# A comprehensive study of nonlinear perturbations in the dynamics of planar crack fronts 

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#### Abstract

The interaction of crack fronts with asperities is central to the criteria of fracture in heterogeneous materials and for predicting fracture surface formation. It is known how dynamic crack fronts respond to small, $1^{\text {st }}$-order, perturbations. However, large and localized disturbances to crack motion induce dynamic and geometric nonlinear effects beyond the existing linear theories. Because the determination of the 3D elastic fields surrounding perturbed crack fronts is a necessary step towards any theoretical study of crack front dynamics, we develop a $2^{\text {nd }}$-order perturbation theory for the asymptotic fields of planar crack fronts. Based on previous work, we consider two models of fracture. In the so-called scalar elastic model, which is analogous to anti-plane (Mode III) fracture, the stress and displacement fields are obtained through matched asymptotic expansions. A self-consistent expansion is used to resolve the fields near tensile (Mode I) crack fronts. Both methods can be extended to higher perturbation orders. The main results of this work are the explicit $2^{\text {nd }}$-order expressions of the local dynamic energy-release-rates for perturbations of straight fronts. These general formulae recover the known energy-release-rates of curved quasi-static fronts and of straight dynamic fronts.


## 1. Introduction

The prediction of crack propagation in heterogeneous media is a central problem in fracture mechanics (Ponson, 2009; Démery et al., 2014; Steinhardt and Rubinstein, 2022; Schmittbuhl and Måløy, 1997; Ponson, 2009; Chopin et al., 2015; Lebihain et al., 2020; Albertini et al., 2021; Roch et al., 2023; Stanchits et al., 2015; Lubomirsky and Bouchbinder, 2023), in designing advanced materials (Xia et al., 2012; Mirkhalaf et al., 2014), and in frictional and earthquake mechanics (Latour et al., 2013; Lebihain et al., 2021; Gounon et al., 2022; Svetlizky and Fineberg, 2014; Bayart et al., 2018; Bedford et al., 2022). Computational frameworks for 3D dynamic fracture, such as the spectral boundary integral method (Geubelle and Rice, 1995; Roch et al., 2022, 2023), phase-field simulations (Pons and Karma, 2010; Henry and Adda-Bedia, 2013; Chen et al., 2015; Bleyer and Molinari, 2017; Henry, 2019; Lubomirsky and Bouchbinder, 2023), and atomistic models (Heizler and Kessler, 2015; Möller and Bitzek, 2015), perform the intensive task of numerically determining the elastodynamic fields. A complementary approach, pursued in this work, reduces the complexity of the 3D problem to the motion of 1D crack fronts. Cracks in brittle materials are described by the theory of Linear Elastic Fracture Mechanics (LEFM). The cornerstone of LEFM is that the near-crack region is dominated by a universal $r^{-1 / 2}$ stress field characterized by a single intensity factor. The stress intensity factor (SIF), then, determines the crack dynamics. This approach was adopted in several works (Rice et al., 1994; Willis and Movchan, 1995, 1997; Ramanathan and Fisher, 1997; Morrissey and Rice, 1998, 2000; Norris and Abrahams, 2007; Willis, 2013; Adda-Bedia et al., 2013) which considered small perturbations to straight crack fronts. The usefulness of this approach was demonstrated by the discovery of crack front waves (Ramanathan and Fisher, 1997; Morrissey and Rice, 1998, 2000), which were later observed experimentally (Sharon et al., 2001, 2002; Fineberg et al., 2003; Sagy et al., 2004). When applied to mixed mode loading configurations, this approach yielded possible instability mechanisms responsible for the generation of corrugation waves (Adda-Bedia et al., 2013) and crack front segmentation (Pons and Karma, 2010; Leblond et al., 2011; Chen et al., 2015). For heterogeneous materials, in which the toughness contrasts are of order unity, however, the linear theory is of little use. Understanding nonlinear perturbations is also needed to determine how heterogeneity affects energy dissipation in fracture, since $1^{\text {st }}$-order perturbations have a net zero contribution to dissipation. The importance of nonlinear effects has been sketched in a recent work (Kolvin et al., 2017). There, a first-principles nonlinear equation of motion for crack fronts was derived in the context of scalar elasticity and their response to externally induced perturbations was
computed. It was suggested that nonlinear front focusing coupled with the rate-dependence of fracture energy dissipation may govern the transition to micro-branching.

In the framework of LEFM, the modeling of the crack front dynamics necessitates the knowledge of the variation of dynamic SIF with the crack front geometry. The mathematical foundations of nonlinear perturbations of static crack fronts are well-developed (Leblond et al., 2012; Vasoya et al., 2016; AddaBedia et al., 2006). While the general methods for dynamic cracks were developed by Movchan et al. (1998), Ramanathan and Fisher (1997); Ramanathan (1997),Norris and Abrahams (2007) and Willis (2013), these works did not provide explicit, tractable formulas that can be applied to concrete crack evolution. Morrissey and Rice (2000), however, obtained explicit $1^{\text {st }}$-order formula for the energy-release-rate of Mode I crack fronts, and numerically studied crack propagation in hetereogeneous solids. Based on these works, we compute the $2^{\text {nd }}$-order corrections to the energy-release-rate in the scalar model of elasticity and conventional "vectorial" elastodynamics. The energy balance between the elastic energy flux into the crack front and the dissipation can then be used to obtain equations of motion for dynamic planar crack fronts.


Figure 1: Geometry of the problem. The crack front (red) subtends two semi-infinite planar crack faces (blue). The front propagates in the $y=0$ plane of a linear elastic solid at a velocity $V$ with a superimposed spatiotemporal perturbation $f(z, t)$.

Consider an infinite body containing a semi-infinite planar crack that is driven by remote loading (Fig. 1). The leading edge of the propagating crack front experiences perturbations in space and time around a steady motion at a constant velocity $V$. The coordinate system is defined as follows: $x$ is the crack propagation direction, the $y$ axis is perpendicular to the fracture $x z$ plane, and $t$ signifies time. The instantaneous crack front position is $h(z, t)=V t+f(z, t)$, where $f$ is the crack front perturbation. The task at hand is to obtain an expression for the local energy-release-rate $G(z, t)$ along the front. Two different methods are used: matched asymptotic expansions, in the case of scalar elasticity, and a self-consistent expansion in the case of Mode I fracture. Both methods can be generalized to higher-order perturbations.

In the following, we outline the main results by providing the explicit expressions of the $2^{\text {nd }}$-order perturbations of the energy-release-rate. Afterward, we detail, for the interested reader, the calculations of both scalar and tensile crack problems.

## 2. Summary of the results

### 2.1. Derivation of $G$ in the scalar elasticity model

In section 3, we solve for the asymptotic displacement and stress fields surrounding a crack in a model scalar elastic solid, extending the calculations of Norris and Abrahams (2007). Fracture in scalar elasticity is identical to anti-plane shear (Mode III) fracture in a solid body that is constrained to displace along one dimension only, which is described by the displacement $\mathbf{u}=\left(0,0, u_{z}\right)$. The theory is characterized by a single wave speed $(b=1)$. We assume that the crack is driven at velocity $V$ by remote loading conditions that are characterized by a distance $l$ from the crack front. The elastic fields are expanded in the small parameter $\epsilon \equiv\|f\| / l \ll 1$ in two separate perturbation series: the terms of the first series are centered at the perturbed crack front $x=h(z, t)$; the terms of the second series are centered at the unperturbed, imaginary, position of the crack front $x=V t$. Solutions of the equation of elasticity contain unknown integration constants that are determined by matching the two expansions. The main result of this calculation is an expression for $G$ that is exact up to $O\left(f^{2}\right)$ and $O\left(\epsilon^{2}\right)$. When $\epsilon \rightarrow 0$, the expansion of the energy-release-rate becomes

$$
\begin{equation*}
G(z, t)=G_{r} g\left(V_{\perp}\right)\left(1+H^{(1)}[f]+H^{(2)}[f, f]+O\left(f^{3}\right)\right), \tag{1}
\end{equation*}
$$

where the rest energy-release-rate $G_{r}$ is determined by the loading conditions, $g\left(V_{\perp}\right)$ is the dynamical contribution given by (Rice et al., 1994),

$$
\begin{equation*}
g\left(V_{\perp}\right)=\sqrt{\frac{1-V_{\perp}}{1+V_{\perp}}}, \tag{2}
\end{equation*}
$$

and $V_{\perp}(z, t)=\left(V+f_{t}\right) / \sqrt{1+f_{z}^{2}}$ is the local normal front velocity of the crack front. Subscripted functions $f_{t}, f_{z}$ denote partial derivatives. The linear functional $H^{(1)}[f]$ and the $2^{\text {nd }}$-order functional $H^{(2)}[f, f]$ depend only on the crack front history $\left\{f\left(z, t^{\prime}<t\right)\right\}$,

$$
\begin{align*}
& H^{(1)}[f]=-\frac{1}{\alpha^{2}} \Psi[f],  \tag{3}\\
& H^{(2)}[f]=\frac{1}{4 \alpha^{4}} \Psi[f]^{2}+\frac{1}{2 \alpha^{4}} \Psi[f \Psi[f]]-\frac{1-2 V}{4 \alpha^{4}} \Psi_{2}\left[f^{2}\right]-\frac{1+2 V}{2 \alpha^{4}} f \Psi_{2}[f], \tag{4}
\end{align*}
$$

where $\alpha=\sqrt{1-V^{2}}$, and, with the definition of the Fourier transform $\bar{f}(k, t)=\int \mathrm{d} z e^{-i k z} f(z, t)$,

$$
\begin{align*}
\overline{\Psi[f]}(k, t) & =\alpha k \int_{-\infty}^{t} \mathrm{~d} t^{\prime} \frac{J_{1}\left(\alpha k\left(t-t^{\prime}\right)\right)}{t-t^{\prime}} \bar{f}\left(k, t^{\prime}\right),  \tag{5}\\
\overline{\Psi_{2}[f]}(k, t) & =\alpha^{2} k^{2} \int_{-\infty}^{t} \mathrm{~d} t^{\prime} \frac{J_{2}\left(\alpha k\left(t-t^{\prime}\right)\right)}{t-t^{\prime}} \bar{f}\left(k, t^{\prime}\right), \tag{6}
\end{align*}
$$

where $J_{1}$ and $J_{2}$ are the $1^{\text {st. }}$ and $2^{\text {nd }}$-order Bessel functions.

### 2.1.1. Recovery of known limits

To test Eq. (1) against known results, we consider two limiting cases: that of a straight front $\left(f_{z}=0\right)$, and that of a steadily propagating front $\left(f_{t}=0\right)$.

The condition $f_{z}=0$ is equivalent to taking $k \rightarrow 0$ in Eqs. (5-6), which approach $\Psi[f], \Psi_{2}[f] \rightarrow 0$. Then the energy-release-rate, Eq. (1), approaches $G \rightarrow G_{r} g\left(V+f_{t}\right)$, which is identical to the known equation of motion of a 2D crack (Rice et al., 1994).

When $f_{t}=0$, Eqs. (5-6) become $\overline{\Psi[f]}=\alpha|k| \bar{f}(k)$ and $\overline{\Psi_{2}[f]}=\frac{1}{2} \alpha^{2} k^{2} \bar{f}(k)$. Then

$$
\begin{align*}
& \overline{H^{(1)}[f]}=-\frac{1}{\alpha}|k| \bar{f}(k), \\
& \overline{H^{(2)}[f, f]}=  \tag{7}\\
& \frac{1}{4 \alpha^{2}} \int \mathrm{~d} k^{\prime} \bar{f}\left(k-k^{\prime}\right) \bar{f}\left(k^{\prime}\right)\left(\left|k^{\prime}\right|\left|k-k^{\prime}\right|+|k|\left(\left|k^{\prime}\right|+\left|k-k^{\prime}\right|\right)-k^{2}+(1+2 V) k^{\prime}\left(k-k^{\prime}\right)\right) .
\end{align*}
$$

Using these results and expanding $g\left(V_{\perp}\right) \simeq g(V)-\frac{1}{2} V g^{\prime}(V) f_{z}^{2}$, Eq. (1) becomes

$$
\begin{align*}
\bar{G}(k) & =G_{r} g(V)\left\{2 \pi \delta(k)-\frac{1}{\alpha}|k| \bar{f}(k)\right. \\
& \left.+\frac{1}{4 \alpha^{2}} \int \mathrm{~d} k^{\prime} \bar{f}\left(k-k^{\prime}\right) \bar{f}\left(k^{\prime}\right)\left(\left|k^{\prime}\right|\left|k-k^{\prime}\right|+|k|\left(\left|k^{\prime}\right|+\left|k-k^{\prime}\right|\right)-k^{2}+k^{\prime}\left(k-k^{\prime}\right)\right)\right\} \tag{8}
\end{align*}
$$

after symmetrization $k^{\prime} \leftrightarrow k-k^{\prime}$. In the limit $V \rightarrow 0$, Eq. (8) becomes

$$
\begin{align*}
\bar{G}(k) & =G_{r}\{2 \pi \delta(k)-|k| \bar{f}(k)  \tag{9}\\
& \left.+\frac{1}{4} \int \mathrm{~d} k^{\prime} \bar{f}\left(k-k^{\prime}\right) \bar{f}\left(k^{\prime}\right)\left(\left|k^{\prime}\right|\left|k-k^{\prime}\right|+|k|\left(\left|k^{\prime}\right|+\left|k-k^{\prime}\right|\right)-k^{2}+k^{\prime}\left(k-k^{\prime}\right)\right)\right\} .
\end{align*}
$$

The same expression is obtained from the $2^{\text {nd }}$-order approximation of the SIF of quasi-static crack fronts (Leblond et al., 2012; Vasoya et al., 2016)).

### 2.2. Derivation of $G$ in Mode I fracture

In section 4, we determine the asymptotic fields in the vicinity of a crack that propagates under remote tensile loading conditions in an infinite body of Young modulus $E$ and Poisson ratio $v$, extending the calculations of Ramanathan (1997). In the absence of perturbations, the loading conditions produce an asymptotic Mode I stress field. Ahead of the crack front and at the fracture plane, the tensile stress is $\sigma_{y y} \sim K_{0}^{\sigma} / \sqrt{2 \pi(x-V t)}$ with the SIF, $K_{0}^{\sigma}=K_{r}^{\sigma} k(V) . K_{r}^{\sigma}$ is the rest SIF and $k(V)$ is a universal function of the crack velocity defined in Eq. (6.4.26) of Freund (1990). The energy-release-rate is then $G=G_{r} g(V)$
where $G_{r}=\frac{1-v^{2}}{E}\left(K_{r}^{\sigma}\right)^{2}$ is the rest energy-release-rate, the dynamical prefactor is $g(V)=A_{I}(V) k(V)^{2}$, and $A_{I}(V)=(1-v)^{-1}\left(V^{2} \alpha_{a}\right) /\left(R(V) b^{2}\right) ; R(V)=4 \alpha_{a} \alpha_{b}-\left(1+\alpha_{b}^{2}\right)^{2} ; \alpha_{s}=\sqrt{1-(V / s)^{2}}$; and $s=a, b$ are the longitudinal and the shear wave speeds (Freund, 1990).

For space-time-dependent crack fronts, we derive a perturbation expansion of the energy-release-rate

$$
\begin{equation*}
G(z, t)=G_{r} g\left(V_{\perp}\right)\left(1+H^{(1)}[f]+H^{(2)}[f, f]+O\left(f^{3}\right)\right) \tag{10}
\end{equation*}
$$

Explicit expressions for the history functionals $H^{(1)}[f]$ and $H^{(2)}[f, f]$ are found in the space $(k, t)$, which defined via the Fourier transform $\bar{f}=\int \mathrm{d} z e^{-i k z} f(z, t)$. The linear functional is $\overline{H^{(1)}[f]}=2 I_{1}[\bar{f}]$ where $I_{1}[\bar{f}]=-\int_{-\infty}^{t} \mathrm{~d} t^{\prime} A_{1}\left(k, t-t^{\prime}\right) \bar{f}\left(k, t^{\prime}\right)$ and (Morrissey and Rice, 2000)

$$
\begin{equation*}
A_{1}(k, t)=k^{2}\left[-\frac{a}{2} \frac{J_{1}\left(\beta_{a} t\right)}{\beta_{a} t}+c \frac{J_{1}\left(\beta_{c} t\right)}{\beta_{c} t}-\frac{1}{4} \int_{b}^{a} \mathrm{~d} \eta \Theta(\eta)\left(\frac{\eta^{2}+V^{2}}{\eta^{2}-V^{2}} J_{2}\left(\beta_{\eta} t\right)-J_{0}\left(\beta_{\eta} t\right)\right)\right], \tag{11}
\end{equation*}
$$

where $\beta_{s}=\sqrt{s^{2}-V^{2}}|k|$. The $2^{\text {nd }}$-order functional is

$$
\begin{aligned}
\overline{H^{(2)}[f, f]} & =2 I_{1}\left[\bar{f} * I_{1}[\bar{f}]\right]-\frac{1}{2} I_{1}\left[I_{1}[\bar{f} * \bar{f}]\right]-\bar{f} * I_{1}\left[I_{1}[\bar{f}]\right]+I_{1}[\bar{f}] * I_{1}[\bar{f}] \\
& +\frac{V}{2} I_{2}[\bar{f} * \bar{f}]-V \bar{f} * I_{2}[\bar{f}] .
\end{aligned}
$$

where the convolution operator is defined as $f * g=\int \mathrm{d} k^{\prime} f\left(k-k^{\prime}\right) g\left(k^{\prime}\right)$, and $I_{2}[\bar{f}]=-\int_{-\infty}^{t} \mathrm{~d} t^{\prime} A_{2}(k, t-$ $\left.t^{\prime}\right) \bar{f}\left(k, t^{\prime}\right)$ with

$$
A_{2}(k, t)=|k|^{3}\left[-\gamma_{a} \frac{J_{2}\left(\beta_{a} t\right)}{\beta_{a} t}+2 \gamma_{c} \frac{J_{2}\left(\beta_{c} t\right)}{\beta_{c} t}-\frac{1}{4} \int_{b}^{a} \mathrm{~d} \eta \frac{\Theta(\eta)}{\sqrt{\eta^{2}-V^{2}}}\left(\frac{3 \eta^{2}+V^{2}}{\eta^{2}-V^{2}} J_{3}\left(\beta_{\eta} t\right)-J_{1}\left(\beta_{\eta} t\right)\right)\right] .
$$

It is possible to use these expressions to evolve crack fronts in time (Morrissey and Rice, 2000).
It is useful to write $G$ in Fourier space $(k, \omega)$, where $\hat{f}(k, \omega)=\int \mathrm{d} k \mathrm{~d} \omega e^{-i \omega t-i k z} f(z, t)$. We define the convolution $\hat{f} \otimes \hat{g}=\int \mathrm{d} k^{\prime} \mathrm{d} \omega^{\prime} \hat{f}\left(k-k^{\prime}, \omega-\omega^{\prime}\right) \hat{f}\left(k^{\prime}, \omega^{\prime}\right)$. The perturbation expansion of the energy-release-rate is then

$$
\begin{equation*}
\widehat{G}(k, \omega)=G_{r} g(V)\left((2 \pi)^{2} \delta(k) \delta(\omega)+\widehat{\delta G^{(1)}}(k, \omega)+\widehat{\delta G^{(2)}}(k, \omega)+O\left(f^{3}\right)\right) . \tag{12}
\end{equation*}
$$

The $1^{\text {st }}$-order correction is (Ramanathan and Fisher, 1997)

$$
\begin{equation*}
\widehat{\delta G^{(1)}}(k, \omega)=-2|k| P_{1}(\omega /|k| ; V, v) \hat{f}(k, \omega), \tag{13}
\end{equation*}
$$

and the $2^{\text {nd }}$-order correction is

$$
\begin{align*}
& \widehat{\delta G^{(2)}}(k, \omega)=2|k| P_{1}\left\{\hat{f} \otimes\left(|k| P_{1} \hat{f}\right)\right\}-\left\{\frac{1}{2} k^{2} P_{1}^{2}+\frac{i}{2} V \omega|k| P_{2}\right\}(\hat{f} \otimes \hat{f})  \tag{14}\\
& -\hat{f} \otimes\left\{\left(k^{2} P_{1}^{2}-i V \omega|k| P_{2}\right) \hat{f}\right\}+\left(|k| P_{1} \hat{f}\right) \otimes\left(|k| P_{1} \hat{f}\right) .
\end{align*}
$$

The kernel functions $P_{1}$ and $P_{2}$ are

$$
\begin{align*}
P_{1}(s ; V, v)= & -\frac{1}{2} \gamma_{a} \sqrt{1-\frac{\gamma_{a}^{2} s^{2}}{a^{2}}}+\gamma_{c} \sqrt{1-\frac{\gamma_{c}^{2} s^{2}}{c^{2}}} \\
& +\frac{1}{2} \int_{b}^{a} \frac{\left(s^{2}+V^{2}\right)\left(\eta^{2}+V^{2}\right)-2 \eta^{2} V^{2}}{\sqrt{\eta^{2}-\left(s^{2}+V^{2}\right)}\left(\eta^{2}-V^{2}\right)^{2}} \Theta(\eta) \mathrm{d} \eta \tag{15}
\end{align*}
$$

and

$$
\begin{align*}
P_{2}(s ; V, v)= & 2 \frac{\gamma_{c}^{3}}{c^{2}} \sqrt{1-\frac{\gamma_{c}^{2} s^{2}}{c^{2}}}-\frac{\gamma_{a}^{3}}{a^{2}} \sqrt{1-\frac{\gamma_{a}^{2} s^{2}}{a^{2}}}+ \\
& \int_{b}^{a} \frac{\left(s^{2}+V^{2}\right)\left(3 \eta^{2}+V^{2}\right)-2 \eta^{2}\left(\eta^{2}+V^{2}\right)}{\sqrt{\eta^{2}-\left(s^{2}+V^{2}\right)}\left(\eta^{2}-V^{2}\right)^{3}} \Theta(\eta) \mathrm{d} \eta, \tag{16}
\end{align*}
$$

where $\gamma_{s}=1 / \sqrt{1-V^{2} / s^{2}}, c$ is the Rayleigh wave speed, and

$$
\begin{equation*}
\Theta(\eta)=\frac{2}{\pi} \arctan \left[4 \sqrt{1-\frac{\eta^{2}}{a^{2}}} \sqrt{\frac{\eta^{2}}{b^{2}}-1} /\left(2-\frac{\eta^{2}}{b^{2}}\right)^{2}\right] \tag{17}
\end{equation*}
$$

The branch cuts of the square roots in Eqs. $(15,16)$ are defined such that $\sqrt{1-s^{2}}=i \operatorname{sign}(s) \sqrt{s^{2}-1}$ for $|s|>1$. Example curves of the functions $P_{1}$ and $P_{2}$ are depicted in Fig. 2. Particularly, the zeros $s_{1}$ and $s_{2}$ of $P_{1}$ and $P_{2}$ do not coincide (Fig. 3). The roots $s= \pm s_{1}$ of $P_{1}$ signify the existence crack front waves, since $\delta G^{(1)}=0$ at $\omega /|k|= \pm s_{1}$. Because $s_{1} \neq s_{2}, \delta G^{(2)} \neq 0$ at $\omega /|k|= \pm s_{1}$. Crack front waves are expected, therefore, to have a non-zero contribution to the energy-release-rate at the $2^{\text {nd }}$-order.


Figure 2: Real and imaginary parts of $P_{1}(s ; V, v)$ (left) and $P_{2}(s ; V, v)$ (right) for $V=0.5 b$ and $v=0.3$.


Figure 3: The real roots $s_{1}$ of $P_{1}$ and $s_{2}$ of $P_{2}$ normalized by $\sqrt{c^{2}-V^{2}} . v=0.3$

### 2.2.1. Recovery of known limits

Eq. (10) can be reduced to known expressions at the 2 D limit $f=f(t)$ and at the quasi-static limit $f=f(z)$. In the 2D limit, $H^{(1)}[f]=0$ and $H^{(2)}[f, f]=0$ since $\bar{f} \propto \delta(k)$. The 2D expression for the energy-release-rate $G=G_{r} g\left(V+f_{t}\right)$ is recovered.

In the quasi-static limit, $f=f(z)$. Then, $I_{1}[\bar{f}]=-P_{1}(0 ; V)|k|$ and $I_{2}[\bar{f}]=\pi_{1} k^{2}$ where

$$
\begin{equation*}
\pi_{1}=\frac{1}{2} \frac{a}{a^{2}-V^{2}}-\frac{c}{c^{2}-V^{2}}+\frac{1}{2} \int_{b}^{a} \frac{\eta^{2}+V^{2}}{\left(\eta^{2}-V^{2}\right)^{2}} \Theta(\eta) \mathrm{d} \eta \tag{18}
\end{equation*}
$$

The history functionals become

$$
\overline{H^{(1)}[f]}=-2 P_{1}(0 ; V)|k| \bar{f}(k)
$$

and

$$
\begin{aligned}
\overline{H^{(2)}[f]}= & P_{1}(0 ; V)^{2} \int \mathrm{~d} k^{\prime} \bar{f}\left(k-k^{\prime}\right) \bar{f}\left(k^{\prime}\right)\left(|k|\left(\left|k^{\prime}\right|+\left|k-k^{\prime}\right|\right)-k^{2}+k^{\prime}\left(k-k^{\prime}\right)+\left|k^{\prime}\right|\left|k-k^{\prime}\right|\right) \\
& +V \pi_{1}(k \bar{f}) *(k \bar{f})
\end{aligned}
$$

Since $2 \pi_{1}=g^{\prime}(V) / g(V)$, the energy-release-rate expansion becomes

$$
\begin{align*}
\bar{G}(k) & =G_{r} g(V)\left\{2 \pi \delta(k)-2 P_{1}(0 ; V)|k| \bar{f}(k)\right. \\
& \left.+P_{1}(0 ; V)^{2} \int \mathrm{~d} k^{\prime} \bar{f}\left(k-k^{\prime}\right) \bar{f}\left(k^{\prime}\right)\left(|k|\left(\left|k^{\prime}\right|+\left|k-k^{\prime}\right|\right)-k^{2}+k^{\prime}\left(k-k^{\prime}\right)+\left|k^{\prime}\right|\left|k-k^{\prime}\right|\right)+O\left(f^{3}\right)\right\} . \tag{19}
\end{align*}
$$

At the limit $V \rightarrow 0$, Eq. (9) is reproduced.

## 3. Nonlinear perturbation of scalar elastic crack fronts

This section develops a perturbation theory for planar crack fronts that propagate in a solid whose are described by a scalar displacement field $u$. In this model, the stress, a vector field, is defined by $\sigma=\mu \nabla u$ where $\mu$ is the elastic modulus. This formulation is identical to the elastic problem of anti-plane shear deformation when the orthogonal displacement components are artificially set to zero (Rice et al., 1994). In general, freely propagating anti-plane shear (Mode III) cracks experience tensile (Mode I) and in-plane shear (Mode II) stresses when perturbed (Geubelle and Rice, 1995). The scalar model, however, is useful in obtaining a qualitative understanding of dynamic fracture, as it is much simpler than "full" elastodynamics (Perrin and Rice, 1994).

Displacements are governed by Newton's $2^{\text {nd }}$ law

$$
\rho \partial_{t}^{2} u=\mu \nabla \cdot \sigma
$$

where $\rho$ is the density. With the definition of the elastic wave velocity $b=\sqrt{\mu / \rho}$, the displacement satisfies the wave equation

$$
\begin{equation*}
\nabla^{2} u-\frac{1}{b^{2}} \partial_{t}^{2} u=0 \tag{20}
\end{equation*}
$$

In the following we set $b=1$ and we assume that the crack is driven at velocity $V$ by stresses that are applied at a large distance $l$ from the crack front. It is helpful to transform the wave equation to the co-moving frame, $x \rightarrow(x-V t) / l, y \rightarrow y / l, z \rightarrow z / l, t / l \rightarrow t, \sigma \rightarrow \sigma / \mu$, such that

$$
\begin{equation*}
\alpha^{2} u_{x x}+u_{y y}+u_{z z}-u_{t t}+2 V u_{x t}=0 \tag{21}
\end{equation*}
$$

where subscripts denote partial derivatives. The equation of motion is supplemented by the boundary conditions at the fracture plane $y=0$,

$$
\begin{equation*}
u(x>f(z, t), y=0, z, t)=0 ; \quad \sigma_{y}(x<f(z, t), y=0, z, t)=0 . \tag{22}
\end{equation*}
$$

For the simple crack, $f(z, t)=0$, the asymptotic elastic fields that solve this problem are well-known (Eshelby, 1969; Rice et al., 1994; Norris and Abrahams, 2007). The displacement can be written as a Williams expansion

$$
\begin{equation*}
u(x, y, z, t)=\sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{\sqrt{x-V t+i \alpha y}\left[1+\frac{m}{3}(x-V t+i \alpha y)+\frac{n}{5}(x-V t+i \alpha y)^{2}+\ldots\right]\right\} \tag{23}
\end{equation*}
$$

where $i=\sqrt{-1}, \alpha=\sqrt{1-V^{2}}$ and $K_{0}^{u}=K_{0}^{\sigma} / \alpha$. The SIF $K_{0}^{\sigma}$ and the coefficients $m \propto l^{-1}$ and $n \propto l^{-2}$ are determined by the loading conditions. The SIF can be written as a product $K_{0}^{\sigma}=K_{r} k(V)$ where $K_{r}$ is the rest SIF and $k(V)=\sqrt{1-V}$ (Rice et al., 1994). The energy-release-rate for the straight crack is then $G_{0}=G_{r} g(V)$ where $G_{r} \propto K_{r}^{2}$ is the rest energy release and $g(V)=\sqrt{(1-V) /(1+V)}$.

The simple crack solution provides a basis for exploring the effect of arbitrary perturbations. Below, we follow the method of Norris and Abrahams (2007) where the elastic fields are determined by matching two asymptotic expansions. In the "inner" expansion the fields are centered at the crack front position $x=f(z, t)$. In the "outer" expansion, the fields are centered at $x=0$. The coefficients of the inner and outer solutions are then matched to yield the displacement and the stress intensity factors, defined as

$$
\begin{align*}
K^{u} & =\lim _{x-f(z, t) \rightarrow 0^{-}}(x-f(z, t))^{-1 / 2} \sqrt{\frac{\pi}{2}} \mu u(x, y=0, z, t)  \tag{24}\\
K^{\sigma} & =\lim _{x-f(z, t) \rightarrow 0^{+}} \sqrt{2 \pi(x-f(z, t))} \sigma_{y}(x, y=0, z, t) \tag{25}
\end{align*}
$$

The energy release is then $G \propto K^{u} K^{\sigma}$.

### 3.1. The inner solution

We define $\epsilon=\max _{z, t}|f(z, t)| / l$, and the "inner" variables $X=\epsilon x, Y=\epsilon y$. Writing the displacement field as a function of the inner variables $U(X, Y, z ; t)=u(x, y, z ; t)$, Eq. (21) becomes

$$
\begin{equation*}
\alpha^{2} U_{X X}+U_{Y Y}+2 \epsilon V U_{X t}+\epsilon^{2}\left(U_{z z}-U_{t t}\right)=0 \tag{26}
\end{equation*}
$$

Expanding $U=\epsilon^{1 / 2} U^{(1 / 2)}+\epsilon^{3 / 2} U^{(3 / 2)}+\epsilon^{5 / 2} U^{(5 / 2)}+\ldots$ and substituting the expansion in Eq. (26), $U$ is found order-by-order in $\epsilon$. Since we are seeking $2^{\text {nd }}$-order corrections to $G$, we will need to compute the expansion up to the $\epsilon^{5 / 2}$ term. The zeroth order solution corresponds to the dominant term in the simple crack field

$$
\begin{equation*}
U^{(1 / 2)}=\sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{S^{1 / 2}\right\} \tag{27}
\end{equation*}
$$

where $S=X-f+i \alpha Y$.
To find the next order terms, we will use the identity

$$
\begin{equation*}
\left(\alpha^{2} \partial_{X}^{2}+\partial_{Y}^{2}\right) w_{1}(\bar{S}) w_{2}(S)=4 \alpha^{2} w_{1}^{\prime}(\bar{S}) w_{2}^{\prime}(S) \tag{28}
\end{equation*}
$$

where $w_{1}(S), w_{2}(S)$ are arbitrary complex functions and $\bar{S}$ is the complex conjugate of $S$. The $1^{\text {st }}$-order term in the expansion is then

$$
\begin{equation*}
U^{(3 / 2)}=\sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{A^{(3 / 2)} S^{3 / 2}+B^{(3 / 2)} S^{1 / 2}+\frac{V}{4 \alpha^{2}} f_{t} \bar{S} S^{-1 / 2}\right\} \tag{29}
\end{equation*}
$$

$A^{(3 / 2)}$ and $B^{(3 / 2)}$ are coefficients to be determined by matching to the outer solution. The $2^{\text {nd }}$-order term in the expansion is

$$
\begin{align*}
& U^{(5 / 2)}= \\
& \sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{\left(A^{(5 / 2)} S^{5 / 2}+B^{(5 / 2)} S^{3 / 2}+C^{(5 / 2)} S^{1 / 2}\right)+\left(\frac{m v}{2} f_{t}-v B_{t}^{(3 / 2)}+\frac{1}{2} f_{z z}-\frac{1}{2 \alpha^{2}} f_{t t}\right) \frac{1}{2 \alpha^{2}} \bar{S} S^{1 / 2}\right. \\
& \left.+\left(\frac{V}{2} B^{(3 / 2)} f_{t}+\frac{1+V^{2}}{4 \alpha^{2}}\left(f_{t}\right)^{2}-\frac{1}{4}\left(f_{z}\right)^{2}\right) \frac{1}{2 \alpha^{2}} \bar{S} S^{-1 / 2}-\frac{V^{2}}{16 \alpha^{4}} f_{t t} \bar{S}^{2} S^{-1 / 2}-\frac{V^{2}}{32 \alpha^{4}}\left(f_{t}\right)^{2} \bar{S}^{2} S^{-3 / 2}\right\} \tag{30}
\end{align*}
$$

$A^{(5 / 2)}, B^{(5 / 2)}$ and $C^{(5 / 2)}$ are additional coefficients to be determined in the matching.
To make the matching to the outer solution straightforward, we expand Eqs. $(27,29,30)$ around the unperturbed crack front position. Replacing $S=\epsilon^{-1}(s-\epsilon f)$ where $s=x+i \alpha y$, and gathering terms of the same order in $\epsilon$ we find the following expressions.

Terms of the order $\epsilon^{0}$ :

$$
\begin{equation*}
\sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{s^{1 / 2}+\frac{1}{3} m s^{3 / 2}+A^{(5 / 2)} s^{5 / 2}\right\} \tag{31}
\end{equation*}
$$

Terms of the order $\epsilon^{1}$ :

$$
\begin{align*}
& \sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{-\frac{1}{2} f s^{-1 / 2}+\left(B^{(3 / 2)}-\frac{1}{2} m f\right) s^{1 / 2}+\frac{V f_{t}}{4 \alpha^{2}} \bar{s} s^{-1 / 2}+\left(B^{(5 / 2)}-\frac{5}{2} A^{(5 / 2)} f\right) s^{3 / 2}\right. \\
& \left.-\frac{V^{2} f_{t t}}{16 \alpha^{4}} \bar{s}^{2} s^{-1 / 2}+\frac{1}{4}\left(\frac{m V}{\alpha^{2}} f_{t}-\frac{2 V}{\alpha^{2}} B_{t}^{(3 / 2)}+\frac{1}{\alpha^{2}} f_{z z}-\frac{1}{\alpha^{4}} f_{t t}\right) \bar{s} s^{1 / 2}\right\} \tag{32}
\end{align*}
$$

Terms of the order $\epsilon^{2}$ :

$$
\begin{align*}
& \sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{-\frac{f^{2}}{8} s^{-3 / 2}+\left(\frac{m}{8} f^{2}-\frac{f B^{(3 / 2)}}{2}-\frac{V}{4 \alpha^{2}} f f_{t}\right) s^{-1 / 2}+\frac{V f f_{t}}{8 \alpha^{2}} \bar{s} s^{-3 / 2}\right. \\
& \left(C^{(5 / 2)}-\frac{3}{2} B^{(5 / 2)} f+\frac{15}{8} A^{(5 / 2)} f^{2}+\frac{V}{2 \alpha^{2}} B_{t}^{(3 / 2)} f-\frac{m V}{4 \alpha^{2}} f f_{t}+\frac{f f_{t t}}{4 \alpha^{4}}-\frac{1}{4 \alpha^{2}} f f_{z z}\right) s^{1 / 2} \\
& +\left(\frac{d}{d t}\left(-\frac{1}{16 \alpha^{2}} m V f^{2}+\frac{1}{4 \alpha^{2}} V B^{(3 / 2)} f+\frac{\left(1+V^{2}\right)}{16 \alpha^{4}} \frac{d}{d t} f^{2}\right)-\frac{1}{16 \alpha^{2}} \frac{d^{2}}{d z^{2}} f^{2}\right) \bar{s} s^{-1 / 2}  \tag{33}\\
& \left.-\frac{V^{2}}{64 \alpha^{4}} \frac{d^{2}}{d t^{2}} f^{2} \bar{s}^{2} s^{-3 / 2}\right\} .
\end{align*}
$$

The matching procedure is further simplified by introducing the polar coordinates $s=r e^{i \theta}$ and taking the imaginary part in the above expressions:
Terms of the order $\epsilon^{0}$ :

$$
\begin{equation*}
\sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu}\left(\sqrt{r} \sin \frac{\theta}{2}+A^{(3 / 2)} r^{3 / 2} \sin \frac{3 \theta}{2}+A^{(5 / 2)} r^{5 / 2} \sin \frac{5 \theta}{2}\right) \tag{34}
\end{equation*}
$$

Terms of the order $\epsilon^{1}$ :

$$
\begin{align*}
& \sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu}\left\{\frac{f}{2} r^{-1 / 2} \sin \frac{\theta}{2}+\left(B^{(3 / 2)}-\frac{3}{2} A^{(3 / 2)} f\right) r^{1 / 2} \sin \frac{\theta}{2}-\frac{V f_{t}}{4 \alpha^{2}} r^{1 / 2} \sin \frac{3 \theta}{2}\right. \\
& +\left(\frac{V B_{t}^{(3 / 2)}}{2 \alpha^{2}}-\frac{3 A^{(3 / 2)} V f_{t}}{4 \alpha^{2}}+\frac{f_{t t}}{4 \alpha^{4}}-\frac{f_{z z}}{4 \alpha^{2}}\right) r^{3 / 2} \sin \frac{\theta}{2}  \tag{35}\\
& \left.+\left(B^{(5 / 2)}-\frac{5}{2} A^{(5 / 2)} f\right) r^{3 / 2} \sin \frac{3 \theta}{2}+\frac{V^{2} f_{t t}}{16 \alpha^{4}} r^{3 / 2} \sin \frac{5 \theta}{2}\right\}
\end{align*}
$$

Terms of the order $\epsilon^{2}$ :

$$
\begin{align*}
& \sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu}\left\{\frac{f^{2}}{8} r^{-3 / 2} \sin \frac{3 \theta}{2}-\frac{V f f_{t}}{8 \alpha^{2}} r^{-1 / 2} \sin \frac{5 \theta}{2}+\left(\frac{1}{2} B^{(3 / 2)} f-\frac{3 A^{(3 / 2)} f^{2}}{8}+\frac{V f f_{t}}{4 \alpha^{2}}\right) r^{-1 / 2} \sin \frac{\theta}{2}\right. \\
& +\left(\frac{15}{8} f^{2} A^{(5 / 2)}-\frac{3}{2} f B^{(5 / 2)}+C^{(5 / 2)}+\frac{V f B_{t}^{(3 / 2)}}{2 \alpha^{2}}-\frac{3 A^{(3 / 2)} V f f_{t}}{4 \alpha^{2}}+\frac{f f_{t t}}{4 \alpha^{4}}-\frac{f f_{z z}}{4 \alpha^{2}}\right) r^{1 / 2} \sin \frac{\theta}{2} \\
& +\left(-\frac{V f B_{t}^{(3 / 2)}}{4 \alpha^{2}}-\frac{V f_{t} B^{(3 / 2)}}{4 \alpha^{2}}+\frac{3 A^{(3 / 2)} V f f_{t}}{8 \alpha^{2}}-\frac{\left(1+V^{2}\right)\left(f_{t}^{2}+f f_{t t}\right)}{8 \alpha^{4}}+\frac{f f_{z z}}{8 \alpha^{2}}+\frac{f_{z}^{2}}{8 \alpha^{2}}\right) r^{1 / 2} \sin \frac{3 \theta}{2}  \tag{36}\\
& \left.+\frac{V^{2}\left(f_{t}^{2}+f f_{t t}\right)}{32 \alpha^{4}} r^{1 / 2} \sin \frac{7 \theta}{2}\right\} .
\end{align*}
$$

### 3.2. The outer solution

We derive an asymptotic solution for the outer problem, where the fields are expanded around the position of the unperturbed crack front. Let us consider an expansion involving a single Fourier component $u=u^{(0)}+\epsilon u^{(1)}+\epsilon^{2} u^{(2)}+\ldots=u^{(0)}+\sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{\left(\epsilon q^{(1)}(x, y)+\epsilon^{2} q^{(2)}(x, y)\right) e^{i(k z+\omega t)}\right\}$, where superscripts mark the order of each term in $\epsilon$. Here

$$
\begin{equation*}
u^{(0)}=\sqrt{\frac{2}{\pi}} \frac{K_{0}^{u}}{\mu} \operatorname{Im}\left\{s^{1 / 2}+\frac{m}{3} s^{3 / 2}+\frac{n}{5} s^{5 / 2}\right\} \tag{37}
\end{equation*}
$$

where $s, m$ and $n$ are the same as in the previous sections. The expansion is substituted in Eq. (21), which is then solved term-by-term. Once a solution is found, the fields due to arbitrary perturbations can be computed by superposing Fourier components. To affect the matching of the inner and outer solutions, explicit expressions of $q^{(i)}$ will be equated with expressions (34-36). Each $q^{(i)}(x, y)$ satisfies the equation

$$
\begin{equation*}
\alpha^{2} q_{x x}+q_{y y}+\left(\omega^{2}-k^{2}\right) q+2 i V \omega q_{x}=0 \tag{38}
\end{equation*}
$$

A general solution to this equation is

$$
\begin{equation*}
q=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} \xi \hat{q}(\xi) e^{i \omega \xi x+\omega \gamma(\xi) y}, \tag{39}
\end{equation*}
$$

where $\gamma^{2}=\alpha^{2} \xi^{2}+2 V \xi+(k / \omega)^{2}-1$. Without loss of generality, we assume that $\omega>0$. To ensure the convergence of the integral (39), we must take the branch of $\gamma$ which has a positive real part. $\gamma$ has two branch points, $\xi=-\lambda_{+}$and $\xi=\lambda_{-}$where

$$
\begin{equation*}
\lambda_{ \pm}=\frac{1}{\alpha^{2}}\left(1-\alpha^{2} k^{2} / \omega^{2}\right)^{1 / 2} \pm \frac{V}{\alpha^{2}} . \tag{40}
\end{equation*}
$$

It is convenient, then, to take one branch cut extending from $\xi=-\lambda_{+}$to $-i \infty$ and the other from $\xi=\lambda_{-}$to $+i \infty$. As we shall see, the exact shape of the branch cuts does not affect our calculations.

To determine $\hat{q}(\xi)$, we employ the boundary conditions Eq. (22). In terms of $q$ these boundary conditions are

$$
\begin{array}{r}
q(x, 0)=0 ; x>0,  \tag{41}\\
q_{y}(x, 0)=0 ; x<0 .
\end{array}
$$

Fourier transforming the boundary conditions results in the relations

$$
\begin{align*}
\hat{q}(\xi) & =\int_{-\infty}^{0} \mathrm{~d} x q(x, 0) e^{-i \omega \xi x}  \tag{42}\\
-\omega \gamma(\xi) \hat{q}(\xi) & =\int_{0}^{\infty} \mathrm{d} x q_{y}(x, 0) e^{-i \omega \xi x}
\end{align*}
$$

These relations provide us with information about the analytical domain of $\hat{q}(\xi)$ and $\gamma(\xi) \hat{q}(\xi)$. The function $\hat{q}(\xi)$ is analytic for $\operatorname{Im}\{\xi\}<0$ and the function $\gamma(\xi) \hat{q}(\xi)$ is analytic for $\operatorname{Im}\{\xi\}>0$. We decompose $\gamma=\gamma^{+} \gamma^{-}$, where $\gamma^{ \pm}=\alpha^{1 / 2}\left(\xi \pm \lambda_{ \pm}\right)^{1 / 2}$ and $\gamma^{+}\left(\gamma^{-}\right)$is analytic for $\operatorname{Im}\{\xi\}>0(\operatorname{Im}\{\xi\}<0)$, according to our choice of the branch of $\gamma$. Then, we readily see that $W \equiv \gamma^{-} \hat{q}=\gamma \hat{q} / \gamma^{+}$is an entire function which can generally be represented as a power series in $\xi$. In fact, $\gamma^{-} q^{(n)}(x, y)$ should be a polynomial of order $n-1$. This can be understood from the previous section, where at each order the solution in the inner variables becomes progressively more singular. As $\xi$ is the Fourier conjugate of $x$, stronger singularities of $x$ translate into higher powers of $\xi$.

Let us consider the $1^{\text {st }}$-order term $q^{(1)}(x, y)$, and set $\hat{q}^{(1)}(\xi)=q_{0} / \gamma^{-}$. For convenience, we write the $x, y$ coordinates as $\operatorname{Re}\{s\}, \operatorname{Im}\{s\}$ respectively, and expand the integral in Eq. (39) in powers of $1 / \xi$ (note that $O(s \xi) \sim O(1))$

$$
\begin{align*}
& q^{(1)}(x, y)=\frac{1}{2 \pi} \int \mathrm{~d} \xi \frac{q_{0}}{\alpha^{1 / 2}\left(\xi-\lambda_{-}\right)^{1 / 2}} e^{i \omega \xi \operatorname{Re}\{s\}+\omega \gamma \operatorname{Im}\{s\}}= \\
& \frac{q_{0}}{2 \pi \alpha^{1 / 2}} \int \mathrm{~d} \xi e^{\omega|\xi| \operatorname{Im}\{s\}+i \xi \omega \operatorname{Re}\{s\}}\left(\frac{1}{\sqrt{\xi}}+\frac{\lambda_{-}-\omega|\xi| \operatorname{Im}\{s\}\left(\lambda_{-}-\lambda_{+}\right)}{2 \xi^{3 / 2}}\right.  \tag{43}\\
& \left.+\frac{3 \lambda_{-}^{2}+\xi^{2} \omega^{2} \operatorname{Im}\{s\}^{2}\left(\lambda_{-}-\lambda_{+}\right)^{2}-\omega|\xi| \operatorname{Im}\{s\}\left(3 \lambda_{-}^{2}+\lambda_{+}^{2}\right)}{8 \xi^{5 / 2}}+O\left(\xi^{-7 / 2}\right)\right) .
\end{align*}
$$

We integrate this expression using the identity

$$
\begin{align*}
\frac{1}{2 \pi} \int \mathrm{~d} \xi & \frac{e^{\omega|\xi| \operatorname{Im}\{s\}+i \xi \omega \operatorname{Re}\{s\}}}{\xi^{n+1 / 2}}[1, \operatorname{sgn} \xi]= \\
& -\frac{2^{2 n} n!}{(2 n)!} \frac{e^{i \pi / 4}}{\sqrt{\pi}}(-i)^{n}|\omega s|^{n-1 / 2}\left[\sin \left(n-\frac{1}{2}\right) \theta, i \cos \left(n-\frac{1}{2}\right) \theta\right] \tag{44}
\end{align*}
$$

where $s=r e^{i \theta}$ and $n \geq 0$. A similar identity holds for the transformation $n \rightarrow-n$,

$$
\begin{align*}
\frac{1}{2 \pi} \int \mathrm{~d} \xi & \frac{e^{\omega|\xi| \operatorname{Im}\{s\}+i \xi \omega \operatorname{Re}\{s\}}}{\xi^{-n+1 / 2}}[1, \operatorname{sgn} \xi]= \\
& \frac{(2 n)!}{2^{2 n} n!} \frac{e^{i \pi / 4}}{\sqrt{\pi}} i^{-n}|\omega s|^{-n-1 / 2}\left[\sin \left(n-\frac{1}{2}\right) \theta,-i \cos \left(n-\frac{1}{2}\right) \theta\right] \tag{45}
\end{align*}
$$

Integrating and substituting $s=r e^{i \theta}$, the $1^{\text {st }}$-order solution is

$$
\begin{align*}
& q^{(1)}(x, y)=\frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}\left[\frac{\sin \frac{\theta}{2}}{\sqrt{r}}+\sqrt{r}\left(-\frac{1}{4} i \omega \sin \frac{\theta}{2}\left(3 \lambda_{-}+\lambda_{+}\right)+\frac{1}{4} i \omega \sin \frac{3 \theta}{2}\left(\lambda_{-}-\lambda_{+}\right)\right)\right. \\
& +r^{3 / 2}\left(-\frac{1}{16} \omega^{2} \sin \frac{\theta}{2}\left(5 \lambda_{-}^{2}+2 \lambda_{-} \lambda_{+}+\lambda_{+}^{2}\right)+\frac{1}{32} \omega^{2} \sin \frac{3 \theta}{2}\left(5 \lambda_{-}^{2}-2 \lambda_{-} \lambda_{+}-3 \lambda_{+}^{2}\right)\right.  \tag{46}\\
& \left.\left.-\frac{1}{32} \omega^{2} \sin \frac{5 \theta}{2}\left(\lambda_{-}-\lambda_{+}\right)^{2}\right)\right] .
\end{align*}
$$

Substituting the explicit expressions (40) for $\lambda_{ \pm}$, the $1^{\text {st }}$-order solution becomes

$$
\begin{align*}
& q^{(1)}(x, y)=\frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}\left[r^{-1 / 2} \sin \frac{\theta}{2}+\left(\frac{i V \omega}{2 \alpha^{2}}+\widehat{\Pi}\right) r^{1 / 2} \sin \frac{\theta}{2}-\frac{i V \omega}{2 \alpha^{2}} r^{1 / 2} \sin \frac{3 \theta}{2}\right. \\
& +\left(\frac{k^{2}}{2 \alpha^{2}}-\frac{\omega^{2}}{2 \alpha^{4}}-\frac{V^{2} \omega^{2}}{4 \alpha^{4}}+\frac{i V \omega \widehat{\Pi}}{2 \alpha^{2}}\right) r^{3 / 2} \sin \frac{\theta}{2}  \tag{47}\\
& \left.+\left(\frac{V^{2} \omega^{2}}{8 \alpha^{4}}-\frac{i V \omega \widehat{\Pi}}{2 \alpha^{2}}\right) r^{3 / 2} \sin \frac{3 \theta}{2}-\frac{V^{2} \omega^{2}}{8 \alpha^{4}} r^{3 / 2} \sin \frac{5 \theta}{2}\right]
\end{align*}
$$

Here, we have introduced a new symbol $\widehat{\Pi}$ with the definition

$$
\begin{align*}
& \widehat{\Pi}(k, \omega)=-i \alpha^{-2} \operatorname{sign}(\omega) \sqrt{\omega^{2}-\alpha^{2} k^{2}} ; \omega^{2}>\alpha^{2} k^{2}  \tag{48}\\
& \widehat{\Pi}(k, \omega)=-\alpha^{-2} \sqrt{\alpha^{2} k^{2}-\omega^{2}} ; \omega^{2}<\alpha^{2} k^{2} .
\end{align*}
$$

The $2^{\text {nd }}$-order field $q^{(2)}(x, y)$ is similarly found by writing $\hat{q}(\xi)=\left(q_{1}+q_{2} \xi\right) / \gamma^{-}$, so that

$$
\begin{equation*}
q^{(2)}(x, y)=\frac{1}{2 \pi} \int \mathrm{~d} \xi \frac{q_{1}+q_{2} \xi}{\alpha^{1 / 2}\left(\xi-\lambda_{-}\right)^{1 / 2}} e^{i \omega \xi \operatorname{Re}\{s\}+\omega \gamma \operatorname{Im}\{s\}} \tag{49}
\end{equation*}
$$

This integral can be separated into two parts, proportional to $q_{1}$ and $q_{2}$ respectively. We have already calculated the first part in Eq. (47). The order of the $r^{3 / 2}$ terms is higher than $2^{\text {nd }}$-order since the $2^{\text {nd }}$-order inner solution in Eq. (36) does not contain them.

It remains therefore to calculate

$$
\begin{align*}
& \frac{1}{2 \pi} \int \mathrm{~d} \xi \frac{q_{2} \xi}{\alpha^{1 / 2}\left(\xi-\lambda_{-}\right)^{1 / 2}} e^{i \omega \xi \operatorname{Re}\{s\}+\omega \gamma \operatorname{Im}\{s\}}= \\
& \frac{q_{2}}{2 \pi \alpha^{1 / 2}} \int \mathrm{~d} \xi e^{\omega|\xi| \operatorname{Im}\{s\}+i \xi \omega \operatorname{Re}\{s\}}\left(\sqrt{\xi}+\frac{\lambda_{-}-\omega|\xi| \operatorname{Im}\{s\}\left(\lambda_{-}-\lambda_{+}\right)}{2 \sqrt{\xi}}\right.  \tag{50}\\
& \left.+\frac{3 \lambda_{-}^{2}+\omega^{2}|\xi|^{2} \operatorname{Im}\{s\}^{2}\left(\lambda_{-}-\lambda_{+}\right)^{2}-\omega|\xi| \operatorname{Im}\{s\}\left(3 \lambda_{-}^{2}+\lambda_{+}^{2}\right)}{8 \xi^{3 / 2}}+O\left(\xi^{-5 / 2}\right)\right)
\end{align*}
$$

This is done in the same way as the $1^{\text {st }}$-order and the result in polar coordinates is

$$
\begin{align*}
& -\frac{q_{2} i e^{i \pi / 4}}{2 \sqrt{\pi} \alpha^{1 / 2}(-\omega)^{3 / 2}}\left[r^{-3 / 2} \sin \frac{3 \theta}{2}+\left(\frac{3 i V \omega}{2 \alpha^{2}}+\widehat{\Pi}\right) r^{-1 / 2} \sin \frac{\theta}{2}-\frac{i V \omega}{2 \alpha^{2}} r^{-1 / 2} \sin \frac{5 \theta}{2}\right. \\
& +\left(\frac{k^{2}}{\alpha^{2}}-\frac{\omega^{2}}{\alpha^{4}}-\frac{7 V^{2} \omega^{2}}{8 \alpha^{4}}+\frac{5 i V \omega \widehat{\Pi}}{2 \alpha^{2}}\right) r^{1 / 2} \sin \frac{\theta}{2}+\left(-\frac{k^{2}}{2 \alpha^{2}}+\frac{\omega^{2}}{2 \alpha^{4}}+\frac{3 V^{2} \omega^{2}}{4 \alpha^{4}}-\frac{i V \omega \widehat{\Pi}}{2 \alpha^{2}}\right) r^{1 / 2} \sin \frac{3 \theta}{2}  \tag{51}\\
& \left.-\frac{V^{2} \omega^{2}}{8 \alpha^{4}} r^{1 / 2} \sin \frac{7 \theta}{2}\right] .
\end{align*}
$$

The expression for the outer $2^{\text {nd }}$-order field becomes

$$
\begin{align*}
& q^{(2)}(x, y)=-\frac{q_{2} i e^{i \pi / 4}}{2 \sqrt{\pi} \alpha^{1 / 2}(-\omega)^{3 / 2}}\left[r^{-3 / 2} \sin \frac{3 \theta}{2}+\left(\frac{3 i V \omega}{2 \alpha^{2}}+\widehat{\Pi}\right) r^{-1 / 2} \sin \frac{\theta}{2}-\frac{i V \omega}{2 \alpha^{2}} r^{-1 / 2} \sin \frac{5 \theta}{2}\right. \\
& +\left(\frac{k^{2}}{\alpha^{2}}-\frac{\omega^{2}}{\alpha^{4}}-\frac{7 V^{2} \omega^{2}}{8 \alpha^{4}}+\frac{5 i V \omega}{2 \alpha^{2}}\right) r^{1 / 2} \sin \frac{\theta}{2}+\left(-\frac{k^{2}}{2 \alpha^{2}}+\frac{\omega^{2}}{2 \alpha^{4}}+\frac{3 V^{2} \omega^{2}}{4 \alpha^{4}}-\frac{i V \omega \widehat{\Pi}}{2 \alpha^{2}}\right) r^{1 / 2} \sin \frac{3 \theta}{2}  \tag{52}\\
& \left.-\frac{V^{2} \omega^{2}}{8 \alpha^{4}} r^{1 / 2} \sin \frac{7 \theta}{2}\right] \\
& +\frac{q_{1} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}\left[r^{-1 / 2} \sin \frac{\theta}{2}+\left(\frac{i V \omega}{2 \alpha^{2}}+\widehat{\Pi}\right) r^{1 / 2} \sin \frac{\theta}{2}-\frac{i V \omega}{2 \alpha^{2}} r^{1 / 2} \sin \frac{3 \theta}{2}\right]
\end{align*}
$$

### 3.3. Matching the solutions

In the two previous sections, we have derived the asymptotic solutions up till the $2^{\text {nd }}$-order in $\epsilon$ for the inner fields Eqs. (34-36) and for the outer fields Eqs. $(37,47,52)$. The expressions we have derived contain
five inner coefficients $A^{(3 / 2)}, A^{(5 / 2)}, B^{(3 / 2)}, B^{(5 / 2)}, C^{(5 / 2)}$ and three outer coefficients $q_{0}, q_{1}, q_{2}$, which will be found by matching the inner and outer solutions. We note that, in general, the coefficients may be functions of $(z, t)$.

To match the terms in the two solutions we equate, order-by-order the prefactors of the polar functions $r^{-j / 2} \sin (j \theta / 2)$, where $j$ is an integer. The matching is made less straightforward by the fact that the inner solution is expressed in the real space coordinates $(z, t)$ and the outer solution is expressed in the Fourier coordinates $(k, \omega)$. However, we will stick to the present notation, with the implicit understanding that the outer solution terms must be first Fourier transformed before matching to the corresponding inner solution terms. We also adopt the notation $\Pi[f](z, t)=\frac{1}{(2 \pi)^{2}} \int \mathrm{~d} \omega \mathrm{~d} k e^{i(k z+\omega t)} \widehat{\Pi}(\omega, k) f(\omega, k)$. The final expressions will always be given in the $(z, t)$ space.

| Term | Inner solution | Outer solution |
| :---: | :---: | :---: |
| $\sqrt{r} \sin \frac{\theta}{2}$ | $K_{0}^{u}$ | $K_{0}^{u}$ |
| $r^{3 / 2} \sin \frac{3 \theta}{2}$ | $A^{(3 / 2)}$ | $\frac{m}{3}$ |
| $r^{5 / 2} \sin \frac{5 \theta}{2}$ | $A^{(5 / 2)}$ | $\frac{n}{5}$ |

Table 1: Terms of order $\epsilon^{0}$

| Term | Inner solution | Outer solution |
| :---: | :---: | :---: |
| $r^{-1 / 2} \sin \frac{\theta}{2}$ | $\frac{f}{2}$ | $\frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}$ |
| $r^{1 / 2} \sin \frac{\theta}{2}$ | $B^{(3 / 2)}-\frac{3}{2} A^{(3 / 2)} f$ | $\frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}\left(\frac{i V \omega}{2 \alpha^{2}}+\widehat{\Pi}\right)$ |
| $r^{1 / 2} \sin \frac{3 \theta}{2}$ | $-\frac{V f_{t}}{4 \alpha^{2}}$ | $-\frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}} \frac{i V \omega}{2 \alpha^{2}}$ |
| $r^{3 / 2} \sin \frac{\theta}{2}$ | $B_{t}^{(3 / 2)} \frac{V}{2 \alpha^{2}}-A^{(3 / 2)} \frac{3 V f_{t}}{4 \alpha^{2}}+\frac{f_{t t}}{4 \alpha^{4}}-\frac{f_{z z}}{4 \alpha^{2}}$ | $\frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}\left(\frac{k^{2}}{2 \alpha^{2}}-\frac{\omega^{2}}{2 \alpha^{4}}-\frac{V^{2} \omega^{2}}{4 \alpha^{4}}+\frac{i V \omega \widehat{\Pi}}{2 \alpha^{2}}\right)$ |
| $r^{3 / 2} \sin \frac{3 \theta}{2}$ | $B^{(5 / 2)}-\frac{5}{2} A^{(5 / 2)} f$ | $\frac{q_{0} i \pi / 4}{\sqrt{-\pi \omega} \alpha^{1 / 2}}\left(\frac{V^{2} \omega^{2}}{8 \alpha^{4}}-\frac{i V \omega \bar{\Pi}}{2 \alpha^{2}}\right)$ |
| $r^{3 / 2} \sin \frac{5 \theta}{2}$ | $\frac{V^{2} f_{t t}}{16 \alpha^{4}}$ | $-\frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}} \frac{V^{2} \omega^{2}}{8 \alpha^{4}}$ |

Table 2: Terms of order $\epsilon^{1}$

| Term | Inner solution | Outer solution |
| :---: | :---: | :---: |
| $r^{-3 / 2} \sin \frac{3 \theta}{2}$ | $\frac{f^{2}}{8}$ | $-\frac{q_{2} i e^{i \pi / 4}}{2 \sqrt{\pi} \alpha^{1 / 2}(-\omega)^{3 / 2}}$ |
| $r^{-1 / 2} \sin \frac{\theta}{2}$ | $\frac{1}{2} B^{(3 / 2)} f-A^{(3 / 2)} \frac{3 f^{2}}{8}+\frac{V f f_{t}}{4 \alpha^{2}}$ | $-\frac{q_{2} i e^{i \pi / 4}}{2 \sqrt{\pi} \alpha^{1 / 2}(-\omega)^{3 / 2}}\left(\frac{3 i V \omega}{2 \alpha^{2}}+\widehat{\Pi}\right)+\frac{q_{1} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}$ |
| $r^{-1 / 2} \sin \frac{5 \theta}{2}$ | $-\frac{V f f_{t}}{8 \alpha^{2}}$ | $\frac{q_{2} i e^{i \pi / 4}}{2 \sqrt{\pi} \alpha^{1 / 2}(-\omega)^{3 / 2}} \frac{i V \omega}{2 \alpha^{2}}$ |
| $r^{1 / 2} \sin \frac{\theta}{2}$ | $\frac{15}{8} f^{2} A^{(5 / 2)}-\frac{3}{2} f B^{(5 / 2)}+C B^{(5 / 2)}$ <br> $+\frac{V \alpha^{2}}{(3 / 2)}-\frac{3 A^{(3 / 2)} V f f_{t}}{4 \alpha^{2}}+\frac{f f_{t t}}{4 \alpha^{4}}-\frac{f f_{z z}}{4 \alpha^{2}}$ | $\left.\begin{array}{l}-\frac{q_{2} i e^{i \pi / 4}}{2 \sqrt{\pi} \alpha^{1 / 2}(-\omega)^{3 / 2}}\left(\frac{k^{2}}{\alpha^{2}}-\frac{\omega^{2}}{\alpha^{4}}-\frac{7 V^{2} \omega^{2}}{8 \alpha^{4}}\right. \\ 2 \alpha^{2}\end{array}\right)+\frac{q_{1} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}\left(\frac{i V \omega}{2 \alpha^{2}}+\widehat{\Pi}\right)$ |

Table 3: Terms of order $\epsilon^{2}$
The zeroth order matching in Table 1 yields $A^{(3 / 2)}=\frac{m}{3}$ and $A^{(5 / 2)}=\frac{n}{5}$. The $1^{\text {st }}$-order matching is given in Table 2. The first row yields an expression for $q_{0} ; \frac{q_{0} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}=\frac{f}{2}$. The second row translates into

$$
\begin{equation*}
B^{(3 / 2)}=\frac{m}{2} f+\frac{V}{4 \alpha^{2}} f_{t}+\frac{1}{2} \Pi[f] . \tag{53}
\end{equation*}
$$

Given the expressions for $A^{(3 / 2)}, A^{(5 / 2)}, q_{0}$ and $B^{(3 / 2)}$, the third, fourth and sixth rows of Table 2 become identities. This is a consistency check that our calculations so far have not contained a mistake. The fifth row yields new information:

$$
\begin{equation*}
B^{(5 / 2)}=\frac{n}{2} f-\frac{V^{2}}{16 \alpha^{4}} f_{t t}-\frac{V}{4 \alpha^{2}} \Pi\left[f_{t}\right] . \tag{54}
\end{equation*}
$$

The $2^{\text {nd }}$-order matching in Table 3 provides the three remaining coefficients $q_{1}, q_{2}$ and $C^{5 / 2}$. The first row of the table translates into $-\frac{q_{2} i e^{i \pi / 4}}{2 \sqrt{\pi} \alpha^{1 / 2}(-\omega)^{3 / 2}}=\frac{f^{2}}{8}$. The determination of $q_{2}$ makes the third and fifth rows of Table 3 an identity. The second row gives $q_{1}$ through the equation

$$
\begin{equation*}
\frac{q_{1} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}=\frac{1}{2}\left(\frac{m}{2} f+\frac{V}{4 \alpha^{2}} f_{t}+\frac{1}{2} \Pi[f]\right) f-\frac{m f^{2}}{8}+\frac{V f f_{t}}{4 \alpha^{2}}-\frac{1}{8}\left(\frac{3 V f f_{t}}{\alpha^{2}}+\Pi\left[f^{2}\right]\right), \tag{55}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{q_{1} e^{i \pi / 4}}{\sqrt{-\pi \omega} \alpha^{1 / 2}}=\frac{m f^{2}}{8}+\frac{1}{4} \Pi[f] f-\frac{1}{8} \Pi\left[f^{2}\right] \tag{56}
\end{equation*}
$$

The fourth row of Table 3 is the center of this calculation since it would give the $2^{\text {nd }}$-order correction for the energy-release-rate $G$. Substituting the expressions for $A^{(5 / 2)}$ and $B^{(5 / 2)}$ the inner solution cell of the fourth row is

$$
\begin{equation*}
-\frac{3}{8} n f^{2}+\frac{8+7 V^{2}}{32 \alpha^{4}} f f_{t t}+\frac{5 V}{8 \alpha^{2}} f \Pi\left[f_{t}\right]+C^{(5 / 2)}-\frac{f f_{z z}}{4 \alpha^{2}}, \tag{57}
\end{equation*}
$$

while the outer solution cell is

$$
\begin{align*}
& -\frac{1}{4 \alpha^{2}}\left(f_{z}^{2}+f f_{z z}\right)+\frac{\left(8+7 V^{2}\right)}{32 \alpha^{4}}\left(f_{t}^{2}+f f_{t t}\right)+\frac{V \Pi\left[f f_{t}\right]}{2 \alpha^{2}}+\frac{m V f f_{t}}{8 \alpha^{2}}  \tag{58}\\
& +\frac{V}{8 \alpha^{2}}\left(f \Pi\left[f_{t}\right]+f_{t} \Pi[f]\right)+\frac{m \Pi\left[f^{2}\right]}{8}+\frac{1}{4} \Pi[\Pi[f] f]-\frac{1}{8} \Pi\left[\Pi\left[f^{2}\right]\right]
\end{align*}
$$

Since $\widehat{\Pi}^{2}=-\frac{1}{\alpha^{4}}\left(\omega^{2}-\alpha^{2} k^{2}\right)$, equating these two expressions yields a formula for the last remaining coefficient $C^{(5 / 2)}$ :

$$
\begin{align*}
& C^{(5 / 2)}=\frac{3}{8} n f^{2}+\frac{7 V^{2}}{32 \alpha^{4}} f_{t}^{2}-\frac{1}{4 \alpha^{4}} f f_{t t}-\frac{V}{2 \alpha^{2}} f \Pi\left[f_{t}\right]+\frac{1}{4 \alpha^{2}} f f_{z z}+\frac{V \Pi\left[f f_{t}\right]}{2 \alpha^{2}} \\
& +\frac{m V f f_{t}}{8 \alpha^{2}}+\frac{V}{8 \alpha^{2}} f_{t} \Pi[f]+\frac{m \Pi\left[f^{2}\right]}{8}+\frac{1}{4} \Pi[\Pi[f] f] \tag{59}
\end{align*}
$$

### 3.4. Calculation of the energy-release-rate

The previous sections dealt with the derivation of the asymptotic field $u$ up to the $2^{\text {nd }}$-order in the perturbation to the crack front. The expressions derived in the last section for the undetermined coefficients of the inner solution can now be inserted into Eqs. (34-36) to find the explicit dependence of $u$ on $f$. To calculate the energy-release-rate $G$, it is simpler to use Eqs. (27-30). Then, the displacement and stress intensity factors are given respectively by

$$
\begin{align*}
& K^{u}=\lim _{X \uparrow f(z, t)} \epsilon^{-1 / 2}(X-f(z, t))^{-1 / 2} \sqrt{\frac{\pi}{2}} \mu U(X, 0, z, t)  \tag{60}\\
& K^{\sigma}=\sqrt{2 \pi} \lim _{X \downarrow f(z, t)} \epsilon^{-1 / 2} \sqrt{2 \pi(X-f(z, t))} \mu U_{Y}(X, 0, z, t) \tag{61}
\end{align*}
$$

Some algebra leads to the two $2^{\text {nd }}$-order expressions

$$
\begin{align*}
K^{u}= & K_{0}^{u}\left(1+\epsilon\left(B^{(3 / 2)}+\frac{V f_{t}}{4 \alpha^{2}}\right)\right. \\
& \left.+\epsilon^{2}\left(C^{(5 / 2)}+\frac{V B^{(3 / 2)} f_{t}}{4 \alpha^{2}}+\frac{f_{t}^{2}}{8 \alpha^{4}}+\frac{3 V^{2} f_{t}^{2}}{32 \alpha^{4}}-\frac{f_{z}^{2}}{8 \alpha^{2}}\right)\right)  \tag{62}\\
K^{\sigma}= & \sqrt{\frac{\pi}{2}} \alpha K_{0}^{u}\left(1+\epsilon\left(B^{(3 / 2)}-\frac{3 V f_{t}}{4 \alpha^{2}}\right)\right. \\
& \left.+\epsilon^{2}\left(C^{(5 / 2)}-\frac{3 V B^{(3 / 2)} f_{t}}{4 \alpha^{2}}-\frac{3 f_{t}^{2}}{8 \alpha^{4}}-\frac{5 V^{2} f_{t}^{2}}{32 \alpha^{4}}+\frac{3 f_{z}^{2}}{8 \alpha^{2}}\right)\right) . \tag{63}
\end{align*}
$$

The energy-release-rate is given by their product

$$
\begin{array}{r}
G \propto K^{u} K^{\sigma}=\alpha\left(K_{0}^{u}\right)^{2}\left(1+\epsilon\left(2 B^{(3 / 2)}-\frac{V f_{t}}{2 \alpha^{2}}\right)\right. \\
\left.+\epsilon^{2}\left(\left(B^{(3 / 2)}\right)^{2}+2 C^{(5 / 2)}-\frac{V B^{(3 / 2)} f_{t}}{\alpha^{2}}-\frac{\left(1+V^{2}\right) f_{t}^{2}}{4 \alpha^{4}}+\frac{f_{z}^{2}}{4 \alpha^{2}}\right)\right) \tag{64}
\end{array}
$$

Using Eqs. $(53,59)$ derived in the last section for $B^{(3 / 2)}$ and $C^{(5 / 2)}$, we write

$$
\begin{equation*}
G=G_{r} g(V)\left(1+\epsilon \delta G^{(1)}+\epsilon^{2} \delta G^{(2)}+O\left(f^{3}\right)\right) \tag{65}
\end{equation*}
$$

where the $1^{5 t}$-order correction is

$$
\begin{equation*}
\delta G^{(1)}=m f+\Pi[f] \tag{66}
\end{equation*}
$$

and the $2^{\text {nd }}$-order correction is

$$
\begin{align*}
\delta G^{(2)}= & \frac{1}{4} m^{2} f^{2}+\frac{3}{4} n f^{2}+\frac{1}{2} m f \Pi[f]+\frac{1}{4} m \Pi\left[f^{2}\right] \\
& +\frac{1}{4} \Pi[f]^{2}+\frac{1}{2} \Pi[f \Pi[f]]-\frac{V f \Pi\left[f_{t}\right]}{\alpha^{2}}+\frac{V \Pi\left[f f_{t}\right]}{\alpha^{2}}-\frac{f_{t}^{2}}{4 \alpha^{4}}-\frac{f f_{t t}}{2 \alpha^{4}}+\frac{f_{z}^{2}}{4 \alpha^{2}}+\frac{f f_{z z}}{2 \alpha^{2}} \tag{67}
\end{align*}
$$

The $1^{\text {st }}$-order correction (66) was derived by Norris and Abrahams (2007), where it was shown that the term $m f$ results in wave dispersion. This result is possibly related to the observations of small amplitude wave dispersion reported in Fineberg et al. (2003). Re-introducing dimensions, the coefficient $m$ scales as $1 / l$, and therefore the term $m f \sim O(f / l)$ becomes negligible when $f \ll l$. The term $\Pi[f]$, however, remains, as it does not depend on sample geometry or loading conditions. In the following, we will assume that $f / l \rightarrow 0$ and put $m=n=0$. The expansion of the energy-release-rate (65) can be rewritten as a product of dynamical and "historical" contributions, as in Eq. (1). To show that we follow Morrissey and Rice (2000) and convert $\widehat{\Pi}(k, \omega)$ to the time domain using the identity

$$
\begin{equation*}
\int \mathrm{d} t e^{i \omega t}\left(\partial_{t}+\frac{p \Theta(t) J_{1}(p t)}{t}\right)=-i \operatorname{sign}(\omega) \sqrt{\omega^{2}-p^{2}} \tag{68}
\end{equation*}
$$

where $\Theta$ is the Heaviside function and $J_{1}$ is the $1^{\text {st }}$-order Bessel function. Then, $\Pi(k, t)=-\alpha^{-2}\left(\partial_{t}+\Psi(k, t)\right)$, where $\Psi(k, t)=\alpha k\left(\Theta(t) J_{1}(\alpha k t) / t\right) *$, with $*$ representing a convolution in time. The appearance of the Heaviside function in $\Psi[f]$ signifies the dependence of real space functional $\Pi[f](z, t)$ on the past history of the front. With this transformation Eq. (65) becomes (setting $m=n=0$ and $\epsilon=1$ )

$$
\begin{align*}
G= & G_{r} g(V)\left\{1-\frac{f_{t}}{\alpha^{2}}-\frac{1}{\alpha^{2}} \Psi[f]+\frac{1}{4 \alpha^{4}} \Psi[f]^{2}+\frac{1}{2 \alpha^{4}} \Psi[f \Psi[f]]+\right. \\
& \left.\frac{1-2 V}{2 \alpha^{4}} \Psi\left[f f_{t}\right]+\frac{1+2 V}{2 \alpha^{4}} f \Psi\left[f_{t}\right]+\frac{1}{\alpha^{4}} f_{t} \Psi[f]+\frac{1-2 V}{2 \alpha^{4}} f_{t}^{2}+\frac{1}{4 \alpha^{2}} f_{z}^{2}+\frac{1}{2 \alpha^{2}} f f_{z z}\right\} . \tag{69}
\end{align*}
$$

To write this expression under the form given by Eq. (1), we develop $V_{\perp}=V+f_{t}-\frac{V}{2} f_{z}^{2}+O\left(f^{3}\right)$ so that

$$
\begin{equation*}
g\left(V_{\perp}\right)=g(V)\left(1-\frac{f_{t}}{\alpha^{2}}+\frac{V}{2 \alpha^{2}} f_{z}^{2}+\frac{1-2 V}{2 \alpha^{4}} f_{t}^{2}+O\left(f^{3}\right)\right) \tag{70}
\end{equation*}
$$

Inserting this expression into Eq. (1) we have

$$
\begin{equation*}
G=G_{r} g(V)\left(1-\frac{f_{t}}{\alpha^{2}}+H^{(1)}[f]+\frac{V}{2 \alpha^{2}} f_{z}^{2}+\frac{1-2 V}{2 \alpha^{4}} f_{t}^{2}-\frac{f_{t}}{\alpha^{2}} H^{(1)}[f]+H^{(2)}[f]\right) \tag{71}
\end{equation*}
$$

A direct comparison between Eqs. (69) and (71) then yields Eq. (3) for $H^{(1)}[f]$ and

$$
\begin{equation*}
H^{(2)}[f]=\frac{1}{4 \alpha^{4}} \Psi[f]^{2}+\frac{1}{2 \alpha^{4}} \Psi[f \Psi[f]]+\frac{1-2 V}{2 \alpha^{4}} \Psi\left[f f_{t}\right]+\frac{1+2 V}{2 \alpha^{4}} f \Psi\left[f_{t}\right]+\frac{1-2 V}{4 \alpha^{2}} f_{z}^{2}+\frac{1}{2 \alpha^{2}} f f_{z z} \tag{72}
\end{equation*}
$$

The expression for $H^{(2)}[f]$ can be further simplified through integration by parts in the time domain. Using

$$
\begin{align*}
\Psi\left[f_{t}\right] & =\alpha k \int_{-\infty}^{t} \frac{J_{1}\left(\alpha k\left(t-t^{\prime}\right)\right)}{t-t^{\prime}} f_{t}\left(k, t^{\prime}\right) d t^{\prime}  \tag{73}\\
& =\frac{\alpha^{2} k^{2}}{2} f(k, t)-\alpha^{2} k^{2} \int_{-\infty}^{t} \frac{J_{2}\left(\alpha k\left(t-t^{\prime}\right)\right)}{t-t^{\prime}} f\left(k, t^{\prime}\right) d t^{\prime}
\end{align*}
$$

the $2^{\text {nd }}$-order correction $H^{(2)}[f]$ is then given by Eq. (4).

## 4. Nonlinear perturbation of Mode I elastic crack fronts

In this section, we consider dynamic crack fronts in materials described by 3D elastodynamics which involves three displacement components $u_{i}(x, y, z, t)$ and a system of three scalar wave equations coupled by boundary conditions. The wave equations are a consequence of Newton's $2^{\text {nd }}$ law

$$
\begin{equation*}
\frac{\partial \sigma_{i x}}{\partial x}+\frac{\partial \sigma_{i y}}{\partial y}+\frac{\partial \sigma_{i z}}{\partial z}=\rho \frac{\partial^{2} u_{i}}{\partial t^{2}}, \tag{74}
\end{equation*}
$$

where the 3D stress $\sigma_{i j}$ is linearly related to the strain tensor through the Young modulus $E$ and the Poisson ratio $v$. Under pure Mode I loading, the boundary conditions comprise tensile loads applied at the remote boundaries and traction-free conditions on the crack faces, i.e.

$$
\begin{equation*}
\sigma_{i j} n_{j}=0 \tag{75}
\end{equation*}
$$

The perturbation scheme of planar crack fronts follows the PhD thesis of Ramanathan (1997). Parts of the calculation, that appear in the thesis, are reproduced below for completeness. The explicit $2^{\text {nd }}$-order expressions and the time-dependent formulation are the main contributions of this section. To compute the expansion of $G$ in $f$, Ramanathan's calculation aims at obtaining an asymptotic solution for the elastic fields of a running in-plane crack with a crack front defined by $h(z, t)=V t+f(z, t)$ (Fig. 1). The solution is to be obtained as a perturbation series around a straight front $f(z, t)=0$. More specifically, the $x$-origin of the system of coordinates is locally translated to the instantaneous position of the crack front ( $X \equiv x-h(z, t), y, z)$. Then the contribution of $f$ in the elastic field components is "eliminated". This latter condition is fulfilled by introducing $f(z, t)$ at the functional level through the transformation

$$
\begin{align*}
& u_{i}(X, y, z, t)=e^{f(z, t) \partial_{X}} U_{i}(X, y, z, t) \\
& \sigma_{i j}(X, y, z, t)=e^{f(z, t) \partial_{X}} \Sigma_{i j}(X, y, z, t) \tag{76}
\end{align*}
$$

The advantage of this transformation is that the new fields $U_{i}$ and $\Sigma_{i j}$ still follow the same constitutive linear elastic relations as $u_{i}$ and $\sigma_{i j}$. Moreover, Eq. (74) becomes

$$
\begin{equation*}
\frac{\partial \Sigma_{i x}}{\partial X}+\frac{\partial \Sigma_{i y}}{\partial y}+\frac{\partial \Sigma_{i z}}{\partial z}=\rho\left(V^{2} \frac{\partial^{2} U_{i}}{\partial X^{2}}-2 V \frac{\partial^{2} U_{i}}{\partial X \partial t}+\frac{\partial^{2} U_{i}}{\partial t^{2}}\right) . \tag{77}
\end{equation*}
$$

Using the symmetry of Mode I loading, the boundary conditions for the fields $U_{i}$ and $\Sigma_{i j}$ on the plane $y=0$ are given by

$$
\begin{align*}
\Sigma_{x y}(X, 0, z, t) & =\Sigma_{y z}(X, 0, z, t)=0  \tag{78}\\
U_{y}(X>0,0, z, t) & =0, \Sigma_{y y}(X<0,0, z, t)=0 \tag{79}
\end{align*}
$$

The fields $\Sigma_{i j}$ and $U_{i}$ should be computed from the solution of the elastodynamic equations in the reference frame ( $X, y, z$ ) with the prescribed boundary conditions. This is done by using the general relations obtained by Geubelle and Rice (1995) between the Fourier components of stress and displacement fields in the plane of fracture. Specifically, the stress component $\Sigma_{n}(X, z, t) \equiv \Sigma_{y y}(X, y=0, z, t)$ and the displacement component $U_{n}(X, z, t)=U_{y}\left(X, y=0^{+}, z, t\right)$ are related through

$$
\begin{equation*}
\widehat{\Sigma}_{n}(q, k, \omega)=\widehat{\Upsilon}_{y y}(q, k, \Omega) \widehat{U}_{n}(q, k, \omega), \tag{80}
\end{equation*}
$$

where the Fourier transforms are defined such that $\widehat{F}(q, k, \omega)=\int d X d z d t e^{-i q X-i k z-i \omega t} F(X, z, t)$ and

$$
\begin{equation*}
\hat{\Upsilon}_{y y}(q, k, \Omega)=-\mu \frac{p R\left(\frac{\Omega}{p}\right)}{\frac{\Omega^{2}}{b^{2} p^{2}} \sqrt{1-\frac{\Omega^{2}}{a^{2} p^{2}}}}, \tag{81}
\end{equation*}
$$

where $\mu$ is the shear modulus, $\Omega=\omega-q V, p=\sqrt{q^{2}+k^{2}}$ and $R(\zeta)$ is the Rayleigh function given by

$$
\begin{equation*}
R(\zeta)=4 \sqrt{1-\frac{\zeta^{2}}{a^{2}}} \sqrt{1-\frac{\zeta^{2}}{b^{2}}}-\left(2-\frac{\zeta^{2}}{b^{2}}\right)^{2} \tag{82}
\end{equation*}
$$

Here $a$ and $b$ are the longitudinal and shear wave speeds.

It is left to explicitly compute $\Sigma_{y y}$ and $U_{y}$ by satisfying the boundary condition (79). Notice that $\Sigma_{n}(X<0, z, t)=0$ (resp. $U_{n}(X<0, z, t)=0$ ) is equivalent to $\widehat{\Sigma}_{n}(q, k, \omega)$ (resp; $\left.\widehat{U}_{n}(q, k, \omega)\right)$ having its analytical domain encompassing the upper (resp. lower) $q$-half-plane. Then, given a decomposition $\widehat{\Upsilon}_{y y}=\widehat{\Upsilon}^{+} \widehat{\Upsilon}^{-}$where $\widehat{\Upsilon}^{+}$is analytical for $\operatorname{Im}(q)>0$ and $\widehat{\Upsilon}^{-}$is analytical for $\operatorname{Im}(q)<0$, there exists an analytical function $\widehat{W}(q, k, \omega)$ for all $q$ 's such that $\widehat{\Sigma}_{n}=\widehat{W} \widehat{\Upsilon}^{+}$and $\widehat{U}_{n}=\widehat{W} / \widehat{\Upsilon}^{-}$. Finally, the crux of the method lies in demanding that the fields have the correct physical singularity locally at the front such that

$$
\begin{align*}
& \sigma_{y y}(X, y=0, z, t) \rightarrow K^{\sigma}(z, t)(2 \pi X)^{-1 / 2} \quad \text { as } \quad X \rightarrow 0^{+} \\
& u_{y}\left(X, y=0^{+}, z, t\right) \rightarrow \frac{1-v^{2}}{E} K^{u}(z, t)(-2 X / \pi)^{1 / 2} \text { as } X \rightarrow 0^{-} \tag{83}
\end{align*}
$$

where $K^{\sigma}$ and $K^{u}$ are the stress and displacement intensity factors respectively. For a straight, unperturbed, crack front, $K^{\sigma}=K_{0}^{\sigma}=K_{r}^{\sigma} k(V)$ and $K^{u}=K_{0}^{u}=2 A_{I}(V) K_{0}^{\sigma}$ where the rest SIF $K_{r}^{\sigma}$ is determined by the loading conditions (Freund, 1990) (see also section 2.2 for definitions of $k(V)$ and $A_{I}(V)$ ). For a curved crack front, the demand, Eq. (83) will be met by eliminating all the higher order singularities that appear on the right-hand side of Eq. (76). In $q$-space, this demand imposes that the field $\widehat{u}_{y}$ shall not contain any power of $q$ higher than $q^{-3 / 2}$.

The energy-release-rate can be directly expressed using $K^{\sigma, u}$. Following Freund (1990), we construct a small parallelopiped $P=(\delta x, \delta y, \delta z)$ around the point on the crack front $x=h(z, t)$ that has two sides of parallel to the $x-z$ plane, two sides parallel to the local front tangent and $\hat{y}$, and two sides parallel to the $x-y$ plane. The parallelopiped also travels at velocity $\dot{h}=V+f_{t}$ along the $x$-axis. With these definitions, the local energy-release-rate is given by

$$
\begin{equation*}
G(z, t)=\lim _{P \rightarrow 0}\left\{\frac{1}{\delta z} \frac{1}{\dot{h}} \int_{P} \mathrm{~d} S \sigma_{i j} \hat{n}_{j} \dot{u}_{i}\right\} \tag{84}
\end{equation*}
$$

where $\hat{n}_{i}$ is the local normal to $P$ and $\mathrm{d} S$ is a surface element. Since only $\sigma_{y y}$ and $\dot{u}_{y}$ have singular $X^{-1 / 2}$ behavior close to the crack front, the above integral simplifies into

$$
\begin{equation*}
G(z, t)=\lim _{\delta x \rightarrow 0} \lim _{\delta y \rightarrow 0}\left\{\frac{2}{\dot{h}} \int_{h(z, t)-\delta x}^{h(z, t)+\delta x} \mathrm{~d} x \sigma_{y y} \dot{u}_{y}\right\}=\lim _{\delta x \rightarrow 0} \lim _{\delta y \rightarrow 0}\left\{2 \int_{-\delta x}^{\delta x} \mathrm{~d} X \Sigma_{y y} \frac{\partial U_{y}}{\partial X}\right\} \tag{85}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
G(z, t)=\frac{1-v^{2}}{2 E} K^{u} K^{\sigma} \tag{86}
\end{equation*}
$$

Alternative formulations of the 3D path-independent energy flux integral give the same result (Amestoy et al., 1981; Dodds et al., 1988; Eriksson, 2002; Leguillon, 2014; El Kabir et al., 2018).

### 4.1. Implementation

To compute the energy-release-rate, we begin by applying the Fourier transform $F(q)=\int \mathrm{d} X e^{i q X} F(X)$ to Eqs. (76) ,

$$
\begin{align*}
& \widetilde{u}_{y}\left(q, y=0^{+}, z, t\right)=e^{i q f(z, t)} \widetilde{U}_{n}(q, z, t), \\
& \widetilde{\sigma}_{y y}(q, y=0, z, t)=e^{i q f(z, t)} \widetilde{\Sigma}_{n}(q, z, t) . \tag{87}
\end{align*}
$$

Consider the displacement first. The zeroth order solution in the vicinity of the straight crack front is given by

$$
\begin{equation*}
u_{0}\left(X, y=0^{+}, z, t\right)=\frac{1-v^{2}}{E} K_{0}^{u} \sqrt{\frac{-2 X}{\pi}} \Theta(-X) \tag{88}
\end{equation*}
$$

which transforms into

$$
\begin{equation*}
\widetilde{u}_{0}\left(q, y=0^{+}, z, t\right)=\kappa^{u} q^{-3 / 2} \tag{89}
\end{equation*}
$$

in the sense that $q$ has a vanishing positive imaginary part, and $\kappa^{u}=i^{3 / 2}\left(1-v^{2}\right)(\sqrt{2} E)^{-1} K_{0}^{u}$. The next orders will be given by $\widehat{W} / \widehat{\Upsilon}^{-}$, such that

$$
\begin{equation*}
\widetilde{u}_{y}\left(q, y=0^{+}, z, t\right)=e^{i q f(z, t)}\left[\widetilde{u}_{0}\left(q, y=0^{+}, z, t\right)+\int \frac{d k}{2 \pi} \frac{d \omega}{2 \pi} e^{i k z+i \omega t} \frac{\widehat{W}(q, k, \omega)}{\widehat{\Upsilon}^{-}(q, k, \Omega)}\right], \tag{90}
\end{equation*}
$$

where $\widehat{W}(q, k, \omega)$ is an analytical function of $q$. Now we expand all functions in $q$ :

$$
\begin{align*}
& e^{i q f(z, t)}=\sum_{j=0}^{\infty} \frac{(i q)^{j}}{j!} f^{j}(z, t) \\
& \widetilde{W}(q, z, t)=\sum_{r=1}^{\infty}(i q)^{r-1} W_{r}(z, t)  \tag{91}\\
& \frac{1}{\widetilde{\Upsilon}^{-}(q, z, t)}=\kappa^{u} q^{-1 / 2}\left(1+\sum_{n=1}^{\infty} \frac{\Lambda_{n}^{-}(z, t)}{(i q)^{n}}\right) .
\end{align*}
$$

Multiplying (90) by $q^{3 / 2} / \kappa^{u}$ we obtain

$$
\begin{equation*}
\frac{q^{3 / 2}}{\kappa^{u}} \widetilde{u}_{y}\left(q, y=0^{+}, z, t\right)=\sum_{j=0}^{\infty}(i q)^{j} \frac{f^{j}}{j!}\left[1-i \sum_{r=1}^{\infty}(i q)^{r} W_{r}-i \sum_{r=1}^{\infty} \sum_{n=1}^{\infty}(i q)^{r-n} \Lambda_{n}^{-} \star W_{r}\right], \tag{92}
\end{equation*}
$$

where $\star$ denotes convolution in $z$ and $t$. Eq. (92) can be simplified into

$$
\begin{align*}
& \frac{q^{3 / 2}}{\kappa^{u}} \widetilde{u}_{y}\left(q, y=0^{+}, z, t\right)=1-i \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{f^{(n-r)}}{(n-r)!}\left(\Lambda_{n}^{-} \star W_{r}\right) \\
& +\sum_{m=1}^{\infty}(i q)^{m}\left[\frac{f^{m}}{m!}-i \sum_{r=1}^{m} \frac{f^{(m-r)}}{(m-r)!} W_{r}-i \sum_{n=1}^{\infty} \sum_{r=1}^{m+n} \frac{f^{(m+n-r)}}{(m+n-r)!}\left(\Lambda_{n}^{-} \star W_{r}\right)\right] \tag{93}
\end{align*}
$$

To meet the physical demand given by Eq. (83), we need to equate in Eq. (93) the coefficients of (iq) ${ }^{m}$ with $m \geq 1$ to zero. Therefore we get

$$
\begin{equation*}
i \sum_{n=1}^{m} \frac{f^{(m-n)}}{(m-n)!} W_{n}+i \sum_{n=1}^{\infty} \sum_{r=1}^{m+n} \frac{f^{(m+n-r)}}{(m+n-r)!}\left(\Lambda_{n}^{-} \star W_{r}\right)=\frac{f^{m}}{m!} \tag{94}
\end{equation*}
$$

Eq. (94) should be solved for $W_{m}$ order by order in $f$. Notice that Eq. (94) shows that each $W_{r}$ has at least one term of the order $f^{r}$. Therefore, $W_{r}$ can be written as

$$
\begin{equation*}
W_{r}(z, t)=-i \sum_{j=r}^{\infty} w_{r, j}(z, t) \tag{95}
\end{equation*}
$$

where $w_{r, j}$ is of the order $O\left(f^{j}\right)$. Plugging this expansion in Eq. (94) we find

$$
\begin{align*}
& \sum_{n=1}^{m} \frac{f^{(m-n)}}{(m-n)!} w_{n, n}=\frac{f^{m}}{m!} \quad m \geq 1 \\
& \sum_{n=1}^{m} \frac{f^{(m-n)}}{(m-n)!} w_{n, l+n-m}+\sum_{n=1}^{l-m} \sum_{r=1}^{m+n} \frac{f^{(m+n-r)}}{(m+n-r)!}\left(\Lambda_{n}^{-} \star w_{r, l+r-m-n}\right)=0 \quad l>m \geq 1 \tag{96}
\end{align*}
$$

It is easy to show that this linear system of equations can be solved recursively order by order. For the orders of interest, we find

$$
\begin{align*}
& w_{1,1}=f \\
& w_{2,2}=\frac{f^{2}}{2}-f w_{1,1}=-\frac{f^{2}}{2}  \tag{97}\\
& w_{1,2}=-f\left(\Lambda_{1}^{-} \star w_{1,1}\right)-\Lambda_{1}^{-} \star w_{2,2}=-f\left(\Lambda_{1}^{-} \star f\right)+\Lambda_{1}^{-} \star \frac{f^{2}}{2} .
\end{align*}
$$

It is left to express the SIF as a series in $f$. The expansion of the SIF comprises the remaining non-zero terms in Eq. (93). Expressing these terms with $w_{r, j}$,

$$
\begin{equation*}
\frac{K^{u}}{K_{0}^{u}}=1-\sum_{p=1}^{\infty} C_{p}=1-\sum_{p=1}^{\infty} \sum_{n=1}^{p} \sum_{r=1}^{n} \frac{f^{(n-r)}}{(n-r)!}\left(\Lambda_{n}^{-} \star w_{r, p+r-n}\right) \tag{98}
\end{equation*}
$$

For the orders of interest, we find

$$
\begin{align*}
& \mathcal{C}_{1}=\Lambda_{1}^{-} \star f \\
& C_{2}=-\Lambda_{1}^{-} \star\left\{f\left(\Lambda_{1}^{-} \star f\right)\right\}+\Lambda_{1}^{-} \star \Lambda_{1}^{-} \star \frac{f^{2}}{2}+f\left\{\Lambda_{2}^{-} \star f\right\}-\Lambda_{2}^{-} \star \frac{f^{2}}{2} . \tag{99}
\end{align*}
$$

Finally, the expression for the displacement intensity factor is

$$
\begin{equation*}
\frac{K^{u}}{K_{0}^{u}}=1-\Lambda_{1}^{-} \star f+\Lambda_{1}^{-} \star\left\{f\left(\Lambda_{1}^{-} \star f\right)\right\}-\Lambda_{1}^{-} \star \Lambda_{1}^{-} \star \frac{f^{2}}{2}-f\left(\Lambda_{2}^{-} \star f\right)+\Lambda_{2}^{-} \star \frac{f^{2}}{2} \tag{100}
\end{equation*}
$$

A similar calculation shows that the SIF is

$$
\begin{equation*}
\frac{K^{\sigma}}{K_{0}^{\sigma}}=1-\Lambda_{1}^{+} \star f+\Lambda_{1}^{+} \star\left\{f\left(\Lambda_{1}^{+} \star f\right)\right\}-\Lambda_{1}^{+} \star \Lambda_{1}^{+} \star \frac{f^{2}}{2}-f\left(\Lambda_{2}^{+} \star f\right)+\Lambda_{2}^{+} \star \frac{f^{2}}{2} \tag{101}
\end{equation*}
$$

where the functions $\Lambda_{n}^{+}$are given by the expansion

$$
\begin{equation*}
\widetilde{\Upsilon}^{+}(q, z, t)=(2 i)^{-1 / 2} K_{0}^{\sigma} q^{1 / 2}\left(1+\sum_{n=1}^{\infty} \frac{\Lambda_{n}^{+}(z, t)}{(i q)^{n}}\right) \tag{102}
\end{equation*}
$$

To obtain explicit expressions for the SIF and the energy-release-rate, one should perform the decomposition of $\widehat{\Upsilon}_{y y}(q, k, \omega)$ and its expansion in powers of $q$. This is done in the next section.

### 4.2. Decomposition of $\widehat{\Upsilon}_{y y}(q, k, \omega)$

According to our plan, we seek a Wiener-Hopf decomposition of $\widehat{\Upsilon}_{y y}(q, k, \omega)$ (Eq. (81)) into a product of two functions $\widehat{\Upsilon}^{+}(q, k, \omega)$ and $\widehat{\Upsilon}^{-}(q, k, \omega)$ that are analytical for $\operatorname{Im} q<0$ and $\operatorname{Im} q>0$ respectively. Similarly to section 2.5 in Freund (1990), a function $T(\Omega / p) \propto \widehat{\Upsilon}_{y y}$ is sought that satisfies $T \rightarrow 1$ when $|q| \rightarrow \infty$ and $\log T=\log T^{+}+\log T^{-}$, where $T^{+}\left(T^{-}\right)$is analytical and non-zero in the lower (upper) half of the $q$-plane. To find $T$, let us analyze the Rayleigh function given by Eq. (82). It is known that $R(\zeta)$ has a double root at $\zeta=0$ and two simple roots at $\zeta= \pm c$, where $c<b<a$ is the Rayleigh velocity. To extract these roots from $\widehat{\Upsilon}_{y y}$, we define

$$
\begin{equation*}
T(\zeta)=\frac{V^{2}}{R(V) \gamma_{c}^{2}} \frac{R(\zeta)}{\zeta^{2} \alpha_{c}(\zeta)^{2}} \tag{103}
\end{equation*}
$$

with $\alpha_{c}(\zeta)=\sqrt{1-\frac{\zeta^{2}}{c^{2}}}$ and $\gamma_{c}=1 / \sqrt{1-V^{2} / c^{2}}$. Since $T(V)=1$, we have $T(\Omega / p) \rightarrow 1$ as $|q| \rightarrow \infty$. Then, Eq. (81) is rewritten as

$$
\begin{equation*}
\widehat{\Upsilon}_{y y}(q, k, \omega)=-\mu \frac{b^{2} R(V) \gamma_{c}^{2}}{V^{2}} p \frac{\alpha_{c}(\Omega / p)^{2}}{\alpha_{a}(\Omega / p)} T(\Omega / p) \tag{104}
\end{equation*}
$$

with $\alpha_{a}(\zeta)=\sqrt{1-\frac{\zeta^{2}}{a^{2}}}$. Moreover, we notice that

$$
\begin{equation*}
p \frac{\alpha_{c}(\Omega / p)^{2}}{\alpha_{a}(\Omega / p)}=\frac{q^{2}+k^{2}-(\omega-q V)^{2} / c^{2}}{\sqrt{q^{2}+k^{2}-(\omega-q V)^{2} / a^{2}}} . \tag{105}
\end{equation*}
$$

Both the denominator and the numerator in Eq. (105) contain similar quadratic expressions in $q$ and they may be decomposed in the same manner. For example, one has

$$
\begin{equation*}
q^{2}+k^{2}-\frac{(\omega-q V)^{2}}{c^{2}}=\frac{1}{\gamma_{c}^{2}}\left(q+\frac{V \omega \gamma_{c}^{2}}{c^{2}}-i Q_{c}\right)\left(q+\frac{V \omega \gamma_{c}^{2}}{c^{2}}+i Q_{c}\right) \tag{106}
\end{equation*}
$$

where $Q_{c}=\gamma_{c} \sqrt{k^{2}-\frac{\omega^{2} \gamma_{c}^{2}}{c^{2}}}$. Finally, one has

$$
\begin{equation*}
\widehat{\Upsilon}_{y y}(q, k, \omega)=-\mu \frac{b^{2} R(V)}{V^{2}} \frac{\left(q+\frac{V \omega \gamma_{c}^{2}}{c^{2}}-i Q_{c}\right)\left(q+\frac{V \omega \gamma_{c}^{2}}{c^{2}}+i Q_{c}\right)}{\frac{1}{\gamma_{a}} \sqrt{q+\frac{V \omega \gamma_{a}^{2}}{a^{2}}-i Q_{a}} \sqrt{q+\frac{V \omega \gamma_{a}^{2}}{a^{2}}+i Q_{a}}} T(\Omega / p) \tag{107}
\end{equation*}
$$

and

$$
\begin{align*}
& \widehat{\mathrm{r}}^{+}=\frac{1}{(2 i)^{1 / 2}} K_{0}^{\sigma} \frac{q+\frac{V \omega \gamma_{c}^{2}}{c^{2}}-i Q_{c}}{\sqrt{q+\frac{V \omega \gamma_{\alpha}^{2}}{a^{2}}-i Q_{a}}} T^{+}  \tag{108}\\
& \widehat{\mathrm{r}}^{-}=\frac{2^{1 / 2} E}{i^{3 / 2}\left(1-v^{2}\right) K_{0}^{u}} \frac{q+\frac{V \omega \gamma_{c}^{2}}{c^{2}}+i Q_{c}}{\sqrt{q+\frac{V \omega \gamma_{a}^{2}}{a^{2}}+i Q_{a}}} T^{-} . \tag{109}
\end{align*}
$$

The decomposition of $\hat{\Upsilon}_{y y}$ reduces to resolving $T=T^{+} T^{-}$. Let us identify the singularities of $T$ in the $q$ plane. $R(\zeta)$ has two branch cuts: namely $b<\zeta<a$ and $-a<\zeta<-b$. Since $\zeta=\frac{\omega-q V}{\sqrt{q^{2}+k^{2}}}$, in the $q$ plane these branch cuts are transformed into into linear segments from $q_{a}$ to $q_{b}$ in the upper $q$ plane and from $q_{a}^{*}$ to $q_{b}^{*}$ in the lower $q$ plane, where $q_{a}=-\frac{\omega V \gamma_{a}^{2}}{a^{2}}+i Q_{a}, q_{b}=-\frac{\omega V \gamma_{b}^{2}}{b^{2}}+i Q_{b}$ and $Q_{a}=\gamma_{a} \sqrt{k^{2}-\frac{\omega^{2} \gamma_{a}^{2}}{a^{2}}}$, $Q_{b}=\gamma_{b} \sqrt{k^{2}-\frac{\omega^{2} \gamma_{b}^{2}}{b^{2}}}$.

After Ramanathan (1997), we name the contour circling [ $q_{a}, q_{b}$ ] in the clockwise direction the branch cut $C_{+}$and the contour circling $\left[q_{a}^{*}, q_{b}^{*}\right.$ ] in the clockwise direction the branch cut $C_{-}$. Then, $\log T$ satisfies the conditions for the Wiener-Hopf decomposition: it decays to 0 as $|q| \rightarrow \infty$ everywhere and it has two finite branch cuts. Therefore, one has

$$
\begin{align*}
T^{+} & =\exp \left(\frac{1}{2 \pi i} \oint_{C_{+}} \frac{\log T}{\xi-q} \mathrm{~d} \xi\right)  \tag{110}\\
T^{-} & =\exp \left(\frac{1}{2 \pi i} \oint_{C_{-}} \frac{\log T}{\xi-q} \mathrm{~d} \xi\right) \tag{111}
\end{align*}
$$

Let's tackle the first integral. Writing it more explicitly we have

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint_{C_{+}} \frac{\log T}{\xi-q} \mathrm{~d} \xi=\frac{1}{2 \pi i} \oint_{C_{+}} \frac{\mathrm{d} \xi}{\xi-q} \log (R(\zeta) / B(\zeta)) \tag{112}
\end{equation*}
$$

where $\zeta=\frac{\omega-\xi V}{\sqrt{\xi^{2}+k^{2}}}$ and $B(\zeta)=V^{-2} R(V) \gamma_{c}^{2} \zeta^{2}\left(1-\zeta^{2} / c^{2}\right)$. Then, since $B(\zeta)$ has no poles or zeros inside the contours $C_{ \pm}$, the integral over its logarithm is trivially zero and we have

$$
\begin{equation*}
\log T^{ \pm}=\frac{1}{2 \pi i} \oint_{C_{ \pm}} \frac{\log T}{\xi-q} \mathrm{~d} \xi=\frac{1}{2 \pi i} \oint_{C_{ \pm}} \frac{\mathrm{d} \xi}{\xi-q} \log (R(\zeta)) . \tag{113}
\end{equation*}
$$

Changing the integration variable to $\zeta=\zeta(\xi)$ such that

$$
\begin{equation*}
q_{ \pm}(\zeta)=-\frac{\omega V}{\zeta^{2}-V^{2}} \pm i|k| \frac{\sqrt{\zeta^{2}} \sqrt{\zeta^{2}-H}}{\zeta^{2}-V^{2}} \tag{114}
\end{equation*}
$$

where $H=V^{2}+\omega^{2} / k^{2}$. Here, the $+\operatorname{sign}$ is taken for the $C_{+}$integral and the $-\operatorname{sign}$ is taken for the $C_{-}$ integral. Then

$$
\begin{equation*}
\frac{\mathrm{d} \xi}{\xi-q}=\frac{q_{ \pm}^{\prime}(\zeta) \mathrm{d} \zeta}{q_{ \pm}(\zeta)-q} \tag{115}
\end{equation*}
$$

This change of variables copies the contour $C_{+}$in the $q$-plane to a clockwise contour around the real interval $b<\zeta<a$ and the contour $C_{-}$to a clockwise contour around $-a<\zeta<-b$. To evaluate the contour integrals, a choice for the branch cuts of $R(\zeta)$ has to be made. To ensure that the elastic fields maintain physical behavior far from the crack, $\alpha_{b}$ must always have a non-negative real part (Geubelle and Rice, 1995; Ramanathan, 1997). For $\zeta=x+i \epsilon$ where $x>b$ and $|\epsilon|$ is arbitrarily small, we have $\alpha_{b} \simeq \pm i \sqrt{x^{2} / b^{2}-1}\left(1+i \epsilon x /\left(x^{2}-b^{2}\right)\right)$. Hence, when $\epsilon>0$, we must choose the minus sign, and vice versa when $\epsilon<0$. Accordingly, the integral which follows the clockwise contour $C_{+}$is

$$
\begin{array}{r}
\log T^{+}=\frac{1}{2 \pi i} \oint_{C_{+}} \frac{q_{+}^{\prime}(\zeta) \mathrm{d} \zeta}{q_{+}(\zeta)-q} \log (R(\zeta))= \\
\frac{1}{2 \pi i} \int_{b}^{a} \frac{q_{+}^{\prime}(\zeta) \mathrm{d} \zeta}{q_{+}(\zeta)-q} \log \left[-4 i \sqrt{1-\frac{\zeta^{2}}{a^{2}}} \sqrt{\frac{\zeta^{2}}{b^{2}}-1}-\left(2-\frac{\zeta^{2}}{b^{2}}\right)^{2}\right]+ \\
\frac{1}{2 \pi i} \int_{a}^{b} \frac{q_{+}^{\prime}(\zeta) \mathrm{d} \zeta}{q_{+}(\zeta)-q} \log \left[4 i \sqrt{1-\frac{\zeta^{2}}{a^{2}}} \sqrt{\frac{\zeta^{2}}{b^{2}}-1}-\left(2-\frac{\zeta^{2}}{b^{2}}\right)^{2}\right] . \tag{116}
\end{array}
$$

After a change of variable to $J=\zeta^{2}$ one has

$$
\begin{equation*}
\log T^{+}=-\frac{1}{2 \pi i} \int_{b^{2}}^{a^{2}} \frac{q_{+}^{\prime}(J) \mathrm{d} J}{q_{+}(J)-q}\{\log (1-i z)-\log (1+i z)\}, \tag{117}
\end{equation*}
$$

where $z(J)=4 \sqrt{1-\frac{J}{a^{2}}} \sqrt{\frac{J}{b^{2}}-1} /\left(2-\frac{J}{b^{2}}\right)^{2}$. Using trigonometric identities, one can $\operatorname{simplify} \log T^{+}$further to obtain

$$
\begin{equation*}
\log T^{+}=\frac{1}{\pi} \int_{b^{2}}^{a^{2}} \frac{q_{+}^{\prime}(J) \mathrm{d} J}{q_{+}(J)-q} \arctan \left[4 \frac{\sqrt{1-\frac{J}{a^{2}}} \sqrt{\frac{J}{b^{2}}-1}}{\left(2-\frac{J}{b^{2}}\right)^{2}}\right] . \tag{118}
\end{equation*}
$$

A similar calculation shows that

$$
\begin{equation*}
\log T^{-}=-\frac{1}{\pi} \int_{b^{2}}^{a^{2}} \frac{q_{-}^{\prime}(J) \mathrm{d} J}{q_{-}(J)-q} \arctan \left[4 \frac{\sqrt{1-\frac{J}{a^{2}}} \sqrt{\frac{J}{b^{2}}-1}}{\left(2-\frac{J}{b^{2}}\right)^{2}}\right] \tag{119}
\end{equation*}
$$

Now we are in a position to develop $T^{ \pm}$in powers of $1 /(i q)$. Denoting

$$
\begin{equation*}
\Pi_{n}^{ \pm}=\mp \frac{1}{\pi} \int_{b^{2}}^{a^{2}} i^{n} q_{ \pm}^{\prime}(J) q_{ \pm}(J)^{n-1} \arctan [z(J)] \mathrm{d} J \tag{120}
\end{equation*}
$$

we find that,

$$
\begin{equation*}
\log T^{ \pm}=\sum_{n=1}^{\infty} \Pi_{n}^{ \pm} \frac{1}{(i q)^{n}} \tag{121}
\end{equation*}
$$

Therefore up to the $2^{\text {nd }}$-order,

$$
\begin{equation*}
T^{ \pm}=\exp \left(\sum_{n=1}^{\infty} \Pi_{n}^{ \pm} \frac{1}{(i q)^{n}}\right) \simeq 1+\frac{\Pi_{1}^{ \pm}}{i q}+\left(\Pi_{2}^{ \pm}+\frac{1}{2}\left(\Pi_{1}^{ \pm}\right)^{2}\right) \frac{1}{(i q)^{2}} . \tag{122}
\end{equation*}
$$

Also, note that

$$
\begin{equation*}
\left(\Pi_{n}^{+}\right)^{*}=(-1)^{n+1} \Pi_{n}^{-}, \tag{123}
\end{equation*}
$$

and that

$$
\begin{equation*}
\Pi_{n}^{ \pm}=-\frac{i^{n-1}}{2 \pi} \oint_{C_{ \pm}} \xi^{n-1} \log T \mathrm{~d} \xi \tag{124}
\end{equation*}
$$

We can now substitute the expansions of $T^{ \pm}$back in Eqs. (108). Equating

$$
\begin{equation*}
\widehat{\Upsilon}^{+}=\frac{1}{(2 i)^{1 / 2}} K_{0}^{\sigma} \frac{q-q_{c}}{\sqrt{q-q_{a}}} T^{+}=\frac{1}{(2 i)^{1 / 2}} K_{0}^{\sigma} q^{1 / 2}\left(1+\sum_{n} \frac{\hat{\Lambda}_{n}^{+}}{(i q)^{n}}\right) \tag{125}
\end{equation*}
$$

where $q_{c}=-\frac{V \omega \gamma_{c}^{2}}{c^{2}}+i Q_{c}$. Since

$$
\begin{equation*}
\frac{q-q_{c}}{\sqrt{q-q_{a}}}=q^{1 / 2}\left(1+\frac{i q_{a} / 2-i q_{c}}{i q}+\frac{(1 / 2) q_{a} q_{c}-(3 / 8) q_{a}^{2}}{(i q)^{2}}+O(i q)^{3}\right) \tag{126}
\end{equation*}
$$

we obtain

$$
\begin{align*}
& \hat{\Lambda}_{1}^{+}=i \frac{q_{a}}{2}-i q_{c}+\Pi_{1}^{+} \\
& \hat{\Lambda}_{2}^{+}=\frac{1}{2} q_{a} q_{c}-\frac{3}{8} q_{a}^{2}+\Pi_{2}^{+}+\frac{1}{2}\left(\Pi_{1}^{+}\right)^{2}+\Pi_{1}^{+}\left(i \frac{q_{a}}{2}-i q_{c}\right) . \tag{127}
\end{align*}
$$

The first of these equations correspond to Eq. (A.25) in Ramanathan (1997) and the second corresponds to Eq. (A.35) in the limit $V=0$.

Now considering

$$
\begin{equation*}
\frac{1}{\hat{\Upsilon}^{-}}=\frac{i^{3 / 2}\left(1-v^{2}\right) K_{0}^{u}}{2^{1 / 2} E} \frac{\sqrt{q-q_{a}^{*}}}{q-q_{c}^{*}} \frac{1}{T^{-}}=\frac{i^{3 / 2}\left(1-v^{2}\right) K_{0}^{u}}{2^{1 / 2} E} q^{-1 / 2}\left(1+\sum_{n} \frac{\hat{\Lambda}_{n}^{-}}{(i q)^{n}}\right) \tag{128}
\end{equation*}
$$

we develop in the same way

$$
\begin{equation*}
\frac{\sqrt{q-q_{a}^{*}}}{q-q_{c}^{*}}=q^{-1 / 2}\left(1+\frac{i q_{c}^{*}-i q_{a}^{*} / 2}{i q}+\frac{q_{a}^{*} q_{c}^{*} / 2+\left(q_{a}^{*}\right)^{2} / 8-\left(q_{c}^{*}\right)^{2}}{(i q)^{2}}\right) \tag{129}
\end{equation*}
$$

and therefore

$$
\begin{align*}
& \hat{\Lambda}_{1}^{-}=i q_{c}^{*}-i \frac{q_{a}^{*}}{2}+\Pi_{1}^{-} \\
& \hat{\Lambda}_{2}^{-}=\frac{1}{2} q_{a}^{*} q_{c}^{*}+\frac{1}{8}\left(q_{a}^{*}\right)^{2}-\left(q_{c}^{*}\right)^{2}+\Pi_{2}^{-}+\frac{1}{2}\left(\Pi_{1}^{-}\right)^{2}+\frac{1}{2} \Pi_{1}^{-}\left(i q_{c}^{*}-i \frac{q_{a}^{*}}{2}\right) \tag{130}
\end{align*}
$$

### 4.3. Evaluation of the energy-release-rate

To compute $G$ we can now use Eq. (86) and substitute Eqs. (100,101). Since explicit expressions for $\Lambda_{n}^{ \pm}$were obtained in $(k, \omega)$ space, we will compute the Fourier transform $\widehat{G}(k, \omega)$. First, note that from Eqs. $(100,101)$

$$
\begin{array}{r}
\frac{\widehat{K}_{0}^{\sigma, u}(k, \omega)}{K_{0}^{\sigma, u}}=(2 \pi)^{2} \delta(k) \delta(\omega)-\hat{\Lambda}_{1}^{ \pm} \hat{f}+\hat{\Lambda}_{1}^{ \pm}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{ \pm} \hat{f}\right)\right\}  \tag{131}\\
- \\
-\frac{1}{2}\left(\hat{\Lambda}_{1}^{ \pm}\right)^{2}(\hat{f} \otimes \hat{f})-\hat{f} \otimes\left(\hat{\Lambda}_{2}^{ \pm} \hat{f}\right)+\frac{1}{2} \hat{\Lambda}_{2}^{ \pm}(\hat{f} \otimes \hat{f}),
\end{array}
$$

where

$$
\begin{equation*}
\widehat{(f \cdot g)}(k, \omega)=\widehat{f} \otimes \widehat{g}=\int \frac{d \omega^{\prime}}{2 \pi} \frac{d k^{\prime}}{2 \pi} \hat{f}\left(k-k^{\prime}, \omega-\omega^{\prime}\right) \hat{g}\left(k^{\prime}, \omega^{\prime}\right) . \tag{132}
\end{equation*}
$$

Then, the Fourier transformed Eq. (86) becomes

$$
\begin{align*}
& \frac{\widehat{G}(k, \omega)}{G_{r} g(V)}=(2 \pi)^{2} \delta(k) \delta(\omega)-\left(\hat{\Lambda}_{1}^{+}+\hat{\Lambda}_{1}^{-}\right) \hat{f}+\hat{\Lambda}_{1}^{+}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{+} \hat{f}\right)\right\}+\hat{\Lambda}_{1}^{-}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{-} \hat{f}\right)\right\}  \tag{133}\\
+ & \frac{1}{2}\left(\hat{\Lambda}_{2}^{+}+\hat{\Lambda}_{2}^{-}-\left(\hat{\Lambda}_{1}^{+}\right)^{2}-\left(\hat{\Lambda}_{1}^{-}\right)^{2}\right)(\hat{f} \otimes \hat{f})-\hat{f} \otimes\left\{\left(\hat{\Lambda}_{2}^{+}+\hat{\Lambda}_{2}^{-}\right) \hat{f}\right\}+\left(\hat{\Lambda}_{1}^{+} \hat{f}\right) \otimes\left(\hat{\Lambda}_{1}^{-} \hat{f}\right) .
\end{align*}
$$

Let us write explicit expressions for each of these terms. First, using the expressions of $\hat{\Lambda}_{1}^{+}$and $\hat{\Lambda}_{1}^{-}$, we obtain

$$
\begin{equation*}
\widehat{\delta G^{(1)}}(k, \omega)=\hat{\Lambda}_{1}^{+}+\hat{\Lambda}_{1}^{-}=2|k| P_{1}\left(\frac{\omega}{|k|} ; V, v\right), \tag{134}
\end{equation*}
$$

where

$$
\begin{align*}
P_{1}(s ; V, v) & =-\frac{1}{2} \gamma_{a} \sqrt{1-\frac{\gamma_{a}^{2} s^{2}}{a^{2}}}+\gamma_{c} \sqrt{1-\frac{\gamma_{c}^{2} s^{2}}{c^{2}}} \\
& +\frac{1}{2 \pi} \int_{b^{2}}^{a^{2}} \frac{\left(s^{2}+V^{2}\right)\left(J+V^{2}\right)-2 J V^{2}}{\sqrt{J} \sqrt{J-\left(s^{2}+V^{2}\right)}\left(J-V^{2}\right)^{2}} \arctan [z(J)] \mathrm{d} J . \tag{135}
\end{align*}
$$

The branch cuts of the square roots are defined so that $\sqrt{1-s^{2}}=i \operatorname{sign}(s) \sqrt{s^{2}-1}$ for $|s|>1$. This definition is consistent with the physical demand that elastic waves are radiated away from the crack front (Geubelle and Rice, 1995; Ramanathan, 1997).

Second, separating real and imaginary parts of the term $\hat{\Lambda}_{1}^{+}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{+} \hat{f}\right)\right\}+\hat{\Lambda}_{1}^{-}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{-} \hat{f}\right)\right\}$ yields

$$
\begin{align*}
& \hat{\Lambda}_{1}^{+}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{+} \hat{f}\right)\right\}+\hat{\Lambda}_{1}^{-}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{-} \hat{f}\right)\right\}= \\
& 2 \operatorname{Re}\left(\hat{\Lambda}_{1}^{+}\right)\left\{\hat{f} \otimes\left(\operatorname{Re}\left(\hat{\Lambda}_{1}\right)^{+} \hat{f}\right)\right\}-2 \operatorname{Im}\left(\hat{\Lambda}_{1}^{+}\right)\left\{\hat{f} \otimes\left(\operatorname{Im}\left(\hat{\Lambda}_{1}^{+}\right) \hat{f}\right)\right\} \tag{136}
\end{align*}
$$

Notice that the imaginary part of $\hat{\Lambda}_{1}$ is linear in $\omega$

$$
\begin{equation*}
\operatorname{Im}\left(\hat{\Lambda}_{1}^{+}\right)=V Q_{1}(V) \omega \tag{137}
\end{equation*}
$$

with

$$
\begin{equation*}
Q_{1}(V)=\frac{1}{2\left(a^{2}-V^{2}\right)}-\frac{1}{c^{2}-V^{2}}+\frac{1}{\pi} \int_{b^{2}}^{a^{2}} \frac{1}{\left(J-V^{2}\right)^{2}} \arctan [z(J)] \mathrm{d} J \tag{138}
\end{equation*}
$$

or (Adda-Bedia et al., 2013; Freund, 1990)

$$
\begin{equation*}
Q_{1}(V)=-\frac{1}{2 V} \frac{\mathrm{~d}}{\mathrm{~d} V}\left(\log \left(\frac{V^{2}}{R(V) \gamma_{a} b^{2}}\right)\right), \tag{139}
\end{equation*}
$$

where $R(V)=4 \sqrt{1-\frac{V^{2}}{a^{2}}} \sqrt{1-\frac{V^{2}}{b^{2}}}-\left(2-\frac{V^{2}}{b^{2}}\right)^{2}$. Hence,

$$
\begin{equation*}
\hat{\Lambda}_{1}^{+}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{+} \hat{f}\right)\right\}+\hat{\Lambda}_{1}^{-}\left\{\hat{f} \otimes\left(\hat{\Lambda}_{1}^{-} \hat{f}\right)\right\}=2|k| P_{1}\left\{\hat{f} \otimes\left(|k| P_{1} \hat{f}\right)\right\}-2 V^{2} Q_{1}(V)^{2} \omega\{\hat{f} \otimes(\omega \hat{f})\} . \tag{140}
\end{equation*}
$$

Third, to simplify the terms $\frac{1}{2}\left(\hat{\Lambda}_{2}^{+}+\hat{\Lambda}_{2}^{-}-\left(\hat{\Lambda}_{1}^{+}\right)^{2}-\left(\hat{\Lambda}_{1}^{-}\right)^{2}\right)(\hat{f} \otimes \hat{f})$ and $f \otimes\left\{\left(\hat{\Lambda}_{2}^{+}+\hat{\Lambda}_{2}^{-}\right) \hat{f}\right\}$, we notice that

$$
\begin{equation*}
\hat{\Lambda}_{2}^{+}=\frac{1}{2}\left(\hat{\Lambda}_{1}^{+}\right)^{2}+\Pi_{2}^{+}+\frac{1}{2}\left(q_{c}^{2}-\frac{1}{2} q_{a}^{2}\right), \hat{\Lambda}_{2}^{-}=\frac{1}{2}\left(\hat{\Lambda}_{1}^{-}\right)^{2}+\Pi_{2}^{-}-\frac{1}{2}\left(q_{c}^{2}-\frac{1}{2} q_{a}^{2}\right)^{*} \tag{141}
\end{equation*}
$$

Since $\hat{\Lambda}_{1}^{-}=\left(\hat{\Lambda}_{1}^{+}\right)^{*}$ and $\left(\Pi_{2}^{+}\right)^{*}=-\Pi_{2}^{-}$, we find

$$
\begin{equation*}
\hat{\Lambda}_{2}^{+}+\hat{\Lambda}_{2}^{-}=\operatorname{Re}\left[\left(\hat{\Lambda}_{1}^{+}\right)^{2}\right]+2 i \operatorname{Im}\left[\Pi_{2}^{+}\right]+i \operatorname{Im}\left[q_{c}^{2}-\frac{1}{2} q_{a}^{2}\right] \tag{142}
\end{equation*}
$$

Using the identity $\operatorname{Re}\left[\left(\hat{\Lambda}_{1}^{+}\right)^{2}\right]=\operatorname{Re}\left[\hat{\Lambda}_{1}^{+}\right]^{2}-\operatorname{Im}\left[\hat{\Lambda}_{1}^{+}\right]^{2}=k^{2} P_{1}^{2}-V^{2} Q_{1}^{2} \omega^{2}$ and defining $i V \omega|k| P_{2}(\omega /|k| ; V, v)=$ $-2 i \operatorname{Im}\left[\Pi_{2}^{+}\right]-i \operatorname{Im}\left[q_{c}^{2}-\frac{1}{2} q_{a}^{2}\right]$ we write

$$
\begin{equation*}
\hat{\Lambda}_{2}^{+}+\hat{\Lambda}_{2}^{-}=k^{2} P_{1}^{2}-V^{2} Q_{1}^{2} \omega^{2}-i V \omega|k| P_{2} . \tag{143}
\end{equation*}
$$

Thus, we can write

$$
\begin{equation*}
\frac{1}{2}\left(\hat{\Lambda}_{2}^{+}+\hat{\Lambda}_{2}^{-}-\left(\hat{\Lambda}_{1}^{+}\right)^{2}-\left(\hat{\Lambda}_{1}^{-}\right)^{2}\right)=-\frac{1}{2}\left(k^{2} P_{1}^{2}-V^{2} Q_{1}^{2} \omega^{2}\right)-\frac{i}{2} V \omega|k| P_{2}, \tag{144}
\end{equation*}
$$

where

$$
\begin{align*}
P_{2}(s ; V, v)= & 2 \frac{\gamma_{c}^{3}}{c^{2}} \sqrt{1-\frac{\gamma_{c}^{2} s^{2}}{c^{2}}}-\frac{\gamma_{a}^{3}}{a^{2}} \sqrt{1-\frac{\gamma_{a}^{2} s^{2}}{a^{2}}}  \tag{145}\\
& +\frac{2}{\pi} \int_{b^{2}}^{a^{2}} \frac{\left(s^{2}+V^{2}\right)\left(3 J+V^{2}\right)-2 J\left(J+V^{2}\right)}{\sqrt{J-\left(s^{2}+V^{2}\right)}\left(J-V^{2}\right)^{3}} \arctan [z(J)] \frac{\mathrm{d} J}{2 \sqrt{J}} .
\end{align*}
$$

Finally, using the distributive law, one obtains

$$
\begin{equation*}
\left(\hat{\Lambda}_{1}^{+} \hat{f}\right) \otimes\left(\hat{\Lambda}_{1}^{-} \hat{f}\right)=\left(|k| P_{1} \hat{f}\right) \otimes\left(|k| P_{1} \hat{f}\right)+V^{2} Q_{1}^{2}(\omega \hat{f}) \otimes(\omega \hat{f}) . \tag{146}
\end{equation*}
$$

Collecting all the above expressions, the $2^{\text {nd }}$-order correction to the energy-release-rate can be written as

$$
\begin{align*}
& \widehat{\delta G^{(2)}}(k, \omega)= \\
& 2|k| P_{1}\left\{\hat{f} \otimes\left(|k| P_{1} \hat{f}\right)\right\}-2 Q_{1}^{2} V^{2} \omega\{\hat{f} \otimes(\omega \hat{f})\}-\left\{\frac{1}{2}\left(k^{2} P_{1}^{2}-V^{2} Q_{1}^{2} \omega^{2}\right)+\frac{i}{2} V \omega|k| P_{2}\right\}(\hat{f} \otimes \hat{f})  \tag{147}\\
& -\hat{f} \otimes\left\{\left(k^{2} P_{1}^{2}-V^{2} Q_{1}^{2} \omega^{2}-i V \omega|k| P_{2}\right) \hat{f}\right\}+\left(|k| P_{1} \hat{f}\right) \otimes\left(|k| P_{1} \hat{f}\right)+V^{2} Q_{1}^{2}(\omega \hat{f}) \otimes(\omega \hat{f}) .
\end{align*}
$$

This expression can be further simplified using the identity $\hat{f} \otimes(\omega \hat{f})=\frac{1}{2} \omega(\hat{f} \otimes \hat{f})$. Then,

$$
\begin{align*}
\widehat{\delta G^{(2)}}(k, \omega)= & 2|k| P_{1}\left\{\hat{f} \otimes\left(|k| P_{1} \hat{f}\right)\right\}-\left\{\frac{1}{2} k^{2} P_{1}^{2}+\frac{i}{2} V \omega|k| P_{2}\right\}(\hat{f} \otimes \hat{f})  \tag{148}\\
& -\hat{f} \otimes\left\{\left(k^{2} P_{1}^{2}-i V \omega|k| P_{2}\right) \hat{f}\right\}+\left(|k| P_{1} \hat{f}\right) \otimes\left(|k| P_{1} \hat{f}\right)
\end{align*}
$$

### 4.4. Transformation to $(k, t)$ space

To obtain a real space expression of the form $G=g\left(V_{\perp}\right)\left(1+H^{(1)}[f]+H^{(2)}[f, f]+O\left(f^{3}\right)\right)$ where the functionals $H^{(i)}$ only depend on the history of the front dynamics $\left\{f\left(z, t^{\prime}\right) ; t^{\prime}<t\right\}$, we transform $\widehat{\delta G^{(1)}}(k, \omega)$ and $\widehat{\delta G^{(2)}}(k, \omega)$ from $\omega$ to $t$. Following Morrissey and Rice (2000), we use the Bessel identities

$$
\frac{1}{2 \pi} \int \mathrm{~d} \omega e^{i \omega t} \sqrt{\beta^{2}-\omega^{2}}=\delta^{\prime}(t)+p^{2} \frac{J_{1}(p t)}{p t} \mathcal{H}(t)
$$

and

$$
\frac{1}{2 \pi} \int \mathrm{~d} \omega e^{i \omega t} \frac{1}{\sqrt{\beta^{2}-\omega^{2}}}=J_{0}(p t) \mathcal{H}(t)
$$

where $\mathcal{H}(t)$ is the Heaviside function, to write

$$
\begin{align*}
& \frac{1}{2 \pi} \int \mathrm{~d} \omega e^{i \omega t}|k| P_{1}\left(\frac{\omega}{|k|}\right)=\left(-\frac{1}{2} \frac{a}{a^{2}-V^{2}}+\frac{c}{c^{2}-V^{2}}\right) \delta^{\prime}(t) \\
& -\partial_{t}^{2} \frac{1}{2} \int_{b}^{a} \mathrm{~d} \eta \Theta(\eta) \frac{\eta^{2}+V^{2}}{\left(\eta^{2}-V^{2}\right)^{2}} J_{0}\left(\beta_{\eta} t\right) \mathcal{H}(t)  \tag{149}\\
& +k^{2}\left(-\frac{a}{2} \frac{J_{1}\left(\beta_{a} t\right)}{\beta_{a} t}+c \frac{J_{1}\left(\beta_{c} t\right)}{\beta_{c} t}-\frac{1}{2} \int_{b}^{a} \mathrm{~d} \eta \Theta(\eta) \frac{V^{2}}{\eta^{2}-V^{2}} J_{0}\left(\beta_{\eta} t\right)\right) \mathcal{H}(t)
\end{align*}
$$

where $\beta_{s}=\sqrt{s^{2}-V^{2}}|k|$. From the Bessel identities $J_{0}^{\prime}(x)=-J_{1}(x)$ and $J_{0}^{\prime \prime}(x)=\left(J_{2}(x)-J_{0}(x)\right) / 2$

$$
\begin{align*}
& \frac{1}{2 \pi} \int \mathrm{~d} \omega e^{i \omega t}|k| P_{1}\left(\frac{\omega}{|k|}\right)=C_{1} \delta^{\prime}(t)+B_{1}(k, t) \delta(t)+A_{1}(k, t) \mathcal{H}(t)  \tag{150}\\
C_{1}(k, t)= & -\frac{1}{2} \frac{a}{a^{2}-V^{2}}+\frac{c}{c^{2}-V^{2}}-\frac{1}{2} \int_{b}^{a} \mathrm{~d} \eta \Theta(\eta) \frac{\eta^{2}+V^{2}}{\left(\eta^{2}-V^{2}\right)^{2}} J_{0}\left(\beta_{\eta} t\right) \\
B_{1}(k, t)= & -|k| \int_{b}^{a} \mathrm{~d} \eta \Theta(\eta) \frac{\eta^{2}+V^{2}}{\left(\eta^{2}-V^{2}\right)^{3 / 2}} J_{1}\left(\beta_{\eta} t\right) \\
A_{1}(k, t)= & k^{2}\left[-\frac{a}{2} \frac{J_{1}\left(\beta_{a} t\right)}{\beta_{a} t}+c \frac{J_{1}\left(\beta_{c} t\right)}{\beta_{c} t}-\frac{1}{4} \int_{b}^{a} \mathrm{~d} \eta \Theta(\eta)\left(\frac{\eta^{2}+V^{2}}{\eta^{2}-V^{2}} J_{2}\left(\beta_{\eta} t\right)-J_{0}\left(\beta_{\eta} t\right)\right)\right] .
\end{align*}
$$

Then,

$$
\begin{equation*}
\frac{1}{2 \pi} \int \mathrm{~d} \omega e^{i \omega t}|k| P_{1}\left(\frac{\omega}{|k|}\right) \hat{f}(k, \omega)=-\pi_{1} \bar{f}_{t}-I_{1}[\bar{f}] \tag{151}
\end{equation*}
$$

where $\pi_{1}$ is given by Eq. (18), and $I_{1}[\bar{f}]=-\int_{-\infty}^{t} \mathrm{~d} t^{\prime} A_{1}\left(k, t-t^{\prime}\right) \bar{f}\left(k, t^{\prime}\right)$.
 $\overline{H^{(1)}[f]}=2 I_{1}[\bar{f}]$ Morrissey and Rice (2000).

To extend this result to the $2^{\text {nd }}$-order, we use the Bessel identity $J_{0}^{\prime \prime \prime}(x)=3 J_{1}(x) / 4-J_{3}(x) / 4$ to find

$$
\begin{equation*}
\frac{1}{2 \pi} \int \mathrm{~d} \omega e^{i \omega t} i \omega|k| P_{2}\left(\frac{\omega}{|k|}\right)=D_{2} \delta^{\prime \prime}(t)+C_{2} \delta^{\prime}(t)+B_{2} \delta(t)+A_{2} \mathcal{H}(t) \tag{152}
\end{equation*}
$$

where

$$
\begin{aligned}
D_{2}(k, t) & =2 \frac{c}{\left(c^{2}-V^{2}\right)^{2}}-\frac{a}{\left(a^{2}-V^{2}\right)^{2}}-\int_{b}^{a} \mathrm{~d} \eta \Theta(\eta) \frac{3 \eta^{2}+V^{2}}{\left(\eta^{2}-V^{2}\right)^{3}} J_{0}\left(\beta_{\eta} t\right) \\
C_{2}(k, t) & =3|k| \int_{b}^{a} \mathrm{~d} \eta \Theta(\eta) \frac{3 \eta^{2}+V^{2}}{\left(\eta^{2}-V^{2}\right)^{5 / 2}} J_{1}\left(\beta_{\eta} t\right) \\
B_{2}(k, t) & =k^{2}\left[2 \frac{c}{c^{2}-V^{2}} \frac{J_{1}\left(\beta_{c} t\right)}{\beta_{c} t}-\frac{a}{a^{2}-V^{2}} \frac{J_{1}\left(\beta_{a} t\right)}{\beta_{a} t}\right. \\
& \left.+\frac{1}{2} \int_{b}^{a} \mathrm{~d} \eta \frac{\Theta(\eta)}{\left(\eta^{2}-V^{2}\right)^{2}}\left(\left(5 \eta^{2}+V^{2}\right) J_{0}\left(\beta_{\eta} t\right)-\left(9 \eta^{2}+3 V^{2}\right) J_{2}\left(\beta_{\eta} t\right)\right)\right] \\
A_{2}(k, t) & =|k|^{3}\left[-\gamma_{a} \frac{J_{2}\left(\beta_{a} t\right)}{\beta_{a} t}+2 \gamma_{c} \frac{J_{2}\left(\beta_{c} t\right)}{\beta_{c} t}\right. \\
& \left.-\frac{1}{4} \int_{b}^{a} \mathrm{~d} \eta \frac{\Theta(\eta)}{\sqrt{\eta^{2}-V^{2}}}\left(\frac{3 \eta^{2}+V^{2}}{\eta^{2}-V^{2}} J_{3}\left(\beta_{\eta} t\right)-J_{1}\left(\beta_{\eta} t\right)\right)\right] .
\end{aligned}
$$

Then,

$$
\begin{equation*}
\frac{1}{2 \pi} \int \mathrm{~d} \omega e^{i \omega t} i \omega|k| P_{2}\left(\frac{\omega}{|k|}\right) \hat{f}(k, \omega)=-\pi_{2} \bar{f}_{t t}-\pi_{1} k^{2} \bar{f}-I_{2}[\bar{f}] \tag{153}
\end{equation*}
$$

where

$$
\begin{equation*}
\pi_{2}=\frac{a}{\left(a^{2}-V^{2}\right)^{2}}-2 \frac{c}{\left(c^{2}-V^{2}\right)^{2}}+\int_{b}^{a} \frac{3 \eta^{2}+V^{2}}{\left(\eta^{2}-V^{2}\right)^{3}} \Theta(\eta) \mathrm{d} \eta \tag{154}
\end{equation*}
$$

The second term results from the identity $\partial_{t}^{2} D_{2}(0)-\partial_{t} C_{2}(0)+B_{2}(0)=-\pi_{1} k^{2}$, and the third term is $I_{2}[\bar{f}]=-\int_{-\infty}^{t} \mathrm{~d} t A_{2}\left(k, t-t^{\prime}\right) \bar{f}\left(k, t^{\prime}\right)$.

The $2^{\text {nd }}$-order contribution to $G$ is then,

$$
\begin{equation*}
\overline{\delta G^{(2)}}(k, t)=\left(2 \pi_{1}^{2}+V \pi_{2}\right) \bar{f}_{t} * \bar{f}_{t}+2 \pi_{1} \bar{f}_{t} * \overline{H^{(1)}[f]}+V \pi_{1}(k \bar{f}) *(k \bar{f})+\overline{H^{(2)}[f, f]}, \tag{155}
\end{equation*}
$$

where the convolution operator is defined as $f * g=\int \mathrm{d} k^{\prime} f\left(k-k^{\prime}\right) g\left(k^{\prime}\right)$ and

$$
\begin{aligned}
\overline{H^{(2)}[f, f]} & =2 I_{1}\left[\bar{f} * I_{1}[\bar{f}]\right]-\frac{1}{2} I_{1}\left[I_{1}[\bar{f} * \bar{f}]\right]-\bar{f} * I_{1}\left[I_{1}[\bar{f}]\right]+I_{1}[\bar{f}] * I_{1}[\bar{f}] \\
& +\frac{V}{2} I_{2}[\bar{f} * \bar{f}]-V \bar{f} * I_{2}[\bar{f}] .
\end{aligned}
$$

Using the identities $\pi_{1}^{\prime}(V)=V \pi_{2}$ and $g^{\prime \prime}(V) / 2 g(V)=2 \pi_{1}^{2}+\pi_{1}^{\prime}(V)$ we obtain

$$
\begin{equation*}
G(z, t)=G_{r} g(V)\left(1+\frac{g^{\prime}(V)}{g(V)} f_{t}+\frac{g^{\prime \prime}(V)}{2 g(V)} f_{t t}-\frac{g^{\prime}(V)}{g(V)} \frac{V}{2} f_{z}^{2}+H^{(1)}[f]+H^{(2)}[f, f]+O\left(f^{3}\right)\right), \tag{156}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
G(z, t)=G_{r} g\left(V_{\perp}\right)\left(1+H^{(1)}[f]+H^{(2)}[f, f]+O\left(f^{3}\right)\right) \tag{157}
\end{equation*}
$$

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