arXiv:2404.18623v1 [math.CV] 29 Apr 2024

THE BOHR-TYPE INEQUALITIES FOR HOLOMORPHIC FUNCTIONS WITH LACUNARY SERIES IN COMPLEX BANACH SPACE

SHANKEY KUMAR, SAMINATHAN PONNUSAMY, AND G. BROCK WILLIAMS

Abstract. In this paper, we study the Bohr inequality with lacunary series to the single valued (resp. vector-valued) holomorphic function defined in unit ball of finite dimensional Banach sequence space. Also, we extend the Bohr inequality with an alternating series to the higher-dimensional space.

1. Introduction and Preliminaries

1.1. **The classical Bohr inequality for the class $\mathcal{B}$.** We denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the unit disk in the complex plane. Let $\mathcal{H}$ be the class of all holomorphic functions defined on $\mathbb{D}$. We set $\mathcal{B} = \{f \in \mathcal{H} : |f(z)| \leq 1\}$. Let us first recall a remarkable result of Bohr [8] that opens up a new avenue for research in geometric function theory.

Theorem A. *Let $f \in \mathcal{B}$ be of the form $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then the Bohr sum $B_f(r)$ satisfies the inequality*

$$B_f(r) := \sum_{k=0}^{\infty} |a_k| r^k \leq 1 \quad for \quad |z| = r \leq \frac{1}{3}$$

and the constant $1/3$ is best possible.

Originally, Bohr obtained this inequality only for $r \leq 1/6$, but later M. Riesz, I. Schur and F. W. Wiener independently proved that it holds in this form, and the number $1/3$ is called the Bohr radius. Several other proofs are also known in the literature. Note that there is no extremal function such that the Bohr radius is precisely $1/3$. See [18, Corollary 8.26]. The Bohr inequality contemplates many generalizations and applications. In 1995, Dixon [16] used the Bohr inequality in connection with the long-standing open problem of characterizing Banach algebras satisfying the von Neumann inequality. In addition, Theorem A was extended to alternating series $A_f(r) = \sum_{k=0}^{\infty} (-1)^k |a_k| r^k$, by Ali et al. (cf. [3]). More precisely they have shown that $|A_f(r)| \leq 1$ for $r \leq 1/\sqrt{3}$ under the assumption of Theorem A. Another natural question was to discuss the asymptotic behaviour of the Bohr sum $B_f(r)$. This has led to the search for the best constant $C(r) \geq 1$ such that $B_f(r) \leq C(r)$. Bombieri [10] showed that if $f \in \mathcal{B}$, then

$$B_f(r) \leq \frac{3 - \sqrt{8(1 - r^2)}}{r} \quad \text{for} \quad \frac{1}{3} \leq r \leq \frac{1}{\sqrt{2}}.$$

2020 *Mathematics Subject Classification.* Primary: 32A05, 32A22; Secondary: 46B20, 46G25, 46E50.
Key words and phrases. Bohr radius, Mixed Bohr radius, Banach spaces, Power series, Homogeneous polynomials.

File: KumarSamyBrock2˙2023˙-LucunarySeries.tex, printed: 30-4-2024, 2.46.

For an alternate proof of this inequality, we refer to the recent paper [23]. Moreover, the question raised by Djakov and Ramanujan [17] about p-Bohr radius was answered affirmatively in [23]. However, in the year 2004, Bombieri and Bourgain [11] established that $B_f(r) < \frac{1}{\sqrt{1-r^2}}$ holds for $r > 1/\sqrt{2}$ which in turn implies that $C(r) \asymp (1-r^2)^{-1/2}$ as $r \to 1$. In the same article, the authors proved that for a given $\varepsilon > 0$ there exists a constant c depending on ε such that

$$B_f(r) \geq (1-r^2)^{-1/2} - \left(c \log \frac{1}{1-r}\right)^{3/2+\varepsilon} \quad \text{as } r \to 1.$$

Some recent results on this topic including refinements and generalizations may be found from [4, 21, 23, 24, 26, 30–32].

1.2. **Multi-dimensional Bohr's inequality.** In the recent years, many authors paid attention to multidimensional generalizations of Bohr's theorem and drew many conclusions. For example, denote an n-variables power series by $\sum_{\alpha} a_\alpha z^\alpha$ with the standard multi-index notation; $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N} := \{1, 2, \ldots\}$ $(1 \leq j \leq n)$, $|\alpha|$ denotes the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ of its components, $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$, $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, and $z^\alpha = z^{\alpha_1} z^{\alpha_2} \cdots z^{\alpha_n}$. The n-dimensional Bohr radius K_n is the largest number such that if $\sum_{\alpha} a_\alpha z^\alpha$ converges in the n-dimensional unit polydisk $\mathbb{D}^n$ such that $\left|\sum_{\alpha} a_\alpha z^\alpha\right| < 1$ in $\mathbb{D}^n$, then $\sum_{\alpha} |a_\alpha z^\alpha| \leq 1$ for $\max_{1 \leq j \leq k} |z_j| \leq K_n$. In 1997, Boas and Khavinson [7] showed that for $n > 1$, the $n-$dimensional Bohr radius K_n satisfies

$$\frac{1}{3\sqrt{n}} < K_n < 2\sqrt{\frac{\log n}{n}}.$$

This article became a source of inspiration for many subsequent investigations including connecting the asymptotic behaviour of K_n to problems in the geometry of Banach spaces (cf. [14]). However determining the exact value of the Bohr radius K_n, $n > 1$, remains an open problem. In 2006, Defant and Frerick [12] improved the lower bound as $K_n \geq c\sqrt{\log n/(n \log \log n)}$ whereas Defant et al. [13] used the hypercontractivity of the polynomial Bohnenblust-Hille inequality and showed that

$$K_n = b_n \sqrt{\frac{\log n}{n}} \quad \text{with} \quad \frac{1}{\sqrt{2}} + o(1) \leq b_n \leq 2.$$

In 2014, Bayart et al. [5] established the asymptotic behaviour of K_n by showing that

$$\lim_{n \to \infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1.$$

We would like to mention that Djakov and Ramanujan [17], and Blasco [9] have studied the asymptotic behavior of the holomorphic functions with p-norm as $r \to 1$ in $\mathbb{D}^n$ and Banach spaces. Aizenberg [1, 2] mainly generalized Carathéodory's inequality for functions holomorphic in $\mathbb{C}^n$. In 2021, Liu and Ponnusamy [27] have established several multidimensional analogues of refined Bohr's inequality for holomorphic functions on complete circular domain in $\mathbb{C}^n$. Other aspects and promotion of Bohr inequality in higher dimensions can be obtained from [6, 15, 19, 20, 29]. Moreover, research on Dirichlet series in higher dimensions is also very popular recently (see [14]).

1.3. **Generalizations and Refinements of Bohr's inequality for the disk.** Recently, Kayumov and Ponnusamy [21], and Ponnusamy et al. [30] established several refined versions and improved versions of Bohr's inequality in the planer case. See also [30–32].

Theorem B.([27,30]) *For $f \in \mathcal{B}$, and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we have*

$$\sum_{k=1}^{\infty} |a_k| r^k + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leq \frac{r}{1-r}(1-|a_0|^2).$$

In the planar case, generalizations of Theorem B are abundant, but they are still limited in the higher-dimensional space. In fact, the alternating series version of this result is contained in the following result which is indeed itself a special case of [24, Theorem 5(I)].

Theorem C. *Suppose that $m \in \mathbb{N}_0$, $p \in \mathbb{N}$ and $0 \leq m \leq p$. Let $f \in \mathcal{B}$, and $f(z) = \sum_{k=0}^{\infty} a_{kp+m} z^{kp+m}$. If p is odd, then*

$$\left| \sum_{k=1}^{\infty} (-1)^{kp+m} |a_{kp+m}| r^{kp+m} + (-1)^{m+p} \frac{r^{p+m}}{1-r^{2p}} \sum_{k=0}^{\infty} |a_{kp+m}|^2 r^{2kp} \right| \leq 1$$

holds for $|z| = r \leq r_*$, where r_* is the unique root in $(0,1)$ of $r^p(r^p + r^m) - 1 = 0$. This result is sharp.

1.4. **Bohr radius in higher dimensional setting.** In 2019, Liu and Liu [28] used the Fréchet derivative to establish the Bohr inequality of norm type for holomorphic mappings with lacunary series on the unit polydisk in $\mathbb{C}^n$ under some restricted conditions. The relevant properties of the Fréchet derivative can be seen below (cf. [19]).

Let X and Y be two complex Banach spaces with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. For convenience, we denote both norms by $\|\cdot\|$, when there is no possibility of confusion with the underlying Banach spaces. We set $\mathbb{B}_X := \{x \in X : \|x\| < 1\}$. Let Ω^* be a domain in X, and let $H(\Omega^*, Y)$ denote the set of all holomorphic mappings from Ω^* into Y. It is well-known (cf. [19]) that if $f \in H(\Omega^*, Y)$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(0)(x^k)$$

for all x in some neighborhood of $0 \in \Omega^*$, where $D^k f(x)$ is the k^{th}-Fréchet derivative of f at x, and for $k \geq 1$, we have

$$D^k f(0)(x^k) = D^k f(0)(\underbrace{x, x, \ldots, x}_{k}).$$

Moreover, if $k = 0$, then $D^0 f(0)(x^0) = f(0)$.

2. Key lemmas and their Proofs

In order to establish our main results, we need the following lemmas which play a key role in proving the subsequent results in Section 3. The above theorem has been generalized in [24, Lemma 4].

Lemma D. *Suppose that $m \in \mathbb{N}_0$, $p \in \mathbb{N}$ and $0 \leq m \leq p$. If $f \in \mathcal{B}$ and $f(z) = \sum_{k=0}^{\infty} a_{pk+m} z^{pk+m}$, then we have*

$$\sum_{k=1}^{\infty} |a_{(2k-1)p+m}| r^{(2k-1)p} + \frac{r^{2p}}{1-r^{2p}} \sum_{k=0}^{\infty} |a_{kp+m}|^2 r^{(2k-1)p} \leq \frac{r^p}{1-r^{2p}} \tag{2.1}$$

for $r \in [0,1)$. This result is sharp. Moreover,

$$\sum_{k=1}^{\infty} |a_{2kp+m}| r^{2kp} + \left(\frac{1}{1+|a_m|} + \frac{r^{2p}}{1-r^{2p}} \right) \sum_{k=1}^{\infty} |a_{kp+m}|^2 r^{2kp} \leq (1-|a_m|^2) \frac{r^{2p}}{1-r^{2p}} \tag{2.2}$$

holds for $r \in [0,1)$. This result is sharp for $f(z) = z^m \left(\frac{a-z^p}{1-az^p} \right)$ with $a \in [0,1)$.

Let $n \in \mathbb{N}$, $t \in [1, \infty)$, and $B_{\ell_t^n}$ be the set defined as the collection of complex vectors $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ satisfying $\sum_{i=1}^{n} |z_i|^t < 1$. This set constitutes the open unit ball in the complex Banach space ℓ_t^n where the norm $\|z\|_t$ of z is given by $\left(\sum_{i=1}^{n} |z_i|^t \right)^{1/t}$. In the special case of $B_{\ell_\infty^n}$, the set represents the unit polydisk in $\mathbb{C}^n$ denoted as $B_{\ell_\infty^n} := \mathbb{D}^n$, where $|z_i| < 1$ for $1 \leq i \leq n$. The norm of $z \in \ell_\infty^n$ is defined as $\|z\| := \max\{|z_i| : 1 \leq i \leq n\}$. Note that the unit disk $\mathbb{D}$ is equivalent to $B_{\ell_t^1}$.

Lemma 2.1. *Suppose that $m, p \in \mathbb{N}$, $1 \leq m \leq p$, $1 \leq t \leq \infty$, $f \in H(B_{\ell_t^n}, \overline{\mathbb{D}}^n)$ and*

$$f(z) = zg(z) = \frac{D^m f(0)(z^m)}{m!} + \sum_{k=1}^{\infty} \frac{D^{kp+m} f(0)(z^{kp+m})}{(kp+m)!},$$

where $g \in H(B_{\ell_t^n}, \mathbb{C})$. Then

$$\sum_{k=1}^{\infty} \frac{\left\| D^{2kp+m} f(0)(z^{2kp+m}) \right\|_\infty}{(2kp+m)!} \leq \frac{\|z\|_t^{2p-m}}{1-\|z\|_t^{2p}} \left[\|z\|_t^{2m} - \left(\frac{\|D^m f(0)(z^m)\|_\infty}{m!} \right)^2 \right]$$

holds for all $\|z\|_t = r \in [0,1)$.

Proof. The proof of this lemma follows if we adopt the same approach as [25, Lemma 2.1]. To do this, we fix $z \in B_{\ell_t^n} \backslash \{0\}$, and let $z_0 = \frac{z}{\|z\|_t}$. Then $z_0 \in \partial B_{\ell_t^n}$. Define j such that $|z_j| = \|z\|_\infty = \max_{1 \leq l \leq n} \{|z_l|\}$. Because $f(z) = zg(z)$, a simple calculation yields

$$\frac{D^k f_j(0)(z_0^k)}{k!} = \frac{D^{k-1} g(0)(z_0^{k-1})}{(k-1)!} \frac{z_j}{\|z\|_t} \quad \text{for each } k \in \mathbb{N}.$$

For $\lambda \in \mathbb{D}$, let $h_j(\lambda) = f_j(\lambda z_0)$. Then $h_j \in H(\mathbb{D}, \overline{\mathbb{D}})$ and

$$\begin{aligned} h_j(\lambda) &= \lambda^m \left[\frac{D^m f_j(0)(z_0^m)}{m!} + \sum_{k=1}^{\infty} \frac{D^{kp+m} f_j(0)(z_0^{kp+m})}{(kp+m)!} \lambda^{kp} \right] \\ &= \lambda^m \left[\frac{D^{m-1} g(0)(z_0^{m-1})}{(m-1)!} \frac{z_j}{\|z\|_t} + \sum_{k=1}^{\infty} \frac{D^{kp+m-1} g(0)(z_0^{kp+m-1})}{(kp+m-1)!} \frac{z_j}{\|z\|_t} \lambda^{kp} \right] \\ &=: \lambda^m \varphi(\lambda^p), \end{aligned}$$

where $\varphi(t) = b_0 + \sum_{k=1}^{\infty} b_k t^k$ with

$$b_0 = \frac{D^{m-1}g(0)(z_0^{m-1})}{(m-1)!}\frac{z_j}{\|z\|_t} \quad \text{and} \quad b_k = \frac{D^{kp+m-1}g(0)(z_0^{kp+m-1})}{(kp+m-1)!}\frac{z_j}{\|z\|_t}.$$

Note that $\varphi \in H(\mathbb{D}, \overline{\mathbb{D}})$ and so, $|b_0| \leq 1$. By Wiener's inequality, it follows that $|b_k| \leq 1 - |b_0|^2$ for all $k \geq 1$. As $|z_j| = \|z\|_\infty = \max_{1\leq l\leq n}\{|z_l|\}$, we have

$$\frac{|D^m f_l(0)(z_0^m)|}{m!} = \frac{|D^{m-1}g(0)(z_0^{m-1})|}{(m-1)!}\frac{|z_l|}{\|z\|_t} \leq \frac{|D^{m-1}g(0)(z_0^{m-1})|}{(m-1)!}\frac{|z_j|}{\|z\|_t} \leq 1$$

for all $l = 1, 2, \ldots, n$, so that $\frac{\|D^m f(0)(z_0^m)\|_\infty}{m!} \leq 1$. Moreover, we have

$$\frac{|D^{kp+m-1}g(0)(z_0^{kp+m-1})|}{(kp+m-1)!}\frac{|z_j|}{\|z\|_t} \leq 1 - \left(\frac{|D^{m-1}g(0)(z_0^{m-1})|}{(m-1)!}\frac{|z_j|}{\|z\|_t}\right)^2 \tag{2.3}$$

for all $k \in \mathbb{N}_0$ and $z_0 \in \partial B_{\ell_t^n}$.

On the other hand, if $z_0 \in \partial B_{\ell_t^n}$, then, for all $l = 1, 2, \ldots, n$, we also have

$$\frac{|D^k f_l(0)(z_0^k)|}{k!} = \frac{|D^{k-1}g(0)(z_0^{k-1})|}{(k-1)!}\frac{|z_l|}{\|z\|_t} \quad \text{for all } k \in \mathbb{N}. \tag{2.4}$$

Combining (2.3) and (2.4), we find that

$$\frac{|D^{kp+m} f_l(0)(z_0^{kp+m})|}{(kp+m)!} \leq 1 - \left(\frac{|D^m f_j(0)(z_0^m)|}{m!}\right)^2,$$

where $z_0 \in \partial B_{\ell_t^n}$, $l = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$; that is,

$$\frac{\|D^{kp+m} f(0)(z_0^{kp+m})\|_\infty}{(kp+m)!} \leq 1 - \left(\frac{\|D^m f(0)(z_0^m)\|_\infty}{m!}\right)^2$$

holds for $z_0 \in \partial B_{\ell_t^n}$ and all $k \geq 0$.

Because $z = z_0\|z\|$, by routine calculations, we obtain that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\left\|D^{2kp+m} f(0)(z^{2kp+m})\right\|_\infty}{(2kp+m)!} &= \sum_{k=1}^{\infty} \frac{\left\|D^{2kp+m} f(0)(z_0^{2kp+m})\right\|_\infty}{(2kp+m)!}\|z\|_t^{2kp+m} \\
&\leq \left[1 - \left(\frac{\|D^m f(0)(z_0^m)\|_\infty}{m!}\right)^2\right]\sum_{k=1}^{\infty}\|z\|_t^{2kp+m} \\
&= \frac{\|z\|_t^{2p-m}}{1-\|z\|_t^{2p}}\left[\|z\|_t^{2m} - \left(\frac{\|D^m f(0)(z^m)\|_\infty}{m!}\right)^2\right]
\end{aligned}$$

for all $\|z\|_t = r \in [0, 1)$. This completes the proof. □

For holomorphic mappings from $B_{\ell_t^n}$, $1 \leq t \leq \infty$, to $\overline{\mathbb{D}}$, using the method of proof as in Lemma 2.1, we can easily derive the following, so we omit its proof.

Lemma 2.2. *Suppose that $m \in \mathbb{N}_0$, $p \in \mathbb{N}$, $0 \leq m \leq p$, and $1 \leq t \leq \infty$. If $f \in H(B_{\ell_t^n}, \overline{\mathbb{D}})$ and*

$$f(z) = \frac{D^m f(0)(z^m)}{m!} + \sum_{k=1}^{\infty} \frac{D^{kp+m} f(0)(z^{kp+m})}{(kp+m)!},$$

then

$$\sum_{k=1}^{\infty}\frac{\left|D^{2kp+m}f(0)(z^{2kp+m})\right|}{(2kp+m)!}\leq\frac{\|z\|_t^{2p-m}}{1-\|z\|_t^{2p}}\left[\|z\|_t^{2m}-\left(\frac{|D^m f(0)(z^m)|}{m!}\right)^2\right]$$

holds for all $\|z\|_t=r\in[0,1)$.

3. Bohr inequality for holomorphic mappings with lacunary series

In this section, we will use the Fréchet derivative to extend the Bohr inequality to higher dimensional space.

3.1. **Extension of Theorem B to the holomorphic mappings from $\mathbb{B}_X$ to $\overline{\mathbb{D}}$.**

Theorem 3.1. *Suppose that* $f\in H(\mathbb{B}_X,\overline{\mathbb{D}})$, $f(z)=\sum_{k=0}^{\infty}\frac{D^kf(0)(z^k)}{k!}$ *and* $p>0$. *Then*

$$|f(0)|^p+\sum_{k=1}^{\infty}\frac{\left|D^kf(0)(z^k)\right|}{k!}+\left(\frac{1}{1+|f(0)|}+\frac{\|z\|}{1-\|z\|}\right)\sum_{k=1}^{\infty}\left(\frac{|D^kf(0)(z^k)|}{k!}\right)^2\leq 1 \tag{3.1}$$

holds for all $\|z\|=r\leq r_p$, *where* $r_p=\frac{1-|f(0)|^p}{2-|f(0)|^2-|f(0)|^p}$. *The number* r_p *is best possible.*

Moreover, when $p=1$, (3.1) *holds for* $\|z\leq\frac{1}{2+|f(0)|}$, *and for* $p=2$, *it holds for* $\|z\|\leq 1/2$.

Proof. Fix $z\in\mathbb{B}_X\backslash\{0\}$ and set $z_0=\frac{z}{\|z\|}$. For $\lambda\in\mathbb{D}$, we define $h(\lambda)=f(\lambda z_0)$. Then $h\in H(\mathbb{D},\overline{\mathbb{D}})$ and

$$h(\lambda)=b_0+\sum_{k=1}^{\infty}b_k\lambda^k,$$

where

$$b_0=f(0)=\frac{D^0f(0)(z_0^0)}{0!}\quad\text{and}\quad b_k=\frac{D^kf(0)(z_0^k)}{k!}\ (k\geq 1).$$

By Theorem B, we know that

$$|b_0|^p+\sum_{k=1}^{\infty}|b_k|\,|\lambda|^k+\left(\frac{1}{1+|b_0|}+\frac{|\lambda|}{1-|\lambda|}\right)\sum_{k=1}^{\infty}|b_k|^2|\lambda|^{2k}\leq|b_0|^p+(1-|b_0|^2)\frac{|\lambda|}{1-|\lambda|}.$$

Set $\lambda=\|z\|=r$ and note that $z=z_0\|z\|$ and $|b_0|=|f(0)|\leq 1$. Then the last relation gives

$$\begin{aligned}
&|f(0)|^p+\sum_{k=1}^{\infty}\frac{\left|D^kf(0)(z^k)\right|}{k!}+\left(\frac{1}{1+|f(0)|}+\frac{\|z\|}{1-\|z\|}\right)\sum_{k=1}^{\infty}\left(\frac{|D^kf(0)(z^k)|}{k!}\right)^2\\
&=\ |f(0)|^p+\sum_{k=1}^{\infty}\frac{\left|D^kf(0)(z_0^k)\right|}{k!}\|z\|^k+\left(\frac{1}{1+|f(0)|}+\frac{\|z\|}{1-\|z\|}\right)\sum_{k=1}^{\infty}\left(\frac{|D^kf(0)(z_0^k)|}{k!}\right)^2\|z\|^{2k}\\
&=\ |b_0|^p+\sum_{k=1}^{\infty}|b_k||\lambda|^k+\left(\frac{1}{1+|b_0|}+\frac{|\lambda|}{1-|\lambda|}\right)\sum_{k=1}^{\infty}|b_k|^2|\lambda|^{2k}\\
&\leq\ |f(0)|^p+(1-|f(0)|^2)\frac{r}{1-r},
\end{aligned}$$

which is less than or equal to 1 provided $\|z\|=r\leq r_p$. Thus, (3.1) holds for $\|z\|=r\leq r_p$, where $r_p=\frac{1-|f(0)|^p}{2-|f(0)|^2-|f(0)|^p}$.

Finally, we prove that inequality (3.1) does not hold true for $x \in r_0\mathbb{B}_X$, where $r_0 \in (r_p, 1)$. We know that there exists a $c \in (0,1)$ and $v \in \partial\mathbb{B}_X$ such that $cr_0 > r_p$ and

$$c\sup\{||x|| : x \in \partial\mathbb{B}_X\} < ||v||.$$

Now, we consider a function f_1 on $\mathbb{B}_X$ defined by

$$f_1(x) = L_1\left(\frac{c\psi_v(x)}{||v||}\right),$$

where $L_1(z) = (a-z)/(1-az)$, $z \in \mathbb{D}$, $a \in [0,1)$, ψ_v is a bounded linear functional on X with $\psi_v(v) = ||v||$ and $||\psi_v|| = 1$. Choose $x = r_0 v$ and we obtain that

$$\begin{aligned}
|f_1(0)|^p &+ \sum_{k=1}^{\infty}\frac{\left|D^k f_1(0)(x^k)\right|}{k!} + \left(\frac{1}{1+|f_1(0)|} + \frac{\|x\|}{1-\|x\|}\right)\sum_{k=1}^{\infty}\left(\frac{|D^k f_1(0)(x^k)|}{k!}\right)^2 \\
&= a^p + \sum_{k=1}^{\infty}(1-a^2)a^{k-1}c^k r_0^k + \left(\frac{1}{1+a} + \frac{cr_0}{1-cr_0}\right)\sum_{k=1}^{\infty}(1-a^2)^2 a^{2k-2}c^{2k}r_0^{2k} \\
&= a^p + (1-a^2)\frac{cr_0}{1-cr_0} > 1
\end{aligned}$$

and the proof is complete. ☐

In the following theorem, we determine the Bohr inequality for holomorphic functions, which fix the origin, with a lacunary series.

Theorem 3.2. *Suppose that $m \in \mathbb{N}_0$, $p \in \mathbb{N}$, and $0 \le m \le p$. If $f \in H(\mathbb{B}_X, \overline{\mathbb{D}})$ and*

$$f(z) = \sum_{k=1}^{\infty}\frac{D^{kp+m}f(0)(z^{kp+m})}{(kp+m)!},$$

then

$$\sum_{k=1}^{\infty}\frac{\left|D^{kp+m}f(0)(z^{kp+m})\right|}{(kp+m)!} + \left(\frac{1}{\|z\|^{p+m}+\Lambda} + \frac{\|z\|^{-m}}{1-\|z\|^p}\right)\sum_{k=2}^{\infty}\left(\frac{|D^{kp+m}f(0)(z^{kp+m})|}{(kp+m)!}\right)^2 \le 1 \tag{3.2}$$

holds for all $0 < \|z\| = r \le \tilde{r}_{p,m}$, where $\Lambda = \frac{|D^{p+m}f(0)(z^{p+m})|}{(p+m)!}$, and $\tilde{r}_{p,m}$ is the unique root in $(0,1)$ of $G(r) = 0$, where

$$G(r) = 5r^{2p+m} - 2r^{p+m} + r^m + 4r^p - 4. \tag{3.3}$$

For each p and m, the number $\tilde{r}_{p,m}$ is best possible.

Proof. As with the proof of Theorem 3.1, we fix $z \in \mathbb{B}_X\backslash\{0\}$ and let $z_0 = \frac{z}{\|z\|}$. For $\lambda \in \mathbb{D}$, define $h(\lambda) = f(\lambda z_0)$. Then $h \in H(\mathbb{D}, \overline{\mathbb{D}})$ and

$$h(\lambda) = \lambda^m\sum_{k=1}^{\infty}\frac{D^{kp+m}f(0)(z_0^{kp+m})}{(kp+m)!}(\lambda^p)^k = \lambda^m\sum_{k=1}^{\infty}b_k(\lambda^p)^k =: \lambda^m\varphi(\lambda^p),$$

where $b_k = \frac{D^{kp+m}f(0)(z_0^{kp+m})}{(kp+m)!}$, $\varphi \in H(\mathbb{D}, \overline{\mathbb{D}})$ with

$$\varphi(\lambda) = \sum_{k=1}^{\infty}b_k\lambda^k = \lambda\sum_{k=1}^{\infty}b_k(\lambda)^{k-1} = \lambda\sum_{k=0}^{\infty}B_k\lambda^k,$$

and $B_k = b_{k+1}$. Clearly, $\varphi(\lambda^p) = \sum_{k=0}^{\infty} B_k(\lambda^p)^k \in H(\mathbb{D}, \overline{\mathbb{D}})$. Then, according to Theorem B, we have

$$\sum_{k=0}^{\infty} |B_k|\,|\lambda^p|^k + \left(\frac{1}{1+|B_0|} + \frac{|\lambda|^p}{1-|\lambda|^p}\right)\sum_{k=1}^{\infty} |B_k|^2|\lambda^p|^{2k} \le |B_0| + (1-|B_0|^2)\frac{|\lambda|^p}{1-|\lambda|^p},$$

which implies that

$$\sum_{k=0}^{\infty} \frac{|D^{(k+1)p+m}f(0)(z_0^{(k+1)p+m})|}{((k+1)p+m)!}|\lambda|^{pk} + \left(\frac{1}{1+\frac{|D^{p+m}f(0)z_0^{p+m}|}{(p+m)!}} + \frac{|\lambda|^p}{1-|\lambda|^p}\right) \times$$
$$\sum_{k=1}^{\infty} \left(\frac{|D^{(k+1)p+m}f(0)(z_0^{(k+1)p+m})|}{((k+1)p+m)!}\right)^2 |\lambda|^{2kp} \le |B_0| + (1-|B_0|^2)\frac{|\lambda|^p}{1-|\lambda|^p}.$$

Multiplying by $|\lambda|^{p+m}$ on both sides of the above inequality yields

$$\sum_{k=0}^{\infty} \frac{|D^{(k+1)p+m}f(0)(z_0^{(k+1)p+m})|}{((k+1)p+m)!}|\lambda|^{(k+1)p+m} + \left(\frac{|\lambda|^{2p+2m}}{|\lambda|^{p+m} + \frac{|D^{p+m}f(0)z_0^{p+m}|}{(p+m)!}|\lambda|^{p+m}} + \frac{|\lambda|^{2p+m}}{1-|\lambda|^p}\right) \times$$
$$\sum_{k=1}^{\infty} \left(\frac{|D^{(k+1)p+m}f(0)(z_0^{(k+1)p+m})|}{((k+1)p+m)!}\right)^2 |\lambda|^{2kp} \le |B_0|\,|\lambda|^{p+m} + (1-|B_0|^2)\frac{|\lambda|^{2p+m}}{1-|\lambda|^p}.$$

Taking $|\lambda| = \|z\|$, since $z = z_0\|z\|$, shows that

$$\sum_{k=0}^{\infty} \frac{|D^{(k+1)p+m}f(0)(z^{(k+1)p+m})|}{((k+1)p+m)!} + \left(\frac{1}{\|z\|^{p+m} + \frac{|D^{p+m}f(0)z^{p+m}|}{(p+m)!}} + \frac{\|z\|^{-m}}{1-\|z\|^p}\right) \times$$
$$\sum_{k=1}^{\infty} \left(\frac{|D^{(k+1)p+m}f(0)(z^{(k+1)p+m})|}{((k+1)p+m)!}\right)^2 \le |B_0|\,\|z\|^{p+m} + (1-|B_0|^2)\frac{\|z\|^{2p+m}}{1-\|z\|^p}.$$

It is already obtained in [25, Theorem 3.2] that

$$|B_0|\,\|z\|^{p+m} + (1-|B_0|^2)\frac{\|z\|^{2p+m}}{1-\|z\|^p} \le 1$$

for all $0 < \|z\| = r \le \tilde{r}_{p,m}$ and $\tilde{r}_{p,m}$ is the unique root in $(0,1)$ of $G(r) = 0$, where $G(r)$ is given by (3.3).

Finally, we prove that inequality (3.2) does not hold true for $x \in r_0\mathbb{B}_X$, where $r_0 \in (\tilde{r}_{p,m}, 1)$. We know that there exists a $c \in (0,1)$ and $v \in \partial\mathbb{B}_X$ such that $cr_0 > \tilde{r}_{p,m}$ and

$$c\sup\{||x|| : x \in \partial\mathbb{B}_X\} < ||v||.$$

Now, we consider a function f on $\mathbb{B}_X$ defined by

$$f(x) = L_2\left(\frac{c\psi_v(x)}{||v||}\right),$$

where

$$L_2(z) = z^{p+m}\left(\frac{a-z^p}{1-az^p}\right),\ z \in \mathbb{D} \text{ and } a \in [0,1),$$

ψ_v is a bounded linear functional on X with $\psi_v(v) = ||v||$ and $||\psi_v|| = 1$. Choosing $x = r_0 v$, we get

$$\Lambda = \frac{|D^{p+m} f(0)(x^{p+m})|}{(p+m)!} = a(cr_0)^{p+m}$$

and

$$\frac{|D^{kp+m} f(0)(x^{kp+m})|}{(kp+m)!} = a^{k-2}(1-a^2)(cr_0)^{kp+m}.$$

Thus, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\left|D^{kp+m} f(0)(x^{kp+m})\right|}{(kp+m)!} + \left(\frac{1}{\|x\|^{p+m}+\Lambda} + \frac{\|x\|^{-m}}{1-\|x\|^p}\right) \sum_{k=2}^{\infty} \left(\frac{|D^{kp+m} f(0)(x^{kp+m})|}{(kp+m)!}\right)^2 \\
&= a(cr_0)^{m+p} + (1-a^2)\frac{(cr_0)^{2p+m}}{1-a(cr_0)^p} + \frac{(1-a^2)(1-a)(cr_0)^{3p+m}}{(1-(cr_0)^p)(1-a(cr_0)^p)} \\
&= a(cr_0)^{p+m} + (1-a^2)\frac{(cr_0)^{2p+m}}{1-(cr_0)^p}. \qquad (3.4)
\end{aligned}$$

In the proof of [25, Theorem 3.2], it has been proved that

$$a(x)^{p+m} + (1-a^2)\frac{(x)^{2p+m}}{1-(x)^p} > 1,$$

for $x > \tilde{r}_{p,m}$. This completes the proof of the sharpness of the constant $\tilde{r}_{p,m}$. □

If $m = 0$, then (3.3) reduces to $(r^p+1)(5r^p-3) = 0$ and hence, $\tilde{r}_{p,0} = \sqrt[p]{3/5}$. In particular, the case $m = 0$ and $p = 1$ in Theorem 3.2, gives the following corollary.

Corollary 3.3. *Let $f \in H(\mathbb{B}_X, \overline{\mathbb{D}})$ and $f(z) = \sum_{k=1}^{\infty} \frac{D^k f(0)(z^k)}{k!}$. Then*

$$\sum_{k=1}^{\infty} \frac{|D^k f(0)(z^k)|}{k!} + \left(\frac{1}{\|z\| + |Df(0)(z)|} + \frac{1}{1-\|z\|}\right) \sum_{k=2}^{\infty} \left(\frac{|D^k f(0)(z^k)|}{k!}\right)^2 \le 1,$$

holds for all $0 < \|z\| = r \le 3/5$. The constant $3/5$ is best possible.

3.2. **Bohr type inequality for holomorphic functions with lacunary series from $B_{\ell_t^n}$ to $\mathbb{D}^n$.** The proof of the following theorem uses the method of proof of [25, Theorem 3.5], which will be also used to prove Theorems 4.1 and 4.4, respectively.

Theorem 3.4. *Suppose that $m, p \in \mathbb{N}$, $1 \le m \le p$, and $1 \le t \le \infty$. Let $f \in H(B_{\ell_t^n}, \overline{\mathbb{D}}^n)$ and*

$$f(z) = zg(z) = \frac{D^m f(0)(z^m)}{m!} + \sum_{k=1}^{\infty} \frac{D^{kp+m} f(0)(z^{kp+m})}{(kp+m)!},$$

where $g \in H(B_{\ell_t^n}, \mathbb{C})$. Then

$$\sum_{k=1}^{\infty} \frac{\left\|D^{(2k-1)p+m} f(0)(z^{(2k-1)p+m})\right\|_\infty}{((2k-1)p+m)!} + \frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}} \sum_{k=0}^{\infty} \left(\frac{\|D^{kp+m} f(0)(z^{kp+m})\|_\infty}{(kp+m)!}\right)^2 \le 1 \qquad (3.5)$$

holds for all $\|z\|_t = r \le r_{p,m}$, where $r_{p,m}$ is the unique root in $(0,1)$ of the equation $r^{p+m} + r^{2p} - 1 = 0$. For each p and m, the number $r_{p,m}$ is best possible.

Proof. Fix $z \in B_{\ell_t^n}\setminus\{0\}$, and set $z_0 = \frac{z}{\|z\|_t}$. Then $z_0 \in \partial B_{\ell_t^n}$. Let j be such that $|z_j| = \|z\|_\infty = \max_{1\le l\le n}\{|z_l|\}$. Because $f(z) = zg(z)$, simple calculations yield

$$\frac{D^k f_j(0)(z_0^k)}{k!} = \frac{D^{k-1}g(0)(z_0^{k-1})}{(k-1)!}\frac{z_j}{\|z\|_t} \quad \text{for all } k \in \mathbb{N}.$$

Let $h_j(\lambda) = f_j(\lambda z_0)$ for $\lambda \in \mathbb{D}$. Then $h_j \in H(\mathbb{D}, \overline{\mathbb{D}})$ and

$$\begin{aligned} h_j(\lambda) &= \lambda^m \left[\frac{D^m f_j(0)(z_0^m)}{m!} + \sum_{k=1}^{\infty} \frac{D^{kp+m} f_j(0)(z_0^{kp+m})}{(kp+m)!}\lambda^{kp}\right] \\ &= \lambda^m \left[\frac{D^{m-1}g(0)(z_0^{m-1})}{(m-1)!}\frac{z_j}{\|z\|_t} + \sum_{k=1}^{\infty} \frac{D^{kp+m-1}g(0)(z_0^{kp+m-1})}{(kp+m-1)!}\frac{z_j}{\|z\|_t}\lambda^{kp}\right] \\ &= b_0\lambda^m + \sum_{k=1}^{\infty} b_k \lambda^{kp+m}, \end{aligned}$$

where

$$b_0 = \frac{D^{m-1}g(0)(z_0^{m-1})}{(m-1)!}\frac{z_j}{\|z\|_t} \quad \text{and} \quad b_k = \frac{D^{kp+m-1}g(0)(z_0^{kp+m-1})}{(kp+m-1)!}\frac{z_j}{\|z\|_t} \quad \text{for } k \ge 1.$$

By equation (2.1) in Lemma D, and

$$\frac{|\lambda|^{2p}}{1-|\lambda|^{2p}}\sum_{k=0}^{\infty}|b_k|^2|\lambda|^{(2k-1)p} = \frac{|\lambda|^{p}}{1-|\lambda|^{2p}}\sum_{k=0}^{\infty}|b_k|^2|\lambda|^{2kp},$$

we have

$$\sum_{k=1}^{\infty}|b_{2k-1}|\,|\lambda|^{(2k-1)p} + \frac{|\lambda|^{p}}{1-|\lambda|^{2p}}\sum_{k=0}^{\infty}|b_k|^2|\lambda|^{2kp} \le \frac{|\lambda|^{p}}{1-|\lambda|^{2p}}. \tag{3.6}$$

As $|z_j| = \|z\|_\infty = \max_{1\le l\le n}\{|z_l|\}$, substituting the expression of b_k ($k \ge 0$) into (3.6), we have

$$\begin{aligned} &\sum_{k=1}^{\infty}\frac{|D^{(2k-1)p+m-1}g(0)(z_0^{(2k-1)p+m-1})|}{((2k-1)p+m-1)!}\frac{|z_j|}{\|z\|_t}|\lambda|^{(2k-1)p} \\ &+\frac{|\lambda|^{p}}{1-|\lambda|^{2p}}\sum_{k=0}^{\infty}\left(\frac{|D^{kp+m-1}g(0)(z_0^{kp+m-1})|}{(kp+m-1)!}\frac{|z_j|}{\|z\|_t}\right)^2|\lambda|^{2kp} \le \frac{|\lambda|^{p}}{1-|\lambda|^{2p}} \end{aligned} \tag{3.7}$$

for all $k \in \mathbb{N}_0$ and $z_0 \in \partial B_{\ell_t^n}$.

Moreover, if $z_0 \in \partial B_{\ell_t^n}$, we also have

$$\frac{|D^k f_l(0)(z_0^k)|}{k!} = \frac{|D^{k-1}g(0)(z_0^{k-1})|}{(k-1)!}\frac{|z_l|}{\|z\|_t} \le \frac{|D^{k-1}g(0)(z_0^{k-1})|}{(k-1)!}\frac{|z_j|}{\|z\|_t}, \quad l = 1, 2, \ldots, n. \tag{3.8}$$

Combining (3.7) and (3.8), for $z_0 \in \partial B_{\ell_t^n}$, we have

$$\tag{3.9} \begin{aligned} &\sum_{k=1}^{\infty} \frac{|D^{(2k-1)p+m} f_{l_k}(0)(z_0^{(2k-1)p+m})|}{((2k-1)p+m)!}|\lambda|^{(2k-1)p} \\ &\quad + \frac{|\lambda|^p}{1-|\lambda|^{2p}} \sum_{k=0}^{\infty} \left(\frac{|D^{kp+m} f_{j_k}(0)(z_0^{kp+m})|}{(kp+m)!} \right)^2 |\lambda|^{2kp} \leq \frac{|\lambda|^p}{1-|\lambda|^{2p}} \end{aligned}$$

for all $k \in \mathbb{N}_0$ and where $l_k = 1, 2, \ldots, n$ and $j_k = 1, 2, \ldots, n$.

Multiplying both sides of (3.9) by $|\lambda|^m$ and setting $|\lambda| = \|z\|_t$ so that $z = z_0\|z\|_t$, we have

$$\sum_{k=1}^{\infty} \frac{|D^{(2k-1)p+m} f_{l_k}(0)(z^{(2k-1)p+m})|}{((2k-1)p+m)!} + \frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}} \sum_{k=0}^{\infty} \left(\frac{|D^{kp+m} f_{j_k}(0)(z^{kp+m})|}{(kp+m)!} \right)^2 \leq \frac{\|z\|_t^{p+m}}{1-\|z\|_t^{2p}}$$

for $z \in \overline{B_{\ell_t^n}}$, and $l_k = 1, 2, \ldots, n$ and $j_k = 1, 2, \ldots, n$. In other words,

$$\sum_{k=1}^{\infty} \frac{\left\|D^{(2k-1)p+m} f(0)(z^{(2k-1)p+m})\right\|_\infty}{((2k-1)p+m)!} + \frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}} \sum_{k=0}^{\infty} \left(\frac{\|D^{kp+m} f(0)(z^{kp+m})\|_\infty}{(kp+m)!} \right)^2 \leq \frac{\|z\|_t^{p+m}}{1-\|z\|_t^{2p}}$$

which is less than or equal to 1 provided $r^{p+m} + r^{2p} - 1 \leq 0$, where $r = \|z\|_t$. The desired conclusion (3.5) follows for $\|z\|_t = r \leq r_{p,m}$, where $r_{p,m}$ is as in the statement.

To prove the sharpness, we just consider the functions g and $f = zg \in H(B_{\ell_t^n}, \overline{\mathbb{D}}^n)$ given by

$$g(z) = z_1^{m-1} \frac{a - z_1^p}{1 - a z_1^p} \quad \text{and} \quad f(z) = \left(z_1^m \frac{a - z_1^p}{1 - a z_1^p}, z_2 z_1^{m-1} \frac{a - z_1^p}{1 - a z_1^p}, \ldots, z_n z_1^{m-1} \frac{a - z_1^p}{1 - a z_1^p} \right)',$$

where $z = (z_1, z_2, \ldots, z_n)'$ and $a \in [0, 1)$. In this case, let $z = (z_1, 0, \ldots, 0)'$, which implies that $\|z\|_t = |z_1| = r$, and according to the definition of Fréchet derivative, we have

$$Df(0)(z) = \left(\frac{\partial f_l(0)}{\partial z_j} \right)_{1 \leq l, j \leq n} (z_1, z_2, \ldots, z_n)'.$$

Because $z = (z_1, 0, \ldots, 0)'$, we have $Df(0)(z) = \left(\frac{\partial f_1(0)}{\partial z_1} z_1, 0, \ldots, 0 \right)$, and therefore,

$$\|Df(0)(z)\| = \left| \frac{\partial f_1(0)}{\partial z_1} z_1 \right|.$$

With the help of the proof of Theorem [25, Theorem 3.5], we obtain

$$\frac{\|D^{kp+m} f(0)(z^{kp+m})\|_\infty}{(kp+m)!} = \left| \frac{\partial^{kp+m} f_1(0)}{\partial z_1^{kp+m}} \frac{z_1^{kp+m}}{(kp+m)!} \right| \quad \text{for } k \geq 0.$$

Therefore, we compute that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\left\|D^{(2k-1)p+m} f(0)(z^{(2k-1)p+m})\right\|_\infty}{((2k-1)p+m)!} + \frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}} \sum_{k=0}^{\infty} \left(\frac{\|D^{kp+m} f(0)(z^{kp+m})\|_\infty}{(kp+m)!} \right)^2 \\
&= \sum_{k=1}^{\infty} (1-a^2) a^{2(k-1)} r^{(2k-1)p+m} + \frac{r^{p-m}}{1-r^{2p}} a^2 r^{2m} + \frac{r^{p-m}}{1-r^{2p}} \sum_{k=1}^{\infty} (1-a^2)^2 a^{2(k-1)} r^{2kp+2m} \\
&= (1-a^2) r^{p+m} \left[\frac{(1-r^{2p}) - (1-a^2 r^{2p}) + (1-a^2) r^{2p}}{(1-r^{2p})(1-a^2 r^{2p})} \right] + \frac{r^{p+m}}{1-r^{2p}} = \frac{r^{p+m}}{1-r^{2p}},
\end{aligned}$$

which is clearly bigger than 1 whenever $r > r_{p,m}$. This completes the proof. $\square$

Using the method of proof Theorems 3.1 and 3.4, we may verify the proof of the following, and so we omit its proof.

Theorem 3.5. *Suppose that $m \in \mathbb{N}_0$, $p \in \mathbb{N}$, $0 \le m \le p$, and $1 \le t \le \infty$. If $f \in H(B_{\ell_t^n}, \overline{\mathbb{D}})$ and*

$$f(z) = \frac{D^m f(0)(z^m)}{m!} + \sum_{k=1}^{\infty} \frac{D^{kp+m} f(0)(z^{kp+m})}{(kp+m)!},$$

then

$$\sum_{k=1}^{\infty} \frac{\left|D^{(2k-1)p+m} f(0)(z^{(2k-1)p+m})\right|}{((2k-1)p+m)!} + \frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}} \sum_{k=0}^{\infty} \left(\frac{|D^{kp+m} f(0)(z^{kp+m})|}{(kp+m)!} \right)^2 \le 1 \tag{3.10}$$

holds for all $\|z\|_t = r \le r_{p,m}$, where $r_{p,m}$ is the same as in Theorem 3.4. The constant $r_{p,m}$ is best possible for each p and m.

4. Bohr inequality for holomorphic mappings with alternating series

In this section, we will use the Fréchet derivative to extend the Bohr inequality with alternating series to the higher-dimensional space, and obtain the higher-dimensional generalizations of Theorem C.

Theorem 4.1. *Suppose that $m, p \in \mathbb{N}$, p is odd, $1 \le m \le p$, and $1 \le t \le \infty$. If $f \in H(B_{\ell_t^n}, \overline{\mathbb{D}}^n)$ and*

$$f(z) = zg(z) = \frac{D^m f(0)(z^m)}{m!} + \sum_{k=1}^{\infty} \frac{D^{kp+m} f(0)(z^{kp+m})}{(kp+m)!},$$

where $g \in H(B_{\ell_t^n}, \mathbb{C})$, then

$$\begin{aligned}
&\Bigg| \sum_{k=1}^{\infty} (-1)^{kp+m} \frac{\left\|D^{kp+m} f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!} \\
&\qquad + (-1)^{m+p} \frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}} \sum_{k=0}^{\infty} \left(\frac{\|D^{kp+m} f(0)(z^{kp+m})\|_\infty}{(kp+m)!} \right)^2 \Bigg| \le 1
\end{aligned} \tag{4.1}$$

holds for all $\|z\|_t = r \le r_{p,m}$, where $r_{p,m}$ is the same as in Theorem 3.4. The constant $r_{p,m}$ is best possible for each p and m.

Proof. (i) Assume first that p is odd and m is even. Then $(-1)^{m+p}=-1$, and

$$\sum_{k=1}^{\infty}(-1)^{kp+m}\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!}=\sum_{k=1}^{\infty}\frac{\left\|D^{2kp+m}f(0)(z^{2kp+m})\right\|_\infty}{(2kp+m)!}-\sum_{k=1}^{\infty}\frac{\left\|D^{(2k-1)p+m}f(0)(z^{(2k-1)p+m})\right\|_\infty}{((2k-1)p+m)!}. \tag{4.2}$$

Now, to find the lower bound, by Theorem 3.4 and (4.2), we see that

$$\begin{aligned}
&\sum_{k=1}^{\infty}(-1)^{kp+m}\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!}-\frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}}\sum_{k=0}^{\infty}\left(\frac{\|D^{kp+m}f(0)(z^{kp+m})\|_\infty}{(kp+m)!}\right)^2\\
\geq\ & -\sum_{k=1}^{\infty}\frac{\left\|D^{(2k-1)p+m}f(0)(z^{(2k-1)p+m})\right\|_\infty}{((2k-1)p+m)!}-\frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}}\sum_{k=0}^{\infty}\left(\frac{\|D^{kp+m}f(0)(z^{kp+m})\|_\infty}{(kp+m)!}\right)^2\\
\geq\ & -1
\end{aligned}$$

holds for all $\|z\|_t=r\leq r_{p,m}$, where $r_{p,m}$ is the unique root in $(0,1)$ of the equation $r^{p+m}+r^{2p}-1=0$.

To find the upper bound, according to Lemma 2.1 and the method of proof of Lemma 2.1, we see that $\frac{\|D^m f(0)(z_0^m)\|_\infty}{m!}\leq 1$ and

$$\begin{aligned}
&\sum_{k=1}^{\infty}(-1)^{kp+m}\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!}-\frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}}\sum_{k=0}^{\infty}\left(\frac{\|D^{kp+m}f(0)(z^{kp+m})\|_\infty}{(kp+m)!}\right)^2\\
\leq\ & \sum_{k=1}^{\infty}\frac{\left\|D^{2kp+m}f(0)(z^{2kp+m})\right\|_\infty}{(2kp+m)!}\leq\frac{\|z\|_t^{2p-m}}{1-\|z\|_t^{2p}}\left[\|z\|_t^{2m}-\left(\frac{\|D^m f(0)(z^m)\|_\infty}{m!}\right)^2\right]\\
=\ & \frac{\|z\|_t^{2p+m}}{1-\|z\|_t^{2p}}\left[1-\left(\frac{\|D^m f(0)(z_0^m)\|_\infty}{m!}\right)^2\right]\leq\frac{\|z\|_t^{2p+m}}{1-\|z\|_t^{2p}}\leq\frac{r^{p+m}}{1-r^{2p}},
\end{aligned} \tag{4.3}$$

which is less than or equal to 1 whenever $\frac{r^{p+m}}{1-r^{2p}}\leq 1$, that is, whenever $r^{p+m}+r^{2p}-1\leq 0$. Thus, combining with the value of the upper and lower bounds, (4.1) holds for all $\|z\|_t=r\leq r_{p,m}$, where $r_{p,m}$ is the unique root in $(0,1)$ of the equation $r^{p+m}+r^{2p}-1=0$.

To prove the sharpness, we just consider the functions $g(z)=z_1^{m+p-1}$, and $f\in H(B_{\ell_t^n},\overline{\mathbb{D}}^n)$ given by $f(z)=zg(z)=(z_1^{m+p},z_2z_1^{m+p-1},\ldots,z_nz_1^{m+p-1})'$, where $z=(z_1,z_2,\ldots,z_n)'$.

Let $z=(z_1,0,\ldots,0)'$. Then $\|z\|_t=|z_1|=r$. In this case, just like the method of proof of Theorem 3.4, we have

$$\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!}=\left|\frac{\partial^{kp+m}f_1(0)}{\partial z_1^{kp+m}}\frac{z_1^{kp+m}}{(kp+m)!}\right|\quad\text{for all } k\geq 0.$$

Since $f_1(z) = z_1^{m+p}$, we have, $\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!} = 0$ (when $k \neq 1$), and $m+p$ is odd. Therefore,

$$\left|\sum_{k=1}^{\infty}(-1)^{kp+m}\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!} + (-1)^{m+p}\frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}}\sum_{k=0}^{\infty}\left(\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!}\right)^2\right|$$
$$= \left|-r^{m+p} - \frac{r^{p-m}}{1-r^{2p}}r^{2(p+m)}\right| = \frac{r^{p+m}}{1-r^{2p}},$$

which shows that the left hand of (4.1) is bigger than 1 whenever $r > r_{p,m}$. This completes the proof of Case (i).

(ii) Next, suppose that both p and m are odd. Then we see that $(-1)^{m+p} = 1$ and

$$\sum_{k=1}^{\infty}(-1)^{kp+m}\frac{\left\|D^{kp+m}f(0)(z^{kp+m})\right\|_\infty}{(kp+m)!} = -\sum_{k=1}^{\infty}\frac{\left\|D^{2kp+m}f(0)(z^{2kp+m})\right\|_\infty}{(2kp+m)!}$$
$$+\sum_{k=1}^{\infty}\frac{\left\|D^{(2k-1)p+m}f(0)(z^{(2k-1)p+m})\right\|_\infty}{((2k-1)p+m)!}.$$

Therefore, combined with Lemma 2.1 and Theorem 3.4, the rest of the proof is similar to the first part above, except that the corresponding formula has the opposite sign. This completes the proof. $\square$

In the following case, since $f((0,0,\ldots,0)') = (0,0,\ldots,0)'$ $g((0,0,\ldots,0)') = (0,0,\ldots,0)'$, Corollary 4.2 is an extension of Theorem B and Corollary 3.3.

Corollary 4.2. *Let $f \in H(B_{\ell_t^n}, \overline{\mathbb{D}}^n)$ and $f(z) = zg(z) = \sum_{k=1}^{\infty}\frac{D^kf(0)(z^k)}{k!}$, where $g \in H(B_{\ell_t^n}, \mathbb{C})$, and j satisfies $|z_j| = \|z\|_\infty = \max_{1\le l\le n}\{|z_l|\}$. Then*

(4.4)
$$\sum_{k=1}^{\infty}\frac{\|D^kf(0)(z^k)\|_\infty}{k!} + \left(\frac{1}{\|z\|_t + \|Df(0)(z)\|_\infty} + \frac{1}{1-\|z\|_t}\right)\sum_{k=2}^{\infty}\left(\frac{\|D^kf(0)(z^k)\|_\infty}{k!}\right)^2 \le 1,$$

holds for all $0 < \|z\|_t = r \le 3/5$. This result is sharp.

Proof. Fix $z \in B_{\ell_t^n}\backslash\{0\}$, and set $z_0 = \frac{z}{\|z\|_t} \in \partial B_{\ell_t^n}$. Because $f(z) = zg(z)$, through simple calculations, we have

$$\frac{D^kf_l(0)(z_0^k)}{k!} = \frac{D^{k-1}g(0)(z_0^{k-1})}{(k-1)!}\frac{z_l}{\|z\|_t}$$

for all $l = 1,2,\ldots,n$ and $k \in \mathbb{N}_0$. Since $|z_j| = \|z\|_\infty = \max_{1\le l\le n}\{|z_l|\}$, it follows that

$$\frac{|D^kf_l(0)(z_0^k)|}{k!} \le \frac{|D^{k-1}g(0)(z_0^{k-1})|}{(k-1)!}\frac{|z_j|}{\|z\|_t} = \frac{|D^kf_j(0)(z_0^k)|}{k!}$$

for all $l = 1,2,\ldots,n$ and $k \in \mathbb{N}_0$. Therefore,

$$\frac{\|D^kf(0)(z_0^k)\|_\infty}{k!} = \frac{|D^kf_j(0)(z_0^k)|}{k!} \text{ for all } k \in \mathbb{N}_0.$$

Multiplying both sides of the above equation by $\|z^k\|_t$ gives

$$\frac{\|D^kf(0)(z^k)\|_\infty}{k!} = \frac{|D^kf_j(0)(z^k)|}{k!} \text{ for all } k \in \mathbb{N}_0.$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{\|D^kf(0)(z^k)\|_\infty}{k!}+\left(\frac{1}{\|z\|_t+\|Df(0)(z)\|_\infty}+\frac{1}{1-\|z\|_t}\right)\sum_{k=2}^{\infty}\left(\frac{\|D^kf(0)(z^k)\|_\infty}{k!}\right)^2$$
$$=\sum_{k=1}^{\infty}\frac{|D^kf_j(0)(z^k)|}{k!}+\left(\frac{1}{\|z\|_t+|Df_j(0)(z)|}+\frac{1}{1-\|z\|_t}\right)\sum_{k=2}^{\infty}\left(\frac{|D^kf_j(0)(z^k)|}{k!}\right)^2.$$

Since $f_j\in H(B_{\ell_t^n},\overline{\mathbb{D}})$, by Corollary 3.3, we find that (4.4) holds for all $0<\|z\|_t=r\le 3/5$.

To prove the sharpness, we just consider the function $g(z)=\frac{a-z_1}{1-az_1}$ and $f\in H(B_{\ell_t^n},\overline{\mathbb{D}}^n)$ given by

$$f(z)=zg(z)=\left(z_1\frac{a-z_1}{1-az_1},z_2\frac{a-z_1}{1-az_1},\dots,z_n\frac{a-z_1}{1-az_1}\right)'.$$

In this case, let $z=(z_1,0,\dots,0)'$. The rest of the proof is similar as in Theorem 3.4. □

Corollary 4.3. *Suppose that $m,p\in\mathbb{N}$, p is odd, $1\le m\le p$, and $1\le t\le\infty$. Let $f\in H(B_{\ell_t^n},\overline{\mathbb{D}}^n)$ be given by*

$$f(z)=zg(z)=\frac{D^mf(0)(z^m)}{m!}+\sum_{k=1}^{\infty}\frac{D^{kp+m}f(0)(z^{kp+m})}{(kp+m)!},$$

where $g\in H(B_{\ell_t^n},\mathbb{C})$. Then

$$\left|A_{f_m}(r)+(-1)^m\left(\frac{1}{\|z\|_t^m+\Gamma}+\frac{\|z\|_t^{2p-m}}{1-\|z\|_t^{2p}}\right)\sum_{k=0}^{\infty}\left(\frac{\|D^{kp+m}f(0)(z^{kp+m})\|_\infty}{(kp+m)!}\right)^2\right|\le 1$$

holds for $0<\|z\|_t=r\le\tilde{R}_{p,m}$, where

$$A_{f_m}(r)=\sum_{k=1}^{\infty}(-1)^{kp+m}\frac{\|D^{kp+m}f(0)(z^{kp+m})\|_\infty}{(kp+m)!},\quad \Gamma=\frac{\|D^mf(0)(z^m)\|_\infty}{m!},$$

and $\tilde{R}_{p,m}$ is the unique root in $(0,1)$ of the equation $r^{2p+m}+2r^{2p}-1=0$. For each p,m, the constant $\tilde{R}_{p,m}$ is best possible.

Proof. First, use the methods of proofs of Lemma 2.1 and Theorem 3.4 to obtain the upper bounds of odd and even terms respectively, and then use (2.2) of Lemma D and combine the methods of proofs Corollary 4.2 and Theorem 4.1 to get quickly the proof of Corollary 4.3, so we skip the details. □

Remark. When $f\in H(B_{\ell_t^n},\overline{\mathbb{D}})$, $m,p\in\mathbb{N}_0$, p is odd, $0\le m\le p$, and $1\le t\le\infty$, then the same conclusion as Corollary 4.3 can be obtained. In the case of $m=0$, we have $\|z\|_t=r\le 1/\sqrt[2p]{3}$, and thus this case can be regarded as an extension of Theorem 1.2 in [3].

Combining Lemma 2.2 and Theorem 3.5, and using the method of proof of Theorem 4.1, we can easily obtain the proof of the following theorem.

Theorem 4.4. *Suppose that $m\in\mathbb{N}_0$, $p\in\mathbb{N}$, p is odd, $0\le m\le p$ and $1\le t\le\infty$. If $f\in H(B_{\ell_t^n},\overline{\mathbb{D}})$ and*

$$f(z)=\frac{D^mf(0)(z^m)}{m!}+\sum_{k=1}^{\infty}\frac{D^{kp+m}f(0)(z^{kp+m})}{(kp+m)!},$$

then

$$\left|\sum_{k=1}^{\infty}(-1)^{kp+m}\frac{\left|D^{kp+m}f(0)(z^{kp+m})\right|}{(kp+m)!}+(-1)^{m+p}\frac{\|z\|_t^{p-m}}{1-\|z\|_t^{2p}}\sum_{k=0}^{\infty}\left(\frac{|D^{kp+m}f(0)(z^{kp+m})|}{(kp+m)!}\right)^2\right|\leq 1$$

holds for all $\|z\|_t = r \leq r_{p,m}$, where $r_{p,m}$ is the same as in Theorem 3.4. For each p and m the number $r_{p,m}$ is best possible.

Remark. When $n = 1$ (that is $f \in H(\mathbb{D}, \overline{\mathbb{D}})$), through comparison, it can be easily found that the results of Theorems 4.1 and 4.4 are the same as that of Theorem C.

Acknowledgements. The work of the first author is supported by the institute postdoctoral fellowship of IIT Madras, India. All authors contribute equally.

Conflict of Interests. The authors declare that they have no conflict of interest, regarding the publication of this paper.

Data Availability Statement. The authors declare that this research is purely theoretical and does not associate with any datas.

References

[1] Aizenberg, L.: Multidimensional analogues of Bohr's theorem on power series, *Proc. Amer. Math. Soc.*, **128**, 1147–1155 (2000)

[2] Aizenberg, L.: Generalization of Caratheodory's inequality and the Bohr radius for multidimensional power series, *Selected topics in complex analysis,* 87–94, Oper. Theory Adv. Appl., 158, Birkhäuser, Basel, 2005.

[3] Ali, R. M., Barnard, R .W., Solynin, A. Yu.: A note on the Bohr's phenomenon for power series, *J. Math. Anal. Appl.*, **449**(1), 154–167 (2017)

[4] Arora, V.: Bohr's phenomenon for holomorphic and harmonic functions with lacunary series in complex Banach spaces, *Complex Var. Elliptic Equ.*, (2022) `DOI: 10.1080/17476933.2022.2146106`

[5] Bayart F., Pellegrino D., Seoane-Sepúlveda J.B.: The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$, *Adv. Math.*, **264**, 726–746 (2014)

[6] Bénéteau, C., Dahlner, A., Khavinson, D.: Remarks on the Bohr phenomenon, *Comput. Methods Funct. Theory*, **4**(1), 1–19 (2004)

[7] Boas, H. P., Khavinson, D.: Bohr's power series theorem in several variables, *Proc. Amer. Math. Soc.*, **125**, 2975–2979 (1997)

[8] Bohr, H.: A theorem concerning power series, *Proc. Lond. Math. Soc.*, **2**(13), 1–5 (1914)

[9] Blasco, O.: The Bohr radius of a Banach space, *Vector measures, integration and related topics,* 59–64, Oper. Theory Adv. Appl., 201, Birkhäuser Verlag, Basel, 2010.

[10] Bombieri, E: Sopra un teorema di H. Bohr e G. Ricci sulle funzioni maggioranti delle serie di potenze, *Boll. Unione Mat. Ital.*, **17**, 276–282 (1962)

[11] Bombieri, E., Bourgain, J.: A remark on Bohr's inequality, *Int. Math. Res. Not.* **80**, 4307–4330 (2004)

[12] Defant A., Frerick L.: A logarithmic lower bound for multi-dimensional bohr radii, *Israel J. Math.*, **152**(1), 17–28 (2006)

[13] Defant A., Frerick L., Ortega-Cerdà J., Ounaïes M., Seip K.: The Bohnenblust-Hille inequality for homogenous polynomials is hypercontractive, *Ann. of Math.*, **174**(2), 512–517 (2011)

[14] Defant, A., Garca, D., Maestre, M., Sevilla-Peris, P.: Dirichlet series and holomorphic functions in higher dimensions. New Mathematical Monographs, 37. Cambridge University Press, Cambridge, xxvii+680 pp (2019).

[15] Defant, A., Prengel, C.: Christopher Harald Bohr meets Stefan Banach. Methods in Banach space theory, 317-339, London Math. Soc. Lecture Note Ser., 337, Cambridge University Press, Cambridge, (2006)

[16] Dixon P.G.: Banach algebras satisfying the non-unital von Neumann inequality, *Bull. London Math. Soc.*, **27**(4), 359–362 (1995)
[17] Djakov, P. B., Ramanujan, M. S.: A remark on Bohr's theorems and its generalizations, *J. Analysis*, **8**, 65–77 (2000).
[18] S. R. Garcia, J. Mashreghi and W. T. Ross, *Finite Blaschke products and their connections*, Springer, Cham, (2018).
[19] Graham, I., Kohr, G.: Geometric Function Theory in One and Higher Dimensions. Marcel Dekker, New York, (2003)
[20] Hamada, H., T. Honda, T., Kohr, G.: Bohr's theorem for holomorphic mappings with values in homogeneous balls, *Israel J. Math.*, **173**, 177–187 (2009)
[21] Kayumov, I.R., Khammatova, D.M., Ponnusamy, S.: Bohr–Rogosinski phenomenon for analytic functions and Cesáro operators, *J. Math. Anal. and Appl.,* 496(2) (2021), 17 pages; Article 124824.
[22] Kayumov, I. R., Ponnusamy, S.: Bohr's inequality for analytic functions with Lacunary series and harmonic functions, *J. Math. Anal. Appl.*, **465**(2), 857–871 (2018)
[23] Kayumov, I. R., Ponnusamy,S.: On a powered Bohr inequality, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **44**, 301–310 (2019)
[24] Lin, R. Y., Liu, M. S., Ponnusamy, S.: Generalization of Bohr-type inequality in analytic functions, *Acta Math. Sin. (Chinese. Ser.)*, **66**(3), 455–474 (2023)
[25] Lin, R., Liu, M., and Ponnusamy, S.: The Bohr-type inequalities for holomorphic mappings with a lacunary series in several complex variables, *Acta Mathematica Scientia*, **44B**(1), 63-79 (2023)
[26] Liu, M. S., Shang, Y. M., Xu, J. F.: Bhor-type inequalities of analytic functions, *J. Inequal. Appl.*, **13**, Art.345 (2018)
[27] Liu, M. S., Ponnusamy, S.: Multidimensional analogues of refined Bohr's inequality, *Proc. Amer. Math. Soc.*, **149**(5), 2133–2146 (2021)
[28] Liu, X. S., Liu, T. S.: The Bohr's inequality for holomorphic mappings with lacunary series in several complex variables, *J. Math. Anal. Appl.*, **485**(2), Art.123844 (2020)
[29] Liu, T., Wang, J. F.: An absolute estimate of the homogeneous expansions of holomorphic mappings, *Pacific J. Math.*, **231**, 155–166 (2007)
[30] Ponnusamy, S., Vijayakumar, R., Wirths, K.-J.: New inequalities for the coefficients of unimodular bounded functions, *Results Math.*, **75**, Art.107 (2020)
[31] Ponnusamy, S., Vijayakumar, R., Wirths, K.-J.: Modifications of Bohr's inequality in various settings, *Houston J. Math.*, **47**(4), 807–831 (2021)
[32] Ponnusamy, S., Vijayakumar, R., Wirths, K.-J.: Improved Bohr's phenomenon in quasi-subordination classes *J. Math. Anal. and Appl.,* **506**(1), Article 125645 (2022).

Shankey Kumar, Department of Mathematics, Indian Institute of Technology Madras, Chennai, 600036, India.
Email address: `shankeygarg93@gmail.com`

Saminathan Ponnusamy, Department of Mathematics, Indian Institute of Technology Madras, Chennai, 600036, India.

Lomonosov Moscow State University, Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia.
Email address: `samy@iitm.ac.in`

G. Brock Williams, Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX, 79409, USA.
Email address: `Brock.Williams@ttu.edu`