Symmetric Entropy Regions of Degrees Six and Seven

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***Abstract*—In this paper, we classify all G-symmetric almost entropic regions according to their Shannon-tightness, that is, whether they can be fully characterized by Shannon-type inequalities, where G is a permutation group of degree 6 or 7.**

I. Introduction

Let $N=\{1,2,\cdots,n\}$ and $(X_i, i\in N)$ be a random vector indexed by N. The set function $\mathbf{h}: 2^N \to \mathbb{R}$ defined by

$$\mathbf{h}(A)=H(X_A), \quad A\subseteq N$$

is called the *entropy function* of $(X_i, i\in N)$, The Euclidean space $\mathcal{H}_n \triangleq \mathbb{R}^{2^N}$ where entropy functions live in is called the *entropy space* of degree n. The set of all entropy functions, denoted by Γ_n^*, is called the *entropy region* of degree n, and its closure $\overline{\Gamma_n^*}$, is called the *almost entropic region* of degree n [1].

It is known that for all A, $B\subseteq N$, an entropy function satisfies the following three polymatroidal axioms:

$$\mathbf{h}(A)\geq 0 \tag{1}$$

$$\mathbf{h}(A)\leq \mathbf{h}(B) \text{ if } A\subseteq B \tag{2}$$

$$\mathbf{h}(A)+\mathbf{h}(B)\geq \mathbf{h}(A\cap B)+\mathbf{h}(A\cup B). \tag{3}$$

That is, every entropy function $\mathbf{h}$ is (the rank function of) a polymatroid [2]. It has been proved that the basic information inequalities, i.e., the nonnegativity of Shannon information measures, are equivalent to the polymatroidal axioms. So the region Γ_n bounded by Shannon-type information inequalities is also called *polymatroidal region* of degree n and $\Gamma_n^*\subseteq\Gamma_n$. It has been proved that $\Gamma_2^*=\Gamma_2$, $\Gamma_3^*\subsetneq\Gamma_3$ but $\overline{\Gamma_3^*}=\Gamma_3$ [3]–[5]. Due to the existence of infinitely many linear non-Shannon-type inequalities [6], [7], when $n\geq 4$, $\overline{\Gamma_n^*}$ is a non-polyhedral convex cone with $\overline{\Gamma_n^*}\subsetneq\Gamma_n$, and the full characterizations of Γ_n^* and $\overline{\Gamma_n^*}$ are extremely difficult.

By imposing symmetric constraints on entropy regions, Chen and Yeung [8] introduced symmetric entropy functions, and they proved that for the closure of a partition-symmetric entropy region, it can be fully characterized by Shannon-type inequalities if and only if the partition is the 1-partition or a 2-partition with one of its blocks being a singleton. In [9], Apte, Chen and Walsh classifies all G-symmetric almost entropic regions according to whether they are equal to the G-symmetric polymatroidal regions, where G is a permutation group of degree 4 or 5.

In this paper, we do the same classification to the G-symmetric almost entropic regions when G are permutation groups of degree 6 or 7.

The rest of this paper is organized as follows. Section II gives the preliminaries on orbit structures of permutation groups acting on 2^N and symmetric entropy functions. The main theorems of the paper about the classifications of the G-symmetric almost entropic regions according to whether they can be fully characterized by Shannon information inequalities are in Section III.

II. Preliminaries

In this section, we give brief preliminaries on orbit structures of permutation groups acting on 2^N, and the symmetries in the entropy space.

A. Orbit structure

Let S_n be the symmetric group on N. Given a subgroup $G\leq S_n$, for any $\sigma\in G$ and $A\subseteq N$, an action of G on 2^N is defined by

$$\sigma(A)=\{\sigma(i): i\in A\}. \tag{4}$$

Let $\mathcal{O}_G(A)=\{\sigma(A):\sigma\in G\}$ be an *orbit* of G that contains A, and $\mathfrak{O}_G=\{\mathcal{O}_G(A): A\subseteq N\}$ be the set of all orbits of G on 2^N. We call $\mathfrak{O}_G$ the *orbit structure* of G. Each orbit structure owns a partial order defined below.

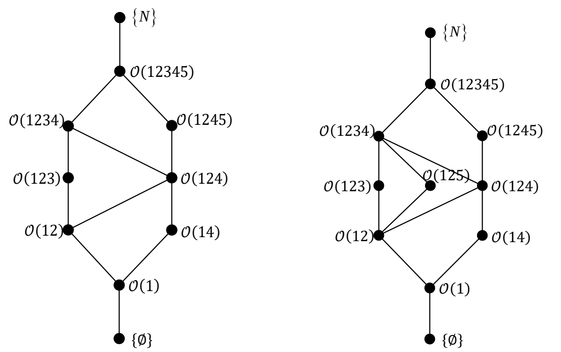
Definition 1. *For $\mathcal{O}_1,\mathcal{O}_2\in\mathfrak{O}_G$, $\mathcal{O}_1\leq\mathcal{O}_2$ if for any $A\in\mathcal{O}_1$, there exists $B\in\mathcal{O}_2$ such that $A\subseteq B$.*

The partial order among subgroups of a symmetric group yields a partial order among the orbit structures they induced.

Definition 2. *$\mathfrak{O}_{G_2}$ is a refinement of $\mathfrak{O}_{G_1}$ if for $G_1,G_2\leq S_n$, $\mathfrak{O}_{G_2}\leq\mathfrak{O}_{G_1}$.*

Example 1. *Let $G_1=S_2\mathrm{wr}_3S_3=\langle(123456),(16)(34)\rangle$*[1]*. $G_2=\langle(123)(456),(1563)(24)\rangle$. The Hasse diagram of the orbit structures of G_1 and G_2 are depicted in Figs. 1(a) and 1(b) respectively. By Definition 2, we can see that $\mathfrak{O}_{\langle(123)(456),(1563)(24)\rangle}\leq\mathfrak{O}_{S_2\mathrm{wr}_3S_3}$.*

[1] $S_2\mathrm{wr}_3S_3$ is the wreath product of S_2 and S_3, see the definition of wreath product in [10, Section 2.6],



(a) $\mathfrak{O}_{S_2 \mathrm{wr}_3 S_3}$ (b) $\mathfrak{O}_{\langle(123)(456),(1563)(24)\rangle}$

Fig. 1. Orbit structures

B. Symmetries in the entropy space

The group action of a permutation group $G \leq S_n$ on 2^N induces a group action on $\mathcal{H}_n$, that is

$$\sigma(\mathbf{h})(A) = \mathbf{h}(\sigma(A)) \tag{5}$$

for any $\sigma \in G$ and $\mathbf{h} \in \mathcal{H}_n$. Let fix_G be the subspace of $\mathcal{H}_n$ fixed by the action, i.e.,

$$\mathrm{fix}_G = \{\mathbf{h} \in \mathcal{H}_n : \mathbf{h}(A) = \mathbf{h}(B) \text{ if } A, B \in \mathcal{O}, \mathcal{O} \in \mathfrak{O}_G\}. \tag{6}$$

It is defined in [8] that G-symmetric entropy region

$$\Psi^*_G = \Gamma^*_n \cap \mathrm{fix}_G \tag{7}$$

and its outer bound, the G-symmetric polymatroidal region

$$\Psi_G = \Gamma_n \cap \mathrm{fix}_G. \tag{8}$$

For self-containment, in the following, we list the three theorems about the relations between G-symmetric entropy regions and their outer bound G-symmetric polymatroidal regions in [8] and [9], which will facilitate our proofs in the rest of this paper.

Theorem 1. [8, Thm.1] *Let $p = \{N_1, N_2, \cdots, N_t\}$ be a partition of N, that is, N_i, $i = 1, 2, \cdots, t$ are disjoint and $\bigcup_{i=1}^{t} N_i = N$. Let $G_p = S_{N_1} \times S_{N_2} \times \cdots \times S_{N_t}$, where S_{N_i} are symmetric groups on N_i. For $|N| \geq 4$,*

$$\overline{\Psi^*_{G_p}} = \Psi_{G_p}$$

if and only if $p = \{N\}$ or $\{\{i\}, N\backslash\{i\}\}$ for some $i \in N$.

Theorem 2. [9, Thm.2] *For $G_1, G_2 \leq S_n$ with $\mathfrak{O}_{G_1} \leq \mathfrak{O}_{G_2}$,*

1) *if $\overline{\Psi^*_{G_1}} = \Psi_{G_1}$, then $\overline{\Psi^*_{G_2}} = \Psi_{G_2}$;*
2) *if $\overline{\Psi^*_{G_2}} \subsetneq \Psi_{G_2}$, then $\overline{\Psi^*_{G_1}} \subsetneq \Psi_{G_1}$.*

In the following theorem, C_n and D_n are cyclic group and dihedral group of degree n, respectively.

Theorem 3. [9, Thm.5] *For $n \geq 6$, $\overline{\Psi^*_{C_n}} \subsetneq \Psi_{C_n}$, $\overline{\Psi^*_{D_n}} \subsetneq \Psi_{D_n}$, $\overline{\Psi^*_{S_1 \times C_{n-1}}} \subsetneq \Psi_{S_1 \times C_{n-1}}$, $\overline{\Psi^*_{S_1 \times D_{n-1}}} \subsetneq \Psi_{S_1 \times D_{n-1}}$.*

III. Symmetric entropy functions of degree six and seven

By definition, Ψ^*_G and Ψ_G are uniquely determined by the orbit structures $\mathfrak{O}_G$. Moreover, if two permutation groups are in the same conjugacy class, their orbit structures are isomorphic up to the permutations of the indices in N. Therefore, permutation groups can be considered equivalent if their orbit structures are isomorphic. To determine whether $\overline{\Psi^*_G} = \Psi_G$, we only need to consider one representative in each equivalence class. For $G \leq S_n$, $n = 6$, 7, by GAP [11], we see that there exsit 56 and 96 conjugacy classes, respectively. According to their orbit structure, they can be further classified into 35 and 51 equivalence classes. By Definition 2, these equivalence classes form two partialy ordered sets P_6 and P_7, whose Hasse diagrams are depicted in Figures 7 and 8 in Appendix B, respectively.

By Theorem 2, to determine the Shannon tightness of $\overline{\Psi^*_G}$, we only need to consider the antichains of all minimal equivalence classes in P_6 and P_7 such that $\overline{\Psi^*_G} = \Psi_G$ with $\mathfrak{O}_G$ in such an equivalence class, and maximal equivalence classes in P_6 and P_7 such that $\overline{\Psi^*_G} \subsetneq \Psi_G$ with $\mathfrak{O}_G$ in such an equivalence class. We call these equivalence classes critical and they are characterized in Theorems 4 and 5, and depicted in Figures 2 and 3, respectively.

A. Symmetric entropy functions of 6 random variables

Theorem 4. *For any $G \leq S_6$, $\overline{\Psi^*_G} = \Psi_G$ if and only if $G =$*

- $S_6 : \langle(123456), (12)\rangle$,
  $A_6 : \langle(12345), (456)\rangle$,
  $\mathrm{PGL}_2(5) : \langle(123456), (16)(23)\rangle$[2],
- $\mathrm{PSL}_2(5) : \langle(123)(456), (14623)\rangle$,
- $S_1 \times S_5 : \langle(23456), (23)\rangle$,
  $S_1 \times A_5 : \langle(23456), (234)\rangle$,
  $S_1 \times \mathrm{AGL}_1(5) : \langle(23456), (2546)\rangle$.

Proof. To prove the theorem, by Theorem 1, we only need to determine the minimal equivalence classes of subgroups of S_6 in P_6 such that $\overline{\Psi^*_G} = \Psi_G$ and the maximal equivalence classes of subgroups of S_6 in P_6 such that $\overline{\Psi^*_G} \subsetneq \Psi_G$. According to the Hasse diagram of P_6 depicted in Figure 7, we will prove in the following that these two families of equivalence classes of subgroups are

- $\mathrm{PSL}_2(5) : \langle(123)(456), (14623)\rangle$,
- $S_1 \times S_5 : \langle(23456), (23)\rangle$,
  $S_1 \times A_5 : \langle(23456), (234)\rangle$,
  $S_1 \times \mathrm{AGL}_1(5) : \langle(23456), (2546)\rangle$,

and

- $S_2 \times S_4 : \langle(12)(3456), (34)\rangle$,
  $S_2 \times A_4 : \langle(12)(345), (34)(56)\rangle$,
  $\langle(12)(3456), (12)(36)\rangle$,
- $S_3 \times S_3 : \langle(123)(56), (23)(456)\rangle$,
  $S_3 \times A_3 : \langle(123), (12), (456)\rangle$,

[2] We adopt the notations of permutation groups in [10] in this paper. The generating element representation of these groups can be found in Figs. 2 and 3.

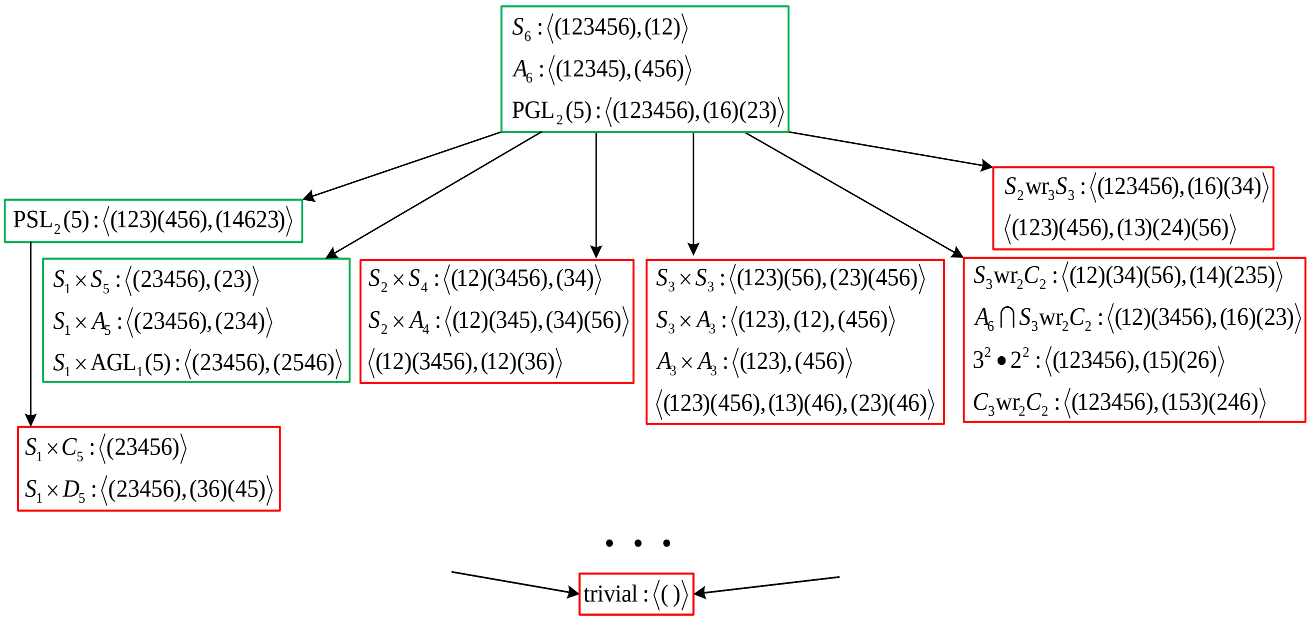


Fig. 2. The Hasse diagram of critical equivalence classes of P_6. The "green" nodes indicate $\overline{\Psi_G^*} = \Psi_G$, and "red" nodes indicate $\overline{\Psi_G^*} \subsetneq \Psi_G$.

$A_3 \times A_3 : \langle (123), (456) \rangle$,
$\langle (123)(456), (13)(46), (23)(46) \rangle$,

- $S_1 \times C_5 : \langle (23456) \rangle$,
  $S_1 \times D_5 : \langle (23456), (36)(45) \rangle$,
- $S_2\mathrm{wr}_3S_3 : \langle (123456), (16)(34) \rangle$,
  $\langle (123)(456), (13)(24)(56) \rangle$,
- $S_3\mathrm{wr}_2C_2 : \langle (12)(34)(56), (14)(235) \rangle$,
  $A_6 \cap S_3\mathrm{wr}_2C_2 : \langle (12)(3456), (16)(23) \rangle$,
  $3^2 \cdot 2^2 : \langle (123456), (15)(26) \rangle$,
  $C_3\mathrm{wr}_2C_2 : \langle (123456), (153)(246) \rangle$.

respectively. The orbit structures of them are depicted in Appendix A.

$\overline{\Psi_G^*} = \Psi_G$ for $G = \mathrm{PSL}_2(5)$:

Reducing the redundancies in (1)-(3), inequalities determining Γ_n are in the following elemental form [1, Section 14.1]

1) $\mathbf{h}(N) \geq \mathbf{h}(N \backslash \{i\})$, $i \in N$
2) $\mathbf{h}(i \cup K) + \mathbf{h}(j \cup K) \geq \mathbf{h}(K) + \mathbf{h}(\{i,j\} \cup K)$, distinct $i, j \in N$, $K \subseteq N \backslash \{i,j\}$,

with each corresponds to a facet of Γ_n. According to the orbit structure of G, $\overline{\Psi_G^*}$ and Ψ_G live in a subspace $\mathbb{R}^{\mathfrak{O}_G}$ of $\mathcal{H}_n$, which is indexed by $(s_{\mathcal{O}} : \mathcal{O} \in \mathfrak{O}_G)$. Imposing the symmetric constraints in (6), we have the elemental form of the inequalities bounding Ψ_G, i.e., the H-representation [12] of Ψ_G. Input these inequalities into lrs [13], we have the V-representation of Ψ_G, i.e., the extreme rays of Ψ_G. The H-representation and V-representation of Ψ_G can be found in Appendix A.

To prove $\overline{\Psi_G^*} = \Psi_G$, it is sufficient to show that all the extreme rays of Ψ_G are almost entropic. Among the 8 extreme rays of Ψ_G, 6 are those containing uniform matroids $U_{i,n}$, $i = 1, 2, \cdots, 6$, which are representative and so almost entropic. The remaining 2 extreme rays are symmetric by index permutation, so we only need to show one of them is almost entropic. Consider the one containing polymatroid

$$\mathbf{h_1}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 4 & \text{if } |A| = 2, \\ 5 & \text{if } A \in \mathcal{O}(123), \\ 6 & \text{otherwise.} \end{cases}$$

It can be shown that it is multi-linear representable over $\mathrm{GF}(11)$. The following is a 6×12 matrix over $\mathrm{GF}(11)$, and each $i \in N$ labels a 6×2 submatrix. It can be checked that it represents $\mathbf{h_1}$, that is, for each subset $A \subseteq N = \{1, 2, \cdots, 6\}$, the rank of the $6 \times 2|A|$ submatrix whose columns are labeled by A is equal to $\mathbf{h_1}(A)$.

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cc:cc:cc:cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 9 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$\overline{\Psi_G^*} = \Psi_G$ for $G = S_1 \times S_5$:

By Theorem 1, $\overline{\Psi_G^*} = \Psi_G$ for $G = S_1 \times S_5$. As $S_1 \times A_5$ and $S_1 \times \mathrm{AGL}_1(5)$ have the same orbit structures with $S_1 \times S_5$, $\overline{\Psi_G^*} = \Psi_G$ for $G = S_1 \times A_5$, $\mathrm{AGL}_1(5)$.

$\overline{\Psi_G^*} \neq \Psi_G$ for $G = S_2 \times S_4$, $S_3 \times S_3$, $S_1 \times C_5$:

It is immediately implied by Theorems 1 and 3.

$\overline{\Psi_G^*} \neq \Psi_G$ for $G = S_3\mathrm{wr}_2C_2$:

By a similar treatment for $\mathrm{PSL}_2(5)$, we obtain the H-representation and V-representation of Ψ_G, $G = S_3\mathrm{wr}_2C_2$, which can be find in Appendix A. Among the extreme ray of Ψ_G, there is one containing the following polymatroid

$$\mathbf{h_2}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 4 & \text{if } |A| = 2, \\ 5 & \text{if } A \in \mathcal{O}(124), \\ 6 & \text{otherwise.} \end{cases}$$

By contracting $\{6\}$ from $\mathbf{h_2}$, then restricting it on $\{1,2,3,4\}$, we have

$$\widetilde{\mathbf{h_2}}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 3 & \text{if } |A| = 2 \text{ and } |A| \neq \{1,4\}, \\ 4 & \text{if } |A| = \{1,4\} \text{ or } |A| \geq 3, \end{cases}$$

which violates Zhang-Yeung inequality [6], and is therefore non-entropic. Thus $\overline{\Psi_G^*} \neq \Psi_G$.

$\overline{\Psi_G^*} \neq \Psi_G$ for $G = S_2\mathrm{wr}_3S_3$:

Among the extreme rays of Ψ_G, one finds the polymatroid given by

$$\mathbf{h_3}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 3 & \text{if } A \in \mathcal{O}(12), \\ 4 & \text{otherwise.} \end{cases}$$

By restricting the polymatroid on {2,3,4,5}, we have

$$\widetilde{\mathbf{h_3}}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 3 & \text{if } |A| = 2 \text{ and } A \neq \{2,5\}, \\ 4 & \text{if } A = \{2,5\} \text{ or } |A| \geq 3, \end{cases}$$

which violates Zhang-Yeung inequality, and is therefore non-entropic. Thus $\overline{\Psi_G^*} \neq \Psi_G$. ☐

B. Symmetric entropy functions of 7 random variables

Lemma 1. *For symmetric group S_{N_1} and S_{N_2}, with N_1 and N_2 disjoint, let $G_1 \leq S_{N_1}$ and $G_2 \leq S_{N_2}$, then $\overline{\Psi_{G_1}^*} \neq \Psi_{G_1}$ implies $\overline{\Psi_{G_1\times G_2}^*} \neq \Psi_{G_1\times G_2}$.*

Proof. As $\overline{\Psi_{G_1}^*} \neq \Psi_{G_1}$, there exsits $\mathbf{h_1} \in \Psi_{G_1}$ such that $\mathbf{h_1} \notin \overline{\Psi_{G_1}^*} = \overline{\Gamma_{n_1}^*} \cap \mathrm{fix}_{G_1}$. Since $\mathbf{h_1} \in \Psi_{G_1} = \Gamma_{n_1} \cap \mathrm{fix}_{G_1}$, $\mathbf{h_1} \in \mathrm{fix}_{G_1}$ which implies $\mathbf{h_1} \notin \overline{\Gamma_{n_1}^*}$. By adding $|N_2|$ loops to $\mathbf{h_1}$, we obtain $\mathbf{h_1}'$ on $N \triangleq N_1 \cup N_2$ with its rank function

$$\mathbf{h_1}'(A) = \mathbf{h_1}(A \cap N_1), \quad \forall A \subseteq N.$$

For any $\mathbf{h_2} \in \Psi_{G_2}$, we obtain $\mathbf{h_2}'$ similarly. Let $\mathbf{h} = \mathbf{h_1'} + \mathbf{h_2'}$, then $\mathbf{h} \in \Psi_{G_1\times G_2}$ but $\mathbf{h} \notin \overline{\Psi_{G_1\times G_2}^*}$. Therefore $\overline{\Psi_{G_1\times G_2}^*} \neq \Psi_{G_1\times G_2}$, which proves the lemma. ☐

Theorem 5. *For any $G \leq S_7$, $\overline{\Psi_G^*} = \Psi_G$ if and only if $G =$*

- $S_7 : \langle(1234567), (12)\rangle$,
  $A_7 : \langle(1234567), (123)\rangle$,
- $S_1 \times S_6 : \langle(234567), (23)\rangle$,
  $S_1 \times A_6 : \langle(234567), (567)\rangle$,
  $\langle(234567), (27)(34)\rangle$,
- $\mathrm{AGL}_1(7) : \langle(1234567), (163247)\rangle$.

Proof. By the same argument in Theorem 4, the equivalence classes of subgroups of S_7 that need to be considered, are the following two families

- $S_1 \times S_6 : \langle(234567), (23)\rangle$,
  $S_1 \times A_6 : \langle(234567), (567)\rangle$,
  $\langle(234567), (27)(34)\rangle$,
- $\mathrm{AGL}_1(7) : \langle(1234567), (163247)\rangle$,

and

- $S_2 \times S_5 : \langle(12), (34567), (34)\rangle$,
  $S_2 \times A_5 : \langle(34567), (12), (345)\rangle$,
  $\langle(12)(345)(67), (12)(3765)\rangle$,
  $\langle(12)(34567), (12)(3651)\rangle$,
- $S_3 \times S_4 : \langle(123)(47), (13)(4567)\rangle$,
  $\langle(123)(45)(67), (13)(4567), (13)(47)\rangle$,
  $\langle(123)(457), (13)(47)(56)\rangle$,
  $\langle(123)(47), (123)(4567)\rangle$,
  $\langle(123)(47)(56), (123)(457)\rangle$,
- $S_1 \times \mathrm{PSL}_2(5) : \langle(234)(567), (25734)\rangle$,
- $S_1 \times 3^2 \cdot 2^2 : \langle(234567), (26)(37)\rangle$,
  $S_1 \times S_3\mathrm{wr}_2C_2 : \langle(23)(45)(67), (25)(346)\rangle$,
  $S_1 \times A_6 \cap S_3\mathrm{wr}_2C_2 : \langle(4567)(23), (27)(34)\rangle$,
  $S_1 \times C_3\mathrm{wr}_2C_2 : \langle(234567), (264)(357)\rangle$,
- $S_1 \times S_2\mathrm{wr}_3S_3 : \langle(234567), (27)(45)\rangle$,
  $\langle(234)(567), (24)(35)(67)\rangle$,
- $\mathrm{PGL}_3(2) : \langle(1234567), (1367)(45)\rangle$.

$\overline{\Psi_G^*} = \Psi_G$ for $G = S_1 \times S_6$:

It is immediately implied by Theorem 1.

$\overline{\Psi_G^*} = \Psi_G$ for $G = \mathrm{AGL}_1(7)$:

Through lrs we can obtain all the 13 extreme rays of Ψ_G, among which 7 are uniform matroids. We now show that the remaining 6 extreme rays contain representable polymatroids and so are almost entropic. For the ray containing

$$\mathbf{h_1}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 4 & \text{if } |A| = 2, \\ 5 & \text{if } A \in \mathcal{O}(124), \\ 6 & \text{otherwise,} \end{cases}$$

as it is the sum of two Fano matroids[3], which are representable, $\mathbf{h_1}$ is representable as well. Now consider the ray containing

$$\mathbf{h_2}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 4 & \text{if } |A| = 2, \\ 6 & \text{if } |A| = 3, \\ 7 & \text{if } A \in \mathcal{O}(1235), \\ 8 & \text{otherwise.} \end{cases}$$

[3]The definition and its representation of Fano matroid can be found in [14, Chapter 6]

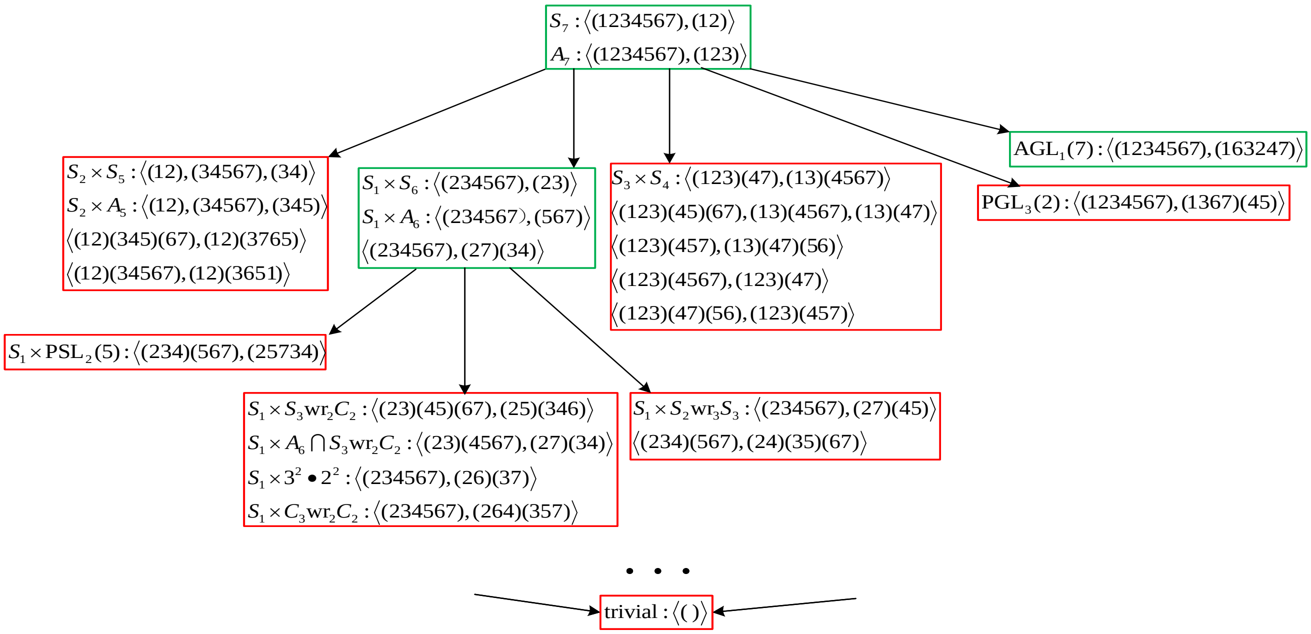


Fig. 3. The Hasse diagram of critical part of P_7

Note that $\mathbf{h_2} = \mathbf{r_1} + \mathbf{r_2}$, where

$$
\mathbf{r_1}(A) = \begin{cases} 1 & \text{if } |A| = 1, \\ 2 & \text{if } |A| = 2, \\ 3 & \text{if } |A| = 3 \text{ or } A = \{1,2,3,5\}, \{1,2,4,7\}, \\ & \{1,3,6,7\}, \{1,4,5,6\}, \{2,3,4,6\}, \{2,5,6,7\}, \\ & \{3,4,5,7\}, \\ 4 & \text{otherwise}, \end{cases}
$$

$$
\mathbf{r_2}(A) = \begin{cases} 1 & \text{if } |A| = 1, \\ 2 & \text{if } |A| = 2, \\ 3 & \text{if } |A| = 3 \text{ or } A = \{1,2,3,6\}, \{1,2,5,7\}, \\ & \{1,3,4,5\}, \{1,4,6,7\}, \{2,3,4,7\}, \{2,4,5,6\}, \\ & \{3,5,6,7\}, \\ 4 & \text{otherwise}, \end{cases}
$$

are both matroids with 7 elements. By [14, Prop. 6.4.10], $\mathbf{r_1}$ and $\mathbf{r_2}$ are representable and then so is $\mathbf{h_2}$. For the remaining 4 extreme rays, due to the page limitation, we list the polymatroids they contain in Appendix A. Using ITAP [15], we obtain the representations of the four polymatroids over GF(2), as shown in the following matrices.

$$
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{cc:cc:cc:cc:cc:cc:cc}
1&0&0&0&0&0&1&0&0&0&1&0&1&1\\
0&1&0&0&1&0&0&0&0&1&1&1&0&0\\
0&0&1&0&1&0&1&0&0&0&0&0&1&1\\
0&0&0&1&0&0&0&0&1&0&0&1&1&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0&1&0&1
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{cc:cc:cc:cc:cc:cc:cc}
1&0&0&0&1&0&1&0&0&0&1&1&1&1\\
0&1&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&1&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&1&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&1&0&0&1\\
0&0&0&0&1&0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&1&0&1
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccc:ccc:ccc:ccc:ccc:ccc:ccc}
1&1&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&1\\
0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&1&0&1&0&1&0\\
0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0&1&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccccc} \;\;1\;\; & \;\;2\;\; & \;\;3\;\; & \;\;4\;\; & \;\;5\;\; & \;\;6\;\; & \;\;7\;\; \end{array} \\
\left[\begin{array}{ccc:ccc:ccc:ccc:ccc:ccc:ccc}
1&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&1\\
0&1&0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&1
\end{array}\right]
\end{array}
$$

$\overline{\Psi_G^*} \neq \Psi_G$ for $G = S_2 \times S_5,\ S_3 \times S_4$:

It is immediately implied by Theorem 1.

$\overline{\Psi_G^*} \neq \Psi_G$ for $G = S_1 \times \mathrm{PSL}_2(5)$:

Among the extreme rays of Ψ_G, one of them containing the following polymatroid

$$
\mathbf{h_3}(A) = \begin{cases} 2 & \text{if } |A| = 1 \text{ and } A \neq \{1\}, \\ 3 & \text{if } A = \{1\}, \\ 4 & \text{if } |A| = 2, \\ 5 & \text{if } A \in \mathcal{O}(123) \text{ or } \mathcal{O}(234), \\ 6 & \text{otherwise.} \end{cases}
$$

By contracting $\{2\}$ from $\mathbf{h_3}$, then restricting it on $\{1,3,4,5\}$, we obtain $\widetilde{\mathbf{h_3}}$, which violates Zhang-Yeung inequality. Thus, it is non-entropic, $\overline{\Psi_G^*} \neq \Psi_G$.

$\overline{\Psi_G^*} \neq \Psi_G$ for $G = S_1 \times S_3\mathrm{wr}_2C_2,\ S_1 \times S_2\mathrm{wr}_3S_3$:

Since $\overline{\Psi_{S_3\mathrm{wr}_2C_2}^*} \neq \Psi_{S_3\mathrm{wr}_2C_2}$, $\overline{\Psi_{S_2\mathrm{wr}_3S_3}^*} \neq \Psi_{S_2\mathrm{wr}_3S_3}$, which immediately implies the result by Lemma 1.

$\overline{\Psi_G^*} \neq \Psi_G$ for $G = \mathrm{PGL}_3(2)$:

Among the extreme rays of Ψ_G, there exsits one containing the following polymatroid

$$
\mathbf{h_4}(A) = \begin{cases} 2 & \text{if } |A| = 1, \\ 4 & \text{if } |A| = 2, \\ 5 & \text{if } A \in \mathcal{O}(123), \\ 6 & \text{otherwise.} \end{cases}
$$

By contracting $\{1\}$ from $\mathbf{h_4}$, then restricting it on $\{2,3,4,5\}$, we obtain $\widetilde{\mathbf{h_4}}$, which violates Zhang-Yeung inequality. Thus, it is non-entropic and $\overline{\Psi_G^*} \neq \Psi_G$. $\square$

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APPENDIX

A. More details about the symmetric polymatroidal regions

In this subsection of Appendix, we provide more details about the symmetric polymatroidal regions, which are used in the proofs in Section III, i.e., orbit structures, H-representation and V-representation of Ψ_G.

1). $G = \mathrm{PSL}_2(5) : \langle(123)(456), (14623)\rangle$

We depicted the orbit structure of $\mathrm{PSL}_2(5)$ in Fig. 1.

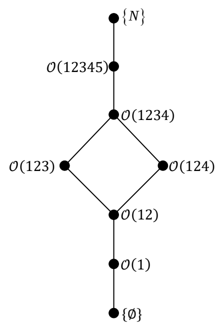

Fig. 1. $\mathfrak{O}_{\mathrm{PSL}_2(5)}$

Note that each orbit in $\mathfrak{O}_{\mathrm{PSL}_2(5)}$ corresponds a dimension in $\mathrm{fix}_{\mathrm{PSL}_2(5)}$, so

$$\begin{aligned}\Psi_{\mathrm{PSL}_2(5)} = \{\mathbf{s} \in \mathrm{fix}_{\mathrm{PSL}_2(5)} : & 2\mathbf{s}_{12} \geq \mathbf{s}_1 + \mathbf{s}_{123}, \\ & 2\mathbf{s}_{12} \geq \mathbf{s}_1 + \mathbf{s}_{124}, \\ & 2\mathbf{s}_{124} \geq \mathbf{s}_{12} + \mathbf{s}_{1234}, \\ & \mathbf{s}_{123} + \mathbf{s}_{124} \geq \mathbf{s}_{12} + \mathbf{s}_{1234}, \\ & 2\mathbf{s}_{123} \geq \mathbf{s}_{12} + \mathbf{s}_{1234}, \\ & 2\mathbf{s}_{1234} \geq \mathbf{s}_{123} + \mathbf{s}_{12345}, \\ & 2\mathbf{s}_{1234} \geq \mathbf{s}_{124} + \mathbf{s}_{12345}, \\ & 2\mathbf{s}_{12345} \geq \mathbf{s}_{1234} + \mathbf{s}_{123456}, \\ & \mathbf{s}_{123456} \geq \mathbf{s}_{12345}, \\ & 2\mathbf{s}_1 \geq \mathbf{s}_{12}\}.\end{aligned}$$

For simplicity, we rewrite each inequality by its coefficient, and so the H-representation of $\Psi_{\mathrm{PSL}_2(5)}$ is

- $-1\ 2\ -1\ 0\ 0\ 0\ 0$
- $-1\ 2\ 0\ -1\ 0\ 0\ 0$
- $0\ -1\ 0\ 2\ -1\ 0\ 0$
- $0\ -1\ 1\ 1\ -1\ 0\ 0$
- $0\ -1\ 2\ 0\ -1\ 0\ 0$
- $0\ 0\ -1\ 0\ 2\ -1\ 0$
- $0\ 0\ 0\ -1\ 2\ -1\ 0$
- $0\ 0\ 0\ 0\ -1\ 2\ -1$
- $0\ 0\ 0\ 0\ 0\ -1\ 1$
- $2\ -1\ 0\ 0\ 0\ 0\ 0$

where the order of the indices is ($\mathcal{O}(1)$,$\mathcal{O}(12)$,$\mathcal{O}(123)$, $\mathcal{O}(124)$,$\mathcal{O}(1234)$, $\mathcal{O}(12345)$,$\mathcal{O}(123456)$).

For the V-representation, for those extreme rays containing polymatroid other than uniform matroid, we represent it by a vector indexed by the orbits.

- $U_{1,6}$, $U_{2,6}$, $U_{3,6}$, $U_{4,6}$, $U_{5,6}$, $U_{6,6}$;
- $2\ 4\ 5\ 6\ 6\ 6\ 6$;
- $2\ 4\ 6\ 5\ 6\ 6\ 6$;

In the following, for a G-symmetric polymatroidal region Ψ_G, we represent its orbit structure by its Hasse diagram in the figure, the H-representation by inequality coefficients and V-representation by vector of polymatroid except the uniform matroid.

2). $G = S_3\mathrm{wr}_2C_2 : \langle(12)(34)(56), (14)(235)\rangle$

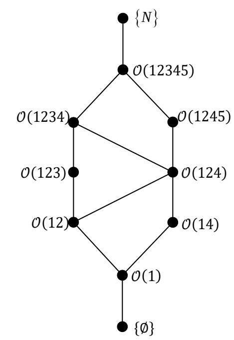

Fig. 2. $\mathfrak{O}_{S_3\mathrm{wr}_2C_2}$

The order of indices: ($\mathcal{O}(1)$,$\mathcal{O}(12)$,$\mathcal{O}(14)$,$\mathcal{O}(124)$,$\mathcal{O}(146)$, $\mathcal{O}(1234)$,$\mathcal{O}(1246)$,$\mathcal{O}(12345)$,$\mathcal{O}(123456)$).

H-representation:

- $-1\ 0\ 2\ -1\ 0\ 0\ 0\ 0\ 0$
- $-1\ 1\ 1\ -1\ 0\ 0\ 0\ 0\ 0$
- $-1\ 2\ 0\ 0\ -1\ 0\ 0\ 0\ 0$
- $0\ -1\ 0\ 1\ 1\ -1\ 0\ 0\ 0$
- $0\ -1\ 0\ 2\ 0\ 0\ -1\ -1\ 0$
- $0\ -1\ 0\ 2\ 0\ 0\ -1\ 0\ 0$
- $0\ 0\ -1\ 2\ 0\ -1\ 0\ 0\ 0$
- $0\ 0\ -1\ 2\ 0\ 0\ -1\ -1\ 0$
- $0\ 0\ -1\ 2\ 0\ 0\ -1\ 0\ 0$
- $0\ 0\ 0\ -1\ 0\ 0\ 2\ -1\ 0$
- $0\ 0\ 0\ -1\ 0\ 0\ 2\ 0\ 0$
- $0\ 0\ 0\ -1\ 0\ 1\ 1\ -1\ 0$
- $0\ 0\ 0\ -1\ 0\ 1\ 1\ 0\ 0$
- $0\ 0\ 0\ 0\ -1\ 2\ 0\ -1\ 0$
- $0\ 0\ 0\ 0\ 0\ -1\ 0\ 2\ -1$
- $0\ 0\ 0\ 0\ 0\ 0\ -1\ 1\ -1$
- $0\ 0\ 0\ 0\ 0\ 0\ -1\ 2\ -1$
- $0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 1$
- $2\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0$
- $2\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 0$

V-representation:

- $U_{1,6}$, $U_{2,6}$, $U_{3,6}$, $U_{4,6}$, $U_{5,6}$,$U_{6,6}$;
- $2\ 3\ 4\ 4\ 4\ 4\ 4\ 4\ 4$
- $2\ 4\ 3\ 4\ 4\ 4\ 4\ 4\ 4$

- 2 4 4 5 6 6 6 6 6
- 4 7 8 10 12 12 12 12 12
- 1 2 2 3 2 3 3 3 3
- 1 2 1 2 1 2 2 2 2
- 3 5 6 7 9 8 9 9 9
- 4 8 8 10 12 11 12 12 12
- 3 5 6 7 8 8 9 9 9
- 2 4 4 6 6 7 8 8 8
- 1 2 2 3 2 4 3 4 4
- 2 4 3 5 4 6 5 6 6
- 2 4 4 6 6 8 7 8 8

3). $S_2\mathrm{wr}_3S_3 : \langle(123456), (16)(34)\rangle$

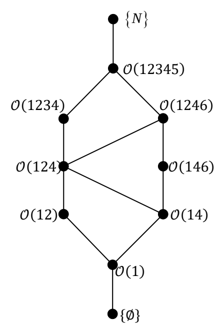

Fig. 3. $\mathfrak{D}_{S_2\mathrm{wr}_3S_3}$

The order of indices: ($\mathcal{O}(1)$,$\mathcal{O}(12)$,$\mathcal{O}(14)$,$\mathcal{O}(123)$,$\mathcal{O}(124)$, $\mathcal{O}(1234)$,$\mathcal{O}(1245)$,$\mathcal{O}(12345)$,$\mathcal{O}(123456)$).

H-representation:

- -1 1 1 0 -1 0 0 0 0
- -1 2 0 -1 0 0 0 0 0
- -1 2 0 0 -1 0 0 0 0
- 0 -1 0 0 2 0 -1 0 0
- 0 -1 0 1 1 -1 0 -1 0
- 0 -1 0 1 1 -1 0 0 0
- 0 -1 0 2 0 -1 0 -1 0
- 0 -1 0 2 0 -1 0 0 0
- 0 0 -1 0 2 -1 0 -1 0
- 0 0 -1 0 2 -1 0 0 0
- 0 0 -1 0 2 0 -1 0 0
- 0 0 0 -1 0 2 0 -1 0
- 0 0 0 -1 0 2 0 0 0
- 0 0 0 0 -1 1 1 -1 0
- 0 0 0 0 -1 1 1 0 0
- 0 0 0 0 -1 2 0 -1 0
- 0 0 0 0 -1 2 0 0 0
- 0 0 0 0 0 -1 0 1 -1
- 0 0 0 0 0 -1 0 2 -1
- 0 0 0 0 0 0 -1 2 -1
- 0 0 0 0 0 0 0 -1 1
- 2 -1 0 0 0 0 0 0 0
- 2 0 -1 0 0 0 0 0 0

V-representation:

- $U_{1,6}$,$U_{2,6}$, $U_{3,6}$, $U_{4,6}$, $U_{5,6}$, $U_{6,6}$;
- 2 3 4 4 4 4 4 4 4
- 1 2 1 2 2 2 2 2 2
- 4 6 8 7 8 8 8 8 8
- 2 4 4 5 6 6 6 6 6
- 2 4 4 6 5 6 6 6 6
- 2 4 3 6 5 6 6 6 6
- 2 4 4 6 6 7 8 8 8
- 4 8 8 11 12 14 16 16 16
- 2 4 4 6 5 6 5 6 6
- 1 2 1 3 2 3 2 3 3
- 2 4 2 5 4 6 4 6 6
- 1 2 2 3 3 4 3 4 4

4). $\mathrm{AGL}_1(7) : \langle(1234567), (163247)\rangle$

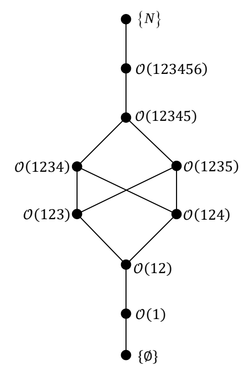

Fig. 4. $\mathfrak{D}_{\mathrm{AGL}_1(7)}$

The order of indices: ($\mathcal{O}(1)$,$\mathcal{O}(12)$,$\mathcal{O}(123)$,$\mathcal{O}(124)$,$\mathcal{O}(1234)$, $\mathcal{O}(1235)$,$\mathcal{O}(12345)$,$\mathcal{O}(123456)$,$\mathcal{O}(1234567)$).

H-representation:

- -1 2 -1 0 0 0 0 0 0
- -1 2 0 -1 0 0 0 0 0
- 0 -1 0 2 -1 0 0 0 0
- 0 -1 1 1 -1 0 0 0 0
- 0 -1 1 1 0 -1 0 0 0
- 0 -1 2 0 -1 0 0 0 0
- 0 -1 2 0 0 -1 0 0 0
- 0 0 -1 0 0 2 -1 0 0
- 0 0 -1 0 1 1 -1 0 0
- 0 0 -1 0 2 0 -1 0 0
- 0 0 0 -1 1 1 -1 0 0
- 0 0 0 -1 2 0 -1 0 0
- 0 0 0 0 -1 0 2 -1 0
- 0 0 0 0 0 -1 2 -1 0
- 0 0 0 0 0 0 -1 2 -1
- 0 0 0 0 0 0 0 -1 1
- 2 -1 0 0 0 0 0 0 0

V-representation:

- $U_{1,7}$, $U_{2,7}$, $U_{3,7}$, $U_{4,7}$,$U_{5,7}$, $U_{6,7}$, $U_{7,7}$;
- 2 4 4 5 6 6 6 6 6
- 2 4 4 6 8 7 8 8 8
- 2 4 4 6 6 6 6 6 6

- 3 6 6 9 10 9 10 10 10
- 2 4 4 6 7 8 8 8 8
- 3 6 6 8 10 11 11 11 11

5). $\Psi_{\mathrm{PGL}_3(2)} : \langle(1234567),(1367)(45)\rangle$

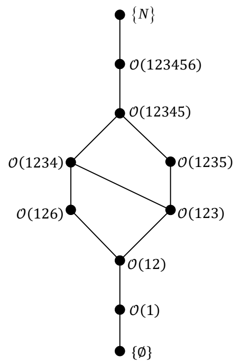

Fig. 5. $\mathfrak{D}_{\mathrm{PGL}_3(2)}$

The order of indices: ($\mathcal{O}(1)$,$\mathcal{O}(12)$,$\mathcal{O}(123)$,$\mathcal{O}(126)$,$\mathcal{O}(1234)$, $\mathcal{O}(1235)$,$\mathcal{O}(12345)$,$\mathcal{O}(123456)$,$\mathcal{O}(1234567)$).

H-representation:

- −1 2 −1 0 0 0 0 0 0
- −1 2 0 −1 0 0 0 0 0
- 0 −1 1 1 −1 0 0 0 0
- 0 −1 2 0 −1 0 0 0 0
- 0 −1 2 0 0 −1 0 0 0
- 0 0 −1 0 1 1 −1 0 0
- 0 0 −1 0 2 0 −1 0 0
- 0 0 0 −1 2 0 −1 0 0
- 0 0 0 0 −1 0 2 −1 0
- 0 0 0 0 0 −1 2 −1 0
- 0 0 0 0 0 0 −1 2 −1
- 0 0 0 0 0 0 0 −1 1
- 2 −1 0 0 0 0 0 0 0

V-representation:

- $U_{1,7}$, $U_{2,7}$, $U_{3,7}$, $U_{4,7}$,$U_{5,7}$, $U_{6,7}$, $U_{7,7}$;
- 2 4 5 6 6 5 6 6 6
- 1 2 3 2 3 3 3 3 3
- 1 2 3 3 4 3 4 4 4
- 2 4 5 6 6 6 6 6 6
- 2 4 6 5 7 8 8 8 8
- 2 4 6 6 7 8 8 8 8

6). $\Psi_{S_1\times\mathrm{PSL}_2(5)} : \langle(234)(567),(25734)\rangle$

The order of indices: ($\mathcal{O}(1)$, $\mathcal{O}(2)$, $\mathcal{O}(12)$, $\mathcal{O}(23)$, $\mathcal{O}(123)$, $\mathcal{O}(234)$, $\mathcal{O}(235)$, $\mathcal{O}(1234)$, $\mathcal{O}(1235)$, $\mathcal{O}(2345)$, $\mathcal{O}(12345)$, $\mathcal{O}(23456)$,$\mathcal{O}(123456)$,$\mathcal{O}(1234567)$).

H-representation:

- −1 0 2 0 −1 0 0 0 0 0 0 0 0 0
- 0 −1 0 2 0 −1 0 0 0 0 0 0 0 0
- 0 −1 0 2 0 0 −1 0 0 0 0 0 0 0
- 0 −1 1 1 −1 0 0 0 0 0 0 0 0 0

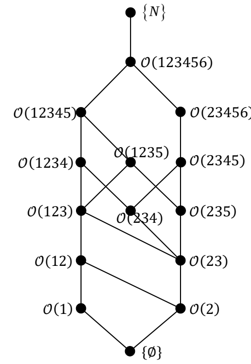

Fig. 6. $\mathfrak{D}_{S_1\times\mathrm{PSL}_2(5)}$

- 0 0 −1 0 2 0 0 −1 0 0 0 0 0 0
- 0 0 −1 0 2 0 0 0 −1 0 0 0 0 0
- 0 0 0 −1 0 0 2 0 0 −1 0 0 0 0
- 0 0 0 −1 0 1 1 0 0 −1 0 0 0 0
- 0 0 0 −1 0 2 0 0 0 −1 0 0 0 0
- 0 0 0 −1 1 0 1 0 −1 0 0 0 0 0
- 0 0 0 −1 1 1 0 −1 0 0 0 0 0 0
- 0 0 0 0 −1 0 0 0 2 0 −1 0 0 0
- 0 0 0 0 −1 0 0 1 1 0 −1 0 0 0
- 0 0 0 0 −1 0 0 2 0 0 −1 0 0 0
- 0 0 0 0 0 −1 0 0 0 2 0 −1 0 0
- 0 0 0 0 0 −1 0 1 0 1 −1 0 0 0
- 0 0 0 0 0 0 −1 0 0 2 0 −1 0 0
- 0 0 0 0 0 0 −1 0 1 1 −1 0 0 0
- 0 0 0 0 0 0 0 −1 0 0 2 0 −1 0
- 0 0 0 0 0 0 0 0 −1 0 2 0 −1 0
- 0 0 0 0 0 0 0 0 0 −1 0 2 −1 0
- 0 0 0 0 0 0 0 0 0 −1 1 1 −1 0
- 0 0 0 0 0 0 0 0 0 0 −1 0 2 −1
- 0 0 0 0 0 0 0 0 0 0 0 −1 2 −1
- 0 0 0 0 0 0 0 0 0 0 0 0 −1 1
- 0 2 0 −1 0 0 0 0 0 0 0 0 0 0
- 1 1 −1 0 0 0 0 0 0 0 0 0 0 0

V-representation:

- $U_{1,7}$, $U_{2,7}$, $U_{3,7}$, $U_{4,7}$,$U_{5,7}$, $U_{6,7}$;
- 1 0 1 0 1 0 0 1 1 0 1 0 1 1
- 0 1 1 2 2 3 3 3 3 4 4 5 5 6
- 6 1 6 2 6 3 3 6 6 4 6 5 6 6
- 2 1 2 2 2 2 2 2 2 2 2 2 2 2
- 5 1 5 2 5 3 3 5 5 4 5 5 5 5
- 3 1 3 2 3 3 3 3 3 3 3 3 3 3
- 4 1 5 2 6 3 3 6 6 4 6 5 6 6
- 0 1 1 2 2 2 2 2 2 2 2 2 2 2
- 3 1 4 2 5 3 3 5 5 4 5 5 5 5
- 5 1 6 2 6 3 3 6 6 4 6 5 6 6
- 4 1 5 2 5 3 3 5 5 4 5 5 5 5
- 0 1 1 1 1 1 1 1 1 1 1 1 1 1
- 2 1 3 2 3 3 3 3 3 3 3 3 3 3
- 3 1 4 2 5 3 3 6 6 4 6 5 6 6

- 2 1 3 2 4 3 3 5 5 4 5 5 5 5
- 0 1 1 2 2 3 3 3 3 3 3 3 3 3
- 6 2 6 4 6 5 6 6 6 6 6 6 6 6
- 4 1 4 2 4 3 3 4 4 4 4 4 4 4
- 2 2 4 4 6 5 6 6 6 6 6 6 6 6
- 2 1 3 2 4 3 3 4 4 4 4 4 4 4
- 4 2 6 4 6 5 6 6 6 6 6 6 6 6
- 3 1 4 2 4 3 3 4 4 4 4 4 4 4
- 3 2 4 4 5 5 6 6 6 6 6 6 6 6
- 1 2 3 4 5 5 6 6 6 6 6 6 6 6
- 2 2 4 4 5 5 6 6 6 6 6 6 6 6
- 6 3 8 6 10 8 9 12 11 10 12 11 12 12
- 4 3 7 6 10 8 9 12 11 10 12 11 12 12
- 5 3 8 6 10 8 9 12 11 10 12 11 12 12
- 4 3 6 6 8 8 9 10 9 10 10 10 10 10
- 2 2 4 4 6 6 6 8 7 8 8 8 8 8
- 2 3 5 6 8 8 9 10 9 10 10 10 10 10
- 3 3 6 6 8 8 9 10 9 10 10 10 10 10
- 5 3 7 6 9 8 9 11 10 10 11 11 11 11
- 3 3 6 6 9 8 9 11 10 10 11 11 11 11
- 4 3 7 6 9 8 9 11 10 10 11 11 11 11
- 6 2 8 4 10 6 6 11 12 8 12 10 12 12
- 3 2 5 4 7 6 6 8 9 7 9 8 9 9
- 4 2 6 4 8 6 6 9 10 8 10 10 10 10
- 2 2 4 4 6 6 6 7 8 7 8 8 8 8
- 0 2 2 4 4 5 6 5 6 6 6 6 6 6
- 2 2 4 4 6 6 6 7 8 8 8 8 8 8
- 0 1 1 2 2 3 3 3 3 3 4 4 4 4 4
- 6 3 8 6 10 9 8 11 12 10 12 11 12 12
- 4 3 7 6 10 9 8 11 12 10 12 11 12 12
- 5 3 8 6 10 9 8 11 12 10 12 11 12 12
- 4 3 6 6 8 9 8 9 10 10 10 10 10 10
- 1 2 3 4 5 6 6 6 7 7 7 7 7 7
- 2 3 5 6 8 9 8 9 10 10 10 10 10 10
- 3 3 6 6 8 9 8 9 10 10 10 10 10 10
- 5 3 7 6 9 9 8 10 11 10 11 11 11 11
- 3 3 6 6 9 9 8 10 11 10 11 11 11 11
- 4 3 7 6 9 9 8 10 11 10 11 11 11 11
- 2 1 3 2 4 3 3 5 5 4 6 5 6 6
- 6 2 6 4 6 6 5 6 6 6 6 6 6 6
- 2 2 4 4 6 6 5 6 6 6 6 6 6 6
- 4 2 6 4 6 6 5 6 6 6 6 6 6 6
- 3 2 4 4 5 6 5 6 6 6 6 6 6 6
- 1 2 3 4 5 6 5 6 6 6 6 6 6 6
- 2 2 4 4 5 6 5 6 6 6 6 6 6 6
- 3 2 5 4 7 6 6 9 8 7 9 8 9 9
- 6 2 8 4 10 6 6 12 11 8 12 10 12 12
- 2 2 4 4 6 6 6 8 7 7 8 8 8 8
- 4 2 6 4 8 6 6 10 9 8 10 10 10 10
- 0 2 2 4 4 6 5 6 5 6 6 6 6 6
- 1 2 3 4 5 6 6 7 6 7 7 7 7 7
- 0 1 1 2 2 3 3 3 3 3 4 4 5 5 5

B. The Hasse diagram of P_6 and P_7

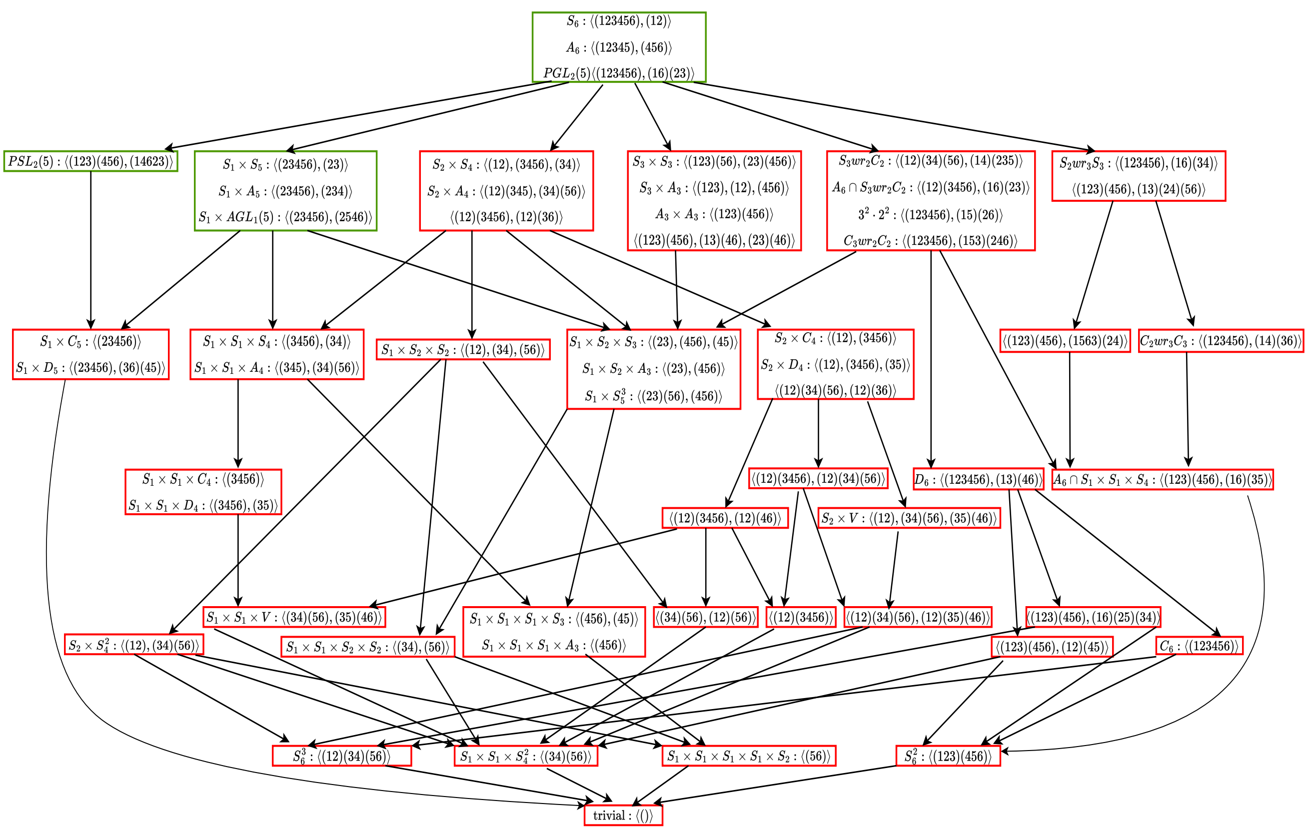

Fig. 7. The Hasse diagram of P_6

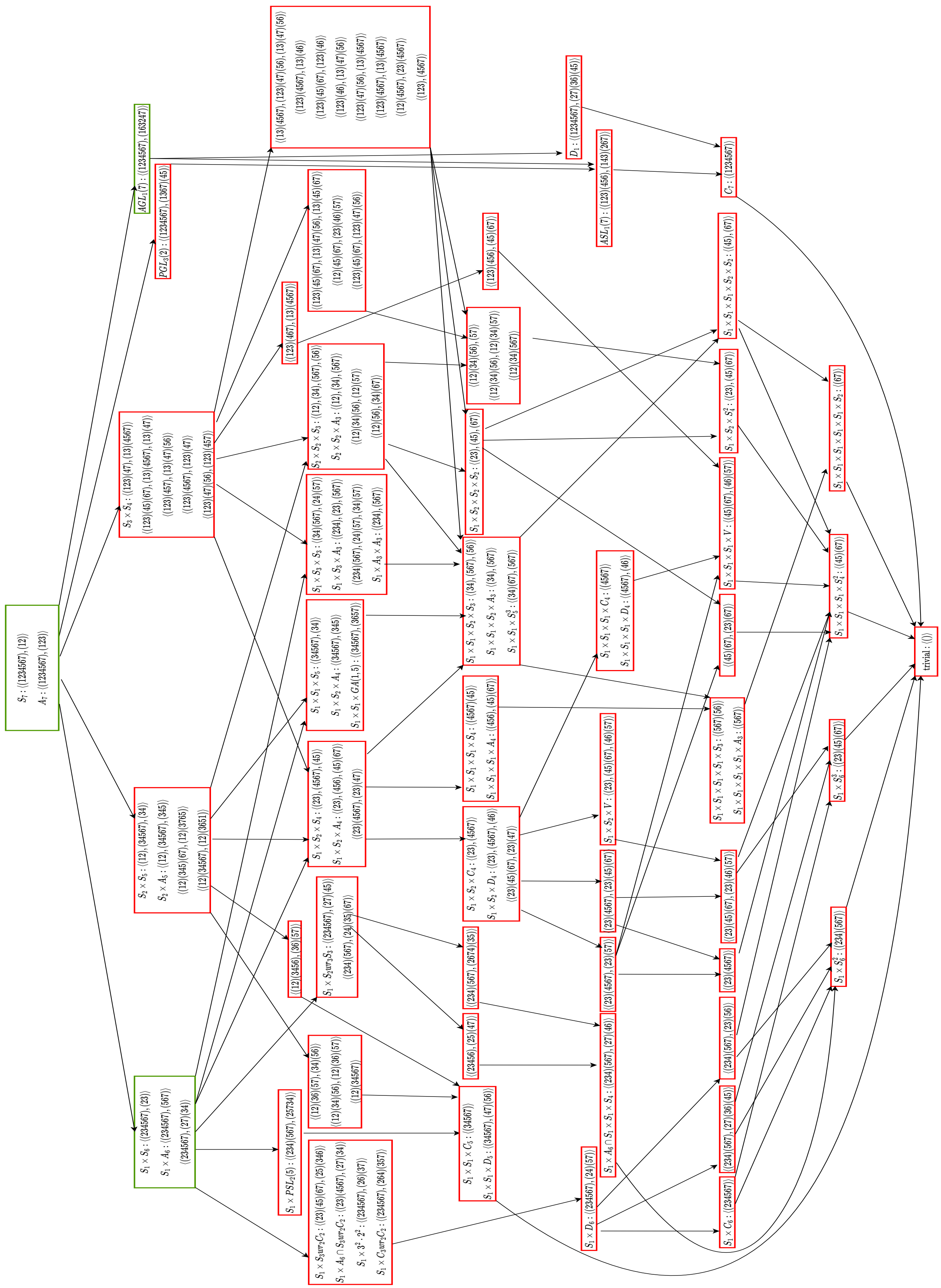

Fig. 8. The Hasse diagram of P_7