HARMONIC LOCUS AND CALOGERO–MOSER SPACES

GIOVANNI FELDER AND ALEXANDER P. VESELOV

ABSTRACT. We study the harmonic locus consisting of the monodromy-free Schrödinger operators with rational potential and quadratic growth at infinity. It is known after Oblomkov that it can be identified with the set of all partitions via the Wronskian map for Hermite polynomials. We show that the harmonic locus can also be identified with the subset of the Calogero–Moser space introduced by Wilson, which is fixed by the symplectic action of $\mathbb{C}^\times$. As a corollary, for the multiplicity-free part of the locus we effectively solve the inverse problem for the Wronskian map by describing the partition in terms of the spectrum of the corresponding Moser matrix. We also compute the characters of the $\mathbb{C}^\times$-action at the fixed points, proving, in particular, a conjecture of Conti and Masoero.

1. INTRODUCTION

One of the most important open problems in quantum integrable systems is the classification of Schrödinger operators with trivial monodromy. The corresponding singular set is called locus configuration.

In dimension one we have the Schrödinger equation with a meromorphic potential $V(z)$, $z \in \mathbb{C}$:

$$(-D^2 + V(z))\psi = \lambda\psi, \quad D = \frac{d}{dz}. \tag{1}$$

We say that such operator has *trivial monodromy* if all the solutions ψ of the corresponding equation (1) are meromorphic in $z \in \mathbb{C}$ for *all* λ. The general problem is to describe all such potentials.

The first results in this direction were found by Duistermaat and Grünbaum [13], who solved the problem in the class of rational potentials decaying at infinity

$$V = \sum_{i=1}^{n} \frac{m_i(m_i+1)}{(z-z_i)^2}.$$

They have shown that the corresponding parameters m_i must be integer and that all such potentials are the results of Darboux transformations applied to the zero potential. The potentials are therefore are given by the Burchnall–Chaundy (or Adler–Moser) explicit formulas [2, 8]. The corresponding configurations of poles z_i are very special: in the case when all the parameters $m_i = 1$ they are none other than the (complex) equilibriums of the Calogero–Moser system with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i\neq j}^{n} \frac{1}{(z_i - z_j)^2}$$

and are described by the following algebraic system

$$\sum_{j\neq i}^{n} \frac{1}{(z_i - z_j)^3} = 0, \quad i = 1, \dots, n.$$

A remarkable fact discovered first by Airault, McKean and Moser [4]) is that this system has no solutions unless the number of particles $n = \frac{m(m+1)}{2}$ is a triangular number, in which case the solutions depend on m arbitrary complex parameters.

Oblomkov [21] generalised the Duistermaat–Grünbaum result to the harmonic case, when the rational potentials have quadratic growth at infinity

$$V = z^2 + \sum_{i=1}^{n} \frac{m_i(m_i+1)}{(z-z_i)^2},\ m_i \in \mathbb{Z}.$$

He proved that all such potentials can be found by applying Darboux transformations to the harmonic oscillator and explicitly described them via the Wronskians of the Hermite polynomials

$$u(z) = z^2 - 2D^2 \log W(H_{k_1}, \dots, H_{k_l}),\ k_1 > k_2 > \dots > k_l > 1.$$

In [17] we studied the geometry of pole configuration of these potentials in relation with the Young diagrams of the corresponding partitions

$$\lambda = (\lambda_1, \dots, \lambda_l),\ \lambda_j = k_j - l + j.$$

We were motivated by the following natural question: *given the pole set of the potential $u(z)$ from the harmonic locus, how can one find the corresponding partition λ?*

For the so-called doubled partitions we observed numerically that the Young diagram can be “seen” directly from the shape of the pole set (see Fig. 2 below).

In this paper we show that the harmonic locus can also be identified with a subset of the Calogero–Moser space introduced by Wilson [27], namely with the subset fixed by a natural symplectic action of $\mathbb{C}^\times = \mathbb{C} \setminus 0$. As a corollary, for the multiplicity-free part of the locus we answer the question above by describing the partition λ explicitly in terms of the spectrum of the corresponding Moser matrix M (see Theorem 1 below).

In the last section we compute the characters of the $\mathbb{C}^\times$-action at the fixed points. As a corollary, we prove a conjecture by Conti and Masoero [12] about spectrum of the Hessian matrix $K(\lambda)$ of the Calogero–Moser potential

$$K_{ij}(\lambda) = \delta_{ij}\left(1 + \sum_{l \neq j} \frac{6}{(z_l - z_j)^4}\right) - (1 - \delta_{ij})\frac{6}{(z_i - z_j)^4},$$

where $z_i = z_i(\lambda)$ are the roots of the corresponding Hermite Wronskian $W_\lambda(z)$, which are assumed to be simple. We prove that in such case the eigenvalues of $K(\lambda)$ have the form

$$\mathit{Spec}\, K(\lambda) = \{(\lambda_{l(\Box)+1} - c(\Box))^2, \quad \Box \in \lambda\},$$

where $l(\Box)$ is the leg length of $\Box \in \lambda$ (the number of boxes of λ below $\Box$). This statement is equivalent to Conjecture 6.3 from [12]. For the roots of the usual Hermite polynomial $H_n(z)$, corresponding to one-row Young diagram, this agrees with the well-known result that the frequencies of the small oscillations of Calogero–Moser systems near the corresponding equilibrium are $1, 2, \dots, n$ (see [24, 3]).

2. The harmonic locus and partitions

By the *harmonic locus* $\mathcal{HL}$ we mean the set of the potentials

$$u(z) = z^2 + \sum_{i=1}^{n} \frac{m_i(m_i+1)}{(z - z_i)^2}, \quad z \in \mathbb{C}, \tag{2}$$

of the Schrödinger operator $L = -D^2 + u(z)$, having trivial monodromy in the complex domain, see [13, 26]. The terminology is inspired by the work of Airault, McKean and Moser [4]. The parameters m_i here must be positive integers called multiplicities. The harmonic locus decomposes as a disjoint union $\mathcal{HL} = \cup_{n=0}^{\infty} \mathcal{HL}_n$ according to the number of poles.

Oblomkov [21] proved that all corresponding potentials have the form

$$u(z) = z^2 - 2D^2 \log W(H_{k_1}, \dots, H_{k_l}),$$

where $k_1 > k_2 > \cdots > k_l$ is a sequence of different positive integers and $W(H_{k_1}, \dots, H_{k_l})$ is the Wronskian $\det((D^{i-1}H_{k_j})_{i,j})$ of the corresponding Hermite polynomials $H_k(z) = (-1)^k e^{z^2} D^k e^{-z^2}$:

$$H_0(z) = 1,\ H_1(z) = 2z,\ H_2(z) = 4z^2 - 2,\ H_3(z) = 8z^3 - 12z, \dots$$

Moreover different sequences correspond to different potentials.

It is convenient [17] to label these Wronskians $W = W_\lambda(z)$ by the partitions $\lambda = (\lambda_1, \dots, \lambda_l)$, $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$, such that

$$k_1 = \lambda_1 + l - 1,\ k_2 = \lambda_2 + l - 2,\ \dots, k_{l-1} = \lambda_{l-1} + 1,\ k_l = \lambda_l.$$

In [17] it was shown that the Wronskians W_λ have the following properties:

1. *$W_\lambda(z)$ is a polynomial in z of degree $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$,*
2. *$W_\lambda(-z) = (-1)^{|\lambda|} W_\lambda(z)$,*
3. *$W_{\lambda^*}(z) = (-i)^{|\lambda|} W_\lambda(iz)$, where λ^* is the conjugate of λ.*

Let $\mathcal{P}$ be the set of all partitions which we also identify with the set of all Young diagrams. By Oblomkov's result we have a well-defined bijection $\mathcal{W}\colon \mathcal{P} \to \mathcal{HL}$,

$$\mathcal{W}\colon \lambda \in \mathcal{P} \to u_\lambda(z) = z^2 - 2D^2 \log W_\lambda(z) \in \mathcal{HL}.$$

We have the following natural question. Given the potential $u(z)$ from the harmonic locus, how can one find the partition λ, such that $u(z) = u_\lambda(z)$? In other words, how to invert the map $\mathcal{W}$?

For the locus potentials (2) with all the multiplicities m_i equal to 1

$$u(z) = z^2 + \sum_{i=1}^{n} \frac{2}{(z - z_i)^2}$$

(which we will call *simple locus potentials*), we have the following answer, conjectured by the second author in 2012.

Recall that the *content* $c(\Box)$ of the box $\Box = (i, j)$ from the Young diagram λ is defined as $j - i$, see Fig. 1. It is easy to see that the multiset of contents

$$C(\lambda) := \{c(\Box),\ \Box \in \lambda\}$$

determines the partition λ uniquely.

Theorem 1. *For a simple locus potential $u(z)$, the corresponding partition λ can be uniquely characterized by the property that the contents of λ coincide with the eigenvalues of Moser's matrix M*

$$C(\lambda) = Spec\, M, \quad M_{ij} = \begin{cases} -\frac{1}{(z_i - z_j)^2} & i \neq j \\ \sum_{k \neq j}^{n} \frac{1}{(z_k - z_j)^2} & i = j. \end{cases}$$

One of the motivations for this answer came from the results of the paper [3], where it was shown that when $z_1, \dots, z_n$ are the zeros of the Hermite polynomial $H_n(z)$ the matrix M has the eigenvalues $0, 1, 2, \dots, n-1$, which is the content set of the one-row Young diagram. (Calogero conjectured that the converse is also true for the matrices of this form, but he later proved that this does not hold for $n = 4$, see [11].)

Another motivation came from the following result of Perelomov [23], which explains why we should expect the spectrum of M to be integer.

It is known after Duistermaat and Grünbaum [13] that the poles $z_1, \dots, z_n$ of such potentials satisfy the following *locus conditions* (see [26]):

$$\sum_{j \neq i}^{n} \frac{2}{(z_i - z_j)^3} - z_i = 0,\ i = 1, \dots, n,$$

which are necessary and sufficient conditions for trivial monodromy. Note that these conditions are simply the equilibrium conditions

$$\frac{\partial}{\partial z_i} U(z) = 0,\ i = 1, \dots, n \tag{3}$$

for the Calogero–Moser system with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + U(q), \quad U(q) = \frac{1}{2}\sum_{i=1}^{n} q_i^2 + \sum_{1\le i<j\le n} \frac{1}{(q_i-q_j)^2}. \tag{4}$$

Following [20, 23] introduce matrix L with

$$L_{ij} = \frac{1-\delta_{ij}}{q_i - q_j}$$

and the matrices

$$L^{\pm} := L \pm Q, \quad Q_{ij} = q_i \delta_{ij}.$$

Proposition 2. *(Perelomov [23, 24]) The equilibrium conditions* (3) *are equivalent to the matrix relations*

$$[M, L^{\pm}] = \pm L^{\pm}.$$

The proof is a modification of the direct check by Moser [20].

Thus matrices $L^{\pm}$ can be viewed as raising/lowering operators for M. Since $Me = 0$ for $e = (1, \dots, 1)^T$ the spectrum of M is integer provided e is cyclic vector for the action of $L^{\pm}$.

In the paper we establish a direct link of harmonic locus with the Calogero–Moser spaces introduced by Wilson [27], and as a corollary we provide a proof of Theorem 1.

3. The harmonic locus and Calogero–Moser spaces

The Calogero–Moser spaces were introduced by Wilson [27], who was inspired by the earlier work by Kazhdan, Kostant and Sternberg on the moment map interpretation of the Calogero–Moser systems [18].

Recall that Moser [20] discovered that the system from [9] describing pairwise interacting particles on the line with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j}^{n} \frac{\gamma^2}{(q_i-q_j)^2}$$

(now called *rational Calogero–Moser system*) can be rewritten in matrix form as

$$\dot{L} = [L, M],$$

where

$$L = \begin{pmatrix} p_1 & \frac{\gamma i}{q_1-q_2} & \frac{\gamma i}{q_1-q_3} & \dots & \dots & \frac{\gamma i}{q_1-q_n} \\ -\frac{\gamma i}{q_1-q_2} & p_2 & \frac{\gamma i}{q_2-q_3} & & \dots & \frac{\gamma i}{q_2-q_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{\gamma i}{q_1-q_n} & -\frac{\gamma i}{q_2-q_n} & -\frac{\gamma i}{q_3-q_n} & \dots & -\frac{\gamma i}{q_{n-1}-q_n} & p_n \end{pmatrix}$$

$$M = -i\gamma \begin{pmatrix} a_{11} & \frac{1}{(q_1-q_2)^2} & \frac{1}{(q_1-q_3)^2} & \dots & \dots & \frac{1}{(q_1-q_n)^2} \\ \frac{1}{(q_1-q_2)^2} & a_{22} & \frac{1}{(q_2-q_3)^2} & \dots & \dots & \frac{1}{(q_2-q_n)^2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(q_1-q_n)^2} & \frac{1}{(q_2-q_n)^2} & \frac{1}{(q_3-q_n)2} & \dots & \frac{1}{(q_{n-1}-q_n)^2} & a_{nn} \end{pmatrix}$$

with $a_{ii} = -\sum_{i\neq j}^n \frac{1}{(q_i-q_j)^2}$.

Following Wilson [27], we will consider the case when $\gamma = -i$ with attractive (rather than repulsive) potential and

$$H = \frac{1}{2}\sum_{i=1}^n p_i^2 - \sum_{i<j}^n \frac{1}{(q_i-q_j)^2},$$

when the particles may collide. One can view this also as changing time $t \to it$, $i = \sqrt{-1}$.

Note that the corresponding matrix L satisfies the commutation relation

$$[L, Q] = I - ee^T,$$

where Q is the diagonal matrix with $Q_{ij} = q_i\delta_{ij}$, I is the identity matrix and the vector e has all coordinates equal to 1, so $e^T = (1, 1, \dots, 1)$. Moreover, if Q has simple spectrum, then this commutation relation determines the form of the off-diagonal elements of L uniquely as $L_{ij} = (q_i - q_j)^{-1}$.

Kazhdan, Kostant and Sternberg [18] interpreted this relation as a moment map and described the rational Calogero–Moser system as the corresponding symplectic reduction of the free motion on the Lie algebra, which in the repulsive case is the unitary Lie algebra $\mathfrak{u}_n$.

Wilson considered a natural complex generalisation of this procedure and introduced the *Calogero–Moser space* $\mathcal{C}_n$ as the quotient space

$$\mathcal{C}_n = \{(X, Z, v, w) : [X, Z] + I = vw\}/GL_n(\mathbb{C}),$$

where X and Z are n by n complex matrices, v and w are n-dimensional vector and covector (considered as $n \times 1$ and $1 \times n$ matrices respectively). The element $g \in GL_n$ acts as

$$(X, Z, v, w) \mapsto (gXg^{-1}, gZg^{-1}, gv, wg^{-1}).$$

Wilson showed that $\mathcal{C}_n$ is a smooth irreducible affine algebraic variety of dimension $2n$ with many remarkable properties and can be viewed as the quantisation of the Hilbert scheme of n points in the complex plane.

We introduce now the *modified Calogero–Moser space* $\mathcal{CM}_n$ as the quotient

$$\mathcal{CM}_n = \{\Pi = (L, Q, M, v, w)\}/GL_n(\mathbb{C}),$$

where L, Q, M are n by n complex matrices, v and w are a vector and covector as before, which satisfy the following relations

$$\begin{aligned} (\mathrm{I}): &\quad [L,Q] = I - vw, \\ (\mathrm{II}): &\quad [M,Q] = L, \\ (\mathrm{III}): &\quad [M,L] = Q, \\ (\mathrm{IV}): &\quad Mv = 0, \quad wM = 0. \end{aligned}$$

The group GL_n acts by conjugation on L, Q, M and on v, w as before.

Our main result is the following theorem.

Theorem 3. *The modified Calogero–Moser space $\mathcal{CM}_n$ is discrete and can be identified with the harmonic locus (and thus with the set of partitions of n) via the map $\chi : \mathcal{CM}_n \to \mathcal{HL}_n$,*

$$\chi(\Pi) = z^2 - 2D^2 \log\det(zI - Q).$$

The spectrum of the matrix M is integer and coincides with the content multiset $C(\lambda)$ of the corresponding partition λ.

We start with the following

Proposition 4. *The subset of the modified Calogero–Moser space $\mathcal{CM}_n$ with diagonalisable Q with simple spectrum can be identified with the simple part of the harmonic locus $\mathcal{HL}_n$.*

Proof. Note first that for diagonal Q with distinct diagonal elements $Q_{ii} = q_i$ Moser's matrices

$$L = \begin{pmatrix} 0 & \frac{1}{q_1-q_2} & \dots & \dots & \frac{1}{q_1-q_n} \\ -\frac{1}{q_1-q_2} & 0 & \dots & \dots & \frac{1}{q_2-q_n} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{q_1-q_n} & -\frac{1}{q_2-q_n} & \dots & -\frac{1}{q_{n-1}-q_n} & 0 \end{pmatrix} \tag{5}$$

$$M = \begin{pmatrix} a_{11} & -\frac{1}{(q_1-q_2)^2} & \dots & \dots & -\frac{1}{(q_1-q_n)^2} \\ -\frac{1}{(q_1-q_2)^2} & a_{22} & \dots & \dots & -\frac{1}{(q_2-q_n)^2} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{(q_1-q_n)^2} & -\frac{1}{(q_2-q_n)^2} & \dots & -\frac{1}{(q_{n-1}-q_n)^2} & a_{nn} \end{pmatrix} \tag{6}$$

with $a_{ii} = \sum_{i\neq j}^{n} \frac{1}{(q_i-q_j)^2}$, obviously satisfy the relations (I), (II) and (IV) (with $w = (1,1,\dots,1) = v^T$), while the relation (III) is equivalent to the locus conditions by Perelomov's result.

Conversely, for any $\Pi \in \mathcal{CM}_n$ with diagonal Q with simple spectrum $q_1, \dots, q_n$ the relation (I) implies that $L_{ij} = \frac{v_i w_j}{q_i - q_j}$, $i \neq j$ and that $v_i w_i = 1$ for all $i = 1, \dots, n$, while the relation (II) implies that $L_{ii} = 0$. Conjugation by the diagonal matrix $g \in GL_n(\mathbb{C})$ with diagonal elements $v_1, \dots, v_n$ reduces w to $w = e = (1, \dots, 1) = v^T$ and L to form (5). Now the relations (II) and (IV) imply that M also has Moser's form (6), and relation (III) implies the locus condition. $\square$

Let us now return to Wilson's Calogero–Moser space $\mathcal{C}_n$ and consider the natural action of $\mathbb{C}^\times = \mathbb{C} \setminus 0$ on it defined by

$$X \mapsto \mu X,\ Z \mapsto \mu^{-1} Z,\ v \mapsto v,\ w \mapsto w,\ \ \mu \in \mathbb{C}^\times. \tag{7}$$

Let $\mathcal{C}_n^{\mathbb{C}^\times}$ be the fixed point subset of $\mathcal{C}_n$ under the action (7).

Proposition 5. *The modified Calogero–Moser space $\mathcal{CM}_n$ can be identified with $\mathcal{C}_n^{\mathbb{C}^\times}$ via the map $\nu : (L, Q, M, v, w) \in \mathcal{CM}_n \mapsto (X, Z, v, w) \in \mathcal{C}_n$ such that*

$$X = \frac{1}{2}(L + Q), \quad Z = L - Q.$$

Proof. Since $[X, Z] + I = -[L, Q] + I = vw$, (X, Z, v, w) indeed belongs to the Calogero–Moser space. We have also due to relations (II) and (III) that

$$[M, X] = X, \quad [M, Z] = -Z. \tag{8}$$

They imply that

$$e^{tM} X e^{-tM} = e^t X, \quad e^{tM} Z e^{-tM} = e^{-t} Z,$$

so the pair $(e^t X, e^{-t} Z)$ is GL_n-equivalent to (X, Z) and thus (X, Z) is a fixed point of the action (7).

Conversely, assume that $(e^t X, e^{-t} Z, v, w)$ is GL_n-equivalent to (X, Z, v, w):

$$e^t X = g(t) X g(t)^{-1},\ e^{-t} Z = g(t) Z g(t)^{-1},\ g(t) \in GL_n(\mathbb{C}), g(0) = I,$$

and $g(t)v = v, wg(t)^{-1} = w$. Differentiating this at $t = 0$ and defining $M := \dot{g}(0)$ we see that M satisfies the relations (8) and hence the relations (II) and (III) for

$$L = X + \frac{1}{2} Z, \quad Q = X - \frac{1}{2} Z. \tag{9}$$

Similarly, differentiating the relations $g(t)v = v$, $wg(t)^{-1} = w$ at $t = 0$ we have the relations $Mv = 0, wM = 0$. $\square$

Luckily, the fixed point set $\mathcal{C}_n^{\mathbb{C}^\times}$ was studied in detail by Wilson [27], who identified it with the set $\mathcal{P}_n$ of all partitions of n. Moreover, he explicitly described the bijective map $\kappa \colon \mathcal{P}_n \to \mathcal{C}_n^{\mathbb{C}^\times}$ sending λ to

$$\kappa(\lambda) = (X_\lambda, Z_\lambda, v_\lambda, w_\lambda) \tag{10}$$

constructed as follows.

Let $\lambda = (\lambda_1, \dots, \lambda_l), \lambda_1 + \dots + \lambda_l = n$ be a partition from $\mathcal{P}_n$. Following Wilson, we will use an alternative way to describe a partition

$$\lambda = (a_1, \dots, a_k | l_1, \dots l_k)$$

introduced by Frobenius [15]. Namely, k is the number of the diagonal boxes in the corresponding Young diagram with $a_1, \dots, a_k$ and $l_1, \dots, l_k$ (called Frobenius coordinates) being the lengths of the corresponding "arms" and "legs" respectively, see Fig. 1. Let $n_i = a_i + l_i + 1$ be the length of the corresponding hook and $r_i = l_i + 1$, $i = 1, \dots, k$. Clearly, we have $n_1 + \dots + n_k = n = |\lambda|$.

0	1	2	3
-1	0	1	
-2			

FIGURE 1. The Young diagram of the partition $\lambda = (4, 3, 1)$ of $|\lambda| = 8$ with boxes labeled by their contents. Its Frobenius representation is $\lambda = (3, 1|2, 0)$ and the multiset of contents is $C(\lambda) = \{-2, -1, 0, 0, 1, 1, 2, 3\}$.

Let

$$\Lambda_m = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \end{pmatrix}$$

be the standard $m \times m$ Jordan block, then Z_λ is defined simply as the block-diagonal matrix with Jordan blocks $\Lambda_{n_1}, \dots, \Lambda_{n_k}$ on the diagonal. The block X_{ij} of the matrix X_λ is the unique n_i by n_j matrix with non-zero entries in the diagonal number $r_j - r_i - 1$[1] obeying the commutation relation

$$X_{ij}\Lambda_{n_j} - \Lambda_{n_i}X_{ij} = n_iE(r_i, r_j) - \delta_{ij}I,$$

where $E(r, s)$ is the matrix with the only non-zero element $E(r, s)_{r,s} = 1$. In particular, the block X_{ii} has the form

$$X_{ii} = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 0 & 0 & & & \\ & 2 & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -2 & 0 & 0 \\ & & & & -1 & 0 \end{pmatrix}$$

with the entries $1, 2, \dots, l_i - 1, l_i; -a_i, -a_i + 1, \dots, -2, -1$ on the diagonal number -1 (see [27]). Finally v_λ and w_λ are such that $v_\lambda w_\lambda$ has blocks $n_iE(r_i, r_j)$.

Using this, we can define the matrices L_λ and Q_λ by (9), so we only need to find the corresponding M.

Proposition 6. *In Wilson's basis the corresponding matrix M_λ is diagonal with blocks*

$$M_{ii} = diag\,(-l_i, -l_i + 1, \dots, -2, -1, 0, 1, 2, \dots, a_i - 1, a_i).$$

[1]The diagonal number m of a matrix consists of the entries indexed by i, j such that $j - i = m$

The proof follows from direct check of the relations (II), (III), (IV).

Corollary 7. *The spectrum of M coincides with the content multiset of the partition λ : $Spec\, M = C(\lambda)$.*

Indeed, the eigenvalues of M_{ii} are simply the contents of the boxes in the i-th diagonal hook of the Young diagram of λ.

Proposition 8. *The characteristic polynomial of the matrix $Q_\lambda = X_\lambda - \frac{1}{2} Z_\lambda$ coincides with the Hermite Wronskian*

$$\det(zI - Q_\lambda) = A(\lambda) W_\lambda(z)$$

for some constant $A(\lambda)$.

Proof. Firstly, we use the result from Wilson's paper [27] (see formula (6.14) on page 29), which claims that for some constant $B(\lambda)$

$$\det(X_\lambda - \sum_{i\geq 1} p_i (-Z_\lambda)^{i-1}) = B(\lambda) s_\lambda,$$

where p_i are the power sum symmetric functions and s_λ is the corresponding Schur symmetric function [19].

Then we use the following general fact expressing the Wronskians of Appell polynomials in terms of Schur symmetric functions, see [7].

Recall that the sequence of polynomials $A_k(x), k \geq 0$ is called *Appell* if they satisfy the relation

$$\frac{d}{dx} A_k(x) = k A_{k-1}(x), \quad k \geq 0$$

with $A_0 = 1$. Let $F_A(t) = e^{xt} f_A(t),\ f_A(t) = 1 + \sum_{k\geq 1} a_k \frac{t^k}{k}$ be the exponential generating function of the Appell sequence

$$F_A(t) = e^{xt} f_A(t) = \sum_{k\geq 0} A_k(x) \frac{t^k}{k!}.$$

Consider the expansion

$$\log F_A(t) = xt + \log f_A(t) = xt + \sum_{k\geq 1} b_k \frac{t^k}{k},$$

then from [7], Theorem 4.1 and Proposition 4.3 it follows that the Wronskian

$$A_\lambda(x) = W(A_{k_1}(x), \dots, A_{k_l}(x)), \quad k_i = \lambda_i + l - i,$$

coincides with the Schur symmetric function $s_\lambda(p_1, p_2, \dots)$ if we specialize

$$p_1 = x + b_1,\ p_i = b_i,\ i \geq 2.$$

The Hermite polynomials (in physicists' version) satisfy the relation

$$\frac{d}{dx} H_k(x) = 2k H_{k-1}(x), \quad k \geq 0,$$

so the scaled monic versions $\tilde{H}_k(x) := 2^{-k}H_k(x)$ form an Appell sequence. The corresponding generating function is $F_H(t) = e^{xt-\frac{t^2}{4}}$ (see e.g. Szegő [25]). Since $\log F_H(t) = xt - \frac{1}{4}t^2$ we have $b_1 = 0$, $b_2 = -\frac{1}{2}$ with $b_i = 0, i > 2$.

This implies that the Hermite Wronskian $W_\lambda(z)$ up to a constant multiple coincides with the Schur function $s_\lambda(z, -\frac{1}{2}, 0, 0, \dots,)$ so, by Wilson's result, up to a constant multiple

$$W_\lambda(z) = \det(X_\lambda - zI - \frac{1}{2}Z_\lambda) = \det(Q_\lambda - zI),$$

as claimed. $\square$

We can now prove our Theorem 3. We need to show that the map

$$\chi\colon (L, Q, M, v, w) \mapsto u(z) = z^2 - 2D^2 \log\det(zI - Q)$$

is a bijection from $\mathcal{CM}_n$ to $\mathcal{HL}_n$. We have a diagram of maps

$$\begin{tikzcd} \mathcal{CM}_n \arrow[r, "\nu"] \arrow[d, "\chi"] & \mathcal{C}_n^{\mathbb{C}^\times} \arrow[dl, "\gamma"] \\ \mathcal{HL}_n & \mathcal{P}_n \arrow[l, "\mu"] \arrow[u, "\kappa"'] \end{tikzcd}$$

In this diagram the maps κ, μ, ν are bijections due to Wilson, Oblomkov, and Proposition 5. The map

$$\gamma : (X, Z, v, w) \to z^2 - 2D^2 \log\det(zI - X + \frac{1}{2}Z)$$

makes the lower triangle commutative by Proposition 8. By construction, $\chi = \gamma \circ \nu$ and is thus a bijection, completing the proof of Theorem 3.

Theorem 1 follows then from Proposition 4.

4. Characters of $\mathbb{C}^\times$-action and the Conti–Masoero conjecture

Consider again the simple part of the harmonic locus with the potential

$$u(z) = z^2 + \sum_{i=1}^{n} \frac{2}{(z - z_i)^2}$$

satisfying the locus conditions

$$\sum_{j \neq i}^{n} \frac{2}{(z_i - z_j)^3} - z_i = 0, \ i = 1, \dots, n,$$

which can be interpreted as the equilibrium conditions for the Calogero–Moser system with the potential

$$U(q) = \frac{1}{2}\sum_{i=1}^{n} q_i^2 + \sum_{1 \le i < j \le n} \frac{1}{(q_i - q_j)^2}.$$

Consider the (complex) Hessian of the potential $U(q)$ at the corresponding equilibrium

$$K_{ij} := \frac{\partial^2 U}{\partial q_i \partial q_j}\bigg|_{q=z} = \delta_{ij}\left(1 + \sum_{l \neq j} \frac{6}{(z_l - z_j)^4}\right) - (1-\delta_{ij})\frac{6}{(z_i - z_j)^4}.$$

Since all the solutions of Calogero–Moser system (4) are known to be periodic with period 2π (see [22]), the eigenvalues of K must be the squares of some integers (cf. [24]). Indeed, for the roots of Hermite polynomial $H_n(z)$ the corresponding eigenvalues are known to be $1, 2^2, 3^2, \dots, n^2$, as follows from the formula

$$K = (M + I)^2,$$

relating K to Moser's matrix M (see [23, 24] and Proposition 3.4 in [3]).

However, already for $\lambda = (3, 1)$ one can check that the corresponding matrix K does not commute with M, which means that we can not apply our Theorem 3 to compute the spectrum of K in the general case.

We will use instead our Proposition 5 and Wilson's results [27] to compute these eigenvalues, proving, in particular, a conjectural formula by Conti and Masoero (see Conjecture 6.3 in [12]).

More generally, consider the action of $\mathbb{C}^\times$ on the Calogero–Moser space $\mathcal{C}_n$ defined by (7). We have identified the fixed points of this action with the harmonic locus. Let $(X_\lambda, Y_\lambda, v_\lambda, w_\lambda)$ be the fixed point corresponding to partition $\lambda \in \mathcal{P}_n$ and consider the character of the linearised action of $\mathbb{C}^\times$ at this point:

$$\chi_\lambda(\mu) := \sum_{s \in S_\lambda} \mu^s, \quad \mu \in \mathbb{C}^\times.$$

Note that because the action of $\mathbb{C}^\times$ is symplectic, the weight set S_λ of the action is invariant under the change $s \to -s$:

$$S_\lambda = \{\pm s_1(\lambda), \dots, \pm s_n(\lambda)\}, \quad s_k(\lambda) \in \mathbb{Z}_{>0}.$$

Theorem 9. *The character of the linearised $\mathbb{C}^\times$-action at the fixed point $(X_\lambda, Y_\lambda, v_\lambda, w_\lambda) \in C_n$ has the form*

$$\chi_\lambda(\mu) = (\mu - 2 + \mu^{-1}) G_\lambda(\mu) G_\lambda(\mu^{-1}) + G_\lambda(\mu) + G_\lambda(\mu^{-1}), \tag{11}$$

where

$$G_\lambda(\mu) := \sum_{\square \in \lambda} \mu^{c(\square)}.$$

with $c(\square)$ being, as before, the content of the $\square$ in the Young diagram of λ.

Proof. Denote by $Mat_{k,l}$ the space of complex k by l matrices. The tangent space at a point $(X, Z, v, w) \in \mathcal{C}_n$ is given by the linearized symplectic reduction, namely as the middle cohomology $\mathrm{Ker}(\delta')/\mathrm{Im}(\delta)$ of the complex

$$Mat_{n,n} \xrightarrow{\delta} Mat_{n,n} \oplus Mat_{n,n} \oplus Mat_{n,1} \oplus Mat_{1,n} \xrightarrow{\delta'} Mat_{n,n}. \tag{12}$$

Here δ' is the differential of the moment map

$$\delta' \colon (\xi, \zeta, \rho, \sigma) \mapsto [\xi, Z] + [X, \zeta] + \rho w + v\eta$$

and δ is the infinitesimal action of GL_n

$$\delta \colon \alpha \mapsto ([\alpha, X], [\alpha, Z], \alpha v, -w\alpha).$$

The GL_n-orbit of $\kappa(\lambda) = (X_\lambda, Z_\lambda, v_\lambda, w_\lambda)$ is fixed by the $\mathbb{C}^\times$-action (7). By the relations (II)–(IV) and Proposition 5, $\kappa(\lambda)$ itself is a fixed point for the twisted action of $\mathbb{C}^\times$

$$(X, Z, v, w) \mapsto (\mu^{1-M} X \mu^M, \mu^{-1-M} Z \mu^M, \mu^{-M} v, w\mu^M), \quad \mu \in \mathbb{C}^\times. \tag{13}$$

The maps δ, δ' are $\mathbb{C}^\times$-equivariant for this action on the middle term of (12) and for the action $\alpha \mapsto \mu^{-M} \alpha \mu^M$, $\alpha \in Mat_{n,n}$, $\mu \in \mathbb{C}^\times$ on the extreme terms.

Since δ is injective and δ' is surjective (see [27], Corollary 1.4 and Proposition 1.7), the character of the $\mathbb{C}^\times$-action on the cohomology is equal to the character-valued Euler characteristic of the complex, namely as the character of the middle term minus the sum of characters of the extreme terms.

By Corollary 7 the eigenvalues of M are contents of λ and the eigenvalues of $[M, -]$ on $Mat_{n,n}$ are differences of contents. Thus the character of the middle term for the $\mathbb{C}^\times$-action (13) is

$$\mu G_\lambda(\mu^{-1}) G_\lambda(\mu) + \mu^{-1} G_\lambda(\mu^{-1}) G_\lambda(\mu) + G_\lambda(\mu^{-1}) + G_\lambda(\mu).$$

From this we subtract twice the character $G_\lambda(\mu^{-1}) G_\lambda(\mu)$ from the extreme terms and obtain formula (11). $\square$

At the simple part of the harmonic locus the set S_λ determines the spectrum of the Hessian $K(\lambda)$ at the corresponding equilibrium by the formula

$$\mathit{Spec}\, K(\lambda) = \{s_1(\lambda)^2, \ldots, s_n(\lambda)^2\}.$$

In the paper [12] Conti and Masoero conjectured a recursive procedure to compute the eigenvalues of K, which can be shown to be equivalent to the following formula for the spectrum of K:

$$\mathit{Spec}\, K(\lambda) = \{(\lambda_{l(\square)+1} - c(\square))^2, \quad \square \in \lambda\}, \tag{14}$$

where $l(\square)$ is the leg length of $\square \in \lambda$.

Proposition 10. *For any partition $\lambda \in \mathcal{P}_n$ we have the identity*

$$\chi_\lambda(\mu) = \sum_{\square \in \lambda} \mu^{\lambda_{l(\square)+1} - c(\square)} + \mu^{-\lambda_{l(\square)+1} + c(\square)}. \tag{15}$$

The proof is by induction in the length of the partition using the recursive procedure from [12].

In particular, for the one-hook Young diagram $\lambda = (a|l)$ (in Frobenius notations) we have

$$G_{(a|l)}(\mu) = \frac{\mu^{a+1} - \mu^{-l}}{\mu - 1},$$

$$\chi_{(a|l)}(\mu) = -(\mu^{a+1} - \mu^{-l})(\mu^{-(a+1)} - \mu^{l}) + \frac{\mu^{a+1} - \mu^{-l}}{\mu - 1} + \frac{\mu^{-(a+1)} - \mu^{l}}{\mu^{-1} - 1}$$
$$= \sum_{j=-l,\, j\neq 0}^{a} (\mu^{j} + \mu^{-j}) + \mu^{a+l+1} + \mu^{-(a+l+1)}.$$

Corollary 11. *The Conti–Masoero conjecture holds for any simple locus potential $u_\lambda(z)$.*

5. Concluding remarks

For the simple part of the harmonic locus we have now an effective way to invert the Wronskian map $\lambda \to u_\lambda(z)$ by computing the spectrum of the corresponding Moser matrix M. The question about a correct analogue of the matrix M when we have multiplicities is still open. Note that there is a conjecture saying that for the harmonic locus only $z = 0$ may have multiplicity larger than 1 (see [17]), but in spite of a large numerical evidence and some recent progress in this direction [14, 16] the general proof is still to be found.

In the case of the so-called doubled partitions $\lambda^2 = (\lambda_1, \lambda_1, \dots, \lambda_l, \lambda_l)$ we have observed in [17] a surprising relation between the shape of the Young diagram of λ and the pattern of zeroes of the corresponding Hermite Wronskian, which allows to "see" 2 copies of the corresponding partition λ (see Fig. 2).

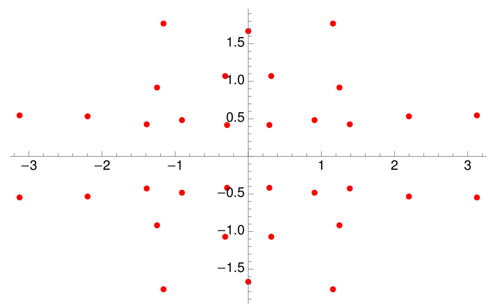

FIGURE 2. Zeroes of the Wronskian W_{λ^2} for the doubled partition λ^2 with $\lambda = (10, 4, 3)$

Note that due to the results of M.G. Krein and V.E. Adler [1] such partitions can be characterised by the property that the corresponding Hermite Wronskian has no real roots (and thus conjecturally has only simple roots).

The harmonic locus is related to the so-called *monster potentials*

$$V(x) = \frac{L}{x^2} + x^{2\alpha} - 2D^2 \sum_{k=1}^{n} \log(x^{2\alpha+2} - z_k)$$

introduced by Bazhanov, Lukyanov and Zamolodchikov [6]. When $\alpha = 1$ and $L = m(m+1)$ they coincide with the simple locus potentials $u_\lambda(x)$

corresponding to the symmetric Young diagrams $\lambda = \lambda^*$. There are several outstanding conjectures about the number of monster potentials, which are open even in this case (see [12] for some recent results in this direction). Our results might be useful in this setting as well.

6. Acknowledgements

APV is grateful to FIM, ETH Zurich for the hospitality in April 2024 when this work was completed.

References

[1] V.E. Adler *A modification of Crum's method.* Theor. Math. Phys. **101 (3)** (1994), 1381-87.

[2] M. Adler, J. Moser *On a class of polynomials connected with the Korteweg-de Vries equation.* Commun. Math. Phys., **61**, 1-30, (1978).

[3] S. Ahmed, M. Bruschi, F. Calogero, M.A. Olshanetsky and A.M. Perelomov *Properties of the zeros of the classical polynomials and of the Bessel functions.* Il Nuovo Cimento **49(2)** (1979), 173-199.

[4] H. Airault, H.P. McKean and J. Moser *Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem.* Comm. Pure Appl. Math. **30** (1977), 95–178.

[5] P. Appell *Sur une classe de polynomes.* Annales Sci. École Norm. Supér. 2e Sèrie. **9** (1880), 119-144.

[6] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov *Higher level eigenvalues of Q operators and Schrödinger equation.* Adv. Theor. Math. Phys. **7** (2003), 711.

[7] N. Bonneux, Z. Hamker, J. Stembridge, M. Stevens *Wronskian Appell polynomials and symmetric functions.* Advances in Applied Math. **111** (2019), 101932.

[8] J.L. Burchnall, T.W. Chaundy *A set of differential equations which can be solved by polynomials*, Proc. London Math. Soc. **30** (1929-30), 401–414.

[9] F. Calogero *Solution of the one-dimensional n-body problems with quadratic and/or inversely quadratic pair potentials.* J. Math. Phys. **12** (1971), 419-436.

[10] F. Calogero *Equilibrium configuration of the one-dimensional n-body problem with quadratic and inversely quadratic pair potentials.* Lett. Nuovo Cimento, **20** (1978), 251.

[11] F. Calogero *Disproof of the conjecture.* Lett. Al Nuovo Cimento, **35(6)** (1982), 181-185.

[12] R. Conti, D. Masoero *Counting monster potentials.* J High Energy Physics, JHEP02 (2021), 059. ArXiv ePrint: 2009.14638.

[13] J.J.Duistermaat, F.A.Grünbaum *Differential equations with the spectral parameter.* Comm. Math. Phys. **103** (1986), 177-240.

[14] A.J. Duran *A proof of the Veselov Conjecture for segments.* Proc. Amer. Math. Soc. **149** (2021), 173-188.

[15] G. Frobenius *Über die Charaktere der symmetrischen Gruppe.* Sitzungsberichte der Akademie der Wiss. zu Berlin, pages 516–534, 1900.

[16] C. Grosu, C. Grosu *The irreducibility of some Wronskian Hermite polynomials.* Indagationes Mathematicae **32:2** (2021), 456–497.

[17] G. Felder, A.D. Hemery, A.P. Veselov *Zeros of Wronskians of Hermite polynomials and Young diagrams*, Physica D **241** (2012), 2131-2137.

[18] D. Kazhdan, B. Kostant, S. Sternberg *Hamiltonian group actions and dynamical systems of Calogero type.* Comm. Pure Appl. Math. **31** (1978), 481-507.

[19] I.G. Macdonald *Symmetric functions and Hall polynomials.* Oxford Univ. Press, 1995.

[20] J. Moser *Three integrable Hamiltonian systems connected with isospectral deformations.* Advances in Math. 16 (1975), 197–220.

[21] A.A. Oblomkov *Monodromy-free Schrödinger operators with quadratically increasing potentials.* Theor. Math. Phys., **121** (1999), 1574-1584.

[22] M.A. Olshanetsky, A.M. Perelomov *Classical integrable finite-dimensional systems related to Lie algebras.* Physics Reports **71:5** (1981), 313-400.

[23] A.M. Perelomov, Preprint ITEP, N27, 1976.

[24] A.M. Perelomov *Equilibrium configurations and small oscillations of some dynamical systems.* Annales de l'I. H. P., section A, tome 28, no 4 (1978), p. 407-415.

[25] G. Szegő *Orthogonal Polynomials.* American Mathematical Society, New York (1939).

[26] A.P. Veselov *On Stieltjes relations, Painlevè-IV hierarchy and complex monodromy.* J. Phys. A 34 (2001), no. 16, 3511–3519.

[27] G. Wilson *Collisions of Calogero–Moser particles and an adelic Grassmannian (With an Appendix by I.G. Macdonald).* Invent. Math. **133** (1998), 1-41.

Department of Mathematics, ETH Zurich, Switzerland
Email address: giovanni.felder@math.ethz.ch

Department of Mathematical Sciences, Loughborough University, Loughborough, UK
Email address: A.P.Veselov@lboro.ac.uk