Thermodynamic Stability Versus Chaos Bound Violation in D-dimensional RN Black Holes: Angular Momentum Effects and Phase Transitions

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Abstract

We compute the Lyapunov exponents for test particles orbiting in unstable circular trajectories around D-dimensional Reissner-Nordström (RN) black holes, scrutinizing instances of the chaos bound violation. Notably, we discover that an increase in particle angular momentum exacerbates the breach of the chaos bound. Our research centrally investigates the correlation between black hole thermodynamic phase transitions and the breaking of the chaos limit. Findings suggest that the chaos bound can only be transgressed within thermodynamically stable phases of black holes. Specifically, in the four-dimensional scenario, the critical point of the thermodynamic phase transition aligns with the threshold condition that delineates the onset of chaos bound violation. These outcomes underscore a deep-rooted link between the thermodynamic stability of black holes and the constraints imposed by the chaos bound on particle dynamics.

1. Introduction

Black hole thermodynamics stands as a cornerstone in the broader landscape of general relativity, intertwining its threads with information theory, quantum attributes of black holes, and statistical physics principles[1, 2]. Over the past several decades, substantial research efforts have delved into the multifaceted thermodynamics of black holes. A seminal contribution by Davies [3] revealed that Kerr and Reissner-Nordström (RN) black holes exhibit a discontinuous divergence in their heat capacity at constant charge or angular momentum, marking a critical phase transition point, colloquially referred to as the Davies point. This discovery significantly enriches our comprehension of the complex and nuanced properties of black holes. Recent investigations have turned their focus towards elucidating the intricate ties between the Lyapunov exponent governing particle motion and the thermodynamic phase structures of black holes[4–7]. Numerical evidence in [4] demonstrates a compelling correspondence between the quasinormal modes and the Davies point in black hole systems. This association gains further traction in [5], where quasinormal modes are shown to have a direct bearing on the phase transitions of regular black holes, with the Davies point serving as a central pivot. The intrigue deepens when the Lyapunov exponent enters the fray; works in [6] and [7] convincingly establish a fundamental link between the phase transitions of black holes and their associated Lyapunov exponents.

Moreover, the Lyapunov exponent of a black hole system operates under an established upper limit, famously known as the chaos bound [8]. Given this intimate connection between the Lyapunov exponent and black hole phase transitions, probing the chaos bound offers a novel lens through which to explore the intricacies of black hole thermodynamics and the dynamical behavior of particles in proximity to the event horizon. Hence, this paper embarks on a comprehensive examination of the thermodynamic stability of black holes and the implications of the chaos bound on particle motion, aiming to shed light on the fundamental interplay between these two seemingly disparate yet inherently intertwined aspects of black hole physics.

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The Lyapunov exponent in black hole systems is fundamentally constrained by a celebrated upper limit, recognized as the *chaos bound* [8], where $\lambda \leq \frac{2\pi T}{\hbar}$, indicative of the maximal degree of chaos in thermal quantum systems. Maldacena, Shenker, and Stanford expounded upon this through quantum field theory and via shock wave analyses near black hole horizons [8]. Hashimoto and Tanahashi subsequently generalized this notion to test particles in close proximity to the horizon, setting the bound as $\lambda \leq \kappa$, consistent with the thermodynamic identity $\kappa = 2\pi T$, thereby solidifying the harmony between the chaos bound in particle dynamics and quantum field theory results [9]. They exposed the inherent connection between the instability of particle motion within black holes and the chaos bound. Building on these findings, Dalui and Majhi extended the chaos bound to Kerr black holes [10] and Rindler horizons [11], yet this remains an evolving chapter in the narrative. Recent scholarly pursuits have scrutinized the chaos bound by examining static equilibrium states of test particles exterior to various charged black holes, discovering that under certain exceptional black hole configurations, the bound $\lambda \leq \kappa$ can be surpassed [12, 13]. Such breaches imply heightened instability in particle motion within spacetime. Further investigation has incorporated the influence of angular momentum, leading to extensive discussions on the chaos bound across a broad spectrum of black holes, including RN(-AdS) [14], Kerr-Newman(-AdS/dS) [15–17], Einstein-Maxwell-Dilaton-Axion [18], and numerous others [19–23], wherein the angular momentum of particles presents additional avenues for chaos bound violation, primarily concentrating on the Lyapunov exponent associated with particles' unstable circular orbits.

Insights into the relationship between chaotic particle/string trajectories and the chaos bound can be found in [24, 25]. Meanwhile, noteworthy research has highlighted the profound physical implications of the Lyapunov exponent in particle motion, revealing its intimate ties to particle momentum and energy [26], causality [27], minimal length effects[28, 29] and the null energy condition [30, 31]. Moreover, the interplay between near-horizon instability and quantum thermal properties of the black hole horizon has been systematically studied [32, 33].

Despite these advancements, the complete physical meaning of the chaos bound and its violation in particle motion, a ubiquitous feature of black holes, remains elusive. It has frequently been observed that within specific parameter domains, the chaos bound can indeed be broken, yet the underlying rationale remains obscure. What is more, the exact physical mechanism facilitating this violation, the microscopic degrees of freedom accountable for it, and the subsequent repercussions are yet to be deciphered. These inquiries are crucial for advancing our comprehension of the fundamental unity between seemingly distinct yet fundamentally interconnected facets of black hole physics and chaos.

So, the points which are mentioned above are one of the grey areas of black hole physics which are not well explained for decades. In this work, for the first time, we have tried to give a possible reason based on our findings for such a feature. Here, we show a new perspective on the intrinsic connection between the thermodynamic stability and the violation of chaos bound $\lambda \leq \kappa$ by analyzing the heat capacity C_Q of D dimensional RN black holes. Here we focus on the asymptotically flat case, and refrain from discussing the AdS and dS cases. We discover that the Lyapunov exponent of unstable circular orbits always increases with the angular momentum of the test particles, thereby increasing the likelihood of exceeding the upper bound on the Lyapunov exponent. With the large angular momentum limit, we identify a threshold value $\bar{r}_c$ that indicates the conditions under which $\lambda \leq \kappa$ are violated. This parameter signifies that when the black hole's event horizon radius r_+ satisfies $r_+ > \bar{r}_c$, the condition $\lambda \leq \kappa$ is always met. By comparing the black hole's thermodynamic phase transition point r_D with $\bar{r}_c$, we observe that violations of the chaos bound only occur in the phase of small black holes with thermodynamic stability. Therefore, the whole analysis establishes the fact that there must be some intriguing connection between the thermal stability of the black hole and the violation of the chaos bound and thereby demands a possible candidate to explore the underlying reason for this fascinating phenomena.

The rest is organized as follows. In the next section, we provide a brief review of the thermodynamics of D dimensional RN black holes. In Sec.3, we calculate the Lyapunov exponent of the unstable circular trajectory of test particles and discuss the relationship between the thermodynamic stability of the black hole and the violation of the chaos bound. Finally, we discuss and summarize in Sec.4.

2. D dimensional Reissner-Nordström (RN) black hole and its thermodynamics

Let us start with the D-dimensional RN black hole which can be reproduced from the charged Myers-Perry black hole [34, 35]. The black hole metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + d\Omega_{D-2}^2, \tag{1}$$

where $f(r)$ is the metric function given by $f(r) = 1 - \frac{2\bar{M}}{r^{D-3}} + \frac{\bar{Q}}{r^{2(D-3)}}$. The largest root for $f(r) = 0$ is the outer horizon r_+ of the black hole. The parameters $\bar{M}$ and $\bar{Q}$ are related to the Arnowitt-Deser-Misner(ADM) mass M and charge Q of black hole

$$\begin{aligned} M =& \frac{(D-2)\omega_{D-2}}{8\pi}\bar{M}, \\ Q^2 =& \frac{(D-2)(D-3)\omega_{D-2}}{8\pi}\bar{Q}^2, \end{aligned} \tag{2}$$

where $\omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$ is the area of the unit sphere in $D-2$ dimensions. The corresponding electromagnetic potential A_t satisfies

$$A_t(r) = \frac{Q}{(D-3)r^{D-3}}. \tag{3}$$

With $f(r_+) = 0$, the black hole mass M can be expressed as

$$M = \frac{Q^2 r_+^{3-D}}{2D-6} + \frac{(D-2)\pi^{\frac{D-3}{2}} r_+^{D-3}}{8\Gamma(\frac{D-1}{2})}. \tag{4}$$

With the Bekenstein-Hawking formula, we can obtain the black hole entropy S

$$S = \frac{A_H}{4} = \frac{\omega_{D-2}}{4} r_+^{D-2}, \tag{5}$$

where A_H is the area of horizon. The D dimensional RN black hole satisfies the first law of thermodynamics

$$\delta M = T\delta S + \Phi\delta Q, \tag{6}$$

where T denotes the temperature of black hole, and Φ is the potential of the black hole. Based on this thermodynamic relationship, we can obtain the temperature T, the potential Φ and the heat capacity C_Q

$$\begin{aligned} T =& \frac{1}{4\pi r_+}\left(D-3-\frac{4\pi^{\frac{1}{2}(3-D)} r_+^{6-2D}\Gamma(\frac{D-2}{2})Q^2}{D-2}\right), \\ \Phi =& \frac{Q}{(D-3)r_+^{D-3}}, \\ C_Q =& \frac{2(D-2)^2\pi^{D+\frac{1}{2}} r_+^{3D} T}{4(2D-5)\pi^{\frac{3}{2}} Q^2 r_+^7 \Gamma^2(\frac{D-1}{2}) - (D-2)(D-3)\pi^{\frac{D}{2}} r_+^{2D+1}\Gamma(\frac{D-1}{2})}, \end{aligned} \tag{7}$$

where C_Q is obtained at the constant Q.

To avoid the naked singularity, it is necessary to satisfy the inequality

$$r_+ \geq r_e = 2^{\frac{1}{2(D-3)}} \left(\frac{\pi^{\frac{1}{4}(3-D)}\sqrt{\Gamma(\frac{D-3}{2})}}{\sqrt{D-2}}\right)^{\frac{1}{D-3}}, \tag{8}$$

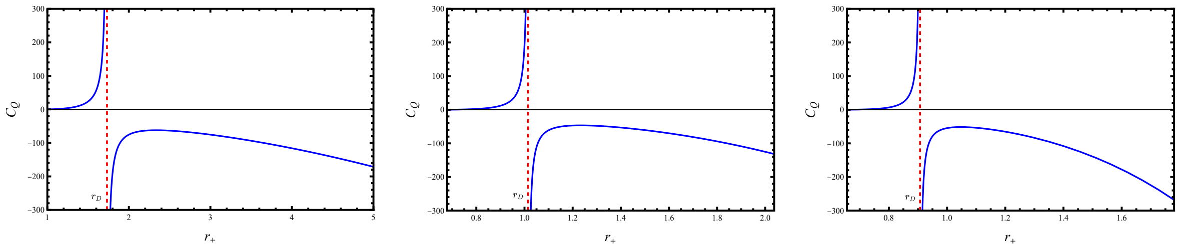
where r_e is the radius of extremal black hole. For the non-extremal cases, there are some singularities in its heat capacity C_Q. From the expression of C_Q, we can obtain that C_Q diverges when satisfying

$$4(2D-5)\pi^{\frac{3}{2}}Q^2r_+^6\Gamma(\frac{1}{2}(D-1)) - (D-2)(D-3)\pi^{\frac{D}{2}}r_+^{2D} = 0. \tag{9}$$

We mark the phase transition point as $r_+ = r_D$, which is also called as Davies points. The phase transition point r_D is

$$r_D = \left(\frac{\sqrt{D-2}\pi^{\frac{D+1}{4}}}{2\sqrt{(2D-5)Q^2\Gamma(\frac{D-3}{2})}}\right)^{\frac{1}{3-D}} \tag{10}$$

We plot the heat capacity at constant charge C_Q as a function of the radius of horizon r_+ for D dimensional RN black holes in Fig. 1, here we take $D = 4,\ 5,\ 6$ and the charge of black hole $Q = 1$. The phase transition point r_D is marked in the red dashed line. In the range of $r_+ < r_D$, the heat capacity C_Q is positive, which means that the black hole is thermodynamically stable till this point, whereas for $r_+ > r_D$ the specific heat C_Q is negative means from this point on the black hole is in the region of thermodynamically unstable phase.



(a) $D = 4$ and $r_D = \sqrt{3} \approx 1.732$. (b) $D = 5$ and $r_D = \left(\frac{10}{3\pi}\right)^{1/4} \approx 1.015$. (c) $D = 6$ and $r_D = \left(\frac{7}{4\pi}\right)^{1/6} \approx 0.907$.

Fig. 1: Heat capacity C_Q at constant charge $Q = 1$ with r_+ for D dimensional RN black holes. The figure (a), (b), (c) correspond to $D = 4,\ 5,\ 6$, respectively.

3. Thermodynamic stability and the violation of chaos bound

We next review the Lyapunov exponent of the particle's circular motion and explore the relationship between thermodynamic stability and the upper bound on the Lyapunov exponent. However, before jumping into that discussion let us first discuss briefly the Lyapunov exponent for a particle's circular motion and its connection to the angular momentum of that same test particle.

3.1. The Lyapunov exponent

The motion of test particles near the black hole can be effectively represented by an inverse harmonic oscillator potential. At the maximum of the effective potential, the test particle has an unstable equilibrium in the radial direction (we consider here as the circular motion in $(r,\ \phi)$ space)

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \lambda^2(r - r_0), \tag{11}$$

where r_0 is the radial position of circular orbits, and the parameter λ is related to the form of the effective potential. The equation of motion corresponding to the circular motion has the general solution given by

$$r = r_0 + C_1e^{\lambda t} + C_2e^{-\lambda t}, \tag{12}$$

where C_1 and C_2 are integration constants. This result shows that when the circular motion is perturbed by ϵ, the perturbation ϵ grows exponentially as $\epsilon \sim e^{\lambda t}$, with the exponent λ representing the Lyapunov exponent.

Details of the calculation on the Lyapunov exponent for unstable circular orbits can be found in Appendix A. The Lyapunov exponent λ satisfies the following equation at the radial equilibrium position $r = r_0$

$$\lambda^2 = \frac{1}{4}\left[f^{'2} - \frac{4L^2 f^2 \left(A_t^{'}\left(2L^2+3r^2\right) + rA_t^{''}\left(L^2+r^2\right)\right)}{r^2\left(L^2+r^2\right)^2 A_t^{'}} + f\left\{f^{'}\left(\frac{4L^2}{rL^2+r^3} + \frac{2A_t^{''}}{A_t^{'}}\right)\right\} - 2f^{''}\right]\Bigg|_{r=r_0} \tag{13}$$

For discussion convenience, here we take the positive angular momentum of test particles ($L > 0$).

To see the effect of the angular momentum L of test particles, we can consider the near-horizon expansion. We can expand λ^2 near the horizon r_+

$$\lambda^2 = \kappa^2 + \gamma_1\left(r - r_+\right) + \gamma_2\left(r - r_+\right)^2 + \mathcal{O}\left(\left(r - r_+\right)^3\right), \tag{14}$$

where the first-order expand parameter γ_1[1]

$$\gamma_1 = 4\kappa^2\left(\frac{L^2}{L^2 r_+ + r_+^3} + \frac{A_t^{''}(r_+)}{2A_t^{'}(r_+)}\right). \tag{15}$$

We can know from the expression of γ_1 that with the increase of L, γ_1 grows too, which means the Lyapunov exponent of unstable circular orbits near the horizon is larger. So the larger L will result in more possibility for violating the chaos bound $\lambda \leq \kappa$.

However, the relationship between angular momentum and chaos bound is not limited to the near-horizon region, it is a more general inference, and the results in some papers also support it [14–16, 23, 31]. To understand the violation of chaos bound more, we can discuss the Lyapunov exponent of test particles in the limit of large angular momentum. With the large angular momentum limit, we can obtain the corresponding Lyapunov exponent $\bar{\lambda}$ by

$$\bar{\lambda}^2 = \frac{1}{4}\left[f^{'2} - \frac{4f^2\left(2A_t^{'} + rA_t^{''}\right)}{r^2 A_t^{'}} + f\left\{f^{'}\left(\frac{2}{r} + \frac{2A_t^{''}}{A_t^{'}}\right) - 2f^{''}\right\}\right]\Bigg|_{r=r_0}. \tag{16}$$

3.2. The numerical results of $\lambda^2 - \kappa^2$

Next, we explore the relationship between the Lyapunov exponent of unstable circular orbits of test particles outside D-dimensional Reissner-Nordström (RN) black holes and the chaos bound $\lambda \leq \kappa$. To more clearly present the results, we use the sign of $\lambda^2 - \kappa^2$ as an indicator of whether the chaos bound is violated. When $\lambda^2 - \kappa^2 > 0$, it indicates that the chaos bound is violated.

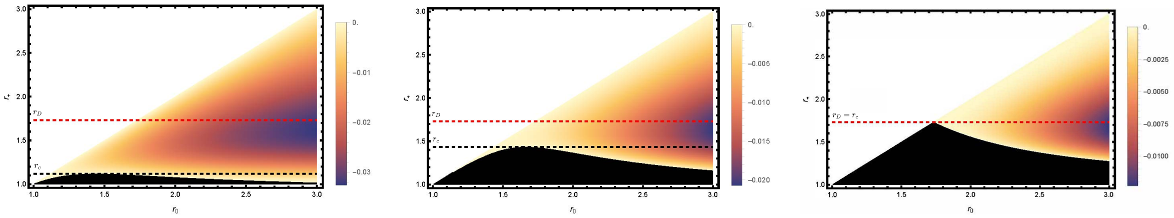
In Fig. 2 and Fig. 3, we plot the density map of the numerical results for $\lambda^2 - \kappa^2$ on the (r_0, r_+) plane, where the circular motion position r_0 and the black hole horizon radius r_+ both vary within the range of $(r_e, 5r_e)$, with r_e denoting the horizon radius corresponding to an extremal black hole. In these figures, we fix the black hole charge

$$Q = 1.$$

We focus on particle movements outside the black hole horizon, requiring that the circular orbit radius r_0 and the horizon radius r_+ satisfy $r_+/r_0 < 1$. Results for this scenario are shown in the bottom right part of each figure. Areas in white, indicating $r_+/r_0 > 1$, are not considered in our study.

To enhance the comprehensibility of the information presented in the figures, we use colored sections to denote results where $\lambda^2 - \kappa^2 < 0$, indicating that the chaos bound is not exceeded; black sections correspond to areas where $\lambda^2 - \kappa^2 > 0$, signifying that the bound $\lambda \leq \kappa$ is violated. Additionally, we draw the thermodynamic phase transition point of the black hole, r_D, with a red dashed line in the figures, and mark a threshold value r_c with a black dashed line, which demonstrates that chaos bound violations no longer occur when $r_+ > r_c$. By comparing the positions of r_D (red dashed line) and r_c (black dashed line) in the figures, we discuss the relationship between violations of the chaos bound and thermodynamic phase transition points in D-dimensional RN black holes.

[1]We write the formula of the second-order expansion parameter γ_2 in Appendix B and discuss its effect.



(a) $D = 4$ and $L = 1$.

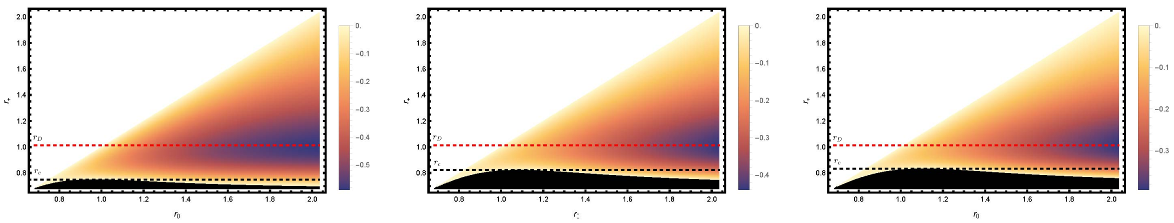
(b) $D = 4$ and $L = 5$.

(c) $D = 4$ and the large angular momentum limit.

Fig. 2: The numerical results of $\lambda^2 - \kappa^2$ in 4-dimensional RN black holes.

We present the results for 4-dimensional RN black holes Fig. 2, where Fig. 2(a), Fig. 2(b) and Fig. 2(c) correspond to particle angular momentum values of $L = 1$, $L = 5$ and the large angular momentum limit, respectively. Comparing these three figures, we observe that the range of the black regions, which indicate the violation of the chaos bound, expands as the particle's angular momentum increases. This implies that the particle's angular momentum indeed enhances the breaking of the chaos bound. Furthermore, the black regions only appear when $r_+ < r_D$. In Fig. 2(c), we also see that the threshold value r_c, beyond which the chaos bound can be violated in the large angular momentum limit, coincides with r_D. This suggests a consistency between the thermodynamic phase transition behavior and the violation of the chaos bound by particle circular orbits in 4-dimensional RN black holes, hinting at a close connection between chaotic dynamics in particle motion and black hole thermodynamics.

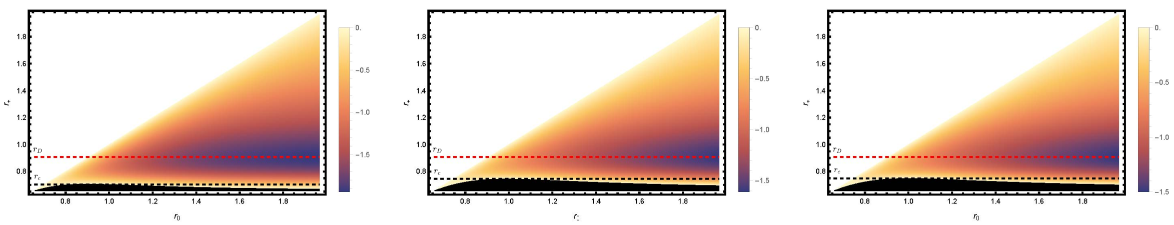
In Fig. 3, we display the results for 5-dimensional and 6-dimensional RN black holes. The first row presents the 5-dimensional case, where Fig. 3(a), Fig. 3(b) and Fig. 3(c) respectively correspond to results of $L = 1$, $L = 5$, and the large angular momentum limit. The bottom row shows the 6-dimensional case. The figures show that, in higher-dimensional scenarios, the increase in particle angular momentum still leads to a stronger violation of the chaos bound. A notable difference, however, is that the consistency between the violation of the chaos bound at the large angular momentum limit and the black hole's thermodynamic phase transition no longer holds. Nonetheless, it is consistently observed that the threshold value r_c, associated with the chaos bound violation, is always less than the thermodynamic phase transition point r_D.



(a) $D = 5$ and $L = 1$.

(b) $D = 5$ and $L = 5$.

(c) $D = 5$ and the large angular momentum limit.



(d) $D = 6$ and $L = 1$.

(e) $D = 6$ and $L = 5$.

(f) $D = 6$ and the large angular momentum limit.

Fig. 3: The numerical results of $\lambda^2 - \kappa^2$ in 5-dimensional and 6-dimensional RN black holes.

The values of threshold parameter r_c for the chaos bound violations discussed here can be found in Tab. 1. From the table, it can be seen that the value of r_c decreases as the spacetime dimension D increases. More importantly, r_c increases with the angular momentum L, confirming our analysis by the near-horizon expansion that angular momentum increases the likelihood of violating the chaos bound. This result is not confined to the near-horizon region.

Angular momentum L / Dimension	$L = 1$	$L = 5$	Large L limit
D=4	$r_c = 1.116$	$r_c = 1.433$	$r_c = 1.732$
D=5	$r_c = 0.750$	$r_c = 0.823$	$r_c = 0.833$
D=6	$r_c = 0.703$	$r_c = 0.744$	$r_c = 0.748$

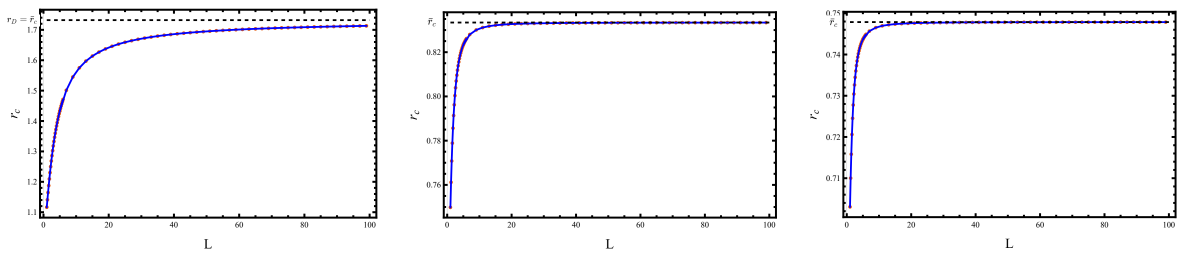
Table 1: The values of threshold parameter r_c for violating the chaos bound in different cases.

3.3. More discussions

Our numerical study on the upper bound of the Lyapunov exponent and its violations in D-dimensional RN black holes reveals that as the angular momentum of the particle increases, the violation of the chaos bound becomes more pronounced. We further analyze the relationship between the violation of the bound and the properties of black holes. To study the violation on $\lambda \leq \kappa$, we analyze the threshold value r_c here. As mentioned previously, r_c delineates the parameter range within which the chaos bound can be violated; for $r_+ > r_c$, the bound remains unviolated.

We plot the threshold value r_c as a function of the test particle's angular momentum L in Fig. 4. The topmost black dashed line in these plots represents the result for the large angular momentum limit, denoted by $\bar{r_c}$, and it also can be defined by $\bar{r_c} = \max(r_c)$. In the figures, it is observed that r_c increases with L, approaching $\bar{r_c}$ in the case of large angular momentum. This result indicates that $\lambda > \kappa$ is only possible within the range $r_+ < \bar{r_c}$.

In our results about D-dimensional RN black holes ($D = 4,\ 5,\ 6$), there is always $r_D \geq \bar{r}_c$, when $D = 4$ it takes equal. That means the chaos bound only can be violated in the stable thermodynamic phase near the extremal black hole.



(a) $D = 4$. (b) $D = 5$. (c) $D = 6$.

Fig. 4: The threshold value r_c for violating chaos bound as a function of the angular momentum L in D-dimensional RN black holes. The figures (a), (b), (c) correspond to $D = 4,\ 5,\ 6$, respectively.

4. Conclusions

In this work, we explore the relationship between the violation of the chaos bound in particle motion outside D-dimensional RN black holes and the thermodynamic stability of black holes. We calculated the Lyapunov exponents for unstable circular orbits of particles outside black holes and discussed the cases for $D = 4, 5, 6$, showing that the violation of $\lambda \leq \kappa$ becomes increasingly apparent with the increase in the angular momentum of particles. In higher-dimensional cases, the chaos bound is violated within a smaller region. In the limit of large angular momentum, we derived the threshold value $\bar{r}_c$ from the numerical results of the Lyapunov exponent, indicating that the chaos bound is no longer broken when $r_+ > \bar{r}_c$. By comparing with the thermodynamic phase transition point r_D of RN black holes, we find $\bar{r}_c \leq r_D$, suggesting that the violation of the chaos bound only occurs in the thermodynamically stable phase near extremal black holes with positive heat capacity.[2] This result reveals the possible intrinsic connection between the chaos bound in particle motion and the thermodynamic properties of black holes.

Previous studies have reported on the violation of the chaos bound in particle motion, yet the physical significance of this violation has remained elusive. The relationship between thermodynamic stability and dynamic stability warrants further in-depth investigation, especially in the context of black hole physics. Considering the recent findings that reveal a potential connection between the Lyapunov exponent and the temperature within thermodynamically stable regions of D-dimensional RN black holes, the question arises whether such a correlation between the Lyapunov exponent and temperature holds a universal character across different systems. To establish this as a general principle, more empirical evidence and theoretical scrutiny are required to substantiate the existence of a persistent link between these two key variables in the broader scope of thermodynamic and dynamical stabilities. Although it is not a conclusive remark, we need to study it further more.

This finding merits further investigation, and it would be interesting to verify the universality of this conclusion across other black holes. We also highlight several issues worth further discussion, such as whether the cosmological constant, black hole spin, and other factors influence the relationship between black hole thermodynamics and the chaos bound. Research into these questions will deepen our understanding of the chaos bound in particle motion and inspire insights into black hole thermodynamics, quantum information, and related topics.

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Appendix A. Lyapunov exponent

We consider test particles with mass m and charge e moving on the equatorial plane of black hole, and write the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(-f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\phi}^2\right) - qA_t(r)\dot{t}, \tag{A.1}$$

where $q = e/m$ and " $\cdot$ " denotes the derivative of the proper time τ. With the definition of generalized momentum $\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$, we can write the component

$$\begin{aligned}
\pi_t =& \frac{\partial \mathcal{L}}{\partial \dot{t}} = -f(r)\dot{t} - qA_t(r) = -E = \text{Constant}, \\
\pi_r =& \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\dot{r}}{f(r)}, \\
\pi_\phi =& \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi} = L = \text{Constant},
\end{aligned} \tag{A.2}$$

where E and L represent the energy and angular momentum of test particles, respectively.

Using the relation $\mathcal{H} = \pi_\mu \dot{x}^\mu - \mathcal{L}$, we can obtain the Hamiltonian

$$\mathcal{H} = \frac{\pi_\phi^2 f + r^2(\pi_r^2 f^2 - (\pi_t + qA_t)^2)}{2r^2 f}. \tag{A.3}$$

[2] In the 4-dimensional case, we have $\bar{r}_c = r_D$, which demonstrates the consistency between thermodynamic phase transitions and the violation of $\lambda \leq \kappa$ in 4-dimensional RN black holes.

The equations of motion in proper time can be given by

$$\dot{x}^{\mu} = \frac{\partial \mathcal{H}}{\partial \pi_{\mu}}, \qquad \dot{\pi}_{\mu} = -\frac{\partial \mathcal{H}}{\partial x^{\mu}}. \tag{A.4}$$

Then let us write the radical motion in coordinate time t

$$\begin{aligned}\frac{dr}{dt} &= \frac{\dot{r}}{\dot{t}} = \frac{\pi_r f^2}{E - qA_t},\\ \frac{d\pi_r}{dt} &= \frac{\dot{\pi}_r}{\dot{t}} = \frac{f^{'}\left(qA_t - E\right)}{2f} + \frac{f\left(2L^2 - r^3\pi_r^2 f^{'}\right)}{2r^3\left(E - qA_t\right)} - qA_t^{'}.\end{aligned} \tag{A.5}$$

Taking (r, π_r) as the phase space variables, we can obtain the Jacobian matrix of particle motion, here we mark $\frac{dr}{dt} = F_1$ and $\frac{d\pi_r}{dt} = F_2$ for convenience

$$K_{ij} = \begin{pmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \pi_r} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \pi_r} \end{pmatrix}. \tag{A.6}$$

For the circular motion at $r = r_0$, the Jacobian matrix can be reduced. We consider the circular motion condition $\pi_r = \frac{d\pi_r}{dt} = 0$ and the 4-velocity normalization condition $\dot{x}^{\mu}\dot{x}_{\mu} = -1$, these conditions can be rewritten as

$$\begin{aligned} q =& \frac{2L^2 f - rf^{'}\left(L^2 + r^2\right)}{2r^2 A_t^{'}\sqrt{f\left(L^2 + r^2\right)}},\\ E =& \frac{2rfA_t^{'}\left(L^2 + r^2\right) + A_t\left(2fL^2 - rf^{'}\left(L^2 + r^2\right)\right)}{2r^2 A_t^{'}\sqrt{f(L^2 + r^2)}}.\end{aligned} \tag{A.7}$$

Then, the components of the Jacobian matrix are

$$\begin{aligned} K_{11} =& 0,\\ K_{12} =& -\left.\frac{f^2}{qA_t - E}\right|_{r=r_0},\\ K_{21} =& \left(\frac{(qA_t - E)f^{'}}{2f}\right)^{'} - qA_t^{''} - \left.\left(\frac{L^2 f}{r^3(qA_t - E)}\right)^{'}\right|_{r=r_0},\\ K_{22} =& 0.\end{aligned} \tag{A.8}$$

The Lyapunov exponent of test particles' circular orbits can be given by $\lambda^2 = K_{12}K_{21}$ from the above functions.

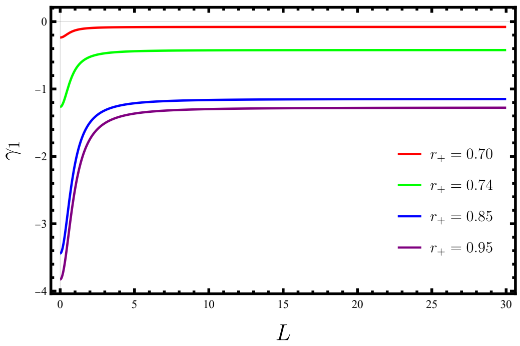
$$\lambda^2 = \frac{1}{4}\left.\left(f^{'2} - \frac{4L^2 f^2\left(A_t^{'}\left(2L^2 + 3r^2\right) + rA_t^{''}\left(L^2 + r^2\right)\right)}{r^2\left(L^2 + r^2\right)^2 A_t^{'}} + f\left(f^{'}\left(\frac{4L^2}{rL^2 + r^3} + \frac{2A_t^{''}}{A_t^{'}}\right)\right) - 2f^{''}\right)\right|_{r=r_0}$$

Appendix B. The high-order effect in near-horizon expansion

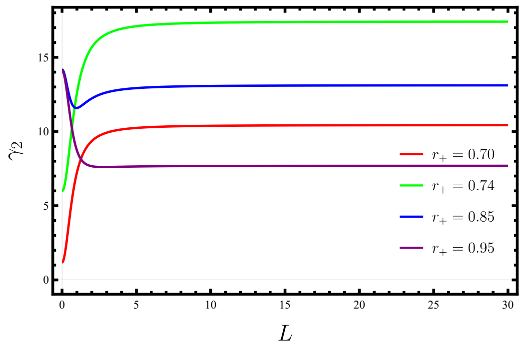
With the near-horizon expansion $\lambda^2 = \kappa^2 + \gamma_1\left(r - r_+\right) + \gamma_2\left(r - r_+\right)^2 + \mathcal{O}\left(\left(r - r_+\right)^3\right)$, we can obtain the second-order expansion parameter γ_2 in the expression of λ^2

$$\begin{aligned}\gamma_2 =& \frac{1}{4}f^{'}(r_+)\left(\frac{6L^2\left(r_+(L^2 + r_+^2)f^{''}(r_+) - 2(L^2 + 2r_+^2)f^{'}(r_+)\right)}{r_+^2(L^2 + r_+^2)^2} - f^{(3}(r_+)\right)\\ &- \frac{f^{'}(r_+)^2 A_t^{''}(r_+)^2}{2A^{'}(r_+)^2} + \frac{3f^{'}(r_+)f^{''}(r_+)A_t^{''}(r_+) + 2f^{'}(r_+)^2\left(A_t^{(3)}(r_+) - \frac{2L^2 A_t^{''}(r_+)}{r_+L^2 + r_+^3}\right)}{4A_t^{'}(r_+)}.\end{aligned} \tag{B.1}$$

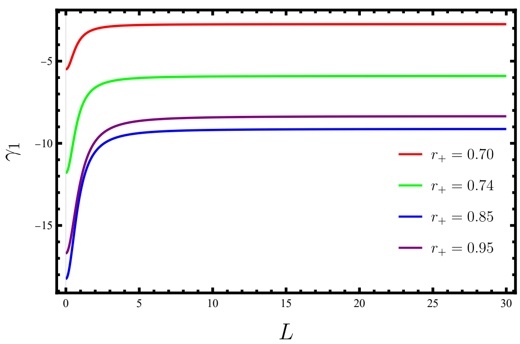
In our previous work on the 4-dimensional case, we discussed how γ_1 approaches zero with increasing angular momentum [14]. Therefore, it becomes necessary to consider the impact of the second-order expansion coefficient γ_2 the limit of large angular momentum. Then we obtained the threshold value for the violation of the chaos bound in 4-dimensional RN black holes as $\bar{r}_c = \sqrt{3}Q$.



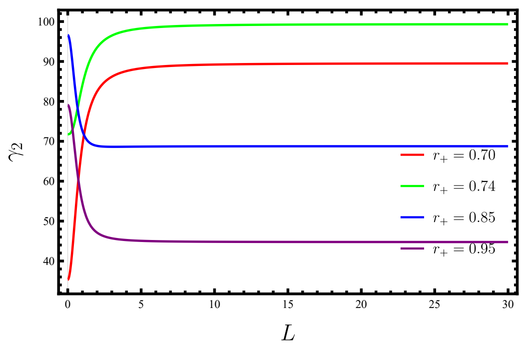
(a) $D = 5$, the first-order expansion coefficient γ_1.



(b) $D = 5$, the second-order expansion coefficient γ_2.



(c) $D = 6$, the first-order expansion coefficient γ_1.



(d) $D = 6$, the second-order expansion coefficient γ_2.

Fig. 5: The near-horizon expansion coefficients γ_1 and γ_2 as the function of particle angular momentum L for 5-dimensional and 6-dimensional RN black holes with the black hole charge $Q = 1$.

Here we study the near-horizon expansion behavior in 5 and 6 dimensional cases. We plot the expansion coefficients γ_1 and γ_2 as the function of angular momentum L in Fig. 5, and the coefficients can be obtained by Eq. (15) and Eq. (B.1). In these figures, we set the radius of horizon $r_+ = 0.70,\ 0.74,\ 0.85,\ 0.95$ with the unit black hole charge $Q = 1$, the different results are plotted in red, green, blue and purple, respectively. In particular, the red and green lines represent examples where the chaos bound can be violated, while the blue and purple lines correspond to cases without the violation of the bound. In Fig. 5(a) and Fig. 5(c), we observe that γ_1 is always negative and increases with angular momentum, ultimately approaching a negative value. This indicates that the first-order term does not lead to the violation of $\lambda \leq \kappa$. For the second-order term, we see in Fig. 5(b) and Fig. 5(d) that γ_2 is positive, offering the possibility to exceed the Lyapunov exponent upper bound. With $r_+ = 0.70, 0.74$, γ_2 increases with L, while for $r_+ = 0.85, 0.95$, it seems decreases with the increase of L. It's observable that for the red and green lines, $|\gamma_1|$ always trends towards smaller values ($\gamma_1 < 0$), and their corresponding γ_2 tends towards larger values. This can explain why, in their associated black hole parameter spaces, the chaos bound can be violated. Therefore, as previous work and this work have shown, higher-order terms in the near-horizon expansion can provide the possibility of exceeding the Lyapunov exponent upper bound. Moreover, further research has revealed that the behavior of violating the chaos bound is not limited solely to the near-horizon region.

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