On some linear equations associated with dispersionless integrable systems

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Abstract

We use a recently proposed scheme of matrix extension of dispersionless integrable systems for the Abelian case, in which it leads to linear equations, connected with the initial dispersionless system. In the examples considered, these equations can be interpreted in terms of Abelian gauge fields on the geometric background defined by the dispersionless system. They are also connected with the linearisation of initial systems. We construct solutions to these linear equations in terms of wave functions of the Lax pair for dispersionless system, which is represented in terms of some vector fields.

Keywords: Dispersionless integrable systems; self-dual conformal structures; Einstein-Weyl geometry; Manakov-Santini system

1 Introduction

Recently we proposed a scheme of matrix extension of the Lax pairs of dispersionless integrable systems (see [1] and references therein), which leads to matrix equations on the background of the initial dispersionless systems. There are important cases in which dispersionless integrable systems describe some geometric structures (self-dual conformal structures, Einstein-Weyl geometry [2]), and in these cases the matrix extension scheme provides equations for gauge fields on the respective geometric background (see [3], [4], [5]).

In the present work, we would like to develop in more detail some observations made in the article [1] about the Abelian case of the matrix extension scheme. In the Abelian case equations of extension become linear, in our examples they can be represented as an action of linear differential operators of the second order (with the coefficients defined through the solutions of the basic dispersionless integrable system) on the scalar function. But nevertheless these equations could be of interest for several reasons. First, in three and four dimension, where we have an interpretation of equations in terms of gauge fields,

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the Abelian case corresponds to electromagnetic fields on geometric background and could be of interest by itself. Second, the arising linear operators are connected with the linearisation of basic dispersionless equations and can be useful for the study of stability of solutions and singularities of these equations. For example, for the second heavenly equation linear operator of Abelian extension is exactly the linearisation of the equation. And finally, the general solution of extension equations in the Abelian case can be found explicitly through the wave functions of the Lax pair. In the paper [1] it was done using the dressing scheme, but here we will not use the dressing scheme, constructing the general solution in a rather elementary way.

Our main examples in this work include the equations of self-dual conformal structure (SDCS) and the Manakov-Santini system, which describes the Einstein-Weyl structures. For convenience of the reader, we provide some basic information about the matrix extension scheme and geometric structures in the Appendices.

2 Abelian extension of SDCS equations

Let us consider the Lax pair [6]

$$\begin{aligned} X_1 &= \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_x\partial_\lambda, \\ X_2 &= \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_y\partial_\lambda. \end{aligned} \tag{1}$$

Commutation relations for this Lax pair give a coupled system of three second-order PDEs for the functions F, G, f

$$\begin{cases} Q(F) = f_y, \\ Q(G) = -f_x, \\ Q(f) = 0, \end{cases} \tag{2}$$

where linear second order differential operator Q is given by

$$\begin{aligned} Q &= (\partial_w + F_y\partial_x + G_y\partial_y)\partial_x + (\partial_z + F_x\partial_x + G_x\partial_y)\partial_y \\ &= \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y. \end{aligned} \tag{3}$$

System (2) can be rewritten in the form of a coupled system of third order PDEs for the functions F, G

$$\begin{cases} \partial_x(Q(F)) + \partial_y(Q(G)) = 0, \\ (\partial_w + F_y\partial_x + G_y\partial_y)Q(G) + (\partial_z + F_x\partial_x + G_x\partial_y)Q(F) = 0, \end{cases} \tag{4}$$

in this form it was introduced in [2] in connection with the self-dual conformal structures (see Appendix 2).

Scalar extension of the Lax pair (1) (see Appendix 1)

$$\begin{aligned} \nabla_{X_1} &= \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_x\partial_\lambda + a_1, \\ \nabla_{X_2} &= \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_y\partial_\lambda + a_2 \end{aligned}$$

generates a linear equation for the potential ϕ, $a_1 = \partial_x \phi$, $a_2 = \partial_y \phi$,

$$Q\phi := (\partial_w \partial_x - \partial_z \partial_y + F_y {\partial_x}^2 - G_x {\partial_y}^2 - (F_x - G_y)\partial_x \partial_y)\phi = 0. \tag{5}$$

Solution through the wave functions.

In [1] we constructed a general solution to linear equation (5) using a dressing scheme. It is easy to obtain this formula directly using the dispersionless Lax pair. Indeed, cross-differentiating over y, x linear equations

$$\begin{aligned}(\partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_x\partial_\lambda)\Psi = 0,\\ (\partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_y\partial_\lambda)\Psi = 0,\end{aligned} \tag{6}$$

we get the relation

$$((\partial_w + F_y\partial_x + G_y\partial_y)\partial_x - (\partial_z + F_x\partial_x + G_x\partial_y)\partial_y)\,\Psi = \partial_\lambda(f_x\partial_y - f_y\partial_x)\Psi. \tag{7}$$

Integration of the r.h.s. with respect to λ over a closed contour gives zero, thus $Q\oint \Psi d\lambda = 0$, and

$$\phi = \frac{1}{2\pi i}\oint \Psi d\lambda \tag{8}$$

gives a solution to linear equation (5) for an arbitrary wave function (analytic in the neihborhood of the contour or given in terms of formal Laurent series). Linear equations (6) possess three basic wave functions Ψ^0, Ψ^1, Ψ^2 [6] and a general wave function is represented as

$$\Psi = f(\Psi^0, \Psi^1, \Psi^2),$$

where f is an arbitrary analytic function. Thus solution (8) possesses a functional freedom of a function of three variables, corresponding to a general solution of linear equation (5).

For trivial background

$$\begin{aligned}&(\partial_w\partial_x - \partial_z\partial_y)\,\phi = 0,\\ &\phi = \frac{1}{2\pi i}\oint f(\lambda, \lambda z + x, \lambda w + y)d\lambda.\end{aligned}$$

This formula is easily recognised as a version of Penrose formula for solutions of the wave equation written for the case of neutral signature.

Considering the linearization of SDCS system (2) (or (4)), we come to the observation that operator Q is contained as a factor in the principal part of linearised equations.

Reductions of the SDCS system

First, let us consider the volume-preserving reduction, that leads to the Dunajski system [7]. This reduction is connected with zero divergence vector fields (1).

In this case functions F, G can be defined through the potential Θ, $F = \Theta_y$, $G = -\Theta_x$, and system (2) takes the form

$$\begin{aligned} &\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f, \\ &Qf = 0, \end{aligned} \tag{9}$$

where the operator Q is expressed in terms of potential Θ as

$$Q = \partial_w\partial_x + \partial_z\partial_y + \Theta_{yy}\partial_x\partial_x + \Theta_{xx}\partial_y\partial_y - 2\Theta_{xy}\partial_x\partial_y,$$

This operator defines an Abelian extension of the Dunajsky system, it also represents a bivector defining a conformal structure. System (9) can be written as one fourth-order equation

$$Q(\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2) = 0$$

Another standard reduction of SDCS system (2) is a linearly degenerate case, which corresponds to hyper CR (Cauchy-Riemann) equations. In this case vector fields (1) do not contain a derivative over spectral variable, $f = 0$, $\Psi^0 = \lambda$ is a wave function of linear operators, and system (2) reads

$$\begin{cases} Q(F) = 0, \\ Q(G) = 0. \end{cases} \tag{10}$$

Operator Q here is of the same form as in general SDSC case, it also defines an abelian extension. However, due to the reduction, some new special features of solutions of this operator arise. Indeed, for the reduced Lax pair

$$\begin{aligned} (\partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y)\Psi = 0, \\ (\partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y)\Psi = 0, \end{aligned}$$

and instead of relation (7) we now have

$$Q\Psi = \left((\partial_w + F_y\partial_x + G_y\partial_y)\partial_x - (\partial_z + F_x\partial_x + G_x\partial_y)\partial_y\right)\Psi = 0. \tag{11}$$

Thus for linearly degenerate case *an arbitrary wave function of the reduced Lax pair satisfies equation (5)*! The reduction also leads to a recursion for solutions of equation (11), defined by the relations

$$\begin{aligned} \partial_x\phi' = (\partial_z + F_x\partial_x + G_x\partial_y)\phi, \\ \partial_y\phi' = (\partial_w + F_y\partial_x + G_y\partial_y)\phi. \end{aligned} \tag{12}$$

The compatibility condition for these relations coincide with equation (11) for ϕ, and cross-action of linear operators of the r.h.s. gives (modulo equation (10)) equation (11) for ϕ'.

Similar observations for operators of linearisation of linearly degenerate equations were done in the work of Segyeyev [8].

Finally, applying both volume-preserving and linearly degenerate case reductions to SDCS system (2), we obtain the famous Plebański second heavenly equation

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0. \tag{13}$$

The operator of Abelian extension is of the same form as for the Dunajski system,

$$Q = \partial_w \partial_x + \partial_z \partial_y + \Theta_{yy}\partial_x\partial_x + \Theta_{xx}\partial_y\partial_y - 2\Theta_{xy}\partial_x\partial_y,$$

it coincides with the linearisation operator for the heavenly equation (13). Solutions to the equation $Q\phi = 0$ are given by arbitrary wave functions of the Lax pair for the heavenly equation, $\phi = \Psi$,

$$\begin{aligned}
&(\partial_z - \lambda\partial_x + \Theta_{xy}\partial_x - \Theta_{xx}\partial_y)\Psi = 0,\\
&(\partial_w - \lambda\partial_y + \Theta_{yy}\partial_x - \Theta_{xy}\partial_y)\Psi = 0.
\end{aligned}$$

Recursion relations for coefficients of expansion of wave functions into the powers of the spectral variable lead to the recursion for solutions of the linearized second heavenly equation $Q\phi = 0$,

$$\begin{aligned}
\partial_x \phi' &= (\partial_z + \Theta_{xy}\partial_x - \Theta_{xx}\partial_y)\phi,\\
\partial_y \phi' &= (\partial_w + \Theta_{yy}\partial_x - \Theta_{xy}\partial_y)\phi.
\end{aligned}$$

This type of recursion was introduced in [8].

3 Abelian extension of the Manakov-Santini system

The Manakov-Santini system [9] is a two-component integrable generalisation of the dispersionless Kadomtsev-Petviashvili (dKP) equation,

$$\begin{aligned}
u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y,\\
v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y
\end{aligned} \tag{14}$$

It correspons to arbitrary vector fields in the Lax pair, instead of Hamiltonian vector fields for the dKP equation,

$$\begin{aligned}
X_1 &= \partial_y - (\lambda - v_x)\partial_x + u_x \partial_\lambda,\\
X_2 &= \partial_t - (\lambda^2 - v_x\lambda + u - v_y)\partial_x + (u_x\lambda + u_y)\partial_\lambda
\end{aligned} \tag{15}$$

For $v = 0$ this system reduces to the dKP (or Khohlov-Zabolotskaya) equation

$$u_{xt} = u_{yy} + (uu_x)_x,$$

reduction $u = 0$ (linearly degenerate case) gives the equation (Mikhalev [10], Pavlov [11])

$$v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y.$$

Abelian extension of the Lax pair (15) (see also Appendix 1)

$$\begin{aligned}\nabla_{X_1} &= \partial_y - (\lambda - v_x)\partial_x + u_x\partial_\lambda + \kappa_x,\\ \nabla_{X_2} &= \partial_t - (\lambda^2 - v_x\lambda + u - v_y)\partial_x + (u_x\lambda + u_y)\partial_\lambda + \lambda\kappa_x + \kappa_y,\end{aligned}\tag{16}$$

leads to linear equation for the scalar function κ

$$Q\kappa := (\partial_t\partial_x - \partial_y\partial_y - (u - v_y)\partial_x\partial_x - v_x\partial_x\partial_y - u_x\partial_x)\kappa = 0.$$

Any wave function $\Psi(\lambda)$ of linear operators (15) given on the contour defines a solution to this equation by the formula analogous to (8).

To derive this formula directlly from the Lax pair (15), we rewrite equations for the wave functions in the form

$$\begin{aligned}&(\partial_y - \lambda\partial_x + v_x\partial_x + u_x\partial_\lambda)\Psi = 0,\\ &(\partial_t - \lambda\partial_y - (u - v_y)\partial_x + u_y\partial_\lambda)\Psi = 0.\end{aligned}$$

Cross-differentiating by respectively y and x and taking the difference, we get

$$Q\Psi = \partial_\lambda(u_x\partial_y - u_y\partial_x)\Psi,\tag{17}$$

then integration over the contour cancels the r.h.s., and we obtain the formula

$$\kappa = \frac{1}{2\pi i}\oint \Psi(\mu)d\mu.\tag{18}$$

In terms of the basic wave functions of the Lax pair (15) $\Psi = F(\Psi^0, \Psi^1)$, and we have a solution with functional freedom of a function of two variables.

The dKP equation case

The case of the dKP equation corresponds to Hamiltonian vector fields in the Lax pair (15), leading to $v = 0$ and

$$u_{xt} = u_{yy} + (uu_x)_x.\tag{19}$$

Commutation relations for the extended Lax pair (reduced extended Lax pair (16))

$$\begin{aligned}\nabla_{X_1} &= \partial_y - \lambda\partial_x + u_x\partial_\lambda + \kappa_x,\\ \nabla_{X_2} &= \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda + \lambda\kappa_x + \kappa_y\end{aligned}$$

imply the dKP equation (19) and linear equation for κ,

$$Q\kappa := (\partial_t\partial_x - \partial_y\partial_y - u\partial_x\partial_x - u_x\partial_x)\kappa = 0.$$

The operator Q doesn't coincide with the linearisation operator for dKP equation (19), which reads

$$P = \partial_t\partial_x - \partial_y\partial_y - u\partial_{xx} - 2u_x\partial_x - u_{xx}$$

However, the two operators are connected via a simple identity $\partial_x Q = P\partial_x$, which implies that $Q\kappa = 0 \Rightarrow P\partial_x\kappa = 0$. In other words, the operator Q corresponds to the linearisation of the potential dKP equation for the function w, $u = w_x$. For κ we have a formula (18),

$$\kappa = \frac{1}{2\pi i}\oint F(\Psi^0, \Psi^1)d\mu,$$

it gives a symmetry of potential dKP equation. The symmetry for the dKP equation (solution of linearised equation) is defined by $\partial_x\kappa$. This is a rather familiar formula for the symmetries of the dKP equation, in standard dKP hierarchy notations $\Psi^0 = L$ (Lax-Sato function), $\Psi^1 = M$ (Orlov function).

Mikhalev-Pavlov equation

Considering the linearly degenerate case of the MS system, for which $u = 0$, we get the equation

$$v_{xt} = v_{yy} + v_x v_{xy} - v_{xx}v_y \tag{20}$$

with the Lax pair

$$\begin{aligned} X_1 &= \partial_y - \lambda\partial_x + v_x\partial_x, \\ X_2 &= \partial_t - (\lambda^2 - v_x\lambda - v_y)\partial_x. \end{aligned} \tag{21}$$

The Abelian extension of the Lax pair

$$\begin{aligned} \nabla_{X_1} &= \partial_y - \lambda\partial_x + v_x\partial_x + \kappa_x, \\ \nabla_{X_2} &= \partial_t - (\lambda^2 - v_x\lambda - v_y)\partial_x + \lambda\kappa_x + \kappa_y \end{aligned} \tag{22}$$

implies linear equation

$$Q\kappa := (\partial_t\partial_x - \partial_y\partial_y + v_y\partial_x\partial_x - v_x\partial_x\partial_y)\kappa = 0. \tag{23}$$

Similar to linearly degenerate case of the SDCS equations (11), *any wave function of linear operators (21)* $\Psi(\lambda)$, $X_1\Psi(\lambda) = 0$, $X_2\Psi(\lambda) = 0$, *satisfies this linear equation*. Indeed, the r.h.s. of the formula (17) in the linearly degenerate case is equal to zero, then $Q\Psi(\lambda) = 0$. In terms of the basic wave functions of linear operators (21), $\Psi = F(\lambda, \Psi^1)$.

Linear operator Q in this case does not coincide with the linearisation operator for equation (20), which is

$$\begin{aligned} P = \partial_t\partial_x - \partial_y\partial_y + v_y\partial_x\partial_x - v_x\partial_x\partial_y + v_{xx}\partial_y - v_{xy}\partial_x \\ = Q + v_{xx}\partial_y - v_{xy}\partial_x. \end{aligned} \tag{24}$$

In the work [8] it was demonstrated that the linearised equation is satisfied by the function Ψ_x^{-1}, $P\Psi_x^{-1} = 0$, where $\Psi(\lambda)$ is a wave function of the Lax pair (21). The recursion for the linearised equation was also constructed.

To derive linearisation operator and solutions for it in terms of the Lax pair, we will use a parametric deformation of the Lax pair described in Appendix 3,

$$
\begin{aligned}
X_{1\alpha} &= \partial_y - \lambda\partial_x + v_x\partial_x + \alpha v_{xx}, \\
X_{2\alpha} &= \partial_t - \lambda\partial_y + v_y\partial_x + \alpha v_{xy}.
\end{aligned}
\tag{25}
$$

This deformation takes us out of the class of vector fields, however, the compatibility conditions remain the same. For $\alpha = 0$ this is a standard Lax pair in terms of vector fields, and $\alpha = 1$ corresponds to the formally adjoint Lax pair. The deformed Lax pair implies a special solution for Abelian extension equation (23) $\kappa = v_x$, that is easily checked by differentiating Mikhalev-Pavlov equation (20). A general wave function of deformed Lax pair (25) in terms of the basic functions of the Lax pair (21) reads (see Appendix 3)

$$
\widetilde{\Psi}_\alpha = (\Psi_x^1)^\alpha F(\lambda, \Psi^1).
$$

Instead of formula (17) we obtain

$$
\partial_y(\partial_y + v_x\partial_x + \alpha v_{xx})\widetilde{\Psi}_\alpha(\lambda) = \partial_x(\partial_t + v_y\partial_x + \alpha v_{xy})\widetilde{\Psi}_\alpha(\lambda),
$$

thus

$$
Q_\alpha\widetilde{\Psi}_\alpha := (\partial_t\partial_x - \partial_y\partial_y + v_y\partial_x\partial_x - v_x\partial_x\partial_y + \alpha(v_{xy}\partial_x - v_{xx}\partial_y))\,\widetilde{\Psi}_\alpha = 0.
$$

For solutions of linear equation $Q_\alpha\phi = 0$ we have a recursion

$$
\begin{aligned}
\phi'_x &= (\partial_y + v_x\partial_x + \alpha v_{xx})\phi, \\
\phi'_y &= (\partial_t + v_y\partial_x + \alpha v_{xy})\phi.
\end{aligned}
$$

The compatibility condition given by cross-differentiation over y, x is the equation $Q_\alpha\phi = 0$, while cross-action of linear operators of the r.h.s. leads (modulo equation (20)) to the equation $Q_\alpha\phi' = 0$. The case of linearisation operator P corresponds to $\alpha = -1$, $P = Q_{-1}$. Symmetries for the Mikhalev-Pavlov equation are given by the expression

$$
\widetilde{\Psi}_{-1} = (\Psi_x^1)^{-1} F(\lambda, \Psi^1),
$$

and recursion for the symmetries is

$$
\begin{aligned}
\phi'_x &= (\partial_y + v_x\partial_x - v_{xx})\phi, \\
\phi'_y &= (\partial_t + v_y\partial_x - v_{xy})\phi.
\end{aligned}
$$

The interpolating reduction

Let us consider also the interpolating reduction of the Manakov-Santini system (14), which is defined by the condition

$$au = v_x,$$

where a is a parameter (see [13], [14]). Under this condition, the Manakov-Santini system can be written as one equation for the function v,

$$v_{xt} = v_{yy} + a^{-1} v_x v_{xx} + v_x v_{xy} - v_{xx} v_y. \tag{26}$$

A remarkable property of this equation, justifying the name 'interpolating' [13], is that its limit for $a \to 0$ leads to the dKP equation, while for $a \to \infty$ it gives the Mikhalev-Pavlov equation. The linear equation of Abelian extension in this case reads

$$Q\kappa := (\partial_t \partial_x - \partial_y \partial_y - (a^{-1} v_x - v_y)\partial_x \partial_x - v_x \partial_x \partial_y - a^{-1} v_{xx} \partial_x)\kappa = 0,$$

and its solution can be obtained using an arbitrary wave function of Lax operators (15) (taking into account reduction condition) by formula (18).

Similar to the case of the Mikhalev-Pavlov equation, linear operator Q does not coincide with the linearisation operator for equation (26), which reads

$$P = Q + v_{xx}\partial_y - v_{xy}\partial_x.$$

To find solutions of equation $P\phi = 0$ through the wave functions of Lax operators, we use the same trick as for the Mikhalev-Pavlov equation case. Introducing a parametric deformation of the Lax pair described in Appendix 3, we arrive to the following formula for solutions of linearised interpolating equation $P\phi = 0$ (symmetries of interpolating equation)

$$\phi = \frac{1}{2\pi i} \oint \mathrm{e}^{a(\mu - \Psi^0)} \Psi(\mu) d\mu.$$

where Ψ is an arbitrary wave function $\Psi = F(\Psi^0, \Psi^1)$, and Ψ^0 is a basic wave function with the expansion $\lambda + u\lambda^{-1} + \dots$ (corresponds to Lax-Sato function L in standard dKP notations).

Appendix 1

Matrix and Abelian extension of multidimensional dispersionless integrable systems

We will give a brief description of the scheme of matrix extension of the Lax pairs of dispersionless integrable systems (see [1] and references therein). Multidimensional dispersionless integrable systems are associated with Lax pairs in terms of vector fields depending on a spectral parameter. We will consider Lax pairs of the type

$$[X_1, X_2] = 0, \tag{27}$$

$$X_1 = \partial_{t_1} + \sum_{i=1}^{N} F_i \partial_{x_i} + F_0 \partial_\lambda, \quad X_2 = \partial_{t_2} + \sum_{i=1}^{N} G_i \partial_{x_i} + G_0 \partial_\lambda. \tag{28}$$

λ - 'spectral parameter', functions F_k, G_k are holomorphic in λ (polynomials, Laurent polynomials) and depend on the variables t_1, t_2, x_n. The class of equations corresponding to such Lax pairs includes dispersionless limits of integrable equations (dKP, dispersionless 2DTL hierarchy), Plebański heavenly equations and their generalizations, hyper-Kähler hierarchies. A scheme of matrix extension leads to gauge covariant Lax pairs of the type

$$\nabla_{X_1} = X_1 + A_1, \quad \nabla_{X_2} = X_2 + A_2, \tag{29}$$

A_1, A_2 are matrix-valued functions of space-time variables holomorphic in λ (polynomials, Laurent polynomials). Lax pairs of this structure were already present in the seminal work of Zakharov and Shabat (1979). The commutator of two covariant (extended) vector fields contains vector field part and matrix (Lie algebraic) part,

$$[\nabla_{X_1}, \nabla_{X_2}] = [X_1, X_2] + X_1 A_2 - X_2 A_1 + [A_1, A_2]$$

Vector fields term of compatibility conditions gives the basic dispersionless system,

$$[X_1, X_2] = 0,$$

while the matrix term provides matrix equations on the dispersionless background

$$X_1 A_2 - X_2 A_1 + [A_1, A_2] = 0. \tag{30}$$

In several important examples the basic system corresponds to some geometric object, and equations (30) are connected with gauge fields on the geometric background (see Appendix 2).

For Abelian gauge fields,

$$\nabla_{X_1} = X_1 + a_1, \quad \nabla_{X_2} = X_2 + a_2, \tag{31}$$

where a_1, a_2 are scalar functions, equations (30) become linear,

$$X_1 a_2 - X_2 a_1 = 0. \tag{32}$$

These equations are the main object of the study in the present work.

Extension of the SDCS equations Matrix extension of the Lax pair (1) reads

$$\begin{aligned}\nabla_{X_1} &= \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_x\partial_\lambda + \Phi_x,\\ \nabla_{X_2} &= \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_y\partial_\lambda + \Phi_y,\end{aligned}$$

it generates an equation for the matrix potential Φ on the background of the SDCS equations,

$$\begin{aligned}&Q\Phi = [\Phi_x, \Phi_y],\\ &Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.\end{aligned} \tag{33}$$

In the Abelian case we have a linear equation for the scalar potential $Q\phi = 0$.

The MS system - extension Matrix extension of the Lax pair for the MS system reads

$$\begin{aligned}\nabla_{X_1} &= \partial_y - (\lambda - v_x)\partial_x + u_x\partial_\lambda + A,\\ \nabla_{X_2} &= \partial_t - (\lambda^2 - v_x\lambda + u - v_y)\partial_x + (u_x\lambda + u_y)\partial_\lambda + \lambda A + B,\end{aligned} \tag{34}$$

where A, B are matrix-valued functions. Vector field part of commutation relations gives the Manakov-Santini system (14), while the matrix part gives a matrix system on the background of the Manakov-Santini system

$$\begin{aligned}&A_y - B_x = 0,\\ &(\partial_y + v_x\partial_x)B - (\partial_t + (v_y - u)\partial_x)A + u_xA + [A, B] = 0\end{aligned}$$

For the potential K, $A = K_x$, $B = K_y$ we have

$$QK = [K_x, K_y], \tag{35}$$

where Q is a linear operator

$$Q = \partial_t\partial_x - \partial_y\partial_y - (u - v_y)\partial_x\partial_x - v_x\partial_x\partial_y - u_x\partial_x.$$

In the Abelian case instead of equation (35) we have linear equation $Q\kappa = 0$.

Appendix 2

Geometric structures

The starting point for the geometric interpretation of the systems considered in this work are two theorems from the article [2], we also refer the reader to this work for more details.

We recall that a conformal structure $[g]$ is called anti-self-dual (ASD) if the self-dual part of the Weyl tensor of any $g \in [g]$ vanishes: $W_+ = \frac{1}{2}(W + *W) = 0$.

Theorem 1 (Dunajski, Ferapontov and Kruglikov (2014)). *There exist local coordinates (z, w, x, y) such that any ASD conformal structure in signature $(2, 2)$ is locally represented by a metric*

$$\tfrac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2, \tag{36}$$

where the functions F, $G : M^4 \to \mathbb{R}$ satisfy a coupled system of third-order PDEs,

$$\begin{aligned} &\partial_x(Q(F)) + \partial_y(Q(G)) = 0, \\ &(\partial_w + F_y\partial_x + G_y\partial_y)Q(G) + (\partial_z + F_x\partial_x + G_x\partial_y)Q(F) = 0, \end{aligned} \tag{37}$$

where

$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

Theorem 2 (Dunajski, Ferapontov and Kruglikov (2014)). *There exists a local coordinate system (x, y, t) on M^3 such that any Lorentzian Einstein-Weyl structure is locally of the form*

$$\begin{aligned} g &= -(dy + v_x dt)^2 + 4(dx + (u - v_y)dt)dt, \\ \omega &= v_{xx}dy + (-4u_x + 2v_{xy} + v_x v_{xx})dt, \end{aligned} \tag{38}$$

where the functions u and v satisfy the Manakov-Santini system

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx}v_y, \\ v_{xt} &= v_{yy} + uv_{xx} + v_x v_{xy} - v_{xx}v_y \end{aligned}$$

Thus the background equations used in this work have a clear geometric meaning: SDCS equations (4) define a general (anti)self-dual conformal structure in signature $(2, 2)$, and the Manakov-Santini system (14) corresponds to a general Lorentzian Einstein-Weyl structure.

We would also like to mention that the interpolating equation (26) was introduced in [13] as "the most general symmetry reduction of the second heavenly equation by a conformal Killing vector with a null self-dual derivative".

Let us consider a gauge potential A, which is a one-form taking its values in some (matrix) Lie algebra, and the two-form $F = \mathrm{d}A + A \wedge A$ (gauge field). Matrix equation (33) represents (anti)self-dual Yang-Mills (SDYM) equations on the background of conformal structure (36),

$$F = \pm{*F}, \tag{39}$$

taken in a special gauge (see [3] for more details). In the Abelian case equations become linear, $F = \mathrm{d}A$, $\mathrm{d}A = \pm{*\mathrm{d}A}$.

Matrix equation (35) corresponds to the equation (taken in a special gauge, see [4] for more details)

$$D\Phi + \tfrac{1}{2}\omega\Phi = *F, \tag{40}$$

where $D\Phi = \mathrm{d}\Phi + [A, \Phi]$, Φ is a function taking values in the Lie algebra (Higgs field, [16]). This equation is considered on the Einstein-Weyl background (38); for Minkowski metric it coincides with the Yang-Mills-Higgs system introduced by Ward [16], leading to integrable chiral model. In the Abelian case this equation becomes linear,

$$\mathrm{d}\Phi + \tfrac{1}{2}\omega\Phi = *\mathrm{d}A.$$

Appendix 3

Adjoint Lax operators and a parametric deformation

To obtain linearisation operator from the Lax pair and construct its solutions, we will use a parametric deformation of the Lax pair introduced in [14],[15]. This deformation is nontrivial only for vector fields with nonzero divergence, and in this case we have a freedom to add some term with first derivatives containing a parameter to the Abelian extension operator Q.

Let us consider a Lax pair of two vector fields of the form (28)

$$X_1 = \partial_{t_1} + \sum_{i=1}^{N} F_i\partial_{x_i} + F_0\partial_\lambda, \quad X_2 = \partial_{t_2} + \sum_{i=1}^{N} G_i\partial_{x_i} + G_0\partial_\lambda. \tag{41}$$

We introduce a basic set of wave functions $\Psi^0, \dots, \Psi^N$, a general wave function is expressed as $\Psi = F(\Psi^0, \dots, \Psi^N)$, $X_1\Psi = 0$, $X_2\Psi = 0$. For the linearly degenerate case the terms with ∂_λ are absent, $F_0 = G_0 = 0$, $\Psi^0 = \lambda$.

A formally adjoint operator to a vector field reads

$$X^* = -(X + \mathrm{div}X),$$

where $\mathrm{div}X = \partial_\lambda F_0 + \sum_{i=1}^{N} \partial_{x_i} F_i$. We should emphasize that an adjoint operator is no more a pure vector field, it contains a term $\mathrm{div}X$ which is a multiplication by the function. In the case of zero divergence (volume-preserving vector field) this term is equal to zero and the vector field is anti-self-adjoint. Compatibility conditions for adjoint vector fields remain the same. It is interesting to note that $a_1 = \mathrm{div}X_1$, $a_2 = \mathrm{div}X_2$ give a special solution to Abelian extension equation (32).

Let us consider linear equations corresponding to adjoint vector fields

$$(X_1 + \mathrm{div}X_1)\widetilde{\Psi} = 0, \quad (X_2 + \mathrm{div}X_2)\widetilde{\Psi} = 0, \tag{42}$$

where by $\widetilde{\Psi}$ we denote a wave function of these equations. A special solution is given by the Jacobian of the basic wave functions for vector fields (41) (see [15])

$$\widetilde{\Psi} = J := \frac{\partial(\Psi^0, \Psi^1, \dots, \Psi^N)}{\partial(\lambda,\ x_1, \dots, x_N\ )}.$$

To construct a general solution, we rewrite equations (42) as nonhomogeneous linear equations for $\ln \widetilde{\Psi}$,

$$X_1 \ln \widetilde{\Psi} + \mathrm{div} X_1 = 0, \quad X_2 \ln \widetilde{\Psi} + \mathrm{div} X_2 = 0.$$

Then, evidently, a general solution reads

$$\ln \widetilde{\Psi} = \ln J_0 + F(\Psi^0, \ldots, \Psi^N),$$

and for equations (42)

$$\widetilde{\Psi} = J_0 f(\Psi^0, \ldots, \Psi^N).$$

Morover, it is possible to rewrite linear equations in terms of the function J^α and obtain a parametric deformation of the Lax pair

$$X \ln(J^\alpha) + \alpha \mathrm{div} X = 0,$$

for equations (42)

$$(X_1 + \alpha \mathrm{div} X_1)\widetilde{\Psi}_\alpha = 0, \quad (X_2 + \alpha \mathrm{div} X_2)\widetilde{\Psi}_\alpha = 0,$$

a general solution to these linear equations is of the form $J^\alpha f(\Psi^0, \ldots, \Psi^N)$. Using parametrically deformed linear problems, we can add linear term to the operator Q, preserving the solvability. For conformal self-duality equations (2) the parametric deformation of the Lax pair reads

$$\begin{aligned}
X_{1\alpha} &= \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_x\partial_\lambda + \alpha(F_x + G_y)_x, \\
X_{2\alpha} &= \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_y\partial_\lambda + \alpha(F_x + G_y)_y.
\end{aligned}$$

Instead of relation (7) we have

$$\begin{aligned}
\big((\partial_w &+ F_y\partial_x + G_y\partial_y + \alpha(F_{xy} + G_{yy}))\partial_x \\
&- (\partial_z + F_x\partial_x + G_x\partial_y + \alpha(F_{xx} + G_{xy}))\partial_y\big)\Psi = \partial_\lambda(f_x\partial_y - f_y\partial_x)\Psi,
\end{aligned}$$

and the parametric deformation of the operator Q looks like

$$Q_\alpha = Q + \alpha((F_{xx} + G_{xy})\partial_y - (F_{xy} + G_{yy})\partial_x)$$

For the Manakov-Santini system the deformed Lax pair reads

$$\begin{aligned}
X_{1\alpha} &= \partial_y - \lambda\partial_x + v_x\partial_x + u_x\partial_\lambda + \alpha v_{xx}, \\
X_{2\alpha} &= \partial_t - \lambda\partial_y + (v_y - u)\partial_x + u_y\partial_\lambda + \alpha v_{xy},
\end{aligned} \tag{43}$$

and the parametric deformation of the operator Q is

$$Q_\alpha = Q + \alpha(v_{xy}\partial_x - v_{xx}\partial_y).$$

For the case of interpolating equation (26) the Jacobian is connected with the basic wave function Ψ^0 having an expansion $\lambda + u\lambda^{-1} + \ldots$ (corresponds to the Lax-Sato function L in standard dKP hierarchy notations) by the relation defining interpolating reduction [15],

$$J = \exp(a(\Psi^0 - \lambda)).$$

The general wave function for the deformed Lax pair with $\alpha = -1$, which is required to construct symmetries for the interpolating equation, is

$$\widetilde{\Psi}_{-1} = \mathrm{e}^{a(\mu - \Psi^0)}\Psi,$$

where Ψ is an arbitrary wave function $\Psi = F(\Psi^0, \Psi^1)$ of the initial Lax pair.

References

[1] L.V. Bogdanov, Matrix extension of multidimensional dispersionless integrable hierarchies, Theoret. Math. Phys., 209(1), 1319-1330 (2021)

[2] M. Dunajski, E.V. Ferapontov and B. Kruglikov, On the Einstein-Weyl and conformal self-duality equations, Journal of Mathematical Physics 56(8), 083501 (2015).

[3] L.V. Bogdanov, SDYM equations on the self-dual background, J. Phys. A 50, 19LT02 (2017)

[4] L.V. Bogdanov, Matrix Extension of the Manakov-Santini System and an Integrable Chiral Model on an Einstein-Weyl Background, Theor. Math. Phys., 201(3), 1701-1709 (2019)

[5] L.V. Bogdanov, Dispersionless integrable systems and the Bogomolny equations on an Einstein–Weyl geometry background, Theoretical and Mathematical Physics 205 (1), 1279-1290 (2020)

[6] L. V. Bogdanov, V. S. Dryuma and S. V. Manakov, Dunajski generalization of the second heavenly equation: dressing method and the hierarchy, J Phys. A: Math. Theor. **40** (2007) 14383–14393

[7] M. Dunajski, Anti-self-dual four–manifolds with a parallel real spinor, Proc. Roy. Soc. Lond. A **458**, 1205 (2002)

[8] A. Sergyeyev, A simple construction of recursion operators for multidimensional dispersionless integrable systems, J. Math. Anal. Appl. **454** (2017) 468—480

[9] S. V. Manakov and P. M. Santini, A hierarchy of integrable PDEs in 2+1 dimensions associated with 2-dimensional vector fields, Theor. Math. Phys. 152 (2007) 1004–1011.

[10] V.G. Mikhalev, On the Hamiltonian formalism for Korteweg-de Vries type hierarchies, Functional Analysis and Its Applications 26 (1992) 140–142.

[11] M.V. Pavlov, Integrable hydrodynamic chains, J. Math. Phys., 44 (2003), 4134–4156.

[12] V.E. Zakharov and A.B. Shabat, Integration of the nonlinear equations of mathematical physics by the method of the inverse scattering problem. II, Funk. Anal. Prilozh. 13 (3) 13-22 (1979) [Funct. Anal. Appl. 13, 166-174 (1979)].

[13] Maciej Dunajski, An interpolating dispersionless integrable system, J. Phys. A: Math. Theor. 41 (2008) 315202

[14] L. V. Bogdanov, On a class of reductions of the Manakov-Santini hierarchy connected with the interpolating system, J Phys. A: Math. Theor. 43 (2010) 115206

[15] L.V. Bogdanov, Interpolating differential reductions of multidimensional integrable hierarchies, Theor. Math. Phys., 167(3), 705-713 (2011)

[16] M. Dunajski, Solitons, Instantons, and Twistors, Oxford University Press (2010)