SZEGŐ RECURRENCE FOR MULTIPLE ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. We investigate polynomials that satisfy simultaneous orthogonality conditions with respect to several measures on the unit circle. We generalize the direct and inverse Szegő recurrence relations, identify the analogues of the Verblunsky coefficients, and prove the Christoffel–Darboux formula. These results stand directly in analogue with the nearest neighbour recurrence relations from the real line counterpart.

1. Introduction

The theory of multiple orthogonal polynomials on the real line (MOPRL for short) deals with polynomials that are orthogonal with respect to several measures on the real line. These polynomials initially appeared in Hermite–Padé approximations of Markov functions. Today MOPRL theory is very well developed with applications in approximation theory, spectral theory, random matrix theory, and integrable probability. See [16] for a recent quick introduction to MORPL and its applications, and [2, 13, 21] for a more thorough treatment.

While the MOPRL theory is well-developed, the theory of multiple orthogonal polynomials *on the unit circle* (MOPUC) is still at its infancy. It was introduced in [18], motivated by applications in approximation theory and prediction theory. In particular, these polynomials appear when studying Hermite–Padé approximations of Carathéodory functions. Since then, MOPUC has only been further studied once, in [7]. The goal of this article is to encourage further development in MOPUC, by deriving the analogues of two MOPRL results that are important milestones in the theory.

The first result is the Christoffel–Darboux formula by Daems and Kuijlaars [8] (see also [1] for the more general setting), which is the starting point of many further applications of MOPRL to random matrix theory and Markov processes for non-colliding particles, see [6, 9, 11, 12, 14, 15].

The other important advance in this theory was the paper [24] by Van Assche showing that MOPRL satisfy the so-called nearest neighbour recurrence relations, which is the generalization of the three-term recurrence relation of the usual orthogonal polynomials on the real line. In particular, these relations became a simple and natural tool for studying asymptotics of MOPRL along *every* direction of $\mathbb{Z}_+^r$, rather than just along the stepline multi-indices (see [5, 17, 19, 20, 23, 25] among others). The recurrences also provide the connection of MOPRL to the spectral

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theory of Jacobi operators on trees, a very recent important development [3, 4, 10] in this area.

We believe that the lack of progress in MOPUC is a consequence of not having the correct analogue of the Szegő recurrence relation from the theory of orthogonal polynomials on the unit circle (OPUC). In this paper we present recurrence relations that not only generalize the recurrence coefficients of OPUC, but are also a perfect analogue of the nearest neighbour recurrence relation from MOPRL. This opens up the theory to further progress, which we illustrate by proving a Christoffel–Darboux formula.

We start by reviewing the basics of OPUC. Let μ be a probability measure on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ having infinite support. The associated inner product on $L^2(\mu)$ is

$$\langle f(z), g(z) \rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, \mathrm{d}\mu(e^{i\theta}). \tag{1}$$

Applying the Gram–Schmidt algorithm to the sequence of monomials $\{z^j\}_{j=0}^{\infty}$ in $L^2(\mu)$, one obtains the sequence of monic orthogonal polynomials $\{\Phi_n(z)\}_{n=0}^{\infty}$ satisfying

$$\langle \Phi_n(z), z^j \rangle = 0, \qquad j = 0, \ldots, n-1. \tag{2}$$

The fundamental result of OPUC is the Szegő recurrence relations given by

$$\Phi_{n+1}(z) = z\Phi_n(z) + \alpha_{n+1}\Phi_n^*(z), \tag{3}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) + \bar{\alpha}_{n+1} z\Phi_n(z), \tag{4}$$

for $n \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ is the reversed polynomial. Equivalently, Φ_n^* can be defined as the unique polynomial that satisfies

$$\langle \Phi_n^*(z), z^p \rangle = 0, \qquad p = 1, \ldots, n, \tag{5}$$

with the normalization $\Phi_n^*(0) = 1$. We also have the inverse Szegő recurrence

$$\Phi_{n+1}(z) = \alpha_{n+1}\Phi_{n+1}^*(z) + \rho_{n+1} z\Phi_n(z), \tag{6}$$

$$\Phi_{n+1}^*(z) = \bar{\alpha}_{n+1}\Phi_{n+1}(z) + \rho_{n+1}\Phi_n^*(z), \tag{7}$$

where $\rho_n = 1 - |\alpha_n|^2$.

The recurrence coefficients α_n (with $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$) belong to complex unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and are called the Verblunsky coefficients of μ (also sometimes referred to as the Schur, Geronimus, or reflection coefficients). Note that we are using α_{n+1} in place where it is traditional (nowadays) to use $-\bar{\alpha}_n$ (see the discussion in [22, p.10]). With such a choice we get $\Phi_n(0) = \alpha_n$ and $\Phi_n^*(z) = \bar{\alpha}_n z^n + o(z^n)$, which will be natural for our purposes later. Note also that our ρ_n is $1 - |\alpha_n|^2$ instead of the traditional $(1 - |\alpha_n|^2)^{1/2}$.

Multiple orthogonal polynomials on the unit circle are polynomials that satisfy orthogonality conditions with respect to a system of measures $\boldsymbol{\mu} = \{\mu_j\}_{j=1}^{r}$. For a multi-index $\boldsymbol{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_+^r$, we write $|\boldsymbol{n}| = n_1 + \ldots + n_r$, and the polynomials $\Phi_{\boldsymbol{n}}$ we want to consider are monic with $\deg \Phi_{\boldsymbol{n}} = |\boldsymbol{n}|$ and

$$\langle \Phi_{\boldsymbol{n}}(z), z^p \rangle_j = 0, \qquad p = 0, \ldots, n_j - 1, \qquad j = 1, \ldots, r, \tag{8}$$

where $\langle \cdot, \cdot \rangle_j$ is the inner product (1) but with μ_j instead of μ. Such polynomials are called the type II multiple orthogonal polynomials. In analogy with (5), we also want to consider polynomials $\Phi^*_{\boldsymbol{n}}$ with $\deg \Phi^*_{\boldsymbol{n}} \le |\boldsymbol{n}|$, $\Phi^*_{\boldsymbol{n}}(0) = 1$ and

$$\left\langle \Phi^*_{\boldsymbol{n}}(z), z^p \right\rangle_j = 0, \qquad p = 1, \dots, n_j, \qquad j = 1, \dots, r. \tag{9}$$

We stress that these are no longer the reversed polynomials (unless $r = 1$).

One of the main results of this paper is that $\Phi_{\boldsymbol{n}}$ and $\Phi^*_{\boldsymbol{n}}$ satisfy the following Szegő-type recurrences:

$$\Phi_{\boldsymbol{n}}(z) = \alpha_{\boldsymbol{n}} \Phi^*_{\boldsymbol{n}}(z) + \sum_{i=1}^{r} \rho_{\boldsymbol{n},i} z \Phi_{\boldsymbol{n}-\boldsymbol{e}_i}(z) \tag{10}$$

$$\Phi^*_{\boldsymbol{n}}(z) = \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k}(z) + \beta_{\boldsymbol{n}} z \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}(z), \tag{11}$$

for some coefficients $\alpha_{\boldsymbol{n}}$, $\beta_{\boldsymbol{n}}$, and $\rho_{\boldsymbol{n},1}, \dots, \rho_{\boldsymbol{n},r}$. These generalize the recurrences (6) and (4). Note that if all the $\alpha_{\boldsymbol{n}}$, $\beta_{\boldsymbol{n}}$, $\rho_{\boldsymbol{n},j}$ are known, then equations (10) and (11) are sufficient to compute all $\Phi_{\boldsymbol{n}}$'s and $\Phi^*_{\boldsymbol{n}}$'s recursively.

The generalizations of (7) and (3) appear as easy consequences of (10) and (11), however these equations seem less natural and the generalization of (3) gives no information in the case $\beta_{\boldsymbol{n}} = 0$. Interestingly, the situation is opposite for type I multiple orthogonal polynomials: we get generalizations of (3) and (7), while the generalizations of (4) and (6) appear less natural, and the generalization of (6) vanishes in the case $\alpha_{\boldsymbol{n}} = 0$.

The coefficients $\{\alpha_{\boldsymbol{n}}\}$ and $\{\beta_{\boldsymbol{n}}\}$ should be viewed as the generalized Verblunsky/reflection coefficients. Indeed, $\Phi_{\boldsymbol{n}}(0) = \beta_{\boldsymbol{n}}$ and $\Phi^*_{\boldsymbol{n}}(z) = \alpha_{\boldsymbol{n}} z^{|\boldsymbol{n}|} + o(z^{|\boldsymbol{n}|})$, similarly to the usual OPUC. Furthermore, for the marginal indices $\boldsymbol{n} = j\boldsymbol{e}_k$, the recurrences (11) and (10) become the usual Szegő recurrences (4) and (6), with $\alpha_{j\boldsymbol{e}_k}$, $\beta_{j\boldsymbol{e}_k}$, $\rho_{j\boldsymbol{e}_k,k}$ reduced to the usual OPUC recurrence coefficients $\alpha_j(\mu_k)$, $\bar{\alpha}_j(\mu_k)$, $\rho_j(\mu_k)$ associated with μ_k, respectively (or $-\bar{\alpha}_{j-1}(\mu_k)$, $-\alpha_{j-1}(\mu_k)$, $\rho_{j-1}(\mu_k)^2$ in the notation of [22]), while $\rho_{j\boldsymbol{e}_k,m} = 0$ for $m \ne k$. We note that the importance of the multiple Verblunsky coefficient $\alpha_{\boldsymbol{n}}$ was also observed in [7] (where it was denoted by $\delta_{n,m}$).

It is worth mentioning that for the recurrence relations (10) and (11) to hold, it is necessary that the polynomials appearing in these equations are uniquely determined by (8) and (9) (we say that the corresponding indices are then *normal*). This is automatic in the one-measure case, but for multiple orthogonality this is not always true, both for MOPRL and MOPUC. In the MOPRL setting, there are several wide classes of systems (Angelesco, AT, Nikishin) where normality can be shown to hold at every index. It would be very valuable to find analogous MOPUC classes since explicit examples of MOPUC systems where normality is proven are currently rare (see [7, Sect 3] and [18, Sect 4]). Note however that normality of an index $\boldsymbol{n}$ can be stated in terms of a certain determinantal condition on the moments of $\{\mu_j\}_{j=1}^r$. Systems of measures that satisfy such a condition therefore form a codimension one submanifold. From this point of view relations (10) and (11) should be regarded to hold generically.

Both the results and the methods in this paper were heavily inspired by Van Assche's MOPRL paper [24] (see also [13]). In order to understand the current paper however, it is not necessary to know any background from MOPRL, as we tried to make the paper self-contained and accessible to a broad audience.

The structure of the paper is as follows. In Section 2 we discuss basic definitions and the question of uniqueness of multiple orthogonal polynomials. In Section 3 we prove Szegő's recurrences for type II and II* polynomials, and in Section 4 we do the same for type I and I*. In Section 5 we show that the recurrence coefficients satisfy a set of partial difference equations very similar to the real line counterpart. Finally, in Section 6 we prove the Christoffel–Darboux formula.

2. Normality

Given a multi-index $\boldsymbol{n} \in \mathbb{Z}_+^r$, a type II multiple orthogonal polynomial is a non-zero polynomial $\Phi_{\boldsymbol{n}}$ such that $\deg \Phi_{\boldsymbol{n}} \leq |\boldsymbol{n}|$ and

$$\left\langle \Phi_{\boldsymbol{n}}, z^p \right\rangle_j = 0, \qquad p = 0, \ldots, n_j - 1, \qquad j = 1, \ldots, r. \tag{12}$$

We also define a type II* multiple orthogonal polynomial to be a non-zero polynomial $\Phi^*_{\boldsymbol{n}}$ such that $\deg \Phi^*_{\boldsymbol{n}} \leq |\boldsymbol{n}|$ and

$$\left\langle \Phi^*_{\boldsymbol{n}}, z^p \right\rangle_j = 0, \qquad p = 1, \ldots, n_j, \qquad j = 1, \ldots, r. \tag{13}$$

A type I multiple orthogonal polynomial, for the multi-index $\boldsymbol{n}$, is a non-zero vector of polynomials $\boldsymbol{\Lambda}_{\boldsymbol{n}} = (\Lambda_{\boldsymbol{n},1}, \ldots, \Lambda_{\boldsymbol{n},r})$ such that $\deg \Lambda_{\boldsymbol{n},j} \leq n_j - 1$ and

$$\sum_{j=1}^{r} \left\langle \Lambda_{\boldsymbol{n},j}, z^p \right\rangle_j = 0, \qquad p = 0, 1, \ldots, |\boldsymbol{n}| - 2. \tag{14}$$

Lastly, we define a type I* multiple orthogonal polynomial, to be a non-zero vector of polynomials $\boldsymbol{\Lambda}^*_{\boldsymbol{n}} = (\Lambda^*_{\boldsymbol{n},1}, \ldots, \Lambda^*_{\boldsymbol{n},r})$ such that $\deg \Lambda^*_{\boldsymbol{n},j} \leq n_j - 1$ and

$$\sum_{j=1}^{r} \left\langle \Lambda^*_{\boldsymbol{n},j}, z^p \right\rangle_j = 0, \qquad p = 1, 2, \ldots, |\boldsymbol{n}| - 1. \tag{15}$$

We may also refer to each $\Lambda_{\boldsymbol{n},j}$ and $\Lambda^*_{\boldsymbol{n},j}$ as type I and type I* polynomials, respectively. Note that $\Lambda_{\boldsymbol{n},j} = \Lambda^*_{\boldsymbol{n},j} = 0$ is the only possibility when $n_j = 0$ (we take the degree of 0 to be $-\infty$). For $\boldsymbol{n} = \boldsymbol{0}$ we just define $\boldsymbol{\Lambda}_{\boldsymbol{0}} = \boldsymbol{0}$ and $\boldsymbol{\Lambda}^*_{\boldsymbol{0}} = \boldsymbol{0}$ as the only type I and type I* polynomials.

Consider the matrix

$$M_{\boldsymbol{n}} = \left(\begin{array}{cccc} \nu_1^0 & \nu_1^1 & \cdots & \nu_1^{|\boldsymbol{n}|-1} \\ \nu_1^{-1} & \nu_1^0 & \cdots & \nu_1^{|\boldsymbol{n}|-2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_1^{1-n_1} & \nu_1^{2-n_1} & \cdots & \nu_1^{|\boldsymbol{n}|-n_1} \\ \hline & & \vdots & \\ \hline \nu_r^0 & \nu_r^1 & \cdots & \nu_r^{|\boldsymbol{n}|-1} \\ \nu_r^{-1} & \nu_r^0 & \cdots & \nu_r^{|\boldsymbol{n}|-2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_r^{1-n_r} & \nu_r^{2-n_r} & \cdots & \nu_r^{|\boldsymbol{n}|-n_r} \end{array} \right), \tag{16}$$

where $\nu_j^p = \int z^p \, \mathrm{d}\mu_j(z)$ are the moments of μ_j. We say that the index $\boldsymbol{n} \neq \boldsymbol{0}$ is normal if $\det M_{\boldsymbol{n}} \neq 0$. This condition ensures uniqueness of our polynomials at the location $\boldsymbol{n}$ if we choose the appropriate normalization. We take $\boldsymbol{n} = \boldsymbol{0}$ to always be normal.

Lemma 2.1. *An index $\boldsymbol{n} \neq \boldsymbol{0}$ is normal if and only if any of the following conditions hold:*

(i) $\deg \Phi_{\boldsymbol{n}} = |\boldsymbol{n}|$ *for every type II polynomial* $\Phi_{\boldsymbol{n}}$.

(ii) $\Phi_{\boldsymbol{n}}^*(0) \neq 0$ *for every type II* polynomial* $\Phi_{\boldsymbol{n}}^*$.

(iii) $\sum_{j=1}^r \left\langle \Lambda_{\boldsymbol{n},j}, z^{|\boldsymbol{n}|-1} \right\rangle_j \neq 0$ *for every non-zero type I polynomial* $\boldsymbol{\Lambda_n}$.

(iv) $\sum_{j=1}^r \left\langle \Lambda_{\boldsymbol{n},j}^*, 1 \right\rangle_j \neq 0$ *for every non-zero type I* polynomial* $\boldsymbol{\Lambda_n^*}$.

Remark 2.2. We show in the proof that normality of $\boldsymbol{n} \neq \boldsymbol{0}$ is also equivalent to any of the following statements:

(a) There is a unique monic type II multiple orthogonal polynomial $\Phi_{\boldsymbol{n}}$ such that $\deg \Phi_{\boldsymbol{n}} = |\boldsymbol{n}|$.

(b) There is a unique type II* multiple orthogonal polynomial $\Phi_{\boldsymbol{n}}^*$ such that $\Phi_{\boldsymbol{n}}^*(0) = 1$.

(c) There is a unique type I multiple orthogonal polynomial $(\Lambda_{\boldsymbol{n},1}, \ldots, \Lambda_{\boldsymbol{n},r})$ such that $\sum_{j=1}^r \left\langle \Lambda_{\boldsymbol{n},j}, z^{|\boldsymbol{n}|-1} \right\rangle_j = 1$.

(d) There is a unique type I* multiple orthogonal polynomial $(\Lambda_{\boldsymbol{n},1}^*, \ldots, \Lambda_{\boldsymbol{n},r}^*)$ such that $\sum_{j=1}^r \left\langle \Lambda_{\boldsymbol{n},j}^*, 1 \right\rangle_j = 1$.

These normalizations will be used in all the future sections. Note that $\{\Phi_{j\boldsymbol{e}_k}\}_{j=0}^\infty$ are the monic orthogonal polynomials with respect to μ_k, and $\Lambda_{(j+1)\boldsymbol{e}_k,k} = \frac{1}{||\Phi_{j\boldsymbol{e}_k}||_k^2} \Phi_{j\boldsymbol{e}_k}$.

Proof. Solving the system (12) for polynomials of the form $c_{|\boldsymbol{n}|-1} z^{|\boldsymbol{n}|-1} + \cdots + c_0$ results in a linear system with coefficient matrix $M_{\boldsymbol{n}}$. Hence $c_{|\boldsymbol{n}|-1} = \cdots = c_0 = 0$ if and only if this matrix is invertible. If we instead solve for $z^{|\boldsymbol{n}|} + c_{|\boldsymbol{n}|-1} z^{|\boldsymbol{n}|-1} + \cdots + c_0$ we again get a linear system with coefficient matrix $M_{\boldsymbol{n}}$. This proves that normality is equivalent to (i) and (a). The system (13) for $c_{|\boldsymbol{n}|} z^{|\boldsymbol{n}|} + \cdots + c_1 z$, as well as $c_{|\boldsymbol{n}|} z^{|\boldsymbol{n}|} + \cdots + c_1 z + 1$, also has the same coefficient matrix. This proves that normality is equivalent to (ii) and (b). Next, the system of equations

$$\sum_{j=1}^r \left\langle \Lambda_{\boldsymbol{n},j}, z^p \right\rangle_j = 0, \qquad p = 0, 1, \ldots, |\boldsymbol{n}| - 1$$

is homogeneous with coefficient matrix $M_{\boldsymbol{n}}^T$, so the existence of $(\Lambda_{\boldsymbol{n},1}, \ldots, \Lambda_{\boldsymbol{n},r}) \neq \boldsymbol{0}$ satisfying this extended set of equations is equivalent to $M_{\boldsymbol{n}}^T$ not being invertible. Similarly, if we change the right hand side to be 1 for $p = |\boldsymbol{n}| - 1$ we see that we have a unique solution exactly when $M_{\boldsymbol{n}}^T$ is invertible. The same argument works for $(\Lambda_{\boldsymbol{n},1}^*, \ldots, \Lambda_{\boldsymbol{n},r}^*)$. □

We will also need the following lemma, cf. [13, Cor. 23.1.1–23.1.2] for MOPRL.

Lemma 2.3. $\boldsymbol{n} + \boldsymbol{e}_k$ *is normal if and only if we have* $\left\langle \Phi_{\boldsymbol{n}}, z^{n_k} \right\rangle_k \neq 0$ *for every type II multiple orthogonal polynomial* $\Phi_{\boldsymbol{n}}$ *for the index* $\boldsymbol{n}$. *Similarly,* $\boldsymbol{n} - \boldsymbol{e}_k$ *is normal if and only if* $\deg \Lambda_{\boldsymbol{n},k} = n_k - 1$ *for every type I multiple orthogonal polynomial* $\Lambda_{\boldsymbol{n},k}$ *for the index* $\boldsymbol{n}$.

Proof. If $\boldsymbol{n} + \boldsymbol{e}_k$ is not normal then for this index there is some non-zero solution $\Phi_{\boldsymbol{n}+\boldsymbol{e}_k}$ of (12) with $\deg \Phi_{\boldsymbol{n}+\boldsymbol{e}_k} \leq |\boldsymbol{n}|$. But then $\Phi_{\boldsymbol{n}} = \Phi_{\boldsymbol{n}+\boldsymbol{e}_k}$ is a solution of (12) for the index $\boldsymbol{n}$ as well, with $\left\langle \Phi_{\boldsymbol{n}}, z^{n_k} \right\rangle_k = 0$. Conversely, if $\left\langle \Phi_{\boldsymbol{n}}, z^{n_k} \right\rangle_k = 0$ and $\Phi_{\boldsymbol{n}}$ is a solution of (12) for the index $\boldsymbol{n}$, then $\Phi_{\boldsymbol{n}}$ is a solution of (12) for the index $\boldsymbol{n} + \boldsymbol{e}_k$. Since $\deg \Phi_{\boldsymbol{n}} < |\boldsymbol{n} + \boldsymbol{e}_k| = |\boldsymbol{n}| + 1$ it follows that $\boldsymbol{n} + \boldsymbol{e}_k$ cannot be normal.

For the second statement, assume $n_k > 0$ (and otherwise the statement is obviously true). If $\boldsymbol{n} - \boldsymbol{e}_k$ is not normal then there is a $(\Lambda_{\boldsymbol{n}-\boldsymbol{e}_k,1}, \ldots, \Lambda_{\boldsymbol{n}-\boldsymbol{e}_k,r}) \neq \boldsymbol{0}$

such that

$$\sum_{j=1}^{r} \left\langle \Lambda_{\boldsymbol{n}-\boldsymbol{e}_k,j}, z^p \right\rangle_j = 0, \qquad p = 0, 1, \ldots, |\boldsymbol{n}| - 2.$$

This is also a solution for the index $\boldsymbol{n}$, so we have a solution $(\Lambda_{\boldsymbol{n},1}, \ldots, \Lambda_{\boldsymbol{n},r}) \neq \mathbf{0}$ with $\deg \Lambda_{\boldsymbol{n},j} < n_j - 1$. Conversely, if $\deg \Lambda_{\boldsymbol{n},j} < n_j - 1$ and $(\Lambda_{\boldsymbol{n},1}, \ldots, \Lambda_{\boldsymbol{n},r}) \neq \mathbf{0}$ then this is a non-zero solution to (14) for $\boldsymbol{n} - \boldsymbol{e}_k$, and

$$\sum_{j=1}^{r} \left\langle \Lambda_{\boldsymbol{n},j}, z^{|\boldsymbol{n}|-2} \right\rangle_j = 0,$$

so $\boldsymbol{n} - \boldsymbol{e}_k$ cannot be normal. □

3. Recurrence Relations for Type II Polynomials

From now on, if an index $\boldsymbol{n}$ is normal then we only work with monic type II polynomials $\Phi_{\boldsymbol{n}}$, and type II* polynomials $\Phi^*_{\boldsymbol{n}}$ with $\Phi^*_{\boldsymbol{n}}(0) = 1$. These polynomials are then unique, as discussed in Remark 2.2 (a) and (b). Also, if $\boldsymbol{n} \notin \mathbb{Z}^r_+$ we always take $\Phi_{\boldsymbol{n}} = 0$. The next theorem is the generalization of the Szegő recurrence relations (6) and (4) from the OPUC theory.

Theorem 3.1. *For each of the equations* (17) *and* (18) *we assume that all the $\mathbb{Z}^r_+$ multi-indices that appear in the corresponding equation are normal.*

(i) There are complex numbers $\alpha_{\boldsymbol{n}}$ and $\rho_{\boldsymbol{n},1}, \ldots, \rho_{\boldsymbol{n},r}$ such that

$$\Phi_{\boldsymbol{n}} = \alpha_{\boldsymbol{n}} \Phi^*_{\boldsymbol{n}} + \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} z \Phi_{\boldsymbol{n}-\boldsymbol{e}_j}. \tag{17}$$

(ii) If $n_k > 0$, there is a complex number $\beta_{\boldsymbol{n}}$, independent of k, such that

$$\Phi^*_{\boldsymbol{n}} = \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k} + \beta_{\boldsymbol{n}} z \Phi_{\boldsymbol{n}-\boldsymbol{e}_k} \tag{18}$$

Remark 3.2. The recurrence coefficients $\alpha_{\boldsymbol{n}}$ and $\beta_{\boldsymbol{n}}$ will be referred to as the multiple Verblunsky coefficients of the system $\boldsymbol{\mu}$. Note that the recurrence relations imply that $\alpha_{\boldsymbol{n}} = \Phi_{\boldsymbol{n}}(0)$ and $\beta_{\boldsymbol{n}}$ is the $z^{|\boldsymbol{n}|}$-coefficient of $\Phi^*_{\boldsymbol{n}}$. Even if the recurrence relations do not hold, we can still define $\alpha_{\boldsymbol{n}}$ and $\beta_{\boldsymbol{n}}$ in this way, as long as $\boldsymbol{n}$ is normal. Also note that the recurrence relations uniquely determine $\alpha_{\boldsymbol{n}}$ and $\beta_{\boldsymbol{n}}$. The same holds for $\rho_{\boldsymbol{n},k}$, assuming $n_k > 0$. If $n_k = 0$ we can adopt the convention $\rho_{\boldsymbol{n},k} = 0$. Furthermore, for indices $\boldsymbol{n}$ of the form $n_k \boldsymbol{e}_k$ we get $\beta_{n_k \boldsymbol{e}_k} = \bar{\alpha}_{n_k \boldsymbol{e}_k}$, which become the usual Verblunsky (reflection) coefficients of μ_k, $\rho_{n_k \boldsymbol{e}_k, j} = 1 - |\alpha_{n_k \boldsymbol{e}_k}|^2$, and $\rho_{n_k \boldsymbol{e}_k, m} = 0$ when $m \neq j$, so that (17) and (18) reduce to the usual Szegő recurrence relations (6) and (4) for the measure μ_j.

Proof. (i) We choose $\alpha_{\boldsymbol{n}}$ such that $(\Phi_{\boldsymbol{n}} - \alpha_{\boldsymbol{n}} \Phi^*_{\boldsymbol{n}})(0) = 0$ and then we get

$$\begin{aligned} \left\langle z^{-1}(\Phi_{\boldsymbol{n}} - \alpha_{\boldsymbol{n}} \Phi^*_{\boldsymbol{n}}), z^p \right\rangle_j &= \left\langle \Phi_{\boldsymbol{n}}, z^{p+1} \right\rangle_j - \alpha_{\boldsymbol{n}} \left\langle \Phi^*_{\boldsymbol{n}}, z^{p+1} \right\rangle_j \\ &= 0, \quad p = 0, \ldots, n_i - 2, \quad j = 1, \ldots, r. \end{aligned} \tag{19}$$

Note that the polynomial $z^{-1}(\Phi_{\boldsymbol{n}} - \alpha_{\boldsymbol{n}} \Phi^*_{\boldsymbol{n}})$ has degree at most $|\boldsymbol{n}| - 1$. Denote $S_{\boldsymbol{n}} = \{j \in \{1, \ldots, r\} : n_j > 0\}$, and let $s_{\boldsymbol{n}}$ be the cardinality of $S_{\boldsymbol{n}}$. Solving (19) for a polynomial $c_{|\boldsymbol{n}|-1} z^{|\boldsymbol{n}|-1} + \cdots + c_0$ results in a homogeneous system with coefficient matrix $M'_{\boldsymbol{n}}$, equal to the matrix $M_{\boldsymbol{n}}$ in (16) but with $s_{\boldsymbol{n}}$ rows removed. Since $\boldsymbol{n}$ is normal the row space of $M'_{\boldsymbol{n}}$ has dimension $|\boldsymbol{n}| - s_{\boldsymbol{n}}$, so the null space of $M'_{\boldsymbol{n}}$ has dimension $s_{\boldsymbol{n}}$. Each $\Phi_{\boldsymbol{n}-\boldsymbol{e}_j}$ with $j \in S_{\boldsymbol{n}}$ solves this system of equations, and we claim they are also linearly independent. To see this, consider the equation

$$\sum_{j \in S_{\boldsymbol{n}}} t_j \Phi_{\boldsymbol{n}-\boldsymbol{e}_j} = 0$$

and suppose $k \in S_{\boldsymbol{n}}$ (so that we have $\Phi_{\boldsymbol{n}-\boldsymbol{e}_k} \neq 0$). Taking the k-th inner product with z^{n_k-1} yields

$$\sum_{j \in S_{\boldsymbol{n}}} t_j \left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_j}, z^{n_k-1} \right\rangle_k = 0.$$

Every term with $j \neq k$ vanishes by the orthogonality relations (12), so we end up with

$$t_k \left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1} \right\rangle_k = 0.$$

The inner product is non-zero by Lemma 2.3, so we must have $t_k = 0$. Hence linear independence follows and $\left\{\Phi_{\boldsymbol{n}-\boldsymbol{e}_j}\right\}_{j \in S_{\boldsymbol{n}}}$ is a basis for the space of polynomials of degree at most $|\boldsymbol{n}| - 1$ with the orthogonality relations (19). This then implies

$$z^{-1}(\Phi_{\boldsymbol{n}} - \alpha_{\boldsymbol{n}} \Phi^*_{\boldsymbol{n}}) = \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} \Phi_{\boldsymbol{n}-\boldsymbol{e}_j}$$

for some constants $\rho_{\boldsymbol{n},1}, \ldots, \rho_{\boldsymbol{n},r}$ (uniquely determined by $\Phi_{\boldsymbol{n}-\boldsymbol{e}_1}, \ldots, \Phi_{\boldsymbol{n}-\boldsymbol{e}_r}$).

(ii) $\Phi^*_{\boldsymbol{n}} - \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k}$ is 0 at $z = 0$, so it divisible by z. Then

$$\begin{aligned}\left\langle z^{-1}(\Phi^*_{\boldsymbol{n}} - \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k}), z^p \right\rangle_k &= \left\langle \Phi^*_{\boldsymbol{n}}, z^{p+1} \right\rangle_k - \left\langle \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{p+1} \right\rangle_k \\ &= 0, \quad p = 0, \ldots, n_k - 2, \quad j = 1, \ldots, r.\end{aligned}$$

Similary, if $j \neq k$ we have the orthogonality relations

$$\begin{aligned}\left\langle z^{-1}(\Phi^*_{\boldsymbol{n}} - \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k}), z^p \right\rangle_j &= \left\langle \Phi^*_{\boldsymbol{n}}, z^{p+1} \right\rangle_j - \left\langle \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{p+1} \right\rangle_j \\ &= 0, \quad p = 0, \ldots, n_j - 1, \quad j = 1, \ldots, r.\end{aligned}$$

Hence we see that $z^{-1}(\Phi^*_{\boldsymbol{n}} - \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k})$ satisfies all the orthogonality conditions in (12) for the index $\boldsymbol{n} - \boldsymbol{e}_k$, while having degree $|\boldsymbol{n}| - 1$ or 0. Hence we get

$$z^{-1}(\Phi^*_{\boldsymbol{n}} - \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k}) = \beta_{\boldsymbol{n},k} \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}.$$

By comparing $z^{|\boldsymbol{n}|-1}$-coefficients, we see that $\beta_{\boldsymbol{n},k}$ is the $z^{|\boldsymbol{n}|}$-coefficient of $\Phi^*_{\boldsymbol{n}}$, hence independent of k, and so (18) follows. □

One can ask what the analogues of (3) and (7) are. We obtain them in Corollary 3.4 and Corollary 3.3, respectively.

Corollary 3.3. *Assume that all the $\mathbb{Z}^r_+$ multi-indices that appear in the next equation are normal. Then*

$$\Phi^*_{\boldsymbol{n}} = \beta_{\boldsymbol{n}} \Phi_{\boldsymbol{n}} + \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_j}. \tag{20}$$

Proof. Multiply both sides of (17) with $\beta_{\boldsymbol{n}}$ and use (18) to get

$$\beta_{\boldsymbol{n}} \Phi_{\boldsymbol{n}} = \alpha_{\boldsymbol{n}} \beta_{\boldsymbol{n}} \Phi^*_{\boldsymbol{n}} + \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} (\Phi^*_{\boldsymbol{n}} - \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_j}) = \left(\alpha_{\boldsymbol{n}} \beta_{\boldsymbol{n}} + \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} \right) \Phi^*_{\boldsymbol{n}} - \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_j},$$

and by comparing the leading coefficients in (17) we see that

$$\alpha_{\boldsymbol{n}}\beta_{\boldsymbol{n}} + \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} = 1. \tag{21}$$

□

Corollary 3.4. *Assume that $\boldsymbol{n}$, $\boldsymbol{n}-\boldsymbol{e}_1, \ldots, \boldsymbol{n}-\boldsymbol{e}_r$ are normal. Define $R_{\boldsymbol{n}}$ to be the $r\times r$ matrix $(\rho_{\boldsymbol{n},k})_{j,k=1}^{r}$.*

(i) Assume that $\beta_{\boldsymbol{n}} \neq 0$. Define $A_{\boldsymbol{n}} = \frac{1}{\beta_{\boldsymbol{n}}}(I - R_{\boldsymbol{n}})$. Then

$$\begin{pmatrix} \Phi_{\boldsymbol{n}} - z\Phi_{\boldsymbol{n}-\boldsymbol{e}_1} \\ \vdots \\ \Phi_{\boldsymbol{n}} - z\Phi_{\boldsymbol{n}-\boldsymbol{e}_r} \end{pmatrix} = A_{\boldsymbol{n}} \begin{pmatrix} \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_1} \\ \vdots \\ \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_r} \end{pmatrix}. \tag{22}$$

(ii) Assume that $\alpha_{\boldsymbol{n}} \neq 0$. Define $A_{\boldsymbol{n}}^{-1} = \frac{1}{\alpha_{\boldsymbol{n}}}\left((1 - \sum_{l=1}^{r} \rho_{\boldsymbol{n},l})I + R_{\boldsymbol{n}}\right)$. Then

$$A_{\boldsymbol{n}}^{-1} \begin{pmatrix} \Phi_{\boldsymbol{n}} - z\Phi_{\boldsymbol{n}-\boldsymbol{e}_1} \\ \vdots \\ \Phi_{\boldsymbol{n}} - z\Phi_{\boldsymbol{n}-\boldsymbol{e}_r} \end{pmatrix} = \begin{pmatrix} \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_1} \\ \vdots \\ \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_r} \end{pmatrix}. \tag{23}$$

Remark 3.5. For $r = 1$, $A_{\boldsymbol{n}}$ becomes $(1 - (1 - |\alpha_n|^2))/\bar{\alpha}_n = \alpha_n$, so that (22) reduces to (3).

Proof. For (i), just plug (18) into (20). To show (ii), first apply elementary row operations to show that $\det(I - R_{\boldsymbol{n}}) = 1 - \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} = \alpha_{\boldsymbol{n}}\beta_{\boldsymbol{n}}$, by (21). Then apply the same approach to compute the classical adjoint of $I - R_{\boldsymbol{n}}$. This produces the formula $A_{\boldsymbol{n}}^{-1} = \frac{1}{\alpha_{\boldsymbol{n}}}\left((1 - \sum_{l=1}^{r} \rho_{\boldsymbol{n},l})I + R_{\boldsymbol{n}}\right)$. □

4. Recurrence Relations for Type I Polynomials

If an index $\boldsymbol{n} \neq \boldsymbol{0}$ is normal then $\boldsymbol{\Lambda}_{\boldsymbol{n}} = (\Lambda_{\boldsymbol{n},1}, \ldots, \Lambda_{\boldsymbol{n},r})$ and $\boldsymbol{\Lambda}^*_{\boldsymbol{n}} = (\Lambda^*_{\boldsymbol{n},1}, \ldots, \Lambda^*_{\boldsymbol{n},r})$ will stand for the unique polynomial vectors with the normalizations described in Remark 2.2 (c) and (d), respectively. The following simple lemma will be used a few times. It is one example of the so-called biorthogonality property, which can be proved in the exact same way as for MOPRL [13, Thm 23.1.6].

Lemma 4.1. *Suppose multi-indices $\boldsymbol{n}$ and $\boldsymbol{n}+\boldsymbol{e}_k$ are normal. Then*

$$\left\langle \Phi_{\boldsymbol{n}}, \Lambda_{\boldsymbol{n}+\boldsymbol{e}_k,k} \right\rangle_k = \sum_{m=1}^{r} \left\langle \Phi_{\boldsymbol{n}}, \Lambda_{\boldsymbol{n}+\boldsymbol{e}_k,m} \right\rangle_m = 1. \tag{24}$$

Proof. For $m \neq k$ we have $\deg \Lambda_{\boldsymbol{n}+\boldsymbol{e}_k,m} \leq n_m - 1$, so that $\left\langle \Phi_{\boldsymbol{n}}, \Lambda_{\boldsymbol{n}+\boldsymbol{e}_k,m} \right\rangle_m = 0$. This proves the first equality, and for the second equality,

$$\sum_{m=1}^{r} \left\langle \Phi_{\boldsymbol{n}}, \Lambda_{\boldsymbol{n}+\boldsymbol{e}_k,m} \right\rangle_m = \overline{\sum_{m=1}^{r} \left\langle \Lambda_{\boldsymbol{n}+\boldsymbol{e}_k,m}, \Phi_{\boldsymbol{n}} \right\rangle_m} = \overline{\sum_{m=1}^{r} \left\langle \Lambda_{\boldsymbol{n}+\boldsymbol{e}_k,m}, z^{|\boldsymbol{n}|} \right\rangle_m} = 1.$$

□

Remark 4.2. Denote $\kappa_{\boldsymbol{n},k}$ to be the leading z^{n_k-1}-coefficient of $\Lambda_{\boldsymbol{n},k}$. Then (24) implies

$$\bar{\kappa}_{\boldsymbol{n}+\boldsymbol{e}_k,k} = \frac{1}{\left\langle \Phi_{\boldsymbol{n}}, z^{n_k} \right\rangle_k}. \tag{25}$$

Note that $\langle \Phi_{\boldsymbol{n}}, z^{n_k} \rangle_k \neq 0 \neq \kappa_{\boldsymbol{n}+\boldsymbol{e}_k,k}$, which can also be seen from Lemma 2.3.

The following result is the type I analogue of Theorem 3.1. From these one can easily deduce the analogue of the remaining two Szegő recurrences, similarly to Corollaries 3.3 and 3.4.

Theorem 4.3.

(i) If $\boldsymbol{n}$ is normal, along with all neighbouring indices $\boldsymbol{n} \pm \boldsymbol{e}_1, \ldots, \boldsymbol{n} \pm \boldsymbol{e}_r$ that belong to $\mathbb{Z}_+^r$, then

$$z\boldsymbol{\Lambda}_{\boldsymbol{n}} = -\bar{\beta}_{\boldsymbol{n}} \boldsymbol{\Lambda}_{\boldsymbol{n}}^* + \sum_{j=1}^{r} \bar{\rho}_{\boldsymbol{n},j} \boldsymbol{\Lambda}_{\boldsymbol{n}+\boldsymbol{e}_j}. \tag{26}$$

(ii) If $\boldsymbol{n}$ and $\boldsymbol{n}+\boldsymbol{e}_k$ are normal, then

$$\boldsymbol{\Lambda}_{\boldsymbol{n}}^* = \boldsymbol{\Lambda}_{\boldsymbol{n}+\boldsymbol{e}_k}^* - \bar{\alpha}_{\boldsymbol{n}} \boldsymbol{\Lambda}_{\boldsymbol{n}+\boldsymbol{e}_k}. \tag{27}$$

Proof. (i) Choose $\delta_{\boldsymbol{n}}$ such that

$$\sum_{m=1}^{r} \langle z\Lambda_{\boldsymbol{n},m} - \delta_{\boldsymbol{n}} \Lambda_{\boldsymbol{n},m}^*, z^p \rangle_m = 0, \qquad p = 0, 1, \ldots, |\boldsymbol{n}| - 1.$$

The orthogonality relations above hold for every choice of $\delta_{\boldsymbol{n}}$ except in the case $p = 0$. In other words, we make the choice

$$\delta_{\boldsymbol{n}} = \sum_{m=1}^{r} \langle z\Lambda_{\boldsymbol{n},m}, 1 \rangle_m.$$

Since $\Phi_{\boldsymbol{n}}^* = \beta_{\boldsymbol{n}} z^{|\boldsymbol{n}|} + \cdots + 1$ we get

$$\begin{aligned} \delta_{\boldsymbol{n}} &= \sum_{m=1}^{r} \langle z\Lambda_{\boldsymbol{n},m}, \Phi_{\boldsymbol{n}}^* \rangle_m - \sum_{m=1}^{r} \langle z\Lambda_{\boldsymbol{n},m}, \beta_{\boldsymbol{n}} z^{|\boldsymbol{n}|} \rangle_m \\ &= \sum_{m=1}^{r} \overline{\langle \Phi_{\boldsymbol{n}}^*, z\Lambda_{\boldsymbol{n},m} \rangle_m} - \bar{\beta}_{\boldsymbol{n}} \sum_{m=1}^{r} \langle \Lambda_{\boldsymbol{n},m}, z^{|\boldsymbol{n}|-1} \rangle_m = -\bar{\beta}_{\boldsymbol{n}}. \end{aligned}$$

Consider the set of vectors of polynomials $(\Xi_1, \ldots, \Xi_r)$ with $\deg \Xi_j \leq n_j$ and

$$\sum_{m=1}^{r} \langle \Xi_m, z^p \rangle_m = 0, \qquad p = 0, 1, \ldots, |\boldsymbol{n}| - 1.$$

Rewriting this as the system of equations for the coefficients of these polynomials results in a homogeneous linear system with coefficient matrix equal to $M_{\boldsymbol{n}}^T$ but with r columns added. Hence the null space of this matrix has dimension r. Note that $(\Lambda_{\boldsymbol{n}+\boldsymbol{e}_j,1}, \ldots, \Lambda_{\boldsymbol{n}+\boldsymbol{e}_j,r})$ is a solution for each $j = 1, \ldots, r$. Moreover, these vectors are linearly independent. To see this, suppose

$$\sum_{j=1}^{r} c_j (\Lambda_{\boldsymbol{n}+\boldsymbol{e}_j,1}, \ldots, \Lambda_{\boldsymbol{n}+\boldsymbol{e}_j,r}) = 0.$$

By Lemma 2.3, $\deg \Lambda_{\boldsymbol{n}+\boldsymbol{e}_m,m} = n_m$, but $\deg \Lambda_{\boldsymbol{n}+\boldsymbol{e}_j,m} = n_m - 1$ if $j \neq m$, so we must have $c_m = 0$. Hence the vectors of the form $(\Lambda_{\boldsymbol{n}+\boldsymbol{e}_j,1}, \ldots, \Lambda_{\boldsymbol{n}+\boldsymbol{e}_j,r})$ form a basis of the solution space of our linear system, and therefore

$$z\boldsymbol{\Lambda}_{\boldsymbol{n}} + \bar{\beta}_{\boldsymbol{n}} \boldsymbol{\Lambda}_{\boldsymbol{n}}^* = \sum_{j=1}^{r} \sigma_{\boldsymbol{n},j} \boldsymbol{\Lambda}_{\boldsymbol{n}+\boldsymbol{e}_j}$$

for some complex numbers $\sigma_{\boldsymbol{n},1}, \dots, \sigma_{\boldsymbol{n},r}$. By comparing the leading coefficients in the above recurrence relation, we see that

$$\kappa_{\boldsymbol{n},k} = \sigma_{\boldsymbol{n},k}\kappa_{\boldsymbol{n}+\boldsymbol{e}_k,k}, \tag{28}$$

where $\kappa_{\boldsymbol{n},j}$ was the z^{n_j-1}-coefficient of $\Lambda_{\boldsymbol{n},j}$. On the other hand, by taking the k-th inner product with respect to z^{n_k} on both sides of (17) we see that

$$\left\langle \Phi_{\boldsymbol{n}}, z^{n_k} \right\rangle_k = \rho_{\boldsymbol{n},k}\left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1} \right\rangle_k. \tag{29}$$

By combining these two relations we get

$$\bar{\sigma}_{\boldsymbol{n},k}\bar{\kappa}_{\boldsymbol{n}+\boldsymbol{e}_k,k}\left\langle \Phi_{\boldsymbol{n}}, z^{n_k} \right\rangle_k = \rho_{\boldsymbol{n},k}\bar{\kappa}_{\boldsymbol{n},k}\left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1} \right\rangle_k.$$

If $\boldsymbol{n}+\boldsymbol{e}_k$ is normal then the left hand side is equal to $\bar{\sigma}_{\boldsymbol{n},k}$, by (25). If $n_k = 0$ then the right hand side vanishes and then $\sigma_{\boldsymbol{n},k} = 0 = \bar{\rho}_{\boldsymbol{n},k}$, and if $\boldsymbol{n}-\boldsymbol{e}_k$ is normal then the right hand side is equal to $\rho_{\boldsymbol{n},k}$ by (25), so $\sigma_{\boldsymbol{n},k} = \bar{\rho}_{\boldsymbol{n},k}$.

(ii) We have the orthogonality relations

$$\sum_{m=1}^{r} \left\langle \Lambda^*_{\boldsymbol{n}+\boldsymbol{e}_k,m} - \Lambda^*_{\boldsymbol{n},m}, z^p \right\rangle_m = 0, \qquad p = 0, 1, \dots, |\boldsymbol{n}| - 1.$$

Since $\deg(\Lambda^*_{\boldsymbol{n}+\boldsymbol{e}_k,m} - \Lambda^*_{\boldsymbol{n},m}) \le n_m - 1$ if $m \ne k$ and $\deg(\Lambda^*_{\boldsymbol{n}+\boldsymbol{e}_k,k} - \Lambda^*_{\boldsymbol{n},k}) \le n_k$ there must be a constant $\epsilon_{\boldsymbol{n},k}$ such that

$$\boldsymbol{\Lambda}^*_{\boldsymbol{n}+\boldsymbol{e}_k} - \boldsymbol{\Lambda}^*_{\boldsymbol{n}} = \epsilon_{\boldsymbol{n},k}\boldsymbol{\Lambda}_{\boldsymbol{n}+\boldsymbol{e}_k}.$$

Now taking inner products with $z^{|\boldsymbol{n}|}$ yields

$$\epsilon_{\boldsymbol{n},k} = -\sum_{m=1}^{r} \left\langle \Lambda^*_{\boldsymbol{n},m}, z^{|\boldsymbol{n}|} \right\rangle_m.$$

Since $\Phi_{\boldsymbol{n}} = z^{|\boldsymbol{n}|} + \dots + \alpha_{\boldsymbol{n}}$ we get

$$\begin{aligned}
\epsilon_{\boldsymbol{n},k} &= -\left(\sum_{m=1}^{r} \left\langle \Lambda^*_{\boldsymbol{n},m}, \Phi_{\boldsymbol{n}} \right\rangle_m - \sum_{m=1}^{r} \left\langle \Lambda^*_{\boldsymbol{n},m}, \alpha_{\boldsymbol{n}} \right\rangle_m \right) \\
&= -\sum_{m=1}^{r} \overline{\left\langle \Phi_{\boldsymbol{n}}, \Lambda^*_{\boldsymbol{n},m} \right\rangle_m} + \bar{\alpha}_{\boldsymbol{n}} \sum_{m=1}^{r} \left\langle \Lambda^*_{\boldsymbol{n},m}, 1 \right\rangle_m = \bar{\alpha}_{\boldsymbol{n}}.
\end{aligned}$$

□

5. Compatibility Conditions

Proposition 5.1 is identical to an important and peculiar feature of MOPRL, that does not appear in OPRL. However for MOPUC, this structure is richer with the presence of both $\Phi_{\boldsymbol{n}}$ and $\Phi^*_{\boldsymbol{n}}$, as Proposition 5.2 shows.

Proposition 5.1. *If $\boldsymbol{n}+\boldsymbol{e}_k$, $\boldsymbol{n}+\boldsymbol{e}_l$, and $\boldsymbol{n}$ are normal, and $k \ne l$, then there is a complex number $\gamma^{kl}_{\boldsymbol{n}}$ such that*

$$\Phi_{\boldsymbol{n}+\boldsymbol{e}_k} - \Phi_{\boldsymbol{n}+\boldsymbol{e}_l} = \gamma^{kl}_{\boldsymbol{n}}\Phi_{\boldsymbol{n}}. \tag{30}$$

Proposition 5.2. *Assume that all the $\mathbb{Z}_+^r$ multi-indices that appear in the corresponding equations are normal. Then*

$$\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_k} - \Phi^*_{\boldsymbol{n}+\boldsymbol{e}_l} = \beta_{\boldsymbol{n}+\boldsymbol{e}_k+\boldsymbol{e}_l} z(\Phi_{\boldsymbol{n}+\boldsymbol{e}_l} - \Phi_{\boldsymbol{n}+\boldsymbol{e}_k}), \tag{31}$$

$$\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_k} - \Phi^*_{\boldsymbol{n}+\boldsymbol{e}_l} = (\beta_{\boldsymbol{n}+\boldsymbol{e}_k} - \beta_{\boldsymbol{n}+\boldsymbol{e}_l}) z\Phi_{\boldsymbol{n}}, \tag{32}$$

$$\beta_{\boldsymbol{n}+\boldsymbol{e}_k}(\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_l} - \Phi^*_{\boldsymbol{n}}) = \beta_{\boldsymbol{n}+\boldsymbol{e}_l}(\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_k} - \Phi^*_{\boldsymbol{n}}). \tag{33}$$

Proof of Propositions 5.1 and 5.2. Assuming $\boldsymbol{n}+\boldsymbol{e}_k$ and $\boldsymbol{n}+\boldsymbol{e}_l$ are normal, then $\Phi_{\boldsymbol{n}+\boldsymbol{e}_k} - \Phi_{\boldsymbol{n}+\boldsymbol{e}_l}$ has degree $\leq |\boldsymbol{n}|$ and satisfies all the orthogonality relations at $\boldsymbol{n}$. This shows (30).

Assuming normality of $\boldsymbol{n}+\boldsymbol{e}_k$, $\boldsymbol{n}+\boldsymbol{e}_l$ and $\boldsymbol{n}+\boldsymbol{e}_k+\boldsymbol{e}_l$, (18) gives

$$\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_k+\boldsymbol{e}_l} = \Phi^*_{\boldsymbol{n}+\boldsymbol{e}_k} + \beta_{\boldsymbol{n}+\boldsymbol{e}_k+\boldsymbol{e}_l} z\Phi_{\boldsymbol{n}+\boldsymbol{e}_k},$$
$$\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_k+\boldsymbol{e}_l} = \Phi^*_{\boldsymbol{n}+\boldsymbol{e}_l} + \beta_{\boldsymbol{n}+\boldsymbol{e}_k+\boldsymbol{e}_l} z\Phi_{\boldsymbol{n}+\boldsymbol{e}_k}.$$

Now combine these to get (31). On the other hand, if $\boldsymbol{n}+\boldsymbol{e}_k$, $\boldsymbol{n}+\boldsymbol{e}_l$ and $\boldsymbol{n}$ are normal, then (18) gives

$$\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_k} = \Phi^*_{\boldsymbol{n}} + \beta_{\boldsymbol{n}+\boldsymbol{e}_k} z\Phi_{\boldsymbol{n}},$$
$$\Phi^*_{\boldsymbol{n}+\boldsymbol{e}_l} = \Phi^*_{\boldsymbol{n}} + \beta_{\boldsymbol{n}+\boldsymbol{e}_l} z\Phi_{\boldsymbol{n}},$$

and by combining these we get (32). If we first multiply by $\beta_{\boldsymbol{n}+\boldsymbol{e}_l}$ in the first equation and $\beta_{\boldsymbol{n}+\boldsymbol{e}_l}$ in the second equation we would instead get (33). □

We could easily derive analogue results for type I polynomials, using similar methods. We only prove the analogue of (30) and leave the analogues of (31), (32), (33) as a quick exercise to the interested reader.

Proposition 5.3. *If all indices appearing below are normal, and $k \neq l$, then*

$$\boldsymbol{\Lambda}_{\boldsymbol{n}-\boldsymbol{e}_k} - \boldsymbol{\Lambda}_{\boldsymbol{n}-\boldsymbol{e}_l} = \bar{\gamma}^{kl}_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l} \boldsymbol{\Lambda}_{\boldsymbol{n}}. \tag{34}$$

Proof. Note that

$$\sum_{m=1}^{r} \left\langle \Lambda_{\boldsymbol{n}-\boldsymbol{e}_k,m} - \Lambda_{\boldsymbol{n}-\boldsymbol{e}_l,m}, z^p \right\rangle_m = 0, \quad p = 0, 1, \ldots, |\boldsymbol{n}| - 2, \tag{35}$$

and $\deg(\Lambda_{\boldsymbol{n}-\boldsymbol{e}_k,m} - \Lambda_{\boldsymbol{n}-\boldsymbol{e}_l,m}) \leq n_m - 1$, so we get

$$\boldsymbol{\Lambda}_{\boldsymbol{n}-\boldsymbol{e}_k} - \boldsymbol{\Lambda}_{\boldsymbol{n}-\boldsymbol{e}_l} = \eta^{kl}_{\boldsymbol{n}} \boldsymbol{\Lambda}_{\boldsymbol{n}}. \tag{36}$$

for some constant $\eta^{kl}_{\boldsymbol{n}}$. If we now compare the degrees in (36) when $m = k$ we get

$$-\kappa_{\boldsymbol{n}-\boldsymbol{e}_l,k} = \kappa_{\boldsymbol{n},k} \eta^{kl}_{\boldsymbol{n}}.$$

On the other hand, if we take the k-th inner product with z^{n_k} in (30), and shift the indices from $\boldsymbol{n}$ to $\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l$ we get

$$-\left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1} \right\rangle_k = \left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}, z^{n_k-1} \right\rangle_k \gamma^{kl}_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}. \tag{37}$$

Combining these relations produces

$$\bar{\kappa}_{\boldsymbol{n},k} \left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1} \right\rangle_k \bar{\eta}^{kl}_{\boldsymbol{n}} = \bar{\kappa}_{\boldsymbol{n}-\boldsymbol{e}_l,k} \left\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}, z^{n_k-1} \right\rangle_k \gamma^{kl}_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}.$$

By (25) we then get $\eta^{kl}_{\boldsymbol{n}} = \bar{\gamma}^{kl}_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}$. □

In MOPRL, the nearest-neighbour recurrence coefficients satisfy a set of partial difference equations (see [24]). The same methods that were used to prove this result can be applied to MOPUC, to get a similar set of equations. However. we choose to give a shorter proof using a different approach.

Theorem 5.4. *We have the compatibility conditions*

$$\beta_{\boldsymbol{n}}(\alpha_{\boldsymbol{n}-\boldsymbol{e}_l} - \alpha_{\boldsymbol{n}-\boldsymbol{e}_k}) = (\beta_{\boldsymbol{n}-\boldsymbol{e}_k} - \beta_{\boldsymbol{n}-\boldsymbol{e}_l})\alpha_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}, \tag{38}$$

$$\alpha_{\boldsymbol{n}}\beta_{\boldsymbol{n}} + \sum_{j=1}^{r} \rho_{\boldsymbol{n},j} = 1, \tag{39}$$

$$(\alpha_{\boldsymbol{n}-\boldsymbol{e}_l} - \alpha_{\boldsymbol{n}-\boldsymbol{e}_k})\alpha_{\boldsymbol{n}-\boldsymbol{e}_l}\rho_{\boldsymbol{n},k} = (\alpha_{\boldsymbol{n}+\boldsymbol{e}_k-\boldsymbol{e}_l} - \alpha_{\boldsymbol{n}})\alpha_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}\rho_{\boldsymbol{n}-\boldsymbol{e}_l,k}, \tag{40}$$

assuming that all the $\mathbb{Z}_+^r$ multi-indices that appear in the corresponding equations are normal, and for (39) *we also need normality of all indices $\boldsymbol{n}-\boldsymbol{e}_1, \dots, \boldsymbol{n}-\boldsymbol{e}_r$ that belong to $\mathbb{Z}_+^r$.*

Proof. By putting $z = 0$ in (30) we see that

$$\alpha_{\boldsymbol{n}+\boldsymbol{e}_k} - \alpha_{\boldsymbol{n}+\boldsymbol{e}_l} = \alpha_{\boldsymbol{n}}\gamma_{\boldsymbol{n}}^{kl}. \tag{41}$$

Similarly, if we combine (30) and (31) and compare leading coefficients we see that

$$\beta_{\boldsymbol{n}+\boldsymbol{e}_k+\boldsymbol{e}_l}\gamma_{\boldsymbol{n}}^{kl} = \beta_{\boldsymbol{n}+\boldsymbol{e}_l} - \beta_{\boldsymbol{n}+\boldsymbol{e}_k}. \tag{42}$$

If we now combine (41) and (42) we get (38), and if we put $z = 0$ in (20) we get (39), so what remains is to prove (40). From (37) we get the inner product formulas

$$\gamma_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}^{kl} = -\frac{\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1}\rangle_k}{\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}, z^{n_k-1}\rangle_k}, \quad \gamma_{\boldsymbol{n}-\boldsymbol{e}_l}^{kl} = -\frac{\langle \Phi_{\boldsymbol{n}}, z^{n_k}\rangle_k}{\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_l}, z^{n_k}\rangle_k}.$$

We can combine these two relations to get

$$\langle \Phi_{\boldsymbol{n}}, z^{n_k}\rangle_k \langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}, z^{n_k-1}\rangle_k \gamma_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}^{kl} = \langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1}\rangle_k \langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_l}, z^{n_k}\rangle_k \gamma_{\boldsymbol{n}-\boldsymbol{e}_l}^{kl}.$$

Dividing by $\langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}, z^{n_k-1}\rangle_k \langle \Phi_{\boldsymbol{n}-\boldsymbol{e}_k}, z^{n_k-1}\rangle_k$ and using (29) we arrive to

$$\rho_{\boldsymbol{n},k}\gamma_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}^{kl} = \rho_{\boldsymbol{n}-\boldsymbol{e}_l,k}\gamma_{\boldsymbol{n}-\boldsymbol{e}_l}^{kl}. \tag{43}$$

Now multiply by $\alpha_{\boldsymbol{n}-\boldsymbol{e}_l}\alpha_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_l}$, and use (41) to get (40). □

Remark 5.5. As we saw in the proof, (40) has the alternative version (43). If we multiply by $\beta_{\boldsymbol{n}}\beta_{\boldsymbol{n}+\boldsymbol{e}_k}$ and use (42) we get another version, of the form

$$(\beta_{\boldsymbol{n}-\boldsymbol{e}_k} - \beta_{\boldsymbol{n}-\boldsymbol{e}_l})\beta_{\boldsymbol{n}+\boldsymbol{e}_k}\rho_{\boldsymbol{n},k} = (\beta_{\boldsymbol{n}} - \beta_{\boldsymbol{n}+\boldsymbol{e}_k-\boldsymbol{e}_l})\beta_{\boldsymbol{n}}\rho_{\boldsymbol{n}-\boldsymbol{e}_l,k}. \tag{44}$$

6. Christoffel–Darboux Formula

Theorem 6.1. *Let $(\boldsymbol{n}_k)_{k=0}^N$ be a path of multi-indices such that $\boldsymbol{n}_0 = \boldsymbol{0}$, and $\boldsymbol{n}_{k+1} - \boldsymbol{n}_k = \boldsymbol{e}_{l_k}$ for some $1 \le l_k \le r$, in particular $|\boldsymbol{n}_k| = k$. Assume all multi-indices on the path are normal, along with all the neighbouring indices that belong to $\mathbb{Z}_+^r$. Then we have the Christoffel–Darboux formula*

$$(1 - z\bar{\zeta})\sum_{k=0}^{N-1} \Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} = \Phi_{\boldsymbol{n}_N}^*(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_N}^*(\zeta)} - \sum_{j=1}^{r} \rho_{\boldsymbol{n}_N,j} z \Phi_{\boldsymbol{n}_N-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_N+\boldsymbol{e}_j}(\zeta)}. \tag{45}$$

Proof. By (26) and (18) we have

$$z\bar{\zeta}\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} = -\beta_{\boldsymbol{n}_{k+1}} z\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_{k+1}}(\zeta)} + \sum_{j=1}^{r} \rho_{\boldsymbol{n}_{k+1},j} z\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)}$$

$$= \Phi^*_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_{k+1}}(\zeta)} - \Phi^*_{\boldsymbol{n}_{k+1}}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_{k+1}}(\zeta)} + \sum_{j=1}^{r} \rho_{\boldsymbol{n}_{k+1},j} z\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)}.$$

By (17) and (27) we also have

$$\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} = \alpha_{\boldsymbol{n}_k}\Phi^*_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} + \sum_{j=1}^{r} \rho_{\boldsymbol{n}_k,j} z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)}$$

$$= \Phi^*_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_{k+1}}(\zeta)} - \Phi^*_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_k}(\zeta)} + \sum_{j=1}^{r} \rho_{\boldsymbol{n}_k,j} z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)}.$$

Putting these together yields

$$\begin{aligned}(1-z\bar{\zeta})\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} &= \Phi^*_{\boldsymbol{n}_{k+1}}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_{k+1}}(\zeta)} - \Phi^*_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_k}(\zeta)} \\ &\quad + \sum_{j=1}^{r} \rho_{\boldsymbol{n}_k,j} z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} \\ &\quad - \sum_{j=1}^{r} \rho_{\boldsymbol{n}_{k+1},j} z\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)}.\end{aligned}$$

By (30) and (34) we can write

$$\begin{aligned}&\sum_{j=1}^{r} \rho_{\boldsymbol{n}_k,j} z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} - \sum_{j=1}^{r} \rho_{\boldsymbol{n}_{k+1},j} z\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)} \\ &\quad = \sum_{j=1}^{r} \rho_{\boldsymbol{n}_k,j} z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\left(\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_k+\boldsymbol{e}_j}(\zeta)} + \gamma^{jl_k}_{\boldsymbol{n}_k}\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)}\right) \\ &\qquad\quad - \sum_{j=1}^{r} \rho_{\boldsymbol{n}_{k+1},j} z\left(\Phi_{\boldsymbol{n}_{k+1}-\boldsymbol{e}_j}(z) - \gamma^{l_kj}_{\boldsymbol{n}_k-\boldsymbol{e}_j}\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\right)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)} \\ &\quad = \sum_{j=1}^{r} \left(\rho_{\boldsymbol{n}_k,j}\gamma^{jl_k}_{\boldsymbol{n}_k} - \rho_{\boldsymbol{n}_{k+1},j}\gamma^{jl_k}_{\boldsymbol{n}_k-\boldsymbol{e}_j}\right) z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)} \\ &\qquad\quad + \sum_{j=1}^{r} \rho_{\boldsymbol{n}_k,j} z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_k+\boldsymbol{e}_j}(\zeta)} - \sum_{j=1}^{r} \rho_{\boldsymbol{n}_{k+1},j} z\Phi_{\boldsymbol{n}_{k+1}-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)}.\end{aligned}$$

Since $\rho_{\boldsymbol{n}_k,j}\gamma^{jl_k}_{\boldsymbol{n}_k} - \rho_{\boldsymbol{n}_{k+1},j}\gamma^{jl_k}_{\boldsymbol{n}_k-\boldsymbol{e}_j} = 0$ (see (43)), we end up with

$$\begin{aligned}(1-z\bar{\zeta})\Phi_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}}(\zeta)} &= \Phi^*_{\boldsymbol{n}_{k+1}}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_{k+1}}(\zeta)} - \Phi^*_{\boldsymbol{n}_k}(z)\overline{\boldsymbol{\Lambda}^*_{\boldsymbol{n}_k}(\zeta)} \\ &\quad + \sum_{j=1}^{r} \rho_{\boldsymbol{n}_k,j} z\Phi_{\boldsymbol{n}_k-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_k+\boldsymbol{e}_j}(\zeta)} \\ &\quad - \sum_{j=1}^{r} \rho_{\boldsymbol{n}_{k+1},j} z\Phi_{\boldsymbol{n}_{k+1}-\boldsymbol{e}_j}(z)\overline{\boldsymbol{\Lambda}_{\boldsymbol{n}_{k+1}+\boldsymbol{e}_j}(\zeta)}.\end{aligned}$$

Now summation over k leads to a telescoping sum resulting in exactly (45). $\square$

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