On completely regular self-dual codes with covering radius $\rho = 2$

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Abstract

We give a complete classification of completely regular self-dual codes with covering radius $\rho = 2$. More precisely, we prove that there are two sporadic such codes, of length 8, and an infinite family, of length 4.

We provide a description of all such codes and give the intersection arrays for all of them.

1 Introduction

Denote by $\mathbb{F}_q^n$ the n-dimensional vector space over the finite field of order q, where q is a prime power. The (Hamming) distance between two vectors $\mathbf{v}, \mathbf{u} \in \mathbb{F}_q^n$ is the number of coordinates in which they differ. The (Hamming) weight of a vector $\mathbf{v} \in \mathbb{F}_q^n$ is the number of nonzero coordinates of $\mathbf{v}$.

A q-ary code C of length n is a subset $C \subseteq \mathbb{F}_q^n$. The elements of C are called codewords. The minimum distance d of C is the minimum distance between any pair of codewords. The minimum weight w of C is the minimum weight of any nonzero codeword. A linear code with parameters $[n, k, d]_q$ is a q-ary code of length n with minimum distance d, such that it is a k-dimensional subspace of $\mathbb{F}_q^n$. For linear codes, the minimum distance and the minimum weight coincide, $d = w$. A t-weight code is a code where the nonzero codewords have t different weights ($t \geq 1$). A linear code of length n is said to be antipodal if there is some codeword of weight n.

The packing radius of a code C is $e = \lfloor (d-1)/2 \rfloor$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its distance to the code C is $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C}\{d(\mathbf{v}, \mathbf{x})\}$ and the covering radius of the code C is $\rho = \max_{\mathbf{v} \in \mathbb{F}_q^n}\{d(\mathbf{v}, C)\}$. Note that $e \leq \rho$. If $e = \rho$, then C is a perfect code. Here we consider only nontrivial linear codes, in particular, $[n, k, d]_q$ codes of dimension $k \geq 2$ and minimum distance $d \geq 3$. It is well known that any nontrivial perfect code has $e \leq 3$ [14, 15]. For $e = 1$, such codes are called Hamming codes which exist for lengths $n = (q^m - 1)/(q - 1)$ ($m \geq 2$), dimension $k = n - m$ and minimum distance $d = 3$.

Given two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{u} = (u_1, \ldots, u_n)$, their inner product is

$$\mathbf{v} \cdot \mathbf{u} = \sum_{i=1}^{n} v_i u_i \in \mathbb{F}_q.$$

For a linear code C, its dual code is $C^\perp = \{\mathbf{x} \in \mathbb{F}_q^n \mid \mathbf{x} \cdot \mathbf{v} = 0,\ \forall \mathbf{v} \in C\}$. The code C is self-dual if $C = C^\perp$. In this case, C and $C^\perp$ have the same dimension $n/2$, hence n must be even.

Denote by $\mathbf{0}$ the all-zero vector. The *support* of a vector $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ is the set of nonzero coordinate positions of x, $\mathrm{supp}(x) = \{i \in \{1, \ldots, n\} \mid x_i \neq 0\}$.

For a given code C of length n and covering radius ρ, define

$$C(i) \;=\; \{\mathbf{x} \in \mathbb{F}_q^n : \; d(\mathbf{x}, C) = i\}, \;\; i = 0, 1, \ldots, \rho.$$

The sets $C(0) = C, C(1), \ldots, C(\rho)$ are called the *subconstituents* of C.

Say that two vectors $\mathbf{x}$ and $\mathbf{y}$ are *neighbors* if $d(\mathbf{x}, \mathbf{y}) = 1$.

Definition 1 ([11]) *A code C of length n and covering radius ρ is* completely regular *(shortly CR), if for all $l \geq 0$ every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Define $a_l = (q-1)\cdot n - b_l - c_l$ and set $c_0 = b_\rho = 0$. The parameters a_l, b_l and c_l $(0 \leq l \leq \rho)$ are called* intersection numbers *and the sequence $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ is called the* intersection array *(shortly* IA*) of C.*

For any $v \in \mathbb{F}_q^n$ and any $t \in \{0, \ldots, n\}$, define $B_{v,t} = |\{x \in C \mid d(v, x) = t\}|$. Then, C is CR if $B_{v,t}$ depends only on t and $d(v, C)$ [8]. The equivalence with Definition 1 can be seen in [11].

Completely regular codes are classical subjects in algebraic coding theory, which are closely connected with graph theory, combinatorial designs and algebraic combinatorics. Existence, construction and enumeration of all such codes are open hard problems (see [6, 7, 10, 11] and references there).

All linear completely regular codes with covering radius $\rho = 1$ are known [3]. The next case, i.e. completely regular codes with $\rho = 2$, was solved for the special case when the dual codes are antipodal [5]. In the present paper, we classify all self-dual completely regular codes with covering radius $\rho = 2$. Such codes are antipodal (hence included in the classification of [5]) with one exception.

In Section 2, we see some definitions and results that we use later. In Section 3 we give the main result of the paper, by proving that the only possible parameters for a self-dual completely regular code with $\rho = 2$ are: $[8, 4, 4]_2$, $[8, 4, 3]_3$, and $[4, 2, 3]_q$ for any $q = 2^r$ with $r > 1$. We identify all such codes and show that, indeed, they are self-dual and completely regular. Moreover, except the case $[8, 4, 3]_3$, the codes are antipodal.

2 Definitions and preliminary results

In this section we see several results we will need in the next section.

The next property is, in fact, well known.

Lemma 1 *The only self-dual Hamming code is the ternary Hamming* $[4, 2, 3]_3$ *code.*

Proof. For any self-dual $[n, k, d]_q$ code, we have that $n = 2k$. Hence, for a Hamming code, $n = 2(n - m)$ and thus $n = 2m$ implying

$$\frac{q^m - 1}{q - 1} = 2m.$$

The only solution is $q = 3$ and $m = 2$. Therefore, $n = 4$ and $k = 2$. $\square$

Definition 2 ([1]) *A code* $C \subseteq \mathbb{F}_q^n$ *with covering radius* ρ *is* uniformly packed in the wide sense (UPWS) *if there exist rational numbers* $\beta_0, \ldots, \beta_\rho$ *such that*

$$\sum_{i=0}^{\rho} \beta_i B_{\mathbf{x},i} = 1, \tag{1}$$

for any $\mathbf{x} \in \mathbb{F}_q^n$. *The numbers* $\beta_0, \ldots, \beta_\rho$ *are called the* packing coefficients.

For UPWS codes, there is a generalized version of the celebrate sphere packing condition for perfect codes.

Lemma 2 ([1]) *Let* $C \subseteq \mathbb{F}_q^n$ *be a UPWS code with covering radius* ρ *and packing coefficients* $\beta_0, \ldots, \beta_\rho$. *Then*

$$|C| = \frac{q^n}{\sum_{i=0}^{\rho} \beta_i (q-1)^i \binom{n}{i}}. \tag{2}$$

For a linear code C, denote by s the number of nonzero weights of $C^\perp$. Following to Delsarte [8], we call *external distance* the parameter s.

Lemma 3 *Let* C *be a linear code with covering radius* ρ*, packing radius* e *and external distance* s.

(i) $\rho \leq s$ *[8].*

(ii) $\rho = s$ *if and only if* C *is UPWS [2].*

(iii) If C *is CR, then* $\rho = s$ *[13].*

(iv) If C *is UPWS and* $\rho = e + 1$*, then* C *is CR [9, 12].*

3 Completely regular self-dual codes $\rho = 2$

Let C be a self-dual CR $[n, k, d]_q$ code with $d \geq 3$ and covering radius $\rho = 2$. In this section we give a full classification of such codes. Note that $n = 2k$ (since C is self-dual) and C is a two-weight code (because $s = \rho = 2$ by Lemma 3). Since $e \leq \rho = 2$, we have that $3 \leq d \leq 6$. But for $d \geq 5$, $e = \rho$ and C would be a perfect 2-error-correcting code, that is, C would be a ternary Golay $[11, 6, 5]_3$ code which obviously is not self-dual (the extended ternary Golay code is self-dual, but with covering radius 3). Hence, C must be a $[2k, k, d]_q$ code with weights $w_1 = d \in \{3, 4\}$ and w_2, where $d < w_2 \leq n$. Since $\rho = e + 1$, the condition to be CR is equivalent to the condition to be UPWS (see Lemma 3).

Now we study separately the cases $d = 3$ and $d = 4$.

3.1 The case $d = 4$

We start with the nonexistence of a particular case of self-dual code.

Lemma 4 *There is no self-dual* $[6, 3, 4]_4$ *code.*

Proof. Let C be a $[6, 3, 4]_4$ code and consider a generator matrix for C:

$$G = \left(\begin{array}{c|c} I_3 & P \end{array} \right),$$

where I_3 is the 3×3 identity matrix and P is a 3×3 matrix with nonzero entries, since C has minimum weight 4. If C is self-dual then any row of G must be self-orthogonal, implying that for any row abc of P, we have $1 + a^2 + b^2 + c^2 = 0$. Thus, $(a+b+c)^2 = 1$ and $a+b+c = 1$. In $\mathbb{F}_4$ and since P has no zero entries, this means that $abc \in \{1xx, x1x, xx1\}$, where $x \neq 0$. If $abc = 111$, then it is not orthogonal to any other row $a'b'c' \in \{1xx, x1x, xx1\}$, where

$x \notin \{0,1\}$. So, the three rows contain exactly one 1. Hence, two rows of P have the same value for x, say α. But such two rows must agree in one position (with 1), since the distance must be two. Therefore, they are not orthogonal and hence the corresponding rows of G are also non-orthogonal. $\square$

Proposition 1 *Let C be a self-dual CR $[2k,k,4]_q$ code with covering radius $\rho = 2$. Then, the packing coefficients verify:*

$$\beta_0 = \beta_1 = 1; \qquad \beta_2 = \frac{q^k - 1 - 2k(q-1)}{(q-1)^2 k(2k-1)},$$

and β_2^{-1} is a natural number.

Proof. Since C is CR, C is also UPWS. Let $\mathbf{x} \in C$, hence, according to (1) in Definition 2, we have:

$$\beta_0 B_{\mathbf{x},0} + \beta_1 B_{\mathbf{x},1} + \beta_2 B_{\mathbf{x},2} = 1.$$

But clearly $B_{\mathbf{x},1} = B_{\mathbf{x},2} = 0$ (since $d = 4$), which implies $\beta_0 = 1$. Take now a vector $\mathbf{y} \in \mathbb{F}_q^n$ such that $d(\mathbf{y}, C) = 1$. In this case, $B_{\mathbf{y},0} = B_{\mathbf{y},2} = 0$, implying $\beta_1 = 1$. Let $\mathbf{z} \in \mathbb{F}_q^n$ such that $d(\mathbf{z}, C) = 2$. Since $B_{\mathbf{z},0} = B_{\mathbf{z},1} = 0$, we have $\beta_2 B_{\mathbf{z},2} = 1$ and β_2^{-1} is a natural number.

Therefore, by Lemma 2, it follows that

$$|C| = q^k = \frac{q^n}{1 + n(q-1) + \beta_2 (q-1)^2 \binom{n}{2}} \implies \beta_2 = \frac{q^k - 1 - 2k(q-1)}{(q-1)^2 k(2k-1)}.$$

$\square$

Corollary 1 *If C is a self-dual CR $[2k,k,4]_q$ code with covering radius $\rho = 2$, then $k = 4$ and $q = 2$.*

Proof. Define the function

$$f(q,k) = \frac{(q-1)^2 k(2k-1)}{q^k - 1 - 2k(q-1)}.$$

By Proposition 1, $f(q,k) = \beta_2^{-1}$ and must be a natural number. For $n = 4$, C cannot be a two-weight code with minimum weight 4. Thus $k \geq 3$. Clearly, for a fixed k, $f(q,k)$ is a decreasing function on q. In fact, it is easy to check that q must be less than 16, otherwise $f(q,k) < 1$. Also, fixing q, $f(q,k)$ is a decreasing function on k and one obtains that $k \leq 6$. Again, $f(q,k) < 1$ for $k > 6$. For all these possible values ($2 \leq q < 16$, $3 \leq k \leq 6$), we have computationally checked that the only natural values of $f(q,k)$ are $f(2,4) = 4$, $f(2,3) = 15$ and $f(4,3) = 3$. But $f(2,3) = 15$ implies $B_{\mathbf{x},2} = 15$ which is not possible for $q = 2$ and $n = 2k = 6$. Indeed, if $\mathbf{x}$ has weight 2, the number of codewords of weight 4 at distance 2 from $\mathbf{x}$ cannot be greater than 2 and, taking into account de zero codeword we have $B_{\mathbf{x},2} \leq 3$.

As a consequence, the only possible values for q and k are $(q,k) \in \{(2,4),(4,3)\}$, but by Lemma 4, the case $(q,k) = (4,3)$ is not possible. $\square$

3.2 The case $d = 3$

For any code C, let C_w be the set of codewords of weight w, $C_w = \{\mathbf{x} \in C \mid \mathrm{wt}(\mathbf{x}) = w\}$.

Lemma 5 *If C is a CR code of length n, containing the zero codeword, and with minimum weight 3, then $\bigcup_{\mathbf{x}\in C_3} \mathrm{supp}(\mathbf{x}) = \{1,\ldots,n\}$.*

Proof. Otherwise taking a one-weight vector $\mathbf{v}$, we would have that $B_{\mathbf{v},2} > 0$ if the nonzero coordinate is in $\bigcup_{\mathbf{x}\in C_3} \mathrm{supp}(\mathbf{x})$, but $B_{\mathbf{v},2} = 0$ if not. Hence, C would not be CR. $\square$

Lemma 6 *If C is a two-weight $[n,k,3]_q$ code with $q > 2$, then*

(i) $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| \neq 1$, for any $\mathbf{x},\mathbf{y} \in C_3$.

(ii) if $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| \in \{0,3\}$ for all $\mathbf{x},\mathbf{y} \in C_3$, then C is not CR.

Proof. (i) Let $\mathbf{x},\mathbf{y} \in C_3$ and assume that $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 1$. By taking different multiples of $\mathbf{x}$ we can have that $\mathrm{wt}(x-y) = 4$ or $\mathrm{wt}(x-y) = 5$. But this is a contradiction since C is a two-weight code.

(ii) In this case, let $\mathbf{x},\mathbf{y} \in C_3$ such that $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 0$, by Lemma 5 such vectors must exist. Then, C has weights 3 and 6. Any other codeword $\mathbf{z} \in C_3$ will have $\mathrm{supp}(\mathbf{z}) = \mathrm{supp}(\mathbf{x})$ or $\mathrm{supp}(\mathbf{z}) = \mathrm{supp}(\mathbf{y})$, otherwise C would have more than two weights.

Without loss of generality, assume that $\mathbf{x} = (x_1, x_2, x_3, 0, 0, 0)$ and $\mathbf{y} = (0, 0, 0, y_1, y_2, y_3)$. Now, the vector $\mathbf{v} = (x_1, v_2, 0, 0, 0, 0)$, where $v_2 \neq x_2$ is clearly at distance 2 to C and, since $d(\mathbf{v}, \mathbf{x}) = d(\mathbf{v}, \mathbf{0}) = 2$, we have $B_{\mathbf{v},2} = 2$. Indeed, a vector $\mathbf{x}' = (x_1, v_2, x_3', 0, 0, 0)$ cannot be a codeword because $d(\mathbf{x}, \mathbf{x}') \leq 2$. Now take $\mathbf{u} = (x_1, 0, 0, y_1, 0, 0)$. Clearly, $d(\mathbf{u}, C) = 2$ but $B_{\mathbf{u},2} = 1$. Therefore, C is not CR. $\square$

Proposition 2 *If C is a self-dual CR two-weight $[n, k, 3]_q$ code, then $n = 4$ or $n = 8$.*

Proof. By Lemmas 5 and 6, there exist codewords $\mathbf{x}, \mathbf{y} \in C_3$, such that $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 2$ and thus $|\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{y})| = 4$. Now, if $\mathbf{z} \in C_3$ has $\mathrm{supp}(\mathbf{z}) \cap (\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{y})) \neq \emptyset$, we claim that $\mathrm{supp}(\mathbf{z}) \subset (\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{y}))$. Otherwise, without loss of generality assume that $\mathbf{x} = (1, x_2, x_3, 0, \ldots, 0)$ and $\mathbf{y} = (1, y_2, 0, y_3, 0, \ldots, 0)$. By (i) of Lemma 6, we can assume that $\mathbf{z} = (z_1, z_2, 0, 0, z_3, 0, \ldots, 0)$. Now, since $q > 2$, we can take a multiple of $\mathbf{z}$, say $\mathbf{z}' = (z_1', z_2', 0, 0, z_3', 0, \ldots, 0)$, such that $z_2' = x_2 - y_2$. Hence, $\mathrm{wt}(\mathbf{x} - \mathbf{y} - \mathbf{z}') = 4$. But we can take another multiple, say $\mathbf{z}''$, such that $z_2'' \neq x_2 - y_2$. In this case, $\mathrm{wt}(\mathbf{x} - \mathbf{y} - \mathbf{z}'') = 5$. So, C has more than two nonzero weights, leading to a contradiction.

As a consequence, we have that C_3 induces a partition of the set of coordinates in 4-subsets, implying that n is a multiple of 4. But for $n > 8$, clearly C would have more than two nonzero weights. Therefore $n = 4$ or $n = 8$. $\square$

3.3 The full classification

Now, from Corollary 1 and Proposition 2, we obtain the main theorem.

Theorem 3.1 *Let C be a self-dual CR $[n, k, d]_q$ code with covering radius $\rho = 2$. Then, C is one of the following:*

(i) The extended Hamming $[8, 4, 4]_2$ code, with weights $w_1 = 4$ and $w_2 = 8$ (so, an antipodal code), and with intersection array

$$\mathrm{IA} = \{8, 7; 1, 4\}.$$

(ii) The direct product of two ternary Hamming codes of length 4, that is, a $[8,4,3]_3$ *code with generator matrix*

$$G = \left(\begin{array}{c|c} H & \mathbf{0} \\ \hline \mathbf{0} & H \end{array}\right),$$

where H *is a generator matrix of a ternary Hamming* $[4,2,3]_3$ *code.* C *has weights* $w_1 = 3$ *and* $w_2 = 6$*, and intersection array*

$$\mathrm{IA} = \{16, 8; 1, 2\}.$$

(iii) A $[4,2,3]_q$ *code, where* $q = 2^r$ *with* $r > 1$*. Such code can have generator matrix*

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \xi_i & \xi_j \end{pmatrix},$$

where $\xi_i, \xi_j \in \mathbb{F}_q^*$ *are two different elements such that* $\xi_i + \xi_j + 1 = 0$*.* C *has weights* $w_1 = 3$ *and* $w_2 = 4$ *(so, an antipodal code), and intersection array*

$$\mathrm{IA} = \{4(q-1), 3(q-3); 1, 12\}.$$

Proof. By Corollary 1 and Proposition 2, the cases (i)–(iii) are the only possible self-dual CR codes with $\rho = 2$. Cases (i) and (iii) correspond to antipodal codes and hence they belong to different families in [5], where all linear CR codes with dual antipodal are classified (see also [3] and [4]).

(i) This is the well-known binary extended Hamming of length 8, which is self-dual. Trivially the weights are 4 and $n = 8$. This code falls into the family (CR.1) in [5], where the intersection array is also specified.

(ii) Let C be a self-dual CR $[8,4,3]_q$ code with covering radius $\rho = 2$. By the argument in the proof of Proposition 2, the set of coordinates $\{1, \ldots, 8\}$ is partitioned into two 4-subsets, say A and B, such that any codeword of weight 3 has its support contained in A or in B. Since C must be a two-weight code, these weights are trivially $w_1 = 3$ and $w_2 = 6$. Therefore C is the direct sum $C = C_1 \oplus C_2$ of two one-weight codes (whose nonzero codewords have weight 3). It is clear that C is self-dual if and only if C_1 and C_2 are self-dual.

On the other hand, if we take a two-weight vector $\mathbf{x}$ with one nonzero coordinate in A and the other one in B, then $d(\mathbf{x}, C) = 2$ and $B_{\mathbf{x},2} = 1$, since the zero codeword is the only one at distance 2 from $\mathbf{x}$. Now, take any two-weight vector $\mathbf{y}$ with both nonzero coordinates in A (or in B). Since $|A| = |B| = 4$, there exist some codeword $\mathbf{z}$ of weight 3 including the support of $\mathbf{y}$ and (taking the appropriate multiple) such that $d(\mathbf{z}, \mathbf{y}) \leq 2$. If $d(\mathbf{z}, \mathbf{y}) = 2$, then $B_{\mathbf{y},2} > 1$ and the code would not be CR. This means that any two-weight vector $\mathbf{y}$ with both nonzero coordinates in A (or in B) is at distance 1 from C. In other words, C_1 and C_2 must be self-dual Hamming codes. By Lemma 1, C_1 and C_2 are ternary Hamming codes of length 4. Therefore, C is the direct sum of two ternary Hamming $[4, 2, 3]_3$ codes.

Indeed, the direct sum of perfect codes is a CR code (see [13] for the binary case or [1, 4] for more general cases).

Now we compute the intersection array. Since $d = 3$ and any codeword has $n(q-1) = 16$ neighbors, we have that $b_0 = 16$. Given a one-weight vector $\mathbf{v} \in C(1)$ with the nonzero coordinate v_i in A (respectively, B) its neighbors in $C(2)$ are all vectors of weight 2 with one nonzero coordinate v_i and the other one in B (respectively, A). Hence, there are $4 \cdot 2 = 8$ such codewords, implying $b_2 = 8$. For a vector $\mathbf{v}$ in $C(1)$, there is exactly one codeword at distance 1 from $\mathbf{v}$ and thus $c_1 = 1$. Finally, given a vector $\mathbf{u}$ of weight 2 in $C(2)$, note that any neighbor of weight 2 or 3 is not in $C(1)$. So, only the neighbors of weight 1 are in $C(1)$. There are two such neighbors of $\mathbf{u}$ and, consequently, $c_2 = 2$. Therefore, the intersection array is $\{16, 8; 1, 2\}$.

(iii) This antipodal code corresponds to the family (CR.3) in [5] (see [3] for the specific case $n = 4$, where self-duality is also justified when, and only when, $q = 2^r$, $r > 1$). The intersection array can also be seen in [5] or [3]. $\square$

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References

[1] L.A. Bassalygo, G.V. Zaitsev, V.A. Zinoviev, "Uniformly packed codes," *Problems Inform. Transmiss.,* vol. 10, no. 1, pp. 9–14, 1974.

[2] L.A. Bassalygo, V.A. Zinoviev, "A note on uniformly packed codes", *Problems Inform. Transmiss.*, vol. 13, no. 3, pp. 22–25, 1977.

[3] J. Borges, J. Rifà, V. A. Zinoviev, "On q-ary linear completely regular codes with $\rho = 2$ and antipodal dual", *Adv. in Maths. of Comms.*, vol. 4, pp. 567–578, 2010.

[4] J. Borges, J. Rifà, V.A. Zinoviev, "On completely regular codes", *Problems Inform. Transmiss.*, vol. 55, no. 1, pp. 1–45, 2019.

[5] J. Borges, V.A. Zinoviev, D.V. Zinoviev, "On the classification of completely regular codes with covering radius two and antipodal duals", *Problems Inform. Transmiss.*, vol. 59, no. 3, pp. 204–216, 2023.

[6] A.E. Brouwer, A.M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer, Berlin, 1989.

[7] E. R. van Dam, J.H. Koolen, H. Tanaka, "Distance-Regular graphs" *The Electronic Journal of Combinatorics.*, DS. no. 22, pp. 1–156.

[8] P. Delsarte, "An Algebraic Approach to the Association Schemes in Coding Theory", *Philips Res. Rep. Suppl.*, 10, 1973.

[9] J.M. Goethals, H.C.A. Van Tilborg, "Uniformly packed codes", *Philips Res.*, vol. 30, pp. 9–36, 1975.

[10] J.H. Koolen, D. Krotov, B. Martin, *Completely regular codes*, 2016, *https://sites.google.com/site/completelyregularcodes.*

[11] A. Neumaier, "Completely regular codes," *Discrete Maths.*, vol. 106/107, pp. 335—360, 1992.

[12] N.V. Semakov, V.A. Zinoviev, G.V. Zaitzev, "Uniformly packed codes", *Problems Inform. Transmiss.*, vol. 7, pp. 38–50, 1971.

[13] P. Solé, "Completely regular codes and completely transitive codes", *Discrete Maths.*, vol.81, no. 2 (1990), pp. 193–201, 1990.

[14] A. Tietäväinen, "On the Nonexistence of Perfect Codes over Finite Fields", *SIAM J. Appl. Math.*, 1973, vol. 24, no. 1, pp. 88-–96, 1993.

[15] V.A. Zinoviev, V.K. Leont'ev, "On Non-existence of Perfect Codes over Galois Fields", *Probl. Upravlen. Teor. Inform.*, 1973, vol. 2, no. 2, pp. 123—132 [Probl. Control Inf. Theory (Engl. Transl.), 1973, vol. 2, no. 2, pp. 16-–24].