Universality of the thermodynamics of a quantum-mechanically radiating black hole departing from thermality

[1,2,3]Christian Corda and [4,1]Carlo Cafaro

April 30, 2024

[1]SUNY Polytechnic Institute, 13502 Utica, New York, USA

[2]Istituto Livi, 59100 Prato, Tuscany, Italy

[3]International Institute for Applicable Mathematics and Information Sciences, B. M. Birla Science Centre, Adarshnagar, Hyderabad 500063, India

[4]University at Albany-SUNY, 12222 Albany, New York, USA

E-mails: ***cordac.galilei@gmail.com, ccafaro@albany.edu***

Abstract

Mathur and Mehta won the third prize in the 2023 Gravity Research Foundation Essay Competition for proving the universality of black hole (BH) thermodynamics. Specifically, they demonstrated that any Extremely Compact Object (ECO) must have the same BH thermodynamic properties regardless of whether or not the ECO possesses an event horizon. The result is remarkable, but it was obtained under the approximation according to which the BH emission spectrum has an exactly thermal character. In fact, strong arguments based on energy conservation and BH back reaction imply that the spectrum of the Hawking radiation cannot be exactly thermal. In this work the result of Mathur and Mehta will be extended to the case where the radiation spectrum is not exactly thermal using the concept of *BH dynamical state*.

In 1975 Hawking [1] discovered that a Schwarzschild BH having mass M radiates at the Hawking temperature (hereafter Planck units will be used, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$)

$$T_H = \frac{1}{8\pi M}, \tag{1}$$

with a corresponding probability of emission

$$\Gamma \sim \exp(-\frac{\omega}{T_H}), \tag{2}$$

where ω is the energy-frequency of the outgoing particle. Hawking's approach was based on a previous work by Bekenstein [2] who associated the BH with an entropy given by

$$S_{BH} = \frac{A}{4} = \pi r_s^2, \tag{3}$$

where A is the horizon area and $r_s = 2M$ is the Schwarzschild radius. These pioneering results of Bekenstein and Hawking led to the creation of the fascinating framework of black hole thermodynamics, which is still the subject of many studies and investigations by theoretical physicists from all over the world. In 1976, however, Hawking showed that this approach led to a non-trivial problem [3], what would later be known as the BH information paradox [4, 26]. Hawking radiation is indeed created by the extreme force of gravity present in the vacuum around the BH horizon. But this type of radiance that is created by the quantum vacuum is not capable of carrying any physical information with it. Thus, it would seem that even if one solves the problem by showing that the BH actually has a different physical structure, i.e. the structure of a normal body which emits radiation from its surface like any other body, one would still be left with the problem of explaining why this body should have the BH thermodynamic properties one expects. This problem has been solved in the remarkable paper of Mathur and Mehta [4]. They showed that any astrophysical ECO having a radius sufficiently close to the Schwarzschild radius will have the same thermodynamic properties as the semiclassical BH considered by Bekenstein and Hawking. On the other hand, the analysis in [4] starts with the assumption that the Hawking radiation spectrum should be strictly thermal. However, strong arguments based on energy conservation and BH back reaction imply that the spectrum of BH radiance cannot be exactly thermal [5]. In fact, by considering the energy conservation, which means that the BH contracts during particle emission, implying a varying BH geometry, the important correction term [5]

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1 - \frac{\omega}{2M})] \tag{4}$$

is obtained. In Eq. (4) the additional term $\frac{\omega}{2M}$ is present. By introducing the *effective temperature* [6]

$$T_E(\omega) \equiv \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)}, \tag{5}$$

one can rewrite Eq. (4) in a Boltzmann-like form similar to Eq. (2)

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp(-\frac{\omega}{T_E(\omega)}). \tag{6}$$

Here $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor,* where

$$\beta_E(\omega) = \frac{1}{T_E(\omega)} = \beta_H \left(1 - \frac{\omega}{2M}\right) \tag{7}$$

with $\beta_H = \frac{1}{T_H}$. Thus, in order to take into due account the BH dynamical state given by the BH varying geometry during particle emission, one must replace T_H with $T_E(\omega)$ in the expression of the probability of emission. One must stress that this happens also in several other fields of Science [7] where the deviation from the thermal spectrum of the emitting body is usually taken into account via the introduction of an effective temperature which represents the temperature of a black body emitting exactly the same amount of radiation as the non-thermal source. $T_E(\omega)$ is a function of the energy-frequency of the emitted radiance and one can quantify the deviation of the radiation spectrum from the strict thermality via the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega} = 1 + \frac{\omega}{2M} + \mathcal{O}\left(\left|\frac{\omega}{2M}\right|^2\right)$. One can also define other dynamical quantities like the BH *effective mass and effective horizon radius* as [6],

$$M_E \equiv M - \frac{\omega}{2} \quad and \quad r_E \equiv 2M_E. \tag{8}$$

M_E and r_E are average quantities. Indeed, r_E is the average of the initial and final horizons, while M_E is the average of the initial and final masses [6]. They are the dynamical BH mass and horizon *during* the BH contraction, i.e. *during* the emission of the particle. This implies that the analysis in [4] can be improved with the replacement $T_H \rightarrow T_E(\omega)$ in the discussion. Of course, despite the fact that this is very intuitive, it must be rigorously justified. This can be done via the Gibbons-Hawking periodicity argument [8]. Following [9], the standard Schwartzschild line-element is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + r^2\left(\sin^2(\theta) d\varphi^2 + d\theta^2\right) + \frac{dr^2}{1 - \frac{2M}{r}}. \tag{9}$$

Setting $\tau \equiv it$ and $x \equiv 4M\sqrt{1 - \frac{2M}{r}}$ [10], the Gibbons-Hawking argument [8–11] permits to write down the Euclidean form of the metric in Eq. (9) as

$$ds_e^2 = x^2 \left[\frac{d\tau}{4M\left(1 - \frac{\omega}{2M}\right)}\right]^2 + \left(\frac{r}{r_E}\right)^2 dx^2 + r^2\left(\sin^2(\theta) d\varphi^2 + d\theta^2\right). \tag{10}$$

In Refs. [9, 10], the corrections to the standard semiclassical Hawking temperature due to the quantum back reaction are estimated by means of the Hamilton-Jacobi method. Specifically, the (complex) semiclassical wave function of a massless particle in spacetime satisfying the Klein-Gordon equation is expressed in terms of a (real) Hamilton-Jacobi function (proportional to the phase of the complex wave function) expanded in powers of $\hbar$. Then, terms from $O(\hbar)$ onwards are treated as quantum corrections over the semiclassical value of the Hamilton-Jacobi function. Within this theoretical scheme, the corrected expression of the BH temperature is $T_h = T_H\left[1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}}\right]^{-1}$, where T_H is the standard semiclassical BH Hawking temperature of Eq. (1). Note that β_i are dimensionless constant parameters and $\beta_i \frac{\hbar^i}{M^{2i}}$ is an adimensional quantity as well since the reduced Planck constant has dimensions equal to the square of a mass. This is a consequence of the fact that units $G = c = k_B = 1$ are used in Refs. [9, 10]. The Euclidean form of the metric of Eq. (10) is regular at $x = 0$ and $r = r_E$, while τ is considered as being an angular variable with period $\beta_E(\omega)$ [8, 9] . Now, if one replaces the quantity $\sum_i \beta_i \frac{\hbar^i}{M^{2i}}$ in [9] with the quantity $-\frac{\omega}{2M}$as in Eq. (4), by following step by step the analysis in [9] one gets the *dynamical Schwartzschild metric* as

$$ds^2 = -\left(1 - \frac{2M_E}{r}\right) dt^2 + r^2 \left(\sin^2(\theta) d\varphi^2 + d\theta^2\right) + \frac{dr^2}{1 - \frac{2M_E}{r}}, \tag{11}$$

with the *dynamical surface gravity* equal to $\frac{1}{4M_E}$. Using Eq. (8) one can also rewrite Eq. (11) as

$$ds^2 = -\left[1 - \frac{2\left(M - \frac{\omega}{2}\right)}{r}\right] dt^2 + r^2 \left(\sin^2(\theta) d\varphi^2 + d\theta^2\right) + \frac{dr^2}{1 - \frac{2\left(M - \frac{\omega}{2}\right)}{r}}. \tag{12}$$

This result implies that the analysis in [4] can be improved by using the dynamical quantities introduced here. We note that also the entropy in Eq. (3) must be replaced with a dynamical (effective) entropy

$$(S_{BH})_E \equiv \frac{A_E}{4} = \pi r_E^2, \tag{13}$$

where $A_E \equiv 4\pi r_E^2$ is the *BH effective horizon.* An ECO was defined in [4] via four properties which here will be modified by taking into account the dynamical Schwarzschild geometry which has been introduced above:

1. The ECO dynamical mass as seen from infinity is M_E, the Schwarzschild BH dynamical mass as in Eq. (8).

2. Denoting $R_{E(ECO)}$ the dynamical radius of the ECO, semiclassical physics is a good approximation to the dynamics at $r \geq R_{E(ECO)}$.

3. The redshift in the semiclassical region reaches a value which is the same as that of the redshift at distances of order of the Planck length outside the dynamical horizon of the Schwarzschild BH. In that case, the ECO is *extremely compact*, in the sense that the dynamical mass contained in the region $r \leq R_{E(ECO)}$ is approximately $M_E - O(M_E)$.

4. The theory governing the physical properties of the ECO does not violate causality, so that, for larger values of $r \geq R_{E(ECO)}$, where semiclassical physics holds, one has
$$\frac{2M_E}{r} < 1. \tag{14}$$

The analysis in [4] is partially based on the previous result [12]. In that work the authors showed that if the ECO being considered was a thin spherical shell supported by its own pressure, and located just outside the horizon, then such a thin spherical shell will be in thermodynamic equilibrium with the local Unruh radiation. This will lead to the Bekenstein-Hawking entropy (3) for the shell. It is worth remembering that, historically, the first to argue that the final state of gravitational collapse was a thin and compact spherical shell of matter was Einstein in 1939 [13]. This is consistent with a notable result obtained by Vaz in 2014 [14]. In Ref. [14], Vaz used the Wheeler-DeWitt equation in a quantum setting to show that the LeMaitre-Tolman-Bondi gravitational collapse [15, 16, 17] caused the collapsing matter to condense naturally on the Schwarzschild surface. This surface, in turn, happened to be an apparent horizon rather than a real one. This result allowed Vaz to win the second prize in the Gravity Research Foundation Essay Competition in 2014. Recently, one of us, CC [18], has shown that these Einstein-Vaz spherical shells can also be obtained from the gravitational collapse of Oppenheimer and Snyder [19] and turn out to be well-defined quantum objects that obey the Schrödinger theory in the non-relativistic approximation, and the Klein-Gordon equation when relativistic corrections are taken into account. We will return to the Einstein-Vaz shells later in this work since the fact that their mass spectrum is discrete has important thermodynamical implications.

Adapting the analysis in [4] we notice that if the dynamical temperature $T_{E(ECO)}$ of the ECO agrees with the BH dynamical temperature in Eq. (5), then the dynamical BH entropy (13) and the probability of emission in Eq. (4) must automatically agree in the same way. Let us start from the entropy [4]. Considering a body with a large number of degrees of freedom, one has $dS_E = T_E dU_E$, where the subscripts E indicate the dynamic values of the various thermodynamic quantities and dU_E is the infinitesimal variation of energy. Hence, if the ECO has dynamical temperature $T_{E(ECO)}$ which matches the BH dynamical temperature in Eq. (5), then one has $d\left(S_{ECO}\right)_E = \frac{dU_E}{T_E} = 8\pi M_E dM_E$, whose integral is

$$\left(S_{ECO}\right)_E = 4\pi M_E^2 = 4\pi\left(M - \frac{\omega}{2}\right)^2 = \frac{A_E}{4}, \tag{15}$$

where ω is the ECO emitted energy. The phyisical meaning of Eq. (15) is the following. Assuming that before the emission of the energy ω the ECO initial mass is M_i, then the ECO initial entropy is

$$\left(S_{ECO}\right)_i = 4\pi M_i^2 = \frac{A_i}{4}, \tag{16}$$

where the subscript i is assigned to the initial values. After the emission of the energy ω, the ECO final mass is $M_f = M_i - \omega$. Then, the ECO final entropy is

$$\left(S_{ECO}\right)_f = 4\pi M_f^2 = 4\pi\left(M_i - \omega\right)^2 = \frac{A_f}{4}. \tag{17}$$

Thus, one understands that the value $\left(S_{ECO}\right)_E$ in Eq. (15) represents the value of the entropy of the ECO during the emission of energy since the mass of the ECO is decreasing due to the emission of energy ω. Thus, any ECO with the dynamical temperature T_E of the BH, defined by Eq. (5), has the same dynamical entropy $(S_{ECO})_E$ of the BH.

The second step in adapting the analysis in [4] to the BH dynamical state is to consider the radiation rate, which is connected with the absorption cross-section as it is required by thermodynamics [1, 4]. Thus, in thermal approximation, quanta with an energy range $(\omega, \omega + d\omega)$ in the spherical harmonic $Y_{l,m}$ are radiated at a rate

$$\Gamma_{BH}(l,m,\omega)\, d\omega = \frac{P(l,m,\omega)\, d\omega}{2\pi\left[\exp(\frac{\omega}{T_H}) - 1\right]}. \tag{18}$$

In this equation, $P(l,m,\omega)$ is the absorption probability for an incoming spherical wave having energy ω in the spherical harmonic $Y_{l,m}$. Eq. (11) permits one to go beyond the thermal approximation by introducing the *dynamical rate*

$$(\Gamma_{BH})_E(l,m,\omega)\, d\omega = \frac{P(l,m,\omega)\, d\omega}{2\pi\left[\exp(\frac{\omega}{T_E}) - 1\right]}. \tag{19}$$

For a general emitting source there is no well-defined relationship between the temperature and the radiation rate. Indeed, the radiation rate depends on the details of the shape, size and composition of the source. But, in the particular case of an emitting ECO, the situation is different [4]. If in a region near and outside the horizon we use the dynamical coordinates

$$s_E = \sqrt{8M_E\,(r - 2M_E)}, \qquad t_E = \frac{t}{4M_E}, \tag{20}$$

then the BH dynamical metric of Eq. (11) becomes the dynamical Rindler space

$$ds^2 \approx -s_E^2 dt_E + ds_E^2 + dx_1^2 + dx_2^2, \tag{21}$$

where x_1, x_2 describe the tangent space to the angular sphere. Note that the line element on the unit 2-sphere $d\Omega^2 \equiv \sin^2(\theta) d\varphi^2 + d\theta^2$ can be written as $d\Omega^2 = d\vec{x} \cdot d\vec{x} = dx_1^2 + dx_2^2$, where $d\vec{x} \equiv dx_1\hat{x}_1 + dx_2\hat{x}_2$ with $dx_1\hat{x}_1 \equiv d\theta\hat{e}_\theta$ and $dx_2\hat{x}_2 \equiv \sin(\theta)d\varphi\hat{e}_\varphi$. The orthonormal vectors $\hat{e}_\theta \equiv \frac{\partial_\theta \vec{r}}{\|\partial_\theta \vec{r}\|} = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), -\sin(\theta))$ and $\hat{e}_\varphi \equiv \frac{\partial_\varphi \vec{r}}{\|\partial_\varphi \vec{r}\|} = (-\sin(\varphi), \cos(\varphi), 0)$ are the tangent vectors that span the tangent space to the unit 2-sphere at the point $P = (r,\ \theta,\ \varphi)$ with vector position $\vec{r} \equiv (r\sin(\theta)\cos(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\theta))$.

In relativistic quantum mechanics, the particle content of a single quantum state in two distinct reference frames is generally different [32]. This difference is due to the fact that distinct reference frames are specified by distinct notions of time. Therefore, the concept of frequency, with particles viewed as excitations of positive frequency modes, is observer-dependent [20–22]. A remarkable effect that emerges from this observer-dependence is the fact that a vacuum in an inertial frame becomes a thermal state to an accelerating observer (this is the so-called Unruh effect [22]). For a clear presentation on how to transform a quantum state that is known in terms of Minkowskian (Rindler) excitations into Rindler (Minkowskian) space, we refer to Ref. [23]. The Boulware vacuum denotes the vacuum with respect to the Schwarzschild time. In the Schwarzschild space, the Boulware vacuum is stable and, in addition, is the lowest-energy state [24, 25]. In BH semiclassical physics, the region near the horizon is locally Minkowskian, with its quantum state being close to the Minkowskian vacuum in [4]. The line element of Eq. (21) concerns the right Rindler wedge of this Minkowskian zone. In a Rindler frame, the Minkowskian vacuum results to be the Rindler vacuum plus a series of non-strictly thermal excitations. In a BH framework, the Rindler vacuum is the Boulware vacuum with a temperature $T(s) \approx \frac{2\pi}{s}$, where s is the proper distance outside the horizon. One notes that, unlike the strictly thermal scenario in the case where one considers deviations from thermality, the horizon cannot be fixed but is subject to quantum oscillations due to the BH back reaction and variable geometry. This means that the temperature $T(s)$ must be replaced by the dynamical temperature $T_E(s_E) \approx \frac{2\pi}{s_E}$ (s_E being the dynamical proper distance outside the horizon) which will be associated with the BH dynamical temperature of Eq. (5), blueshifted to the local orthonormal frame at s_E, instead of the traditional Hawking temperature in Eq. (1):

$$\frac{T_E(\omega)}{\sqrt{-g_{tt}}} = \frac{1}{2\pi s_E}. \tag{22}$$

The left-hand-side of Eq. (22) is the usual gravitational shift law in Eq. (25.26) of [27], adapted to the gravitational blueshift of the dynamical temperature $T_E(\omega)$. Moreover, g_{tt} is given in Eq. (11) and equals $-\left(1-\frac{2M_E}{r}\right)$. Here are some details of the computation. The gravitational blueshift of the dynamical temperature T_E in spherical symmetry is given by $T_E \rightarrow \frac{T_E}{\sqrt{-g_{tt}}}$ [27]. From the first relation in Eq. (20) one gets

$$r = \frac{s_E^2}{8M_E} + 2M_E, \tag{23}$$

where $2M_E = 2M - \omega$ from Eq. (8). Then, we have

$$\frac{T_E(\omega)}{\sqrt{-g_{tt}}} = \frac{1}{4\pi(2M-\omega)} \frac{1}{\sqrt{1-\frac{1}{1+\frac{s_E^2}{16M_E^2}}}} \simeq \frac{1}{8\pi M_E}\frac{4M_E}{s_E} = \frac{1}{2\pi s_E}. \tag{24}$$

In the derivation of the Hawking radiation in Schwarzschild coordinates for a scalar field $\hat{\Phi}$ satisfying $\Box\hat{\Phi} = 0$, one expands the scalar field in modes in the external dynamical region $r > 2M_E$ as [4]

$$\begin{aligned}\hat{\Phi} = \textstyle\sum_{l,m,k}[&\hat{a}_{l,m,k} f_{l,m,k}(r) Y_{l,m}(\theta,\phi) \exp\left(-i\omega_{l,m,k}t\right) \\ &+\hat{a}^{\dagger}_{l,m,k} f^{*}_{l,m,k}(r) Y^{*}_{l,m}(\theta,\phi)\exp\left(i\omega_{l,m,k}t\right)].\end{aligned} \tag{25}$$

The quantity $f_{l,m,k}(r)$ is the effective potential to which the modes of Eq. (25) are subject. The quantities $\hat{a}_{l,m,k}$ and $\hat{a}^{\dagger}_{l,m,k}$ are the creation and annihilation operators, respectively. One has a barrier that separates the region near the horizon from the region at infinity. Since the horizon is not fixed but is oscillating due to successive Hawking quanta emissions, Eq. (11) allows us to use the dynamical horizon in Eq. (8). The dynamical rate in Eq. (19) is obtained as follows [4]:

1. In the region near the dynamical horizon, the local Minkowski vacuum happens to be the local Boulware vacuum plus a quasi-thermal gas of excitations $\hat{a}^{\dagger}_{l,m,k}$ over the Boulware vacuum. The quasi-thermal gas turns out to have a dynamical temperature given by Eq. (22) at a dynamical proper distance s_E from the dynamical horizon.

2. These modes overcome the barrier due to the tunneling effect, reaching infinity with a probability $P(l,m,k)$ and, therefore, generating the (not strictly thermal) Hawking radiation.

3. Since the probability of tunnelling is symmetric for incoming and outgoing modes, this probability $P(l,m,k)$ equals the probability that the same mode is absorbed by the BH. This gives rise to the dynamical rate of emission in Eq. (19).

Following [4], one supposes to replace the horizon by an oscillating ECO (for example, an Einstein-Vaz spherical shell) having the same dynamical BH temperature. Consequently, the ECO's surface should populate the region just outside this surface with particles of the field $\hat{\Phi}$ with the same dynamical temperature as in Eq. (22). Since the dynamical metric in the region outside the ECO is the same as the BH dynamical metric associated to the same dynamical mass M_E (in both cases it is the dynamical Schwarzschild metric in Eq. (12)), the potential barrier is the same as the barrier for the BH. Hence, the ECO dynamical radiation rate is equal to the BH dynamical radiation rate in Eq. (19). The key point that allows one to achieve this result is that the wavelengths of the quasi-thermal particles near the dynamical Schwarzschild surface are very short compared to the length scale of the extension of the effective gravitational potential [4]. This is due to the third of the properties that define the ECO under Eq. (13) and allows one to separate the quasi-thermal physics near the Schwarzschild surface from the dynamics of tunneling through the barrier [4].

From the discussion made so far, it emerges that if an ECO had the same dynamical temperature as a BH, then it must also have the same dynamical entropy along with the same dynamical emission rate. So, to obtain what is proposed in this paper, i.e. to show that the BH thermodynamics is equal to the ECO thermodynamics even in a scenario that is not exactly thermal, one must demonstrate that the BH and the ECO have the same dynamical temperature. Following [4] again, let us consider the *energy densities* near the dynamical gravitational radius $r \approx r_E = 2M_E$. Since for a BH the region around the horizon is a Minkowski vacuum, then, to leading order, the expectation value of the total energy density must be zero because the geometry is smooth at the horizon, and one finds no flux into or out of the horizon. Note that

$\rho_T \equiv - < T_0^0 >$, with T_0^0 being the 00−component of the stress-energy tensor $T^{\mu\nu}$. For more details on the calculation of (renormalized) vacuum expectation values of $T^{\mu\nu}$, we refer to [28]. In the coordinates of Eq. (11), the energy density of excitations at dynamical temperature (24) for the Unruh vacuum is [29] $< 0_U|T_0^0|0_U >= \rho_U = \frac{1}{48\pi^2 s_E^4}$. This energy density is added to the negative Casimir energy density of the Rindler vacuum, which is exactly its opposite, $< 0_R|T_0^0|0_R >= \rho_R = -\frac{1}{48\pi^2 s_E^4}$, so that the total energy density is zero. On the other hand, an ECO at zero dynamical temperature has approximately the same negative Casimir energy [4] $\rho_{ECO,R} \simeq -\frac{1}{48\pi^2 s_E^4}$, at $r \gtrsim R_{E(ECO)}$, where $R_{E(ECO)}$ is the dynamical oscillating radius of the ECO (in the case of the Einstein-Vaz shell this is exactly equal to the dynamic BH gravitational radius of Eq. (8) and obeys a Schrödinger equation completely analogous to that of the s-states of the hydrogen atom [18]). This can be understood by expanding the scalar field $\hat{\Phi}$ in modes in the background of the ECO as [4]

$$\begin{aligned}\hat{\Phi} = \textstyle\sum_{l,m,k}[&\hat{b}_{l,m,k}g_{l,m,k}(r)Y_{l,m}(\theta,\phi)\exp(-i\omega_{l,m,k}t)\\ &+\hat{b}^{\dagger}_{l,m,k}g^{*}_{l,m,k}(r)Y^{*}_{l,m}(\theta,\phi)\exp(i\omega_{l,m,k}t)],\end{aligned} \tag{26}$$

and defining the vacuum state for the ECO $|0 >_{ECO}$ via the condition $\hat{b}_{l,m,k}|0 >_{ECO}=$ 0 for all l, m, k. As the wavemode in the ECO has a large number of oscillations in the region $r > R_{E(ECO)}$ (of order of $\log\left(\frac{GM}{l_p}\right)$ in standard units, with G and l_p being the gravitational constant and the Planck lenght, respectively [4]), one makes well-defined wavepackets out of these modes to study the physics at any point $r \gtrsim R_{E(ECO)}$ without needing to know the form of modes in $r < R_{E(ECO)}$. By using the modes in Eq. (25), Candelas [28] derived the BH Casimir energy density in the Boulware vacuum as

$$\rho_B = -\frac{1}{48\pi^2 s^4}, \tag{27}$$

where s is the standard proper distance outside the horizon. Mathur and Mehta [4] obtained the same result, to an excellent approximation, for the ECO Casimir energy density in the Boulware vacuum as

$$\rho_{ECO,B} \simeq -\frac{1}{48\pi^2 s^4}. \tag{28}$$

Remembering that the external geometry of the ECO is the same as that of the BH (in both cases it is the Schwarzshild one), using the dynamical metric in Eq. (11), one gets finally

$$\rho_{ECO,B} \simeq \rho_B = -\frac{1}{48\pi^2 s_E^4}, \tag{29}$$

where the dynamical proper distance outside the horizon s_E replaced s.

Now, again adapting the analysis in [4], one assumes that the dynamic temperature of the ECO is zero so that the ECO has a corresponding quantum ket $|0>_{ECO}$. The meaning of the Casimir density energy in Eq. (29) is the presence of negative energy in the semiclassical region just outside of the oscillating $R_{E(ECO)}$. One considers the energy of a thin spherical shell having an oscillating, dynamical internal radius, in terms of its dynamical proper distance, $\bar{s}_E = \mu l_P = \mu$, since $l_P \equiv \sqrt{\frac{\hbar G}{c^3}} = 1$ in Planck units, outside $r_E = 2M_E$. The energy of the shell is

$$E_{shell} \approx 4\pi \left(2M_E\right)^2 \int_{\bar{s}_E}^{\infty} \rho_{ECO,B}(s_E) \sqrt{-g_{tt}(s_E)} ds_E, \tag{30}$$

with

$$\sqrt{-g_{tt}(s_E)} = 2\pi s_E T_E(\omega) = \frac{s_E}{4M_E} \tag{31}$$

from Eq (22). Then, one gets

$$E_{shell} \approx 16\pi M_E^2 \int_{\bar{s}_E}^{\infty} \left(-\frac{1}{48\pi^2 s_E^4}\right) \left(\frac{s_E}{4M_E}\right) ds_E = -\frac{M_E}{24\pi \bar{s}_E^2} = -\left(\frac{1}{24\pi\mu^2}\right) M_E = -XM_E, \tag{32}$$

with $X \equiv \frac{1}{24\pi\mu^2}$. One now takes $\mu = \mathcal{O}(1)$ and assumes that $R_{E(ECO)}$ is the Planck length $r_E = 2M_E$ and we can imagine that $\bar{s}_E$ is five times the Planck length [4]. So, one has that also $X = \mathcal{O}(1)$. In this case, we note that the ECO under consideration has a problem [4]. The dynamical mass as seen from infinity is M_E. Hence, in the region $r < R_{E(ECO)}$ the effective mass should be

$$M_E(r < R_{E(ECO)}) = M_E + XM_E = (1 + X)\, M_E. \tag{33}$$

But this dynamical mass implies a dynamical horizon outside $R_{E(ECO)}$, which, in turn, implies a violation of the 4th ECO property associated to Eq. (14). Consequently, an ECO having zero dynamical temperature cannot exist. A similar situation occurs for every ECO dynamical temperature $T_{E(ECO)}$ lower than the BH dynamical temperature T_E. The state corresponding to the dynamical temperature of the ECO is obtained by adding a quasi-thermal bath of excitations $\hat{b}_{l,m,k}$ to the vaccum state $|0>_{ECO}$. Then, the local dynamical temperature near the dynamical horizon reads

$$T_{E(ECO)}(s_E) = \frac{2\pi}{s_E} \frac{T_{E(ECO)}}{T_E}. \tag{34}$$

Thus, one finds the energy in the shell region as

$$E_{shell} \approx -\frac{M_E}{24\pi \bar{s}_E^2} \left[1 - \left(\frac{T_{E(ECO)}}{T_E}\right)^4\right] = -\frac{1}{24\pi\mu^2} \left[1 - \left(\frac{T_{E(ECO)}}{T_E}\right)^4\right] M_E = -X(T_E)M_E, \tag{35}$$

where one still has $X(T_E) = \mathcal{O}(1)$ with $X(T_E) \equiv \frac{1}{24\pi\mu^2} \left[1 - \left(\frac{T_{E(ECO)}}{T_E}\right)^4\right]$. Therefore, even in this case the ECO cannot exist. One now wonders what happens if it is $T_{E(ECO)} > T_E$. Still following the approach in [4] one has no viable solution to the Tolman-Oppenheimer-Volkoff equation which describes the quasi- thermal radiation in the region outside $R_{E(ECO)}$. In fact, the third property that specifies the ECO implies that the redshift reaches $\mathcal{O}(M_E)$ at $R_{E(ECO)}$. One specifies a parameter u_E which allows one to characterize the redshift as $\frac{4M_E}{u_E}$. So, it must be u_E of order of the Planck length. Following the second property that defines the ECO, in the region outside $R_{E(ECO)}$ standard semi-classical physics works. Hence, in this region the local dynamical temperature is $\frac{4M_E}{u_E}T_E$. Then, the local energy density in this region outside $R_{E(ECO)}$ arises from this dynamical temperature and the energy in a shell outside $R_{E(ECO)}$ is

$$E_{shell} \approx \frac{1}{24\pi u_E^2}\left[\left(\frac{T_{E(ECO)}}{T_E}\right)^4 - 1\right] M_E = Y(T_E)M_E, \tag{36}$$

with $Y(T_E) \sim 1$. Thus, one finds a violation of the third property that defines the ECO, which required that the dynamical mass outside $R_{E(ECO)}$ be $\mathcal{O}(M_E)$, that is, negligible compared to M_E. Hence, exactly as in the previous cases, this ECO cannot exist in this case as well.

For a more detailed discussion on the fact that any ECO has, to leading order, the same thermodynamic properties as the semiclassical BH of the same mass, we refer to the recent work by Mathur and Mehta in Ref. [33].

Concluding remarks

The concepts of mass, entropy, and information lead naturally to thermodynamic arguments [31]. A prominent example of such an occurrence is represented by the important work of Mathur and Mehta on the universality of BH thermodynamics [4]. In Ref. [4] Mathur and Mehta showed that any ECO must have the same BH thermodynamic properties regardless of whether the ECO possesses an event horizon or not. In this paper, we use the concept of BH dynamical state developed in Ref. [6] to extend the analysis of Mathur and Mehta in Ref. [4] to the case where the BH radiation spectrum is not exactly thermal. Our results are consistent with the Einstein-Vaz shell approach developed in Refs. [14, 18, 30]. According this approach, in both the LeMaitre-Tolman-Bondi and the Oppenheimer-Snyder gravitational collapses, the collapsing matter condenses naturally on the Schwarzschild surface which, in turn, results to be an apparent horizon rather than a real one. Interestingly, the Einstein-Vaz shells are horizon-sized quantum objects that obey the Schrödinger and Klein-Gordon theories [18] and radiate from their surface like normal bodies having a discrete mass spectrum. Therefore, there is no information loss paradox. Indeed, the time evolution of such Einstein-Vaz shells is governed by a time-dependent Schrödinger equation which guarantees the unitarity of the evaporation process assumed to be not strictly thermal [30], in consistent agreement with the shells discrete mass spectrum.

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