QUANTUM U-CHANNELS ON S-SPACES

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Abstract. If the symmetry, (an operator J satisfying $J = J^* = J^{-1}$) which defines the Krein space, is replaced by a (not necessarily self-adjoint) unitary, then we have the notion of an S-space which was introduced by Szafraniec. In this paper, we consider S-spaces and study the structure of completely U-positive maps between the algebras of bounded linear operators. We first give a Stinespring-type representation for a completely U-positive map. On the other hand, we introduce Choi U-matrix of a linear map and establish the equivalence of the Kraus U-decompositions and Choi U-matrices. Then we study properties of nilpotent completely U-positive maps. We develop the U-PPT criterion for separability of quantum U-states and discuss the entanglement breaking condition of quantum U-channels and explore U-PPT squared conjecture. Finally, we give concrete examples of completely U-positive maps and examples of $3 \otimes 3$ quantum U-states which are U-entangled and U-separable.

1. Introduction

The Gelfand-Naimark-Segal (GNS) construction for a given state on a C^*-algebra provides us a representation of the C^*-algebra on a Hilbert space and a generating vector. A linear map τ from a C^*-algebra $\mathcal{B}$ to a C^*-algebra $\mathcal{C}$ is said to be *completely positive (CP)* if $\sum_{i,j=1}^{n} c_j^* \tau(b_j^* b_i) c_i \geq 0$ whenever $b_1, b_2, \ldots, b_n \in \mathcal{B}$; $c_1, c_2, \ldots, c_n \in \mathcal{C}$ and $n \in \mathbb{N}$. Stinespring's theorem (cf. [18, Theorem 1]), which characterizes operator-valued completely positive maps, is a generalization of the GNS construction. Choi decomposition (cf. [6]) for completely positive maps is a pioneering work in Matrix Analysis.

Dirac [10] and Pauli [14] were among the pioneers to explore the quantum field theory using Krein spaces, defined below. For our study, we require the following important definitions:

Definition 1.1. *Assume $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ to be a Hilbert space and J to be a symmetry, that is, $J = J^* = J^{-1}$. Define a map $[\cdot,\cdot] : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ by*

$$[x, y]_J := \langle Jx, y\rangle \text{ for all } x, y \in \mathcal{K}. \tag{1.1}$$

The tuple $(\mathcal{K}, J)$ is called a Krein space (cf. [3]).

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Definition 1.2. *For each $V \in B(\mathcal{K})$, there exists an operator $V^\natural := JV^*J \in B(\mathcal{K})$ such that*

$$\begin{aligned}[Vx, y]_J &= \langle JVx, y\rangle = \langle x, V^*Jy\rangle = \langle x, J^*JV^*Jy\rangle \\ &= \langle Jx, JV^*Jy\rangle = \langle Jx, V^\natural y\rangle = [x, V^\natural y]_J.\end{aligned}$$

The operator $V^\natural$ is called the J-adjoint of V.

In the definition of the Krein space, if we replace the symmetry J by a (not necessarily self-adjoint) unitary U, then we arrive at the following generalized notion due to Szafraniec [19]:

Definition 1.3. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let U be a unitary on $\mathcal{H}$, that is, $U^* = U^{-1}$. Then we can define a sesquilinear form by*

$$[x, y]_U := \langle x, Uy\rangle \text{ for all } x, y \in \mathcal{H}. \tag{1.2}$$

In this case, we call $(\mathcal{H}, U)$ as an *S*-space.

The following definition is given by Phillipp, Szafraniec and Trunk, see [15, Definition 3.1]:

Definition 1.4. *For each $V \in B(\mathcal{H})$, there exists an operator $V^\# := UV^*U^* \in B(\mathcal{H})$ such that*

$$\begin{aligned}[x, Vy]_U &= \langle x, UVy\rangle = \langle V^*U^*x, y\rangle = \langle U^*UV^*U^*x, y\rangle \\ &= \langle UV^*U^*x, Uy\rangle = [V^\# x, y]_U.\end{aligned}$$

The operator $V^\#$ is called the U-adjoint *of V.*

Phillipp, Szafraniec and Trunk [15] investigated invariant subspaces of self-adjoint operators in Krein spaces by using results obtained through a detailed analysis of S-spaces. Recently, in [16], Felipe-Sosa and Felipe introduced and analyzed the notions of state and quantum channel on spaces equipped with an indefinite metric in terms of a symmetry J. This study was further taken up by Heo, in [11], where equivalence of Choi J-matrices and Kraus J-decompositions was obtained and applications to J-PPT criterion and J-PPT squared conjuncture were discussed. The notion of completely U-positive maps was studied by Dey and Trivedi in [8, 9]. Motivated by these inspiring works, in this paper, we develop structure theory of quantum U-channels and its applications to the entanglement breaking.

The plan of the paper is as follows: In Section 2, we give Stinespring-type representation for a completely U-positive map. In Section 3, Choi U-matrix is introduced and the equivalence of Kraus U-decompositions and Choi U-matrices is established. In Section 4, some properties of nilpotent U-CP maps are discussed. In Sections 5 and 6, we develop U-PPT criterion for separability of quantum U-states and discuss the entanglement breaking condition of quantum U-channels and explore U-PPT squared conjecture. Finally, in Section 7, we give concrete examples of completely U-positive maps and examples of $3 \otimes 3$ quantum U-states which are U-entangled and U-separable.

1.1. **Background and notations.** Let $(\mathcal{H}, U)$ be an S-space. Then, $\mathcal{H}^n$ is the direct sum of n-copies of the Hilbert space $\mathcal{H}$, and we denote by $(\mathcal{H}^n, U^n)$ the S-space with the indefinite inner-product

$$[\mathbf{h}, \mathbf{k}]_{U^n} = \langle \mathbf{h}, U^n \mathbf{k} \rangle = \sum_{j=1}^{n} \langle h_j, U k_j \rangle = \sum_{j=1}^{n} [h_j, k_j]_U \tag{1.3}$$

where $U^n = \operatorname{diag}(U, U, \ldots, U) \in M_n(B(\mathcal{H}))$ and $\mathbf{h} = (h_1, \ldots, h_n),\ \mathbf{k} = (k_1, \ldots, k_n) \in \mathcal{H}^n$.

Definition 1.5. *Let $(\mathcal{H}, U)$ be an S-space with the indefinite inner-product $[\cdot, \cdot]_U$. We denote by $B(\mathcal{H})^{U+}$ the set of all U-positive linear operator V on $\mathcal{H}$, that is,*

$$0 \leq [Vh, h]_U := \langle Vh, Uh \rangle = \langle U^* V h, h \rangle\ , \text{ for all }\ h \in \mathcal{H}.$$

Hence V is U-positive if and only if U^*V is positive with respect to the usual inner product $\langle \cdot, \cdot \rangle$.

Definition 1.6. *Let $(\mathcal{H}_i, U_i)$ $(i = 1, 2)$ be an S-space with the indefinite inner-product $[\cdot, \cdot]_{U_i}$. Let $\phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ be a linear map. Then ϕ is called (U_1, U_2)-*Hermitian *if $\phi(U_1 V^* U_1^*) = U_2 \phi(V^*) U_2^*$ for $V \in B(\mathcal{H}_1)$. We say that a (U_1, U_2)-Hermitian linear map ϕ is*

(1) (U_1, U_2)-positive if $\phi(B(\mathcal{H}_1)^{U+}) \subset B(\mathcal{H}_2)^{U+}$, that is, if $V \in (B(\mathcal{H}_1))^{U+}$ (or V is U_1-positive), then $\phi(V)$ is U_2-positive. In simple words, if $U_1^* V$ is positive with respect to the usual inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$, then $U_2^* \phi(V)$ is positive with respect to the usual inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$.

(2) completely (U_1, U_2)-positive or (U_1, U_2)-CP if for each $l \in \mathbb{N}$ the l-fold amplification $\phi^l : I_l \otimes \phi : M_l(\mathbb{C}) \otimes B(\mathcal{H}_1) \to M_l(\mathbb{C}) \otimes B(\mathcal{H}_2)$ defined by

$$\phi^l([V_{ij}]) = [\phi(V_{ij})], \quad \text{for} \quad [V_{ij}] \in M_l(B(\mathcal{H}_1))$$

satisfies

$$\phi^l(M_l(B(\mathcal{H}_1))^{U+}) \subset M_l(B(\mathcal{H}_2))^{U+},$$

that is, if $V = [V_{ij}]_{i,j} \in M_l(B(\mathcal{H}_1))^{U+}$ (i.e., V is U_1^l-positive), then $\phi^l(V)$ is U_2^l-positive. Here $M_l(B(\mathcal{H}_i))^{U+} = B(\mathcal{H}_i^l)^{U+}$ is the set of all U_i^l-positive linear operators on S-spaces $(\mathcal{H}_i^l, U_i^l)$, and $U_i^l = diag(U, U, \ldots, U) \in M_l(B(\mathcal{H}_i))$ for $i = 1, 2$.

(3) U-positive (and completely U-positive (U-CP))if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $U_1 = U_2 = U$ and it is (U_1, U_2)-positive (and (U_1, U_2)-CP, respectively).

2. Completely U-positive and completely U-co-positive maps

Our main objective in this section is to obtain Stinespring-type theorem for completely U-positive maps. Let $(\mathcal{H}_i, U_i)$ $(i = 1, 2)$ be an S-space with the indefinite inner product $[\cdot, \cdot]_{U_i}$. Suppose $\phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is a linear map. Define a linear map ψ from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$ by $\psi(X) := U_2 \phi(U_1^* X)$ where $X \in B(\mathcal{H}_1)$. For any $l \in \mathbb{N}$ and $V = [V_{ij}] \in$

$M_l(B(\mathcal{H}_1))$, we obtain

$$\psi^l(V) = [\psi(V_{ij})]_{i,j} = [U_2\phi(U_1^*V_{ij})]_{i,j} = \begin{pmatrix} U_2\phi(U_1^*V_{11}) & \cdots & U_2\phi(U_1^*V_{1l}) \\ \vdots & \ddots & \vdots \\ U_2\phi(U_1^*V_{l1}) & \cdots & U_2\phi(U_1^*V_{ll}) \end{pmatrix}$$
$$= \begin{pmatrix} U_2 & & 0 \\ & \ddots & \\ 0 & & U_2 \end{pmatrix} \begin{pmatrix} \phi(U_1^*V_{11}) & \cdots & \phi(U_1^*V_{ll}) \\ \vdots & \ddots & \vdots \\ \phi(U_1^*V_{l1}) & \cdots & \phi(U_1^*V_{ll}) \end{pmatrix} = U_2^l\phi^l(U_1^{l^*}V).$$

Similarly, we can easily show that $\phi^l(V) = U_2^{l^*}\psi(U_1^lV)$ where $\phi(V_{ij}) = U_2^*\psi(U_1V_{ij})$.

The following result is a generalization of [16, Theorem 20] and [11, Proposition 2.2] in the setting of S-spaces:

Proposition 2.1. *Let $(\mathcal{H}_i, U_i)$ $(i = 1, 2)$ be an S-space with the indefinite inner product $[\cdot,\cdot]_{U_i}$. Suppose $\phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is a linear map, then ϕ is CP if and only if the corresponding linear map ψ from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$ defined by $\psi(X) := U_2\phi(U_1^*X)$ is (U_1, U_2)-CP, where $X \in B(\mathcal{H}_1)$.*

Proof. Let ϕ be a linear map from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$. First assume that ϕ is CP. We have to prove that ψ is (U_1, U_2)-CP. For this purpose, let $V = [V_{ij}] \in M_l(B(\mathcal{H}_1))^{U+}$, that is, $U_1^{l^*}V \in M_l(B(\mathcal{H}_1))$ is positive, that is,

$$0 \le [V\mathbf{h}, \mathbf{h}]_{U_1^l} = \langle V\mathbf{h}, U_1^l\mathbf{h}\rangle = \langle U_1^{l^*}V\mathbf{h}, \mathbf{h}\rangle,$$

where $\mathbf{h} \in \mathcal{H}^l$. Consider

$$[\psi^l(V)\mathbf{h}', \mathbf{h}']_{U_2^l} = \langle \psi^l(V)\mathbf{h}', U_2^l\mathbf{h}'\rangle = \langle U_2^{l^*}\psi^l(V)\mathbf{h}', \mathbf{h}'\rangle$$
$$= \langle U_2^l\phi^l(U_1^{l^*}V)\mathbf{h}', U_2^l\mathbf{h}'\rangle = \langle \phi^l(U_1^{l^*}V)\mathbf{h}', \mathbf{h}'\rangle \ge 0,$$

where $\mathbf{h}' \in \mathcal{H}^l$. Therefore $\langle U_2^{l^*}\psi^l(V)\mathbf{h}', \mathbf{h}'\rangle \ge 0$, that is, $U_2^{l^*}\psi^l(V)$ is positive. This proves that $\psi(V)$ is U_2-positive. Thus ψ is (U_1, U_2)-CP.

Conversely, suppose that ψ is (U_1, U_2)-CP. Since $\psi(\cdot) = U_2\phi(U_1^*\cdot)$, we get $\phi(U_1^*\cdot) = U_2^*\psi(\cdot)$. Therefore $\phi(\cdot) = U_2^*\psi(U_1\cdot)$. Let $V = [V_{ij}] \in M_l(B(\mathcal{H}_1))^+$, then we have to show that $\phi^l(V) = [\phi(V_{ij})] \in M_l(B(\mathcal{H}_2))^+$. Now

$$0 \le \langle V\mathbf{h}, \mathbf{h}\rangle = \langle U_1^lV\mathbf{h}, U_1^l\mathbf{h}\rangle = [U_1^lV\mathbf{h}, \mathbf{h}]_{U_1^l},$$

where $\mathbf{h} \in \mathcal{H}^l$, it means, $U_1^lV \in M_l(B(\mathcal{H}_1))^{U+}$. Therefore

$$\langle \phi^l(V)\mathbf{h}', \mathbf{h}'\rangle = \langle U_2^{l^*}\psi(U_1^lV)\mathbf{h}', \mathbf{h}'\rangle = \langle \psi(U_1^lV)\mathbf{h}', U_2^l\mathbf{h}'\rangle$$
$$= [\psi(U_1^lV)\mathbf{h}', \mathbf{h}']_{U_2^l} \ge 0,$$

where $\mathbf{h}' \in \mathcal{H}^l$ and the last inequality follows from the fact that $U_1^lV \in M_l(B(\mathcal{H}_1))^{U+}$ and hence ψ is (U_1, U_2)-CP.

□

Theorem 2.2. *Let $(\mathcal{H}_i, U_i)$ $(i = 1, 2)$ be an S-space. Assume that a linear map ψ from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$ defined by $\psi(V) := U_2\phi(U_1^*V)$ for all $V \in B(\mathcal{H}_1)$ is (U_1, U_2)-CP. Then* *there exist an S-space $(\mathcal{H}, U)$, a $*$-representation π of $B(\mathcal{H}_1)$ on the Hilbert space $\mathcal{H}$ and a bounded linear operator $R : \mathcal{H}_2 \to \mathcal{H}$ such that*

$$\psi(V) = R^{\#}\pi(V)R$$

where $U = \pi(U_1)$, and $R^{\#} := U_2 R^* U^*$. Moreover, if $\psi(U_1) = U_2$, then $R^*R = I_{\mathcal{H}_2}$.

Proof. Suppose a linear map ψ is (U_1, U_2)-CP. Then with the help of Proposition 2.1, we get that ϕ defined by $\phi(V) = U_2^*\psi(U_1V)$ is CP. Then using Stinespring's theorem [18, Theorem 1], there exist a Hilbert space $\mathcal{H}$, a representation (a unital $*$-homomorphism) π of $B(\mathcal{H}_1)$ on the Hilbert space $\mathcal{H}$ and a bounded linear operator $R : \mathcal{H}_2 \to \mathcal{H}$, such that $\phi(V) = R^*\pi(V)R$ for every $V \in B(\mathcal{H}_1)$.

Let $U = \pi(U_1) \in B(\mathcal{H})$, where U is a fundamental unitary, that is, $U^* = U^{-1}$, so that $(\mathcal{H}, U)$ becomes an S-space. Define $R^{\#} := U_2R^*U^*$, then

$$\psi(V) = U_2\phi(U_1^*V) = U_2R^*\pi(U_1^*V)R = U_2R^*U^*\pi(V)R = R^{\#}\pi(V)R.$$

Furthermore, if $\psi(U_1) = U_2$, then

$$U_2 = \psi(U_1) = U_2\phi(U_1^*U_1) = U_2R^*\pi(U_1^*U_1)R = U_2R^*R,$$

hence $R^*R = I_{\mathcal{H}_2}$. $\square$

Theorem 2.3. *Suppose $\phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is a linear map. If ϕ satisfies the following conditions for all $V \in B(\mathcal{H}_1)$:*

$$\phi(U_1^*V) = U_2^*\phi(V) \quad \textit{and} \quad \phi(U_1V) = U_2\phi(V),$$

then ϕ is a CP map if and only if ϕ is (U_1, U_2)-CP.

Proof. First assume ϕ to be a CP map. Let $V = [V_{ij}] \in M_l(B(\mathcal{H}_1))^{U+}$. Observe that

$$\begin{aligned}\phi^l(U_1^{l^*}V) = [\phi(U_1^*V_{ij})]_{i,j} &= \begin{pmatrix} \phi(U_1^*V_{11}) & \cdots & \phi(U_1^*V_{1l}) \\ \vdots & \ddots & \vdots \\ \phi(U_1^*V_{l1}) & \cdots & \phi(U_1^*V_{ll}) \end{pmatrix} \\ &= \begin{pmatrix} U_2^* & & 0 \\ & \ddots & \\ 0 & & U_2^* \end{pmatrix} \begin{pmatrix} \phi(V_{11}) & \cdots & \phi(V_{ll}) \\ \vdots & \ddots & \vdots \\ \phi(V_{l1}) & \cdots & \phi(V_{ll}) \end{pmatrix} = U_2^{l^*}\phi^l(V).\end{aligned}$$

Similarly, we obtain $\phi^l(U_1^lV) = U_2^l\phi^l(V)$. Now consider

$$\begin{aligned}[\phi^l(V)\mathbf{h}', \mathbf{h}']_{U_2^l} &= \langle \phi^l(V)\mathbf{h}', U_2^l\mathbf{h}'\rangle = \langle U_2^{l^*}\phi^l(V)\mathbf{h}', \mathbf{h}'\rangle \\ &= \langle \phi^l(U_1^{l^*}V)\mathbf{h}', \mathbf{h}'\rangle \geq 0,\end{aligned}$$

where $\mathbf{h}' \in \mathcal{H}_2^l$. Therefore $\langle U_2^{l^*}\phi^l(V)\mathbf{h}', \mathbf{h}'\rangle \geq 0$, that is, $U_2^{l^*}\phi^l(V)$ is positive with respect to the usual inner product $\langle \cdot, \cdot \rangle$. This proves that $\phi(V)$ is U_2-positive. Thus ϕ is (U_1, U_2)-CP.

Conversely, suppose that ϕ is (U_1, U_2)-CP. Let $V = [V_{ij}] \in M_l(B(\mathcal{H}_1))^+$. Then we have to show that $\phi^l(V) = [\phi(V_{ij})] \in M_l(B(\mathcal{H}_2))^+$. Since

$$0 \leq \langle V\mathbf{h}, \mathbf{h}\rangle = \langle U_1^lV\mathbf{h}, U_1^l\mathbf{h}\rangle = [U_1^lV\mathbf{h}, \mathbf{h}]_{U_1^l},$$

where $\mathbf{h} \in \mathcal{H}^l$, it means $U_1^l V \in M_l(B(\mathcal{H}_1))^{U+}$. Then

$$\begin{aligned}\langle \phi^l(V)\mathbf{h}', \mathbf{h}'\rangle &= \langle U_2^l \phi(V)\mathbf{h}', U_2^l\mathbf{h}'\rangle = \langle \phi(U_1^l V)\mathbf{h}', U_2^l \mathbf{h}'\rangle \\ &= [\phi(U_1^l V)\mathbf{h}', \mathbf{h}']_{U_2^l} \geq 0,\end{aligned}$$

where $\mathbf{h}' \in \mathcal{H}^l$ and the last inequality follows from the fact that $U_1^l V \in M_l(B(\mathcal{H}_1))^{U+}$ and ϕ is (U_1, U_2)-CP. □

Remark 2.4. *In particular, if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $U_1 = U_2 = U$, and if a linear map $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ satisfies $\phi(U^*V) = U^*\phi(V)$ and $\phi(UV) = U\phi(V)$ for all $V \in B(\mathcal{H}_1)$, then ϕ is CP if and only if ϕ is U-CP.*

Definition 2.5. *Let $(\mathcal{H}_i, U_i)$ $(i = 1, 2)$ be an S-space. Assume that ψ is a linear map from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$. Then*

*(1) for each $l \in \mathbb{N}$, ψ is l-(U_1, U_2)-*co-positive* if $\tau_l \otimes \psi : M_l(\mathbb{C}) \otimes B(\mathcal{H}_1) \to M_l(\mathbb{C}) \otimes B(\mathcal{H}_2)$ is $(I_l \otimes U_1, I_l \otimes U_2)$-positive where τ_l is the transpose map on $M_l(\mathbb{C})$.*
(2) ψ is completely (U_1, U_2)-co-positive *if it is l-(U_1, U_2)-co-positive for each $l \in \mathbb{N}$.*
(3) ψ is (U_1, U_2)-positive partial transpose ((U_1, U_2)-PPT) *if it is (U_1, U_2)-CP and completely (U_1, U_2)-co-positive.*
(4) In particular, if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $U_1 = U_2 = U$, then we simply call it completely U-co-positive (and U-positive partial transpose (U-PPT)) if it is completely (U_1, U_2)-co-positive (and (U_1, U_2)-positive partial transpose, respectively).

Proposition 2.6. *Let $(\mathcal{H}_i, U_i)$ $(i = 1, 2)$ be an S-space. Suppose $\phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is a linear map, then ϕ is completely co-positive if and only if the corresponding linear map ψ from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$ defined by $\psi(X) := U_2\phi(U_1^*X)$ is completely (U_1, U_2)-co-positive, where $X \in B(\mathcal{H}_1)$.*

Proof. Let $V = [V_{ij}] \in M_l(\mathbb{C}) \otimes B(\mathcal{H}_1)$ be such that $(I_l \otimes U_1^*)V \geq 0$. Then

$$\begin{aligned}(\tau_l \otimes \psi)(V) &= \begin{pmatrix} \psi(V_{11}) & \cdots & \psi(V_{l1}) \\ \vdots & \ddots & \vdots \\ \psi(V_{1l}) & \cdots & \psi(V_{ll}) \end{pmatrix} = \begin{pmatrix} U_2\phi(U_1^*V_{11}) & \cdots & U_2\phi(U_1^*V_{l1}) \\ \vdots & \ddots & \vdots \\ U_2\phi(U_1^*V_{1l}) & \cdots & U_2\phi(U_1^*V_{ll}) \end{pmatrix} \\ &= \begin{pmatrix} U_2 & & 0 \\ & \ddots & \\ 0 & & U_2 \end{pmatrix} \begin{pmatrix} \phi(U_1^*V_{11}) & \cdots & \phi(U_1^*V_{l1}) \\ \vdots & \ddots & \vdots \\ \phi(U_1^*V_{1l}) & \cdots & \phi(U_1^*V_{ll}) \end{pmatrix} \\ &= (I_l \otimes U_2)(\tau_l \otimes \phi)(I_l \otimes U_1^*)V.\end{aligned}$$

Hence $(I_l \otimes U_2^*)(\tau_l \otimes \psi)(V)$ is positive as ϕ is completely co-positive map.

Conversely, for any $V = [V_{ij}] \in M_l(B(\mathcal{H}_1))$, we have

$$0 \leq \langle V\mathbf{h}, \mathbf{h}\rangle = \langle U_1^l V\mathbf{h}, U_1^l\mathbf{h}\rangle = [U_1^l V\mathbf{h}, \mathbf{h}]_{U_1^l},$$

where $\mathbf{h} \in \mathcal{H}^l$, it means $U_1^l V \in M_l(B(\mathcal{H}_1))^{U+}$. We obtain

$$(\tau_l \otimes \phi)(V) = \begin{pmatrix} \phi(V_{11}) & \cdots & \phi(V_{l1}) \\ \vdots & \ddots & \vdots \\ \phi(V_{1l}) & \cdots & \phi(V_{ll}) \end{pmatrix} = \begin{pmatrix} U_2^*\psi(U_1V_{11}) & \cdots & U_2^*\psi(U_1V_{l1}) \\ \vdots & \ddots & \vdots \\ U_2^*\psi(U_1V_{1l}) & \cdots & U_2^*\psi(U_1V_{ll}) \end{pmatrix}$$

$$= \begin{pmatrix} U_2^* & & 0 \\ & \ddots & \\ 0 & & U_2^* \end{pmatrix} \begin{pmatrix} \psi(U_1V_{11}) & \cdots & \psi(U_1V_{l1}) \\ \vdots & \ddots & \vdots \\ \psi(U_1V_{1l}) & \cdots & \psi(U_1V_{ll}) \end{pmatrix}$$
$$= U_2^{l^*}(\tau_l \otimes \psi)(U_1^l V).$$

Therefore $(\tau_l \otimes \phi)(V) = U_2^{l^*}(\tau_l \otimes \psi)(U_1^l V)$. Since $U_1^l V \in M_l(B(\mathcal{H}_1))^{U+}$ and ψ is completely (U_1, U_2)-co-positive, ϕ is co-positive.

□

3. Kraus U-decomposition and Choi U-matrix

In this section, we derive Kraus U-decomposition and Choi U-matrix and establish their relation with the completely U-positive maps. Let $M_m(\mathbb{C})$ denote the set of all $m \times m$-complex matrices. Kraus proved that $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a CP map if and only if

$$\phi(V) = \sum_{i=1}^{l} R_i^* V R_i, \tag{3.1}$$

where $V = [V_{ij}]_{i,j} \in M_m(\mathbb{C})$ and for each i, $R_i \in M_{m,n}(\mathbb{C})$. The expression in above equation is called a Kraus decomposition.

Denote $M_A := M_m(\mathbb{C})$ and $M_B := M_n(\mathbb{C})$. Let U_A and U_B be the fundamental unitaries in M_A and M_B, respectively. Define a linear map $\psi : M_A \to M_B$ by

$$\psi(V) := \sum_{i=1}^{l} R_i^{\#_{A,B}} V R_i, \tag{3.2}$$

where $R_i^{\#_{A,B}} = U_B R_i^* U_A^*$. Then ψ is (U_A, U_B)-CP. Indeed, for any $k \in \mathbb{N}$, take a $U_A^{k^*}$-positive matrix $V = [V_{ij}] \in M_k(M_A)^{U+}$. Since $V = [V_{ij}] \in M_k(M_A)^{U+}$, $U_A^{k^*} V \in M_k(M_A)^+$, that is,

$$U_A^{k^*} V = \begin{pmatrix} U_A^* & & 0 \\ & \ddots & \\ 0 & & U_A^* \end{pmatrix} \begin{pmatrix} V_{11} & \cdots & V_{1k} \\ \vdots & \ddots & \vdots \\ V_{k1} & \cdots & V_{kk} \end{pmatrix}$$
$$= \begin{pmatrix} U_A^*V_{11} & \cdots & U_A^*V_{1k} \\ \vdots & \ddots & \vdots \\ U_A^*V_{k1} & \cdots & U_A^*V_{kk} \end{pmatrix} \in M_k(M_A)^+.$$

Consider

$$\psi^k(V) = \psi^k \begin{pmatrix} V_{11} & \cdots & V_{1k} \\ \vdots & \ddots & \vdots \\ V_{k1} & \cdots & V_{kk} \end{pmatrix} = \begin{pmatrix} \psi(V_{11}) & \cdots & \psi(V_{1k}) \\ \vdots & \ddots & \vdots \\ \psi(V_{k1}) & \cdots & \psi(V_{kk}) \end{pmatrix}$$
$$= \sum_{i=1}^{l} \begin{pmatrix} R_i^{\#_{A,B}} V_{11} R_i & \cdots & R_i^{\#_{A,B}} V_{1k} R_i \\ \vdots & \ddots & \vdots \\ R_i^{\#_{A,B}} V_{k1} R_i & \cdots & R_i^{\#_{A,B}} V_{kk} R_i \end{pmatrix}$$

$$
\begin{aligned}
&= \sum_{i=1}^{l} \begin{pmatrix} U_B R_i^* U_A^* V_{11} R_i & \cdots & U_B R_i^* U_A^* V_{1k} R_i \\ \vdots & \ddots & \vdots \\ U_B R_i^* U_A^* V_{k1} R_i & \cdots & U_B R_i^* U_A^* V_{kk} R_i \end{pmatrix} \\
&= \sum_{i=1}^{l} \begin{pmatrix} U_B & & 0 \\ & \ddots & \\ 0 & & U_B \end{pmatrix} \begin{pmatrix} R_i^* & & 0 \\ & \ddots & \\ 0 & & R_i^* \end{pmatrix} U_A^{k^*} V \begin{pmatrix} R_i & & 0 \\ & \ddots & \\ 0 & & R_i \end{pmatrix} \\
&= U_B^k \sum_{i=1}^{l} \begin{pmatrix} R_i^* & & 0 \\ & \ddots & \\ 0 & & R_i^* \end{pmatrix} U_A^{k^*} V \begin{pmatrix} R_i & & 0 \\ & \ddots & \\ 0 & & R_i \end{pmatrix},
\end{aligned}
$$

and since $U_A^{k^*} V \in M_k(M_A)^+$, by using the Kraus decomposition

$$\sum_{i=1}^{l} R_i^{*^k} U_A^{k^*} V R_i^k \in M_k(M_A)^+,$$

we obtain $U_B^{k^*} \psi^k(V) \geq 0$. Hence $\psi^k(V)$ is a U_B-positive matrix, that is, ψ is (U_A, U_B)-CP map.

Theorem 3.1. *Let U_A and U_B be the fundamental unitaries in M_A and M_B, respectively. A linear map $\psi : M_A \to M_B$ is a (U_A, U_B)-CP map if and only if it has a decomposition of the form (3.2).*

Proof. Assume that ψ is a (U_A, U_B)-CP map. Since a linear map $\phi : M_A \to M_B$ defined by $\phi(V) = U_B^* \psi(U_A V)$ is CP, ϕ has a Kraus decomposition, that is,

$$\phi(V) = \sum_{i=1}^{l} R_i^* V R_i,$$

where $V \in M_m(\mathbb{C})$ and for each i, $R_i \in M_{m,n}(\mathbb{C})$. Thus we have

$$\psi(V) = U_B \phi(U_A^* V) = U_B \sum_{i=1}^{l} R_i^* U_A^* V R_i = \sum_{i=1}^{l} U_B R_i^* U_A^* V R_i = \sum_{i=1}^{l} R_i^{\#} V R_i.$$

Therefore ψ is a (U_A, U_B)-CP map if and only if ψ has the expression $\psi(V) = \sum_{i=1}^{l} R_i^{\#} V R_i$, we call ψ has a *Kraus U-decomposition* in this case. $\square$

Suppose $\{e_{ij} \mid 1 \leq i, j \leq m\}$ are the matrix units of $M_m(\mathbb{C})$. We observe that $D = [U_A e_{ij}]_{1 \leq i,j \leq m}$ is $I_m \otimes U_A$-positive. Indeed,

$$
\begin{aligned}
(I_m \otimes U_A^*) D &= \begin{pmatrix} U_A^* & & 0 \\ & \ddots & \\ 0 & & U_A^* \end{pmatrix} \begin{pmatrix} U_A e_{11} & \cdots & U_A e_{1m} \\ \vdots & \ddots & \vdots \\ U_A e_{m1} & \cdots & U_A e_{mm} \end{pmatrix} \\
&= \begin{pmatrix} e_{11} & \cdots & e_{1m} \\ \vdots & \ddots & \vdots \\ e_{m1} & \cdots & e_{mm} \end{pmatrix} \in M_{m^2}^+(\mathbb{C}).
\end{aligned}
$$

It implies from the above proposition that $[\psi(U_A e_{ij})]_{1 \leq i,j \leq m}$ is $I_m \otimes U_B$-positive.

Theorem 3.2. *Let $\psi : M_A \to M_B$ be a linear map. Then ψ is (U_A, U_B)-CP if and only if $[U_B^* \psi(U_A e_{ij})]_{1\le i,j\le m}$ is positive.*

Proof. The proof directly follows from [6, Theorem 2]. $\square$

Let $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Choi [6] defined $C_\phi = \sum_{i,j=1}^m e_{ij} \otimes \phi(e_{ij})$, called the *Choi matrix*, and proved that it is positive if and only if ϕ is a CP map.

Definition 3.3. *Let $\psi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. We define $C_\psi^U := \sum_{i,j=1}^m e_{ij} \otimes \psi(U_A e_{ij})$. The matrix C_ψ^U is called the* Choi U-matrix.

Theorem 3.4. *Let U_A and U_B be the fundamental unitaries in M_A and M_B, respectively, where $M_A = M_m(\mathbb{C})$ and $M_B = M_n(\mathbb{C})$. Then a linear map $\psi : M_A \to M_B$ is a (U_A, U_B)-CP map if and only if C_ψ^U is $I_A \otimes U_B$-positive in $M_A \otimes M_B$.*

Proof. Let $\phi : M_A \to M_B$ be the linear map defined by $\phi(V) := U_B^* \psi(U_A V)$ where $V \in M_A$. Then by Proposition 2.1, ϕ is CP if and only if ψ is a (U_A, U_B)-CP map. It is known from [6] that ϕ is CP if and only if C_ϕ is positive semi-definite. Since, for any $\mathbf{h}, \mathbf{h}' \in \mathbb{C}^{mn}$, we have

$$\begin{aligned}
[C_\psi^U \mathbf{h}, \mathbf{h}']_{U_B^m} &= \langle C_\psi^U \mathbf{h}, U_B^m \mathbf{h}' \rangle = \langle U_B^{m^*} C_\psi^U \mathbf{h}, \mathbf{h}' \rangle \\
&= \langle \begin{pmatrix} U_B^* \psi(U_A e_{11}) & \cdots & U_B^* \psi(U_A e_{1m}) \\ \vdots & \ddots & \vdots \\ U_B^* \psi(U_A e_{m1}) & \cdots & U_B^* \psi(U_A e_{mm}) \end{pmatrix} \mathbf{h}, \mathbf{h}' \rangle \\
&= \langle \begin{pmatrix} \phi(e_{11}) & \cdots & \phi(e_{1m}) \\ \vdots & \ddots & \vdots \\ \phi(e_{1m}) & \cdots & \phi(e_{mm}) \end{pmatrix} \mathbf{h}, \mathbf{h}' \rangle \\
&= \langle C_\phi \mathbf{h}, \mathbf{h}' \rangle,
\end{aligned}$$

that is, C_ϕ is positive if and only if C_ψ^U is $I_A \otimes U_B$-positive in $M_A \otimes M_B$, which completes the proof. $\square$

4. Nilpotent U-CP maps

Nilpotent CP maps were studied by Bhat and Mallick in [2]. Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a CP map. Suppose ϕ is a nilpotent map of order p, that is, $\phi^p = 0$ and $\phi^{p-1} \neq 0$. Define $\mathcal{H}_1 := \ker(\phi(U))$ and $\mathcal{H}_k := \ker(\phi^k(U)) \ominus \ker(\phi^{k-1}(U))$, where $2 \le k \le p$. Then $\cap_{k=1}^p \mathcal{H}_k = \emptyset$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_p$. Let $b_i := \dim(\mathcal{H}_i)$ for $1 \le i \le p$. Then $(b_1, b_2, \ldots, b_p)$ is called the CP nilpotent type of ϕ. In this section, we introduce U-CP nilpotent type of U-CP maps.

Proposition 4.1. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $(\mathcal{H}, U)$ be an S-space with the indefinite inner product $[\cdot,\cdot]_U$. Suppose $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a CP map, then the corresponding linear map ψ from $B(\mathcal{H})$ to $B(\mathcal{H})$ defined by $\psi(X) := U\phi(U^* X)$ is U-CP, with the Kraus U-decomposition $\psi(X) = \sum_{i=1}^l R_i^\# X R_i$, where $X \in B(\mathcal{H})$ and $R_i^\# = U R_i^* U^*$ for each $1 \le i \le l$. Then*

(1) $\ker(\psi(U)) = \cap_{i=1}^l \ker(U R_i)$,

(2) For U-positive X, $\psi(X) = 0$ if and only if ran$(X) \subseteq \cap_{i=1}^{l} \ker(R_i^* U^*)$,
(3) $\{h \in \mathcal{H} \mid \psi(|Uh\rangle\langle h|) = 0\} = \cap_{i=1}^{l} \ker(R_i^* U^*)$,
(4) ran$(\psi(U)) = \overline{span}\{UR_i^* h \mid h \in \mathcal{H},\ 1 \leq i \leq l\}$.

Proof. (1) Consider

$$\begin{aligned}
\ker(\psi(U)) &= \{h \in \mathcal{H} \mid \psi(U)h = 0\} \\
&= \{h \in \mathcal{H} \mid \sum_{i=1}^{l} R_i^{\#} U R_i h = 0\} \\
&= \{h \in \mathcal{H} \mid \sum_{i=1}^{l} [R_i^{\#} U R_i h, h]_U = 0\} \\
&= \{h \in \mathcal{H} \mid \sum_{i=1}^{l} [U R_i h, R_i h]_U = 0\} \\
&= \{h \in \mathcal{H} \mid \sum_{i=1}^{l} \langle U R_i h, U R_i h\rangle = 0\} \\
&= \{h \in \mathcal{H} \mid \sum_{i=1}^{l} \|U R_i h\|^2 = 0\} \\
&= \{h \in \mathcal{H} \mid U R_i h = 0,\ \text{ for each } 1 \leq i \leq l\} \\
&= \bigcap_{i=1}^{l} \ker(U R_i).
\end{aligned}$$

(2) Suppose $\psi(X) = U\phi(U^*X) = 0$ where X is U-positive. It follows that $\phi(U^*X) = 0$, and since ϕ is a CP map, using the Kraus decomposition, we obtain $\sum_{i=1}^{l} R_i^* U^* X R_i = 0$. As X is U-positive (U^*X is positive), we get $R_i^* U^* X R_i = 0$ for each i. Note that $R_i^*(U^*X)^{\frac{1}{2}} = 0$. It implies that $R_i^* U^* X = 0$. Let $h_1 \in \text{ran}(X)$, then there exists $h_2 \in \mathcal{H}$ such that $X(h_2) = h_1$. Now by applying $R_i^* U^*$ on both the sides, we get $R_i^* U^* h_1 = 0$ for each i. Hence $\text{ran}(X) \subseteq \cap_{i=1}^{l} \ker(R_i^* U^*)$.

Conversely, let $\text{ran}(X) \subseteq \cap_{i=1}^{l} \ker(R_i^* U^*)$, then $\psi(X) = \sum_{i=1}^{l} R_i^{\#} X R_i = \sum_{i=1}^{l} U R_i^* U^* X R_i = 0$.

(3) One can easily see that $|Uh\rangle\langle h|$ is U-positive. Indeed, $U^*|Uh\rangle\langle h| = |h\rangle\langle h| \geq 0$. Also, we have $\psi(|Uh\rangle\langle h|) = 0$, and $\text{ran}(|Uh\rangle\langle h|) = \mathbb{C}h$, therefore it directly follows from (2) that $\{h \in \mathcal{H} \mid \psi(|Uh\rangle\langle h|) = 0\} = \cap_{i=1}^{l} \ker(R_i^* U^*)$.

(4) Let $h_1 \in \text{ran}(\psi(U)) = \text{ran}(\sum_{i=1}^{l} R_i^{\#} U R_i) = \text{ran}(\sum_{i=1}^{l} U R_i^* R_i)$. Then $\sum_{i=1}^{l} U R_i^* R_i h_2 = h_1$ for some $h_2 \in \mathcal{H}$. Therefore $h_1 \in \overline{span}\{U R_i^* h \mid h \in \mathcal{H},\ 1 \leq i \leq l\}$. Hence $\text{ran}(\psi(U)) \subseteq \overline{span}\{U R_i^* h \mid h \in \mathcal{H},\ 1 \leq i \leq l\}$.

Conversely, let $h \in \overline{span}\{U R_i^* h \mid h \in \mathcal{H},\ 1 \leq i \leq l\}$. Then $h = \sum_{i=1}^{l} \alpha_i U R_i^* h_i$ where $\alpha_i \in \mathbb{C}$, $h_i \in \mathcal{H}$. We have to show that $h \in \text{ran}(\psi(U)) = \text{ran}(U \sum_{i=1}^{l} R_i^* R_i)$. It is equivalent to show that $h \in \ker(\sum_{i=1}^{l} R_i^* R_i U^*)^{\perp}$, that is, $\langle h, h'\rangle_{\mathcal{H}} = 0$ for all $h' \in \ker(\sum_{i=1}^{l} R_i^* R_i U^*)$.

Consider $h' \in \ker(\sum_{i=1}^{l} R_i^* R_i U^*)$, then we have

$$0 = \sum_{i=1}^{l} [R_i^* R_i U^* h', h']_{U^*} = \sum_{i=1}^{l} \langle R_i^* R_i U^* h', U^* h' \rangle.$$

It follows that $R_i U^* h' = 0$ for each i. Observe that

$$\langle h, h' \rangle = \sum_{i=1}^{l} \alpha_i \langle U R_i^* h_i, h' \rangle = \sum_{i=1}^{l} \alpha_i \langle h_i, R_i U^* h' \rangle = 0,$$

which proves that $\mathrm{ran}(\psi(U)) = \overline{span}\{U R_i^* h \mid h \in \mathcal{H},\ 1 \leq i \leq l\}$. □

Proposition 4.2. *Let $(\mathcal{H}, U)$ be an S-space with the indefinite inner product $[\cdot,\cdot]_U$. Suppose $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a CP map, then the corresponding linear map ψ from $B(\mathcal{H})$ to $B(\mathcal{H})$ defined by $\psi(X) := U\phi(U^*X)$ is U-CP, with the Kraus U-decomposition $\psi(X) = \sum_{i=1}^{l} R_i^{\#} X R_i$, where $X \in B(\mathcal{H})$ and $R_i^{\#} = U R_i^* U^*$ for each $1 \leq i \leq l$. Then the followings are equivalent:*

(1) $\psi^p(X) = 0$ for all $X \in B(\mathcal{H})$;
(2) $R_{i_1} R_{i_2} \cdots R_{i_p} = 0$ for all $i_1, i_2, \ldots, i_p$.

Proof. (1) $\implies$ (2) : Let us assume for each $X \in B(\mathcal{H})$, we have

$$0 = \psi^p(X) = \sum_{i_1, i_2, \ldots, i_p = 1}^{l} R^{\#}_{i_p, \ldots, i_1} X R_{i_1} R_{i_2} \cdots R_{i_p},$$

where $R^{\#}_{i_p, \ldots, i_1} = U R_{i_p}^* R_{i_{p-1}}^* \cdots R_{i_1}^* U^*$. Therefore

$$0 = \psi^p(I) = \sum_{i_1, i_2, \ldots, i_p = 1}^{l} R^{\#}_{i_p, \ldots, i_1} R_{i_1} R_{i_2} \cdots R_{i_p}.$$

Now observe that

$$\begin{aligned} &\{h \in \mathcal{H} \mid \sum_{i_1, i_2, \ldots, i_p = 1}^{l} R^{\#}_{i_p, \ldots, i_1} R_{i_1} R_{i_2} \cdots R_{i_p} h = 0\} \\ =&\{h \in \mathcal{H} \mid \sum_{i_1, i_2, \ldots, i_p = 1}^{l} [R^{\#}_{i_p, \ldots, i_1} R_{i_1} R_{i_2} \cdots R_{i_p} h, h]_U = 0\} \\ =&\{h \in \mathcal{H} \mid \sum_{i_1, i_2, \ldots, i_p = 1}^{l} [R_{i_1} R_{i_2} \cdots R_{i_p} h, R_{i_1} R_{i_2} \cdots R_{i_p} h]_U = 0\}, \end{aligned}$$

which concludes the desired equality (2).

(2) $\implies$ (1) : Trivial. □

Suppose ψ is a U-CP map from $B(\mathcal{H})$ to $B(\mathcal{H})$ defined by $\psi(X) = U\phi(U^*X)$. Let ψ be a nilpotent map of order p. Define $\mathcal{K}_1 := \ker(\psi(U))$ and $\mathcal{K}_k := \ker(\psi^k(U)) \ominus \ker(\psi^{k-1}(U))$, where $2 \leq k \leq p$. Then $\cap_{k=1}^{p} \mathcal{K}_k = \emptyset$ and $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \cdots \oplus \mathcal{K}_p$.

Definition 4.3. *Let $c_i := \dim(\mathcal{K}_i)$ for $1 \leq i \leq p$. Then $(c_1, c_2, \ldots, c_p)$ is called the* U-CP nilpotent type *of ψ.*

5. Quantum U-channels and quantum U-states

The U-states and the quantum U-channel, which are the S-space versions of the states and quantum channel, respectively, are introduced in this section. Together, we introduce U-separable and U-entangled states and present the U-PPT criterion for U-separability of U-states.

Definition 5.1. *Let $\phi : M_A \to M_B$ be a linear map and U_A and U_B be the fundamental unitaries in M_A and M_B, respectively. Then*

(1) ϕ is a quantum channel *if it is CP and trace preserving, that is, Tr$(\phi(V))$ = Tr(V) where $V \in M_A$.*

(2) a linear map ψ from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$ defined by $\psi(V) := U_2\phi(U_1^*V)$ is a quantum (U_A, U_B)-channel *if it is (U_A, U_B)-CP and trace preserving.*

Remark 5.2. *It is well known that ϕ is a quantum channel if and only if there exist $m \times n$-matrices $R_1, \ldots, R_l$ such that*

$$\phi(V) = \sum_{i=1}^{l} R_i^* V R_i \quad \textit{and} \quad \sum_{i=1}^{l} R_i R_i^* = I$$

where $V \in M_A$. Indeed, if ϕ is a quantum channel, then it is a CP map and trace preserving. Therefore by Kraus decomposition (3.1), there exist $m \times n$-matrices $R_1, \ldots, R_l$ such that $\phi(V) = \sum_{i=1}^{l} R_i^* V R_i$, and if ϕ is a trace preserving map, then $\phi^*(V) = \sum_{i=1}^{l} R_i V R_i^*$ is unital (Tr$(X) = \langle I_X, X\rangle =$ Tr$(\phi(X)) = \langle I_X, \phi(X)\rangle = \langle \phi^*(I_X), X\rangle$) which implies $\sum_{i=1}^{l} R_i R_i^* = I$.

Similarly, if ψ is a quantum (U_A, U_B)-channel, then by Kraus U-decomposition (3.2) we have $\psi(V) = \sum_{i=1}^{l} R_i^{\#_{A,B}} V R_i$, where $R_i^{\#_{A,B}} = U_B R_i^* U_A^*$. Since ψ is trace preserving, it means ψ^* is unital and we obtain $I_B = \psi^*(I_A) = \sum_{i=1}^{l} R_i R_i^{\#_{A,B}}$. Moreover,

$$\sum_i R_i U_B^* R_i^{\#_{A,B}} = R_i U_B^* U_B R_i^* U_A^* = U_A^*.$$

A *quantum state* $\rho \in M_n(\mathbb{C})$ is a positive semi-definite matrix with $\mathrm{Tr}(\rho) = 1$.

Definition 5.3. *Let U be a fundamental unitary in $M_n(\mathbb{C})$, then a matrix $\rho \in M_n(\mathbb{C})$ is called a* quantum U-state *if the following conditions hold:*

(1) ρ is U-positive, that is, $U^*\rho$ is positive and

(2) Tr$(U^*\rho) = 1$.

Example 5.4. *Let U be a fundamental unitary in $M_l(\mathbb{C})$, where $l \in \mathbb{N}$. Define $\rho \in M_l(\mathbb{C})$ as $\rho = |Ue\rangle\langle e|$ where $e \in \mathbb{C}^l$ with $\|e\| = 1$. Then*

$$U^*\rho = U^*|Ue\rangle\langle e| = |U^*Ue\rangle\langle e| = |e\rangle\langle e|.$$

It follows that $U^*\rho$ is positive and also note that Tr$(U^*\rho) =$ Tr$(|e\rangle\langle e|) = \langle e, e\rangle = 1$. Hence ρ is a quantum U-state.

Proposition 5.5. *A quantum (U_A, U_B)-channel $\psi : M_A \to M_B$ maps quantum U_A-states into quantum U_B-states.*

Proof. Let V be a quantum U_A-state, that is, V is U_A-positive and $\mathrm{Tr}(U_A^* V) = 1$. Since ψ is a quantum (U_A, U_B)-channel, we have

$$\psi(V) = \sum_{i=1}^{l} R_i^{\#_{A,B}} V R_i = \sum_{i=1}^{l} U_B R_i^* U_A^* V R_i,$$

for some $m \times n$-matrices $R_1, \ldots, R_l$. Since V is U_A-positive, we have $U_A^* V \geq 0$. Therefore $U_B^* \psi(V) = \sum_{i=1}^{l} R_i^* U_A^* V R_i \geq 0$, that is, $\psi(V)$ is U_B-positive. Furthermore, we obtain

$$\begin{aligned}\mathrm{Tr}(U_B^* \psi(V)) &= \mathrm{Tr}(\sum_{i=1}^{l} R_i^* U_A^* V R_i) = \mathrm{Tr}(\sum_{i=1}^{l} U_A^* V R_i R_i^*) = \mathrm{Tr}(U_A^* V \sum_{i=1}^{l} R_i R_i^*) \\ &= \mathrm{Tr}(U_A^* V) = 1,\end{aligned}$$

which proves that $\psi(V)$ is a quantum U_B-state. □

A bipartite quantum state $\rho \in M_A \otimes M_B$ is a *product state* if $\rho = \rho_A \otimes \rho_B$ with $\rho_A \in M_A^+$ and $\rho_B \in M_B^+$ and is *separable* if it is a convex combination of product states. Moreover, it is *entangled* if it is not separable. We define $\tau := t \otimes \mathrm{id} : M_A \otimes M_B \to M_A \otimes M_B$ where t is the transpose on M_A. We call the τ map the *partial transpose* or *the blockwise transpose* and a bipartite quantum state ρ is *positive partial transpose* (PPT) if $\rho^\tau := t \otimes \mathrm{id}(\rho)$ is positive. The *positive partial transpose criterion* says that if ρ is separable, then ρ is positive partial transpose.

Definition 5.6. *Let U_A and U_B be the fundamental unitaries in M_A and M_B, respectively. Let $U_A \otimes U_B$ be the fundamental unitary in $M_A \otimes M_B$ and $\rho \in M_A \otimes M_B$ be a bipartite quantum $U_A \otimes U_B$-state. Then*

(1) ρ is a product $U_A \otimes U_B$-state *if $\rho = \rho_A \otimes \rho_B$ where $\rho_A \in M_A^{U+}$ and $\rho_B \in M_B^{U+}$.*
(2) ρ is $U_A \otimes U_B$-separable *if it is a convex combination of product $U_A \otimes U_B$-states.*
(3) ρ is $U_A \otimes U_B$-entangled *if it is not $U_A \otimes U_B$-separable.*
(4) ρ is $U_A \otimes U_B$-positive partial transpose *if the partial transpose ρ^τ is $U_A^t \otimes U_B$-positive, that is, $(\overline{U}_A \otimes U_B^*)(\rho^\tau)$ is positive.*

Proposition 5.7. *If a bipartite quantum $U_A \otimes U_B$-state $\rho \in M_A \otimes M_B$ is $U_A \otimes U_B$-separable, then ρ is $U_A \otimes U_B$-positive partial transpose.*

Proof. Consider that ρ is $U_A \otimes U_B$-separable, it means we can write it as a convex combination of product $U_A \otimes U_B$-states, that is,

$$\begin{aligned}\rho &= \sum_{i=1}^{l} p_i (U_A \otimes U_B)(|z_i\rangle\langle z_i|) = \sum_{i=1}^{l} p_i (U_A \otimes U_B)(|x_i\rangle \otimes |y_i\rangle)(\langle x_i| \otimes \langle y_i|) \\ &= \sum_{i=1}^{l} p_i (U_A \otimes U_B)(|x_i\rangle\langle x_i| \otimes |y_i\rangle\langle y_i|) = \sum_{i=1}^{l} p_i U_A(|x_i\rangle\langle x_i|) \otimes U_B(|y_i\rangle\langle y_i|),\end{aligned}$$

with $\sum_{i=1}^{l} p_i = 1$, and $|z_i\rangle = |x_i\rangle \otimes |y_i\rangle \in M_A \otimes M_B$. Since $(U_A(|x_i\rangle\langle x_i|))^t = |\overline{x_i}\rangle\langle\overline{x_i}|\overline{U_A^*}$, we obtain

$$\rho^\tau = t \otimes \mathrm{id}(\rho) = \sum_{i=1}^{l} p_i |\overline{x_i}\rangle\langle\overline{x_i}|\overline{U_A^*} \otimes U_B(|y_i\rangle\langle y_i|).$$

Since $\overline{U_A}|\overline{x_i}\rangle\langle\overline{x_i}|\overline{U_A^*}$ is a positive matrix in M_A, $(\overline{U}_A \otimes U_B^*)(\rho^\tau)$ is positive. □

6. U-entanglement breaking maps

In this section, we consider the special class of quantum channels which can be simulated by a classical channel in the following sense: The sender makes a measurement on the input state ρ, and send the outcome k via a classical channel to the receiver who then prepares an agreed upon state R_k. Such channels can be written in the form

$$\phi(\rho) = \sum_k R_k \mathrm{Tr}(E_k \rho),$$

where each R_k is a *density matrix* (density matrices, also called density operators, which conceptually take the role of the state vectors, that is, R_k is a positive semi-definite matrix with $\mathrm{Tr}(R_k) = 1$) and the E_k form a positive operator valued measure ($\{E_k\}_k$ form a *positive operator valued measure* means for each k, E_k is positive semi-definite and $\sum_k E_k = id_A$). We call this the *"Holevo form"* because it was introduced by Holevo in [13]. In this context, it is natural to consider the class of channels which break entanglement.

Definition 6.1. *Let $\phi : M_A \to M_B$ be a quantum channel. If $(id_n \otimes \phi)(S)$ is always separable for all bipartite quantum states $S \in M_n(\mathbb{C}) \otimes M_A$, then we call it an* entanglement breaking map.

Let U_A and U_B be the fundamental unitaries in M_A and M_B, respectively. The family $\{F_k\}_k$ is a U_A*-positive operator valued measure* if each $F_k U_A$ is positive semi-definite and $\sum_k F_k U_A = id_A$ (or $\sum_k F_k = U_A^*$) and D is called U_A*-density matrix* if D is a U_A-positive semi-definite matrix, that is, $U_A^* D$ is positive semi-definite matrix with $\mathrm{Tr}(U_A^* D) = 1$.

Definition 6.2. *Let $\psi : M_A \to M_B$ be a (U_A, U_B)-quantum channel.*

(1) ψ is said to be (U_A, U_B)-entanglement breaking *if $(id_n \otimes \psi)(S)$ is $I_n \otimes U_B$-separable for any $I_n \otimes U_A$-density matrix $S \in M_n(\mathbb{C}) \otimes M_A$.*

(2) ψ is in (U_A, U_B)-Holevo form *if it can be expressed as*

$$\psi(\rho) = \sum_k D_k \, Tr(F_k \rho),$$

where D_k is a U_B-density matrix, that is, $U_B^* D_k$ is positive semi-definite matrix and $Tr(U_B^* D_k) = 1$ and F_k is a U_A-positive operator valued measure in M_A, that is $F_k U_A$ is positive semi-definite and $\sum_k F_k U_A = id_A$.

Theorem 6.3. *Let $\psi : M_A \to M_B$ be a (U_A, U_B)-quantum channel. Then the following statements are equivalent:*

(1) ψ is (U_A, U_B)-entanglement breaking;

(2) ψ is in (U_A, U_B)-Holevo form .

Proof. (1) $\implies$ (2) : Suppose ψ is (U_A, U_B)-entanglement breaking. The map ϕ given by $\phi(V) = U_B^* \psi(U_A V)$ is a quantum channel and we have for each $n \in \mathbb{N}$,

$$id_n \otimes \phi = id_n \otimes (U_B^* \psi(U_A)) = (I_n \otimes U_B^*)(id_n \otimes \psi)(I_n \otimes U_A). \tag{6.1}$$

Let $S \in M_n(\mathbb{C}) \otimes M_A$ be a density matrix. One can easily verify that $(I_n \otimes U_A)S$ is a $(I_n \otimes U_A)$-density matrix, that is, $(I_n \otimes U_A^*)(I_n \otimes U_A)S$ is positive and $\mathrm{Tr}((I_n \otimes U_A^*)(I_n \otimes U_A)S) = 1$ which trivially hold as $(I_n \otimes U_A^*)(I_n \otimes U_A)S = S$. Since $(id_n \otimes \psi)(I_n \otimes U_A)S$ is $(I_n \otimes U_B)$-separable, $(id_n \otimes \phi)(S)$ is separable. This implies that ϕ is an entanglement breaking map. Now using [12, Theorem 4], we can write ϕ in the Holevo form, that is,

$$\phi(\rho) = \sum_k R_k \mathrm{Tr}(E_k \rho),$$

where each R_k is a density matrix and $\{E_k\}_k$ is a positive operator valued measure with $\sum_k E_k = id_A$. Observe that

$$\psi(\rho) = U_B \phi(U_A^* \rho) = \sum_k U_B R_k \mathrm{Tr}(E_k U_A^* \rho) = \sum_k D_k \mathrm{Tr}(F_k \rho),$$

where $D_k := U_B R_k$ and $F_k := E_k U_A^*$. Note that D_k is a U_B-density matrix since $U_B^* D_k = U_B^* U_B R_k = R_k$ and R_k is already a density matrix in M_B and also $\{F_k\}_k$ is a U_A-positive operator valued measure in M_A as $E_k U_A^* U_A = E_k$ is positive semi-definite and $\sum_k E_k U_A^* U_A = id_A$.

(2) $\implies$ (1) : Assume that ψ has the (U_A, U_B)-Holevo form, it means $\psi(\rho) = \sum_k D_k \mathrm{Tr}(F_k \rho)$, where D_k is a U_B-density matrix and $\{F_k\}_k$ is a U_A-positive operator valued measure in M_A. Define ϕ by $\phi(\rho) = U_B^* \psi(U_A \rho)$, where $\rho \in M_A$. We obtain

$$\begin{aligned}\phi(\rho) = U_B^* \psi(U_A \rho) = U_B^* \psi(U_A \rho) &= U_B^* \sum_k D_k \mathrm{Tr}(F_k U_A \rho) \\ &= \sum_k U_B^* D_k \mathrm{Tr}(F_k U_A \rho).\end{aligned}$$

Since D_k is a U_B-density matrix and $\{F_k\}_k$ is a U_A-positive operator valued measure in M_A, ϕ has a Holevo form and by [12, Theorem 4] ϕ is an entanglement breaking map and hence Equation (6.1) implies that ψ is a (U_A, U_B)-entanglement breaking map. $\square$

Remark 6.4. *Let $\phi, \psi : M_A \to M_B$ be linear maps such that $\psi(\rho) = U_B \phi(U_A^* \rho)$, where $\rho \in M_A$. As we know ϕ is positive if and only if ψ is a (U_A, U_B)-positive map. Suppose ϕ is a quantum channel, that is, ψ is a (U_A, U_B)-quantum channel. Note that $\theta \circ \phi$ is a CP map for any CP map $\theta : M_B \to M_C$ if and only if $\omega \circ \psi$ is (U_A, U_C)-CP for any (U_B, U_C)-CP $\omega : M_B \to M_C$. Therefore, it follows from Theorem 6.3 that ϕ is an entanglement breaking map if and only if ψ is a (U_A, U_B)-entanglement breaking map.*

7. Examples of fundamental unitary and U-CP maps

In this section, we provide concrete examples of completely U-positive maps and examples of $3 \otimes 3$ quantum U-states which are U-entangled and U-separable. It is easy to observe that the 2×2 identity matrix I and the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis for $M_2(\mathbb{C})$. That is, for any $A \in M_2(\mathbb{C})$, we have $A = aI + b\sigma_x + c\sigma_y + d\sigma_z$ where $a, b, c, d \in \mathbb{C}$. Any fundamental unitary on the 2-dimensional complex S-space has the form

$$U = \begin{pmatrix} a & b \\ -e^{\iota\phi}\overline{b} & e^{\iota\phi}\overline{a} \end{pmatrix} \tag{7.1}$$

where $\phi \in \mathbb{R}$ and $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. For example, if we choose $a = 1$ and $b = 0$, then we have the unitary

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{\iota\phi} \end{pmatrix}$$

which is called a Phase Gate (see [17]) that represents a rotation about the z-axis by an angle ϕ on the Bloch sphere.

If we define an S-space with respect to the fundamental unitary U as in (7.1), then $U^*A = aU^* + b\sigma_x^U + c\sigma_y^U + d\sigma_z^U$, where $\sigma_x^U = U^*\sigma_x$, $\sigma_y^U = U^*\sigma_y$, and $\sigma_z^U = U^*\sigma_z$, and we call these matrices *U-Pauli matrices*.

Let $U_1 = \begin{pmatrix} 1 & 0 \\ 0 & \iota \end{pmatrix}$ and $U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ be two unitaries which are not symmetries, where U_1 is the Phase gate for $\phi = \frac{\pi}{2}$.

(1) Consider the S-space $(\mathbb{C}^2, U_1)$. For any $A \in M_2(\mathbb{C})$, we have

$$U_1^*A = \left[a\begin{pmatrix} 1 & 0 \\ 0 & -\iota \end{pmatrix} - \iota b\begin{pmatrix} 0 & \iota \\ 1 & 0 \end{pmatrix} - \iota c\begin{pmatrix} 0 & 1 \\ \iota & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & \iota \end{pmatrix}\right]$$

and

$$(U_1^*A)^* = \left[\overline{a}\begin{pmatrix} 1 & 0 \\ 0 & \iota \end{pmatrix} + \overline{b}\begin{pmatrix} 0 & \iota \\ 1 & 0 \end{pmatrix} + \overline{c}\begin{pmatrix} 0 & 1 \\ \iota & 0 \end{pmatrix} + \overline{d}\begin{pmatrix} 1 & 0 \\ 0 & -\iota \end{pmatrix}\right].$$

Comparing U_1^*A and $(U_1^*A)^*$, one may easily find out that A is U_1-self adjoint if and only if $a = \overline{d}$, $-\iota c = \overline{c}$ and $-\iota b = \overline{b}$, that is, A has the form

$$A = \begin{pmatrix} a+d & b-\iota c \\ b+\iota c & a-d \end{pmatrix} = \begin{pmatrix} a+\overline{a} & b+\overline{c} \\ b-\overline{c} & a-\overline{a} \end{pmatrix} = \begin{pmatrix} 2\Re(a) & b+\overline{c} \\ b-\overline{c} & 2\iota\Im(a) \end{pmatrix}$$

and U_1^*A has the form

$$U_1^*A = \begin{pmatrix} a+d & b-\iota c \\ c-\iota b & -\iota(a-d) \end{pmatrix} = \begin{pmatrix} a+\overline{a} & b+\overline{c} \\ c+\overline{b} & \iota(a-\overline{a}) \end{pmatrix} = \begin{pmatrix} 2\Re(a) & b+\overline{c} \\ c+\overline{b} & 2\Im(a) \end{pmatrix}$$

where $a, b, c \in \mathbb{C}$. Further, U_1^*A is positive, that is, A is U_1-positive if and only if

$$0 \le \Re(a) \quad \text{and} \quad 4\Re(a)\Im(a) \ge (b+\overline{c})(\overline{b}+c)$$

Also, U_1^*A is a quantum state, that is, A is a quantum U_1-state if and only if

$$\Re(a) + \Im(a) = \frac{1}{2}.$$

In particular, if $a = \frac{1}{2} \in \mathbb{R}$, $b = t$ and $c = -t$ for all $t \geq 0$, then all the above relations are trivially satisfied. In other words, for $t \geq 0$,

$$A = \rho_t = \begin{pmatrix} 1 & 0 \\ 2t & 0 \end{pmatrix}$$

provides a one parameter family of quantum U_1-states in $M_2(\mathbb{C})$. Similarly, the following provides a one parameter family of quantum $U_1 \otimes U_1$-states

$$\frac{1}{16}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2t & 0 & 0 & 0 \\ 2t & 0 & 0 & 0 \\ 4t^2 & 0 & 0 & 0 \end{pmatrix},$$

where $t \geq 0$.

Since $M_2(\mathbb{C})$ is a unital $*$-algebra, any $*$-homomorphism π from $M_2(\mathbb{C})$ into $M_2(\mathbb{C})$ has the form $\pi(A) = W^*AW$ for some unitary matrix $W \in M_2(\mathbb{C})$. If ϕ is a U_1-CP map defined on $M_2(\mathbb{C})$, then by Theorem 2.2 there exist a $*$-homomorphism π on $M_2(\mathbb{C})$ and a matrix $V \in M_2(\mathbb{C})$ such that

$$\phi(A) = V^{\#}\pi(A)V,$$

where $V^{\#} = U_1 V^* U_1^*$. For example, if we consider $V = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and a unitary $W = \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}$, then we get U_1-CP ϕ in the following form:

$$\begin{aligned} \phi(A) = V^{\#}\pi(A)V = (U_1V^*U_1^*)(W^*AW)V &= \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \overline{\beta} \end{pmatrix} \begin{pmatrix} a_{11} & \overline{\gamma}a_{12}\delta \\ \overline{\delta}a_{21}\gamma & a_{22} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \\ &= \begin{pmatrix} \overline{\alpha}\alpha a_{11} & \overline{\alpha\gamma}\delta\beta a_{12} \\ \overline{\beta\delta}\gamma\alpha a_{21} & \overline{\beta}\beta a_{22} \end{pmatrix}, \end{aligned}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$. Furthermore, if $|\alpha| = |\beta| = 1$, then $\phi(A)$ is of the form

$$\phi(A) = \begin{pmatrix} a_{11} & \overline{\alpha\gamma}\delta\beta a_{12} \\ \overline{\beta\delta}\gamma\alpha a_{21} & a_{22} \end{pmatrix}.$$

(2) Consider the S-space $(\mathbb{C}^2, U_2)$. For any $A \in M_2(\mathbb{C})$, we obtain

$$U_2^* A = \frac{1}{\sqrt{2}}\left[a\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \iota c\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} + d\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \right].$$

Comparing U_2^*A and $(U_2^*A)^*$, one may easily find out that A is U_2-self adjoint if and only if b and d are reals and $c = -\iota\overline{a}$, that is, A has the form

$$\begin{pmatrix} a+d & -\overline{a}+b \\ \overline{a}+b & a-d \end{pmatrix}$$

where $a \in \mathbb{C}$ and $b, d \in \mathbb{R}$. Further, $U_2^* A$ is positive, that is, A is U_2-positive if and only if

$$-(b+d) \leq 2\Re(a) \quad \text{and} \quad b^2 + d^2 \leq 2((\Re(a))^2 - (\Im(a))^2).$$

Also, $U_2^* A$ is a quantum state, that is, A is a quantum U_2-state if and only if

$$\Re(a) = \frac{\sqrt{2}}{4}, \quad -(b+d) \leq \frac{\sqrt{2}}{2} \quad \text{and} \quad b^2 + d^2 \leq \frac{1}{4} - 2(\Im(a))^2.$$

In particular, if $a = \sqrt{2}/4 \in \mathbb{R}$ and $b = d = t/4$, with $-\sqrt{2} \leq t \leq \sqrt{2}$, then all the above relations are trivially satisfied. In other words, for $-\sqrt{2} \leq t \leq \sqrt{2}$,

$$\rho_t = \frac{1}{4} \begin{pmatrix} t+\sqrt{2} & t-\sqrt{2} \\ t+\sqrt{2} & -t+\sqrt{2} \end{pmatrix}$$

provides a one parameter family of quantum U_2-states in $M_2(\mathbb{C})$. Similarly, the following provides a one parameter family of quantum $U_2 \otimes U_2$-states

$$\frac{1}{16} \begin{pmatrix} t^2+2\sqrt{2}t+2 & t^2-2 & t^2-2 & t^2-2\sqrt{2}t+2 \\ t^2+2\sqrt{2}t+2 & -t^2+2 & t^2-2 & -t^2+2\sqrt{2}t-2 \\ t^2+2\sqrt{2}t+2 & t^2-2 & -t^2+2 & -t^2+2\sqrt{2}t-2 \\ t^2+2\sqrt{2}t+2 & -t^2+2 & -t^2+2 & t^2-2\sqrt{2}t+2 \end{pmatrix},$$

where $-\sqrt{2} \leq t \leq \sqrt{2}$.

Also, similar to the earlier example, we get any U_2-CP map ϕ in the following form:

$$\begin{aligned} \phi(A) &= V^{\#}\pi(A)V = (U_2 V^* U_2^*)(W^* A W)V \\ &= \frac{1}{2} \begin{pmatrix} \overline{\alpha}+\overline{\beta} & \overline{\alpha}-\overline{\beta} \\ \overline{\alpha}-\overline{\beta} & \overline{\alpha}+\overline{\beta} \end{pmatrix} \begin{pmatrix} a_{11} & \overline{\gamma} a_{12} \delta \\ \overline{\delta} a_{21} \gamma & a_{22} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} (\overline{\alpha}+\overline{\beta})\alpha a_{11} + (\overline{\alpha}-\overline{\beta})\overline{\delta}\gamma\alpha a_{21} & (\overline{\alpha}+\overline{\beta})\overline{\gamma}\delta\beta a_{12} + (\overline{\alpha}-\overline{\beta})\beta a_{22} \\ (\overline{\alpha}-\overline{\beta})\alpha a_{11} + (\overline{\alpha}+\overline{\beta})\overline{\delta}\gamma\alpha a_{21} & (\overline{\alpha}-\overline{\beta})\overline{\gamma}\delta\beta a_{12} + (\overline{\alpha}+\overline{\beta})\beta a_{22} \end{pmatrix}, \end{aligned}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$. Also if $|\alpha| = |\beta| = 1$, then $\phi(A)$ is of the form

$$\phi(A) = \frac{1}{2} \begin{pmatrix} (1+\overline{\beta}\alpha)a_{11} + (1-\overline{\beta}\alpha)\overline{\delta}\gamma a_{21} & (\overline{\alpha}\beta+1)\overline{\gamma}\delta a_{12} + (\overline{\alpha}\beta-1)a_{22} \\ (1-\overline{\beta}\alpha)a_{11} + (1+\overline{\beta}\alpha)\overline{\delta}\gamma a_{21} & (\overline{\alpha}\beta-1)\overline{\gamma}\delta a_{12} + (\overline{\alpha}\beta+1)a_{22} \end{pmatrix}.$$

(3) Let $\mathbb{C}^3$ be a 3-dimensional S-space with an indefinite metric induced by U_3, where $U_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$. It is easy to observe that the matrices

$$\mu_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu_5 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu_6 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\mu_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}, \quad \mu_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad \mu_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

form a basis for $M_3(\mathbb{C})$. Thus, for any $A \in M_3(\mathbb{C})$, we have $A = \sum_{i=1}^{9} a_i\mu_i$, where $a_i \in \mathbb{C}$. Then, we get

$$A = \begin{pmatrix} a_1 - a_4 & a_2 - a_5 & a_3 - a_6 \\ a_1 + a_4 & a_2 + a_5 & a_3 + a_6 \\ a_7\sqrt{2} & a_8\sqrt{2} & a_9\sqrt{2} \end{pmatrix}. \tag{7.2}$$

Since

$$U_3^*A = \sqrt{2}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix},$$

after comparing U_3^*A and $(U_3^*A)^*$, one may easily find out that A is U_3-self adjoint if and only if a_1, a_5 and a_9 are reals and $a_2 = \overline{a_4}$, $a_3 = \overline{a_7}$ and $a_6 = \overline{a_8}$, that is, U_3^*A has the form

$$U_3^*A = \sqrt{2}\begin{pmatrix} a_1 & a_2 & a_3 \\ \overline{a_2} & a_5 & a_6 \\ \overline{a_3} & \overline{a_6} & a_9 \end{pmatrix}.$$

Further, U_3^*A is positive, that is, A is U_3-positive if and only if the following conditions hold:

$$a_1 \geq 0, \tag{7.3}$$
$$a_1a_5 - |a_2|^2 \geq 0 \tag{7.4}$$
$$\text{and} \quad a_1a_5a_9 - a_1|a_6|^2 - |a_2|^2a_9 - |a_3|^2a_5 + 2\Re(a_2\overline{a_3}a_6) \geq 0. \tag{7.5}$$

Also, U_3^*A is a quantum state, that is, A is a quantum U_3-state if and only if

$$a_1 + a_5 + a_9 = \frac{1}{\sqrt{2}}.$$

In particular, if we choose $a_i = \dfrac{1}{3\sqrt{2}}$ in (7.2), then the matrix $A = \dfrac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$ is a U_3-state, where

$$U_3^*A = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Using this example we give the following quantum separable $U_3 \otimes U_3$-state:

$$\frac{1}{9}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In [7], Choi gave the following entangled state which has positive partial transpose:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider

$$C := \frac{2}{21}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$U_3 \otimes U_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

One may easily check that

$$A = (U_3 \otimes U_3)C = \frac{1}{21}\begin{pmatrix} 2 & -3 & 0 & -\frac{3}{2} & 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -2\sqrt{2} & \sqrt{2} & -\sqrt{2} & 0 \\ 0 & -1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & \frac{3}{2} & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 2\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} & 2\sqrt{2} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 2\sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

is a $U_3 \otimes U_3$-entangled state.

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