$

Thus, we have

$$\|X\|_{\psi_s} \leq 3/\log 2. \quad (69)$$

(c) Without loss of generality, we can assume $\|X\|_{\psi_s} = 1$. By variable substitution $u = t^s$ and (67) we have

$$\begin{aligned} \mathbb{E} |X|^q &= \int_0^\infty q t^{q-1} \mathbb{P}(|X| \geq t) dt \\ &\leq \int_0^\infty q u^{\frac{q-1}{s}} 2 e^{-u} \frac{1}{s} u^{\frac{1}{s}-1} du \\ &= \int_0^\infty \frac{2q}{s} u^{\frac{q}{s}-1} e^{-u} du = \frac{2q}{s} \Gamma\left(\frac{q}{s}\right) = 2\Gamma\left(\frac{q}{s} + 1\right). \end{aligned}$$

Note that for $r > 0$,

$$\Gamma(r+1) = \int_0^\infty (x^r e^{-\frac{x}{2}}) e^{-\frac{x}{2}} dx \leq (2r)^r e^{-r} \int_0^\infty e^{-\frac{x}{2}} dx = 2 \left(\frac{2r}{e}\right)^r, \quad (70)$$

where we used the fact that $x^r e^{-\frac{x}{2}}$ attains maximum at $x = 2r$ as

$$\frac{d}{dx} (x^r e^{-\frac{x}{2}}) = x^{r-1} e^{-\frac{x}{2}} \left(r - \frac{x}{2} \right).$$

Therefore

$$\mathbb{E} |X|^q \leq 4 \left(\frac{2q}{se} \right)^{\frac{q}{s}} = 4 \left(\frac{2}{se} \right)^{\frac{q}{s}} q^{\frac{q}{s}} \leq 4q^{\frac{q}{s}} \leq \left(4q^{\frac{1}{s}} \right)^q.$$

B High-Order Moments

B.1 Proof of Lemma 1

Note that when $q \geq 1$, the triangle inequality yields

$$\begin{aligned} (\mathbb{E} |\phi^* \mathbf{X} \phi|^q)^{1/q} &\leq (\mathbb{E} |\phi^* \mathbf{X} \phi - \mathbb{E} [\phi^* \mathbf{X} \phi]|^q)^{1/q} + (\mathbb{E} |\mathbb{E} [\phi^* \mathbf{X} \phi]|^q)^{1/q} \\ &= (\mathbb{E} |\phi^* \mathbf{X} \phi - \mathbb{E} [\phi^* \mathbf{X} \phi]|^q)^{1/q} + |\text{Tr}(\mathbf{X})| \\ &\leq (\mathbb{E} |\phi^* \mathbf{X} \phi - \mathbb{E} [\phi^* \mathbf{X} \phi]|^q)^{1/q} + \|\mathbf{X}\|_* \end{aligned} \quad (71)$$

By Hanson-Wright inequality in [39], we have that

$$\begin{aligned} &\mathbb{E} |\phi^* \mathbf{X} \phi - \mathbb{E} [\phi^* \mathbf{X} \phi]|^q \\ &= \int_0^\infty q t^{q-1} \mathbb{P} (|\phi^* \mathbf{X} \phi - \mathbb{E} [\phi^* \mathbf{X} \phi]| > t) dt \\ &\leq 2q \left(\int_0^\infty t^{q-1} \exp \left(-c \frac{t^2}{K^4 \|\mathbf{X}\|_F^2} \right) dt + \int_0^\infty t^{q-1} \exp \left(-c \frac{t}{K^2 \|\mathbf{X}\|_{op}} \right) dt \right) \\ &= 2q \left(K^{2q} \|\mathbf{X}\|_F^q \int_0^\infty x^{q-1} \exp(-cx^2) dx + K^{2q} \|\mathbf{X}\|_{op}^q \int_0^\infty x^{q-1} \exp(-cx) dx \right) \\ &\leq 2q K^{2q} \max \{ c^{q/2-1}, c^{q-1} \} \left(\Gamma \left(\frac{q}{2} \right) + \Gamma(q) \right) \cdot \|\mathbf{X}\|_F^q \\ &\leq 4q K^{2q} (\max \{ c, 1 \})^q \Gamma(q) \cdot \|\mathbf{X}\|_F^q \leq (4 \max \{ c, 1 \} q K^2)^q \cdot \|\mathbf{X}\|_F^q. \end{aligned} \quad (72)$$

In the last line, we use the behavior of the Gamma function $\Gamma(x)$ in (70). Thus, by (71) and (72), we finish the proof.

B.2 Proof of Lemma 2

The proof is similar to Lemma 1. Using Hanson-Wright inequality [39] and note that $\mathbb{E} \phi_a^* \mathbf{X} \phi_b = 0$, we have

$$\begin{aligned} (\mathbb{E} |\phi_a^* \mathbf{X} \phi_b|^q)^{1/q} &\leq (\mathbb{E} |\phi_a^* \mathbf{X} \phi_b - \mathbb{E} [\phi_a^* \mathbf{X} \phi_b]|^q)^{1/q} + |\mathbb{E} \phi_a^* \mathbf{X} \phi_b| \\ &= (\mathbb{E} |\phi_a^* \mathbf{X} \phi_b - \mathbb{E} [\phi_a^* \mathbf{X} \phi_b]|^q)^{1/q} \lesssim q K^2. \end{aligned}$$

C Proof of Lemma 4

When $\ell = 1$, for any $\mathbf{u}, \mathbf{v} \in \mathbb{F}^n$, choosing $\theta := \text{Phase}(\mathbf{v}^* \mathbf{u})$, then we have

$$\begin{aligned} \mathcal{M}_1(K, q) &= \sup_{\mathbf{u}, \mathbf{v} \in \mathbb{F}^n} \mathbb{E} \left| \frac{|\langle \phi, \mathbf{u} \rangle| - |\langle \phi, \mathbf{v} \rangle|}{d_1(\mathbf{u}, \mathbf{v})} \right|^q \\ &\leq \sup_{\mathbf{u}, \mathbf{v} \in \mathbb{F}^n} \mathbb{E} \left| \left\langle \phi, \frac{\mathbf{u} - e^{i\theta} \mathbf{v}}{d_1(\mathbf{u}, \mathbf{v})} \right\rangle \right|^q = \sup_{\mathbf{w} \in \mathbb{S}_{\ell_2}} \mathbb{E} |\langle \phi, \mathbf{w} \rangle|^q \lesssim (\sqrt{q}K)^q. \end{aligned}$$

The last inequality follows from Proposition 2.(c) as $|\langle \phi, \mathbf{w} \rangle|$ is subgaussian.

When $\ell = 2$, by Lemma 1, we have

$$\mathcal{M}_2(K, q) = \sup_{\mathbf{u}, \mathbf{v} \in \mathbb{F}^n} \mathbb{E} \left| \left\langle \phi \phi^*, \frac{\mathbf{u} \mathbf{u}^* - \mathbf{v} \mathbf{v}^*}{\|\mathbf{u} \mathbf{u}^* - \mathbf{v} \mathbf{v}^*\|_F} \right\rangle \right|^q \lesssim \left(qK^2 + \sqrt{2} \right)^q \lesssim (qK^2)^q.$$

We have used the facts $\mathbf{u} \mathbf{u}^* - \mathbf{v} \mathbf{v}^*$ is at most rank 2 and $K \gtrsim 1$.

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