Diffusion coefficient matrix for multiple conserved charges: a Kubo approach

Sourav Dey,[1] **Amaresh Jaiswal**[1,2] **and Hiranmaya Mishra**[1]

[1]*School of Physical Sciences, National Institute of Science Education and Research, An OCC of Homi Bhabha National Institute, Jatni-752050, India*

[2]*Institute of Theoretical Physics, Jagiellonian University, ul. St. Łojasiewicza 11, 30-348 Krakow, Poland*

E-mail: sourav.dey@niser.ac.in, a.jaiswal@niser.ac.in, hiranmaya@niser.ac.in

ABSTRACT: The strongly interacting matter created in relativistic heavy-ion collisions possesses several conserved quantum numbers, such as baryon number, strangeness, and electric charge. The diffusion process of these charges can be characterized by a diffusion matrix that describes the mutual influence of the diffusion of various charges. We derive the Kubo relations for evaluating diffusion coefficients as elements of a diffusion matrix. We further demonstrate that in the weak coupling limit, the diffusion matrix elements obtained through Kubo relations reduce to those obtained from kinetic theory with an appropriate identification of the relaxation times. We illustrate this evaluation in a toy model of two interacting scalar fields with two conserved charges.

Contents

1   Introduction

The computation of transport coefficients in relativistic systems holds significant importance across various fields, such as relativistic astrophysics, cosmology, and high-energy heavy-ion physics. In the context of heavy ion collisions, the strongly interacting medium that is produced in such a process shows collective motion and its space-time evolution has been described successfully using relativistic dissipative hydrodynamics along with a modelling for the early stage and the freeze out for the hadrons [1–3]. The dissipative effects like the shear and bulk viscosity play a significant role in the evolution of QGP and influence various observables like flow coefficients [4, 5] and the hadron transverse momentum spectra, the hadron transverse momentum spectrum [6, 7]. Besides the viscosity coefficients, the other transport coefficient that has become relevant in the context of heavy ion collisions is the electrical conductivity of the strongly interacting matter. This has been discussed in the context of charge fluctuations [8], the evolution of electromagnetic fields [9, 10]. Further, it has been suggested that it can be extracted from the flow parameters in heavy ion collisions [11]. There has been a recent interest in the diffusion of conserved charges due to temperature and density gradients, particularly for low energy collisions[12–16]. As

a dissipative phenomenon, diffusion emerges whenever variations in a conserved quantity occur.

Let us note that the fluctuations of conserved charges play an important role to find the critical point of QCD [17–20]. In this context, the process of diffusion plays an important role as the time evolution of conserved charges can be caused by such process. When dealing with a system with multiple conserved charges, such as electric charge, baryon number, and strangeness, the diffusion processes associated with these charges are not independent [21]. Instead, they must be described through a set of coupled diffusion equations. To account for this phenomenon, the conventional diffusion coefficients corresponding to each conserved charge are replaced by a diffusion coefficient matrix [12–16]. This matrix quantifies the coupling between the various conserved quantum numbers. The diagonal elements of the matrix represent the familiar charge diffusion coefficients, while the off-diagonal elements characterize the diffusive coupling between different charge currents.

To estimate the transport coefficients, it must be kept in mind that these coefficients capture the dynamics of the underlying microscopic theory. Thus, any first principle calculation must include the challenges of strong coupling. In the context of diffusion matrix, the initial estimation of the matrix elements of this matrix have been derived within the ambit of kinetic theory, both with Chapman Enskog (CE) expansion[12, 13, 16] and a relaxation time approximation (RTA) [13, 15]. Although the relaxation time approximation allows one to use a much simpler collision kernel for the Boltzman kinetic equation, it is not possible to have a control on the systematic degree of accuracy of the method. Moreover, RTA, with momentum dependent relaxation time, is in contradiction with the macroscopic conservation laws although novel approaches have been proposed to overcome them [22, 23]. Due to its simplicity of estimating the transport coefficients, the RTA has been used widely both for hadronic and partonic matter [24–33]. On the other hand, CE approach is a variational approach that allows one to obtain solutions with arbitrary accuracy depending on the order of approximation used. The CE method is also consistent with macroscopic conservation laws.

In contrast to kinetic theory approach, the other often used method to estimate the transport coefficients is the Green-Kubo correlator approach. Using the linear response theory, the Green-Kubo formula relates the transport coefficients to the spectral functions of the relevant current-current correlators. For example, the shear and bulk viscosities are related to energy-momentum correlators, the electrical conductivity corresponds to the vector currents correlators of the light quarks and the heavy quark diffusion to the color electric field correlators. If the temperature is sufficiently high or the theory is weakly coupled, transport coefficients can be computed in a perturbative expansion using the Kubo relations. On the other hand, such relation is also valid, in general, for strong couplings. This, therefore, opens up the possibility of applying lattice QCD at finite temperature as a nonperturbative tool to compute transport coefficients provided the analytic continuation from eucledian correlators to spectral functions can be done reliably. Indeed, shear and bulk viscosity coefficients have been obtained within the ambit of lattice QCD and Green-kubo relations [34]. The electrical conductivity has also been estimated for a 2+1 flavor system in Ref. [35] using maximum entropy method. Keeping in mind the importance of

cross diffusion coefficients and the nonperturbative features of Green-Kubo formulation, we attempt here to derive the Green-Kubo relations for the diffusion matrix elements.

In Kubo approach, the equilibrium correlation functions can be utilized to compute the response, offering the advantage of evaluating transport coefficients using equilibrium Green's functions formulated in imaginary time. In the present investigation, we employ the correlation-function method developed by Zubarev to derive transport coefficients of a relativistic fluid [36]. In particular, we obtain Kubo formulas for diffusion matrix for a system with multiple conserved charges, such as electric charge, baryon number, and strangeness. The Zubarev method is founded on the concept of a non-equilibrium statistical operator (NESO), which extends the equilibrium Gibbs statistical operator to non-equilibrium states. By expanding the operator using small gradients of the thermodynamic parameters, this approach allows us to derive kinetic transport coefficients from correlation functions. Typically, the state of the system can be described by fields of temperature, chemical potential, and momentum, which may vary in space and time. The specific parameter set required for a complete system description depends on the problem at hand. The transport coefficients are obtained by perturbing the non-equilibrium statistical operator linearly around its equilibrium value. This allows to relate the transport coefficients to retarded equilibrium correlation functions of the microscopic theory. Such an approach has been successfully applied to compute viscosity coefficients of strongly interacting matter [37] also in the presence of magnetic field [38, 39]. We follow a similar approach to estimate diffusion matrix elements for a system of two complex interacting scalar fields.

This paper is organized as follows: In Sec. 2, we review the basics of Zubarev's method of non-equilibrium statistical operators to derive transport coefficients and derive Kubo formulas for transport coefficients, including the diffusion matrix. In Sec. 3, we employ Kubo relations to derive the diffusion coefficient matrix for two interacting charged scalar fields coupled to each other and with self-interaction. We summarize our results in Sec. 4.

We use natural units ($c = \hbar = k_B = 1$) and mostly negative metric sign convention $\eta^{\mu\nu} = \text{diag}\,(+1, -1, -1, -1)$. The bold font denotes three vectors. The scalar products for three- and four-vectors are denoted by a dot, i.e., $a \cdot b = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}$.

2 Non-equilibrium statistical operators and Kubo relations

Hydrodynamics is an effective field theory to describe the collective behaviour of a system, and its universal features are based on conservation laws. Zubarev's method of non-equilibrium statistical operator (NESO) provides a suitable framework for describing systems in the hydrodynamic regime while incorporating thermodynamic parameters such as temperature and chemical potential, which can be defined locally. In this method, the relevant NESO, which characterizes the nonequilibrium state of the system, can be expressed as [36]

$$\hat{\varrho}(t) = \frac{1}{\mathcal{Q}} \exp\left[-\int d^3\mathbf{x}\, \hat{\mathcal{Z}}(\mathbf{x}, t)\right],\tag{2.1}$$

where $\mathcal{Q} = \text{Tr} \exp\left[-\int d^3\mathbf{x}\, \hat{\mathcal{Z}}(\mathbf{x},t)\right]$ is the normalization factor. The operator $\hat{\mathcal{Z}}(\mathbf{x},t)$ in Eq. (2.1) has the form [36]

$$\hat{\mathcal{Z}}(\mathbf{x},t) = \varepsilon \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1-t)} \left[\beta^\nu\left(\mathbf{x},t_1\right)\hat{T}_{0\nu}\left(\mathbf{x},t_1\right) - \sum_a \alpha_a\left(\mathbf{x},t_1\right)\hat{J}_a^0\left(\mathbf{x},t_1\right)\right], \qquad (2.2)$$

where ε tends to zero after taking the thermodynamic limit.

The operators $\hat{T}^{\mu\nu}$ and $\hat{J}_a^\mu$ correspond to the energy-momentum tensor and the charge currents, respectively. They satisfy the conservation equations given by

$$\partial_\mu \hat{T}^{\mu\nu} = 0 \ , \ \ \partial_\mu \hat{J}_a^\mu = 0. \qquad (2.3)$$

The quantities $\beta^\mu(\mathbf{x},t)$ and $\alpha_a(\mathbf{x},t)$ in Eq. (2.2) are related to temperature, chemical potential, and fluid velocity as

$$\beta^\mu(\mathbf{x},t) = \beta(\mathbf{x},t)\, u^\mu(\mathbf{x},t), \qquad (2.4)$$

$$\alpha_a(\mathbf{x},t) = \sum_A q_{aA}\, \alpha_A(\mathbf{x},t) = \beta(\mathbf{x},t) \sum_A q_{aA}\, \mu_A(\mathbf{x},t), \qquad (2.5)$$

where $\beta(\mathbf{x},t)$ represents the inverse temperature, $u^\mu(\mathbf{x},t)$ is the fluid velocity, and $\mu_a(\mathbf{x},t) = \sum_A q_{aA}\, \mu_A(\mathbf{x},t)$ is the chemical potential associated with the species labeled by a. In general, each species can have multiple conserved charges q_{aA} with the associated chemical potential $\mu_A(\mathbf{x},t)$. For instance, in the context of heavy ion collision, $a \in (u,d,s)$ while $A \in (B,Q,S)$. Here (u,d,s) represents up, down and strange quarks, respectively, and (B,Q,S) corresponds to baryon number, electric charge and strangeness. Performing integration by parts in Eq. (2.2), we obtain,

$$\hat{\mathcal{Z}}(\mathbf{x},t) = \beta^\nu\left(\mathbf{x},t\right)\hat{T}_{0\nu}\left(\mathbf{x},t\right) - \sum_a \alpha_a\left(\mathbf{x},t\right)\hat{J}_a^0\left(\mathbf{x},t\right)$$

$$- \varepsilon \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1-t)}\frac{d}{dt_1}\left[\beta^\nu\left(\mathbf{x},t_1\right)\hat{T}_{0\nu}\left(\mathbf{x},t_1\right) - \sum_a \alpha_a\left(\mathbf{x},t_1\right)\hat{J}_a^0\left(\mathbf{x},t_1\right)\right]. \qquad (2.6)$$

Using the conservation equations given in Eq. (2.3) in the above equation, one can write the NESO of Eq. (2.1) as

$$\hat{\varrho}(t) = \frac{1}{\mathcal{Q}}\exp\left[\hat{\mathcal{A}} - \hat{\mathcal{B}}\right]. \qquad (2.7)$$

In the above equation, we have

$$\hat{\mathcal{A}} = \int d^3\mathbf{x}\left[\beta^\nu(\mathbf{x},t)\hat{T}_{0\nu}(\mathbf{x},t) - \sum_a \alpha_a(\mathbf{x},t)\hat{J}_a^0(\mathbf{x},t)\right], \qquad (2.8)$$

$$\hat{\mathcal{B}} = \int d^3\mathbf{x}\int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1-t)}\, \hat{\mathcal{C}}(\mathbf{x},t_1), \qquad (2.9)$$

and

$$\hat{\mathcal{C}}(\mathbf{x},t_1) = \hat{T}^{\mu\nu}\left(\mathbf{x},t_1\right)\partial_\mu \beta_\nu\left(\mathbf{x},t_1\right) - \sum_a \hat{J}_a^\mu\left(\mathbf{x},t_1\right)\partial_\mu \alpha_a\left(\mathbf{x},t_1\right), \qquad (2.10)$$

where $\frac{\partial}{\partial t_1} \to \partial_\mu$ has been performed by adding a surface term.

The expression $\hat{\mathcal{A}}$ in Eq. (2.8) having terms without any gradient of temperature and chemical potentials corresponds to the equilibrium part of the NESO whereas the operator $\hat{\mathcal{B}}$ in Eq. (2.9) corresponds to the non-equilibrium part, with the expression of $\hat{\mathcal{C}}$ in Eq. (2.10) containing gradient of temperature and chemical potentials representing thermodynamic *forces*. The statistical operator in Eq. (2.7) can then be used to derive transport equations. To this end, we will treat the non-equilibrium part $\hat{\mathcal{B}}$ as a perturbation. We will keep up to linear order in $\hat{\mathcal{B}}$ so that the relations between the thermodynamic processes and dissipative currents are linear. Expanding up to linear order in $\hat{\mathcal{B}}$, i.e., first-order gradient of thermodynamic variables, the NESO can be written as

$$\hat{\varrho} = \left[1 + \int_0^1 d\lambda \left(\hat{\mathcal{B}}_\lambda - \langle\hat{\mathcal{B}}_\lambda\rangle_l\right)\right]\hat{\varrho}_l \ , \tag{2.11}$$

where we have used the shorthand notation for any operator $\hat{\mathcal{X}}$ as

$$\hat{\mathcal{X}}_\lambda = e^{-\lambda\hat{\mathcal{A}}}\hat{\mathcal{X}}e^{\lambda\hat{\mathcal{A}}} \ . \tag{2.12}$$

Here, $\hat{\varrho}_l$ is the equilibrium part of NESO given as

$$\hat{\varrho}_l(t) = \mathcal{Q}_l^{-1}\exp\left[-\hat{\mathcal{A}}\right], \tag{2.13}$$

with $\mathcal{Q} = \text{Tr}\exp\left[-\hat{\mathcal{A}}\right]$. Further, in Eq. (2.11), $\langle\hat{O}\rangle_l$ is the average over the local equilibrium that is $\langle\hat{O}\rangle_l = \text{Tr}\left(\hat{\varrho}_l(t)\,\hat{O}\right)$.

Using Eq. (2.11) we can write down the deviation of the statistical average of any arbitrary tensor operator from its equilibrium value as

$$\delta\left\langle\hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_n}(\mathbf{x},t)\right\rangle = \text{Tr}\left[\hat{\varrho}_l\int_0^1 d\lambda\,\hat{\mathcal{B}}_\lambda\,\hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_n}(\mathbf{x},t)\right] - \text{Tr}\left[\hat{\varrho}_l\,\langle\hat{\mathcal{B}}_\lambda\rangle_l\,\hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_n}(\mathbf{x},t)\right] \ . \tag{2.14}$$

Substituting the expression for the non-equilibrium part of $\hat{\mathcal{B}}$ from Eq. (2.9) in Eq. (2.14), we can write the operator deviation as

$$\delta\left\langle\hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_n}(\mathbf{x},t)\right\rangle = \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)}$$
$$\times \int_0^1 d\lambda\left\langle\hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_n}(\mathbf{x},t)\left[\hat{\mathcal{C}}_\lambda\left(\mathbf{x}_1,t_1\right) - \left\langle\hat{\mathcal{C}}_\lambda\left(\mathbf{x}_1,t_1\right)\right\rangle_l\right]\right\rangle_l \ . \tag{2.15}$$

Compactly, the off-equilibrium contribution of a hydro variable $\hat{\mathcal{O}}$ is connected with the gradient forces through the Kubo-Mori-Bogoliubov (KMB) inner product between the operator $\hat{\mathcal{O}}$ and $\hat{\mathcal{C}}$,

$$\delta\left\langle\hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_n}(\mathbf{x},t)\right\rangle = \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1\,e^{\varepsilon(t_1-t)}\left(\hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_n}(\mathbf{x},t),\hat{\mathcal{C}}\left(\mathbf{x}_1,t_1\right)\right) \ , \tag{2.16}$$

where the Kubo-Mori-Bogoliubov (KMB) inner product is given by,

$$\left(\hat{\mathcal{O}}_a(\mathbf{x},t), \hat{\mathcal{O}}_b(\mathbf{x}_1,t_1)\right) \equiv \int_0^1 d\lambda \left\langle \hat{\mathcal{O}}_a(\mathbf{x},t) \left[\hat{\mathcal{O}}_{2\lambda}(\mathbf{x}_1,t_1) - \left\langle \hat{\mathcal{O}}_{2\lambda}(\mathbf{x}_1,t_1)\right\rangle_l\right]\right\rangle_l. \qquad (2.17)$$

We shall use Eq. (2.16) to define the transport coefficients in the calculation of dissipative contributions. To do so, one needs to distinguish between the equilibrium and dissipative contribution.

To separate the dissipative contributions in conserved hydrodynamic currents, we decompose the energy-momentum tensor and the charge currents. In general, we construct the operator, $\hat{T}^{\mu\nu}$, $\hat{J}_a^\mu$ in the form

$$\hat{T}^{\mu\nu} = \hat{\epsilon}u^\mu u^\nu - \hat{P}\Delta^{\mu\nu} + \hat{Q}^\mu u^\nu + \hat{Q}^\nu u^\mu + \hat{\mathcal{T}}^{\mu\nu}, \qquad (2.18)$$

$$\hat{J}_a^\mu = \hat{n}_a u^\mu + \hat{\mathcal{N}}_a^\mu, \qquad (2.19)$$

with u_μ being the fluid velocity and the dissipative operators $\hat{\mathcal{T}}^{\mu\nu}, \hat{Q}^\mu$ and $\hat{\mathcal{N}}_a^\mu$ are orthogonal to u_μ, and $\hat{\mathcal{T}}^{\mu\nu}$ is traceless [38, 40, 41]. For further computation it is convenient to decompose the operator $\hat{\mathcal{C}}$ containing the thermodynamic forces into different dissipative processes as in Eq. (2.18),

$$\hat{\mathcal{C}} = \hat{\epsilon}D\beta - \hat{P}\beta\theta - \sum_a \hat{n}_a D\alpha_a + \hat{Q}_\mu\left(\beta D u^\mu + \partial^\mu\beta\right) - \sum_a \hat{\mathcal{N}}_a^\mu \nabla_\mu \alpha_a + \beta\hat{\mathcal{T}}^{\mu\nu}\nabla_\mu u_\nu, \quad (2.20)$$

where we used the notation $D = u^\mu\partial_\mu$, $\theta = \partial_\mu u^\mu$, $\nabla_\alpha = \Delta_{\alpha\beta}\partial^\beta$ with $\Delta_{\alpha\beta} = \eta_{\alpha\beta} - u_\alpha u_\beta$ being the projector orthogonal to u_α. One can eliminate the derivatives $D\beta$, $D\alpha_a$ and Du^μ using the equations of ideal hydrodynamics. From the conservation of energy-momentum tensor and currents of ideal hydrodynamics $\partial_\mu\hat{T}_{(0)}^{\mu\nu} = 0$, $\partial_\mu\hat{J}_{a\,0}^\mu = 0$, we have

$$D\epsilon = -h\theta, \qquad (2.21)$$

$$Dn_a = -n_a\theta, \qquad (2.22)$$

$$Du_\alpha = \frac{1}{h}\nabla_\alpha P, \qquad (2.23)$$

where, $\theta = \partial_\alpha u^\alpha$ is the expansion scalar and $h = \epsilon + P$ is the enthalpy. One can choose the energy density ϵ and number density n_a as independent thermodynamic variables to write

$$D\beta = \left(\frac{\partial\beta}{\partial\epsilon}\right)_{n_a} D\epsilon + \sum_a \left(\frac{\partial\beta}{\partial n_a}\right)_{\epsilon,n_b\neq n_a} Dn_a \qquad (2.24)$$

$$= -\theta\left[\left(\frac{\partial\beta}{\partial\epsilon}\right)_{n_a} h + \sum_a \left(\frac{\partial\beta}{\partial n_a}\right)_{\epsilon,n_b\neq n_a} n_a\right], \qquad (2.25)$$

$$D\alpha_c = \left(\frac{\partial\alpha_c}{\partial\epsilon}\right)_{n_a} D\epsilon + \sum_a \left(\frac{\partial\alpha_c}{\partial n_a}\right)_{\epsilon,n_b\neq n_a} Dn_a \qquad (2.26)$$

$$= -\theta\left[\left(\frac{\partial\alpha_c}{\partial\epsilon}\right)_{n_a} h + \sum_a \left(\frac{\partial\alpha_c}{\partial n_a}\right)_{\epsilon,n_b\neq n_a} n_a\right]. \qquad (2.27)$$

Next, we use the Gibbs-Duhem relations in thermodynamics, $\beta dP = -h d\beta + \sum_a n_a d\alpha_a$, which leads to

$$h = -\beta \left(\frac{\partial P}{\partial \beta} \right)_{\alpha_a} \;,\; n_a = \beta \left(\frac{\partial P}{\partial \alpha_a} \right)_{\beta, n_b \neq n_a}. \tag{2.28}$$

Substituting the above equation into Eq. (2.25) and (2.27), one can write

$$D\beta = \beta \, \theta \, \gamma = \beta \, \theta \left(\frac{\partial P}{\partial \epsilon} \right)_{n_a}, \tag{2.29}$$

$$D\alpha_c = -\delta_c \, \beta \, \theta = -\beta \, \theta \left(\frac{\partial P}{\partial \alpha_c} \right)_{\epsilon, n_b \neq n_c}. \tag{2.30}$$

Similarly, using the Gibbs-Duhem relation in Eq. (2.23), we have

$$Du_\mu = -T \, \nabla_\mu \beta + T \sum_a \frac{n_a}{\epsilon + P} \, \nabla_\mu \alpha_a. \tag{2.31}$$

Substituting the expression for the differentials in Eqs. (2.29), (2.30) and (2.23) into Eq. (2.20), one obtains the operator $\hat{\mathcal{C}}$ at first order in gradients in a compact form as

$$\hat{\mathcal{C}} = -\beta \, \theta \, \hat{P}^* + \beta \, \hat{\mathcal{T}}^{\mu\nu} \sigma_{\mu\nu} - \sum_a \hat{\mathcal{J}}_a^\mu \, \nabla_\mu \alpha_a. \tag{2.32}$$

Here, $\sigma_{\mu\nu} \equiv \frac{1}{2} \left[\Delta_\mu^\alpha \Delta_\nu^\beta + \Delta_\nu^\alpha \Delta_\mu^\beta - (2/3)\Delta^{\alpha\beta}\Delta_{\mu\nu} \right] \nabla_\alpha u_\beta$ and, the operators $\hat{P}^*$ and $\hat{\mathcal{J}}_a^\mu$ are defined as [1]

$$\hat{P}^* = \hat{P} - \gamma \, \hat{\epsilon} - \sum_a \delta_a \, \hat{n}_a, \tag{2.33}$$

$$\hat{\mathcal{J}}_a^\mu = \hat{\mathcal{N}}_a^\mu - \frac{n_a}{h} \, \hat{\mathcal{Q}}^\mu. \tag{2.34}$$

In the present work, we perform our calculation with the above quantities which are valid in generic fluid frames. Since, hydrodynamic modes need to be subtracted as given in second expression with a proper definition of number density and enthalpy in equilibrium.[40, 43–46]

Let us note that $\hat{\mathcal{J}}_a^\mu$ and the corresponding density $\hat{n}_a$ is defined for the particle species basis. To calculate the cross-conductivity matrix elements (or the diffusion matrix in charge basis), we need to go to the conserved charge basis [47]. The chemical potentials in the basis of species are $\mu_a = \sum_A q_{aA} \mu_A$ where q_{aA} is the corresponding matrix of charges of the species 'a'. Here, we refer to the capital alphabet 'A' for conserved charge quantum numbers. It is easy to check using the fact that $\sum_a \mu_a \hat{n}_a = \sum_A \mu_A \hat{n}_A$, the number densities

---

[1]By matching condition, one can locally demand the densities of thermodynamic parameters to match its expectation value with respect to the local relevant statistical operator. But the relevant statistical operator as a function of non-uniform thermodynamic parameters gives an additional shift; that term has been absorbed into $\hat{P}^\star$ [41]. Moreover, one can define a combination of energy flux and the particle flux, $\hat{\mathcal{J}}_a^\mu$, which remains invariant under any first-order transformation in choosing fluid frame u^μ.[42]

in the species basis are related to those in conserved charge basis as $n_a = \sum_A q_{aA}^{-1} n_A$ [47]. Accordingly, the diffusion current $\hat{\mathcal{J}}_A^\mu$ in the conserved charge basis can be written as

$$\hat{\mathcal{J}}_A^\mu = \sum_a q_{aA} \, \hat{\mathcal{J}}_a^\mu = \hat{\mathcal{N}}_A^\mu - \frac{n_A}{h} \, \hat{\mathcal{Q}}^\mu. \tag{2.35}$$

In the present work, we are interested in evaluating the diffusion matrix in the conserved charge basis, so for that, one needs to employ the transformation matrix q_{aA}, which connects the particle basis to the charge basis.

Now, using Eqs. (2.16) and (2.20), one can extract the off-equilibrium correction to each conserved current. Therefore, we notice that a similar correction to the particle diffusion currents in particle basis, i.e., the linear response of the particle diffusion current can be defined as

$$\mathcal{J}_a^\mu(\mathbf{x}, t) = \left\langle \hat{\mathcal{J}}_a^\mu(\mathbf{x}, t) \right\rangle = - \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1 - t)} \left(\hat{\mathcal{J}}_a^\mu(\mathbf{x}, t), \hat{\mathcal{C}}\left(\mathbf{x}_1, t_1\right) \right) . \tag{2.36}$$

Keeping terms till linear order in gradients, we obtain

$$\mathcal{J}_a^\mu(\mathbf{x}, t) = \left\langle \hat{\mathcal{J}}_a^\mu(\mathbf{x}, t) \right\rangle_{(1)} = - \sum_b \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 \, e^{\varepsilon(t_1 - t)} \, \nabla_\nu \alpha_b\left(\mathbf{x}, t\right)$$
$$\times \left(\hat{\mathcal{J}}_a^\mu(\mathbf{x}, t), \hat{\mathcal{J}}_b^\nu\left(\mathbf{x}_1, t_1\right) \right) + \mathcal{O}(\nabla^2). \tag{2.37}$$

The above equation can be expresses in the form

$$\mathcal{J}_a^\mu(\mathbf{x}, t) = \sum_b \mathcal{K}_{ab}^{\mu\nu} \, \nabla_\nu \alpha_b(\mathbf{x}, t), \tag{2.38}$$

where the diffusion tensor is defined as

$$\mathcal{K}_{ab}^{\mu\nu} = - \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1 - t)} \left(\hat{\mathcal{J}}_a^\mu(\mathbf{x}, t), \hat{\mathcal{J}}_b^\nu\left(\mathbf{x}_1, t_1\right) \right) . \tag{2.39}$$

In order to construct the form of $\mathcal{K}_{ab}^{\mu\nu}$ tensor, we note that heat flow $\hat{\mathcal{Q}}^\mu$ and particle current $\hat{\mathcal{N}}_a^\mu$ satisfy the orthogonality with the time-like vector β^μ, i.e., $\beta^\mu \hat{\mathcal{Q}}_\mu = \beta^\mu \hat{\mathcal{N}}_{a\,\mu} = 0$. This is satisfied by demanding

$$u_\mu \, \mathcal{K}_{ab}^{\mu\nu} = 0 \,. \tag{2.40}$$

Furthermore, the KMB inner product must satisfy the Onsager reciprocal relations which is expressed as [2]

$$\mathcal{K}_{ab}^{\mu\nu} = \mathcal{K}_{ba}^{\nu\mu}. \tag{2.41}$$

Using the properties of diffusion tensor in Eqs. (2.40) and (2.41), we can express it as

$$\mathcal{K}_{a\,b}^{\mu\nu} = \Delta^{\mu\nu} \kappa_{ab}, \tag{2.42}$$

---

[2]The proof of this relation is discussed in details in Refs. [38, 42]

which leads to the expression for diffusion matrix as

$$\kappa_{ab} = \frac{1}{3}\Delta_{\mu\nu}\mathcal{K}_{ab}^{\mu\nu} = -\frac{1}{3}\int d^3\mathbf{x}_1 \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1-t)} \left(\hat{\mathcal{J}}_a^\mu(\mathbf{x},t), \hat{\mathcal{J}}_{b\,\mu}(\mathbf{x}_1,t_1)\right). \tag{2.43}$$

The above expression of diffusion matrix can be converted in the conserved charges basis by employing the conversion matrix as

$$\kappa_{AB} = -\frac{1}{3}\int d^3\mathbf{x}_1 \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1-t)} \left(\hat{\mathcal{J}}_A^\mu(\mathbf{x},t), \hat{\mathcal{J}}_{B\,\mu}(\mathbf{x}_1,t_1)\right) = \sum_{a,b} q_{aA}\, q_{bB}\, \kappa_{ab}. \tag{2.44}$$

This matrix is symmetric subject to the symmetry conditions of the diffusion matrix elements in species basis κ_{ab}. In the present work, however, we shall confine our discussion to the diffusion coefficient κ_{ab} in particle basis respectively. In the next section, we evaluate these diffusion matrix elements in species basis for a system of two complex interacting scalar field theory. We then employ Eq. (2.44) to translate this diffusion matrix into charge basis.

The computation of the dissipative diffusion coefficients boils down to the evaluation of the KMB product of the dissipative diffusive currents. We give a detailed derivation of this evaluation in Appendix-A.2. This leads to the diffusion coefficients getting related to the corresponding retarded Green's function $G^R_{\hat{\mathcal{J}}_a^\mu, \hat{\mathcal{J}}_{\mu\,b}}$, resulting in

$$\kappa_{ab} = \frac{T}{3}\frac{\partial}{\partial\omega}\mathrm{Im}\left(G^R_{\hat{\mathcal{J}}_a^\mu, \hat{\mathcal{J}}_{\mu\,b}}\right)(\mathbf{0},\omega)\bigg|_{\omega\to 0}. \tag{2.45}$$

Using Eq. (A.9) one can further relate κ_{ab} to the spectral function as,

$$\kappa_{ab} = \frac{T}{3}\frac{\rho_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}(\omega,\mathbf{0})}{\omega}\bigg|_{\omega\to 0}, \tag{2.46}$$

Thus the diffusion coefficients are determined by the small frequency limit of the zero momentum spectral functions of corresponding composite operators. These two point correlators are to be evaluated at constant values of thermodynamic parameters, i.e. as if the system is in global thermal equilibrium. In the next section, a similar condition helps to set the thermal field theory with uniform temperature T and chemical potentials μ_a for particle and then extract the transport coefficients.[3]

For the sake of completeness, we also extract the off-equilibrium contribution to the trace-less and trace part of the energy-momentum tensor. A similar linear response correction to the energy-momentum tensor can be written as

$$\pi^{\mu\nu}(\mathbf{x},t) = \left\langle\hat{\mathcal{T}}^{\mu\nu}(\mathbf{x},t)\right\rangle_{(1)}, \quad \Pi(\mathbf{x},t) = \langle P^*(\mathbf{x},t)\rangle_{(1)}. \tag{2.47}$$

---

[3]Here, one may think of uniform values of thermodynamic parameters as an average over a local patch of the fluid cells. The deviation due to the non-uniform profile has been absorbed in the higher order gradient correction than the transport coefficients at a given gradient order and discarded by changing the local equilibrium distribution to a global one. This non-locality in the thermodynamical forces is crucial for causal transport equations of hydrodynamics[40, 44, 48, 49].

Further, up to linear order in gradients, the above relations can be expressed as

$$\pi^{\mu\nu}\left(\mathbf{x},t\right) = \beta\left(\mathbf{x},t\right)\sigma_{\rho\sigma}\left(\mathbf{x},t\right)\int d^3\mathbf{x}_1\int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)}\left(\hat{\mathcal{T}}^{\mu\nu}(\mathbf{x},t),\hat{\mathcal{T}}^{\rho\sigma}\left(\mathbf{x}_1,t_1\right)\right) + \mathcal{O}(\nabla^2),$$
(2.48)

$$\Pi\left(\mathbf{x},t\right) = -\beta\left(\mathbf{x},t\right)\theta\left(\mathbf{x},t\right)\int d^3\mathbf{x}_1\int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)}\left(\hat{P}^*(\mathbf{x},t),\hat{P}^*\left(\mathbf{x}_1,t_1\right)\right) + \mathcal{O}(\nabla^2).$$
(2.49)

From this equation, one can extract the coefficient of shear and bulk viscosity by comparing with the Navier-Stokes equation, which is

$$\pi^{\mu\nu}\left(\mathbf{x},t\right) = 2\eta\,\sigma^{\mu\nu}\left(\mathbf{x},t\right)$$
(2.50)

$$\Pi\left(\mathbf{x},t\right) = -\zeta\,\theta\left(\mathbf{x},t\right)$$
(2.51)

In the isotropic medium, one can expect to calculate the shear and bulk viscosity coefficient from KMB correlation function Eq. (2.17),

$$\eta = \frac{\beta}{10}\int d^3\mathbf{x}_1\int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)}\left(\hat{\mathcal{T}}^{\mu\nu}(\mathbf{x},t),\hat{\mathcal{T}}_{\mu\nu}\left(\mathbf{x}_1,t_1\right)\right),$$
(2.52)

$$\zeta = \beta\int d^3\mathbf{x}_1\int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)}\left(\hat{P}^*(\mathbf{x},t),\hat{P}^*\left(\mathbf{x}_1,t_1\right)\right).$$
(2.53)

Employing Eqs. (A.1) and (A.8), one can represent these two KMB correlators in terms of the corresponding spectral functions (See Appendix-[A.2]) which can be evaluated in global thermal equilibrium.

3   Cross diffusion in a toy model: scalar field theory

Having defined the matrix of diffusion coefficient κ_{ab} in Eq. (2.46) we shall next estimate this perturbatively from field theory using the spectral function $\rho_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}$ as given in Eq. (C.20). We shall explicitly outline this derivation for scalar field theory with quartic interaction for simplicity. We note here that the transport coefficients, such as the viscosity coefficients, have been studied earlier for single component scalar field [50]. On the other hand, in the present work, we discuss the cross-diffusion matrix with multiple conserved charges for multicomponent systems. The dissipative coefficients for diffusion have not been investigated so far. We shall consider a system of two complex interacting scalar fields with a Lagrangian given as

$$\mathcal{L}(x) = \partial\phi^\dagger\partial\phi + \partial\xi^\dagger\partial\xi - V(\phi,\xi),$$
(3.1)

where, the potential part is given by

$$V(\phi,\xi) = m_\phi^2\,\phi^\dagger\phi + m_\xi^2\,\xi^\dagger\xi + \frac{g}{2}\phi^\dagger\phi\,\xi^\dagger\xi + \frac{\lambda_1(\phi^\dagger\phi)^2}{4} + \frac{\lambda_2(\xi^\dagger\xi)^2}{4}.$$
(3.2)

Here, m_ϕ and m_ξ are the masses of the scalar fields ϕ and ξ, respectively, while λ_1 and λ_2 are the coupling for the self-interaction of these two fields, g is the mutual interaction

coupling. In the following, we shall evaluate the spectral function at finite temperature T and chemical potentials μ_a using the imaginary time formalism as outlined in Refs. [51, 52]. Henceforth, we denote Minkowskian space-time coordinates by $\mathcal{X} = (t, \mathbf{x})$ and momenta by $\mathcal{K} = (k^0, \mathbf{k})$, whereas their Euclidean counterparts are denoted by $X = (\tau, \mathbf{x})$ and $K = (k_n, \mathbf{k})$. Scalar products are defined as $\mathcal{K} \cdot \mathcal{X} = k^0 t - \mathbf{k} \cdot \mathbf{x}$ and $K \cdot X = k_n \tau - \mathbf{k} \cdot \mathbf{x}$ in the Minkowski and Euclidean manifold, respectively. Moreover the integral measures are $\int_{\mathcal{X}} = \int dt \int d^3\mathbf{x}$ and $\int_X = \int_0^\beta d\tau \int d^3\mathbf{x}$.

We shall assume the Lagrangian is invariant under two global U(1) symmetry having corresponding chemical potential μ_a with $(a = \phi, \xi)$. We also take the chemical potentials for conserved charges as μ_A with $(A = 1, 2)$. These two pairs of chemical potentials are related as $\mu_a = q_{aA}\,\mu_A$, i.e.,

$$
\begin{pmatrix} \mu_\phi \\ \mu_\xi \end{pmatrix} = \begin{pmatrix} q_{\phi 1} & q_{\phi 2} \\ q_{\xi 1} & q_{\xi 2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}. \tag{3.3}
$$

Using the condition [4]

$$
\mu_\phi n_\phi + \mu_\xi n_\xi = \mu_1 n_1 + \mu_2 n_2, \tag{3.4}
$$

the number densities of the fields can be related to the charge densities as

$$
n_a = q_{aA}^{-1} n_A. \tag{3.5}
$$

We shall calculate here the spectral function in the species basis and relate it to the diffusion matrix κ_{AB} by employing the Eq. (2.44). Consistent with the Kubo-Martin-Schwinger (KMS) condition with finite chemical potential, the field operators can be Fourier expanded in the Eucledian space as

$$
\phi_a(X) = \sum\!\!\!\!\!\!\int_P e^{ip_n\tau - i\mathbf{p}\cdot\mathbf{x}} \tilde{\phi}_a(P), \tag{3.6}
$$

where, we have used the notation

$$
\sum\!\!\!\!\!\!\int_P = T \sum_{p_n} \int \frac{d^3\mathbf{p}}{(2\pi)^3} = T \sum_{p_n} \int d\mathbf{p}, \tag{3.7}
$$

and p_n are the Matsubara bosonic frequency $p_n = 2n\pi T$.

The two-point Eucledian correlation function can be written as

$$
\mathcal{G}_{a,b}^E(X) = \left\langle \phi_a(X)\,\phi_b^\dagger(0) \right\rangle_E = \sum\!\!\!\!\!\!\int_{PQ} e^{i\tilde{p}_n\tau + ip_i\mathbf{x}^i} \left\langle \tilde{\phi}_a(P)\,\tilde{\phi}_b^\dagger(Q) \right\rangle_E, \tag{3.8}
$$

---

[4]Under such transformation, this condition ensures the invariance of energy due to the presence of finite chemical potentials in either particle or charge basis. One may infer that free energy for the Grand canonical partition function is invariant under such transformation, i.e., $\mathcal{Z}(T, \mu_\phi, \mu_\xi) = C \int \mathcal{D}\phi \mathcal{D}\xi\, e^{-S_E} = \mathcal{Z}(T, \mu_1, \mu_2)$.

where, $\tilde{p}_n = p_n + i\mu_a$.[5] The free propagator can be expressed as

$$\langle \tilde{\phi}_a(P)\tilde{\phi}_b^\dagger(Q)\rangle^{\mathrm{f}} = \delta_{ab}\frac{\tilde{\delta}(P-Q)}{(p_n + i\mu_a)^2 + \mathbf{p}^2 + m_a^2}, \tag{3.9}$$

where, 'f' in the superscript denotes the free propagator and $\tilde{\delta}(P-Q)$ is defined as

$$\tilde{\delta}(P-Q) = \int_X e^{i(P-Q)\cdot X} = \int_0^\beta d\tau \, e^{i(p_n - q_n)\tau}\int d^3\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \beta\,(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\,\delta_{p_n,q_n}. \tag{3.10}$$

The two-point Eucledian correlation function can be expressed as

$$\mathcal{G}_{a,b}^{E\,\mathrm{f}}(X) = \delta_{ab}\sum_P\!\!\!\!\!\!\!\int \frac{e^{i\tilde{p}_n\tau + ip_i\mathbf{x}^i}}{(p_n + i\mu_a)^2 + \vec{p}^2 + m_a^2}, \tag{3.11}$$

and the two-point function in Eucledian momentum space can be written as,

$$\mathcal{G}_{a,b}^{E\,\mathrm{f}}(K) = \int_0^\beta d\tau \int_{\mathbf{x}} e^{(ik_n + \mu_a)\tau - i\mathbf{k}\cdot\mathbf{x}}\,\mathcal{G}_{a,b}^{E\,\mathrm{f}}(X). \tag{3.12}$$

The Euclidean correlator is time-ordered by definition $(0 \leq \tau \leq \beta)$.

The two-point function $\mathcal{G}_{a,b}^{E}(K)$ can also be represented in terms of the single free particle spectral function[6], i.e.,

$$\mathcal{G}_{a,b}^{E\,\mathrm{f}}(K) = \delta_{ab}\int_{-\infty}^\infty \frac{dk^0}{\pi}\frac{\rho_a^{\mathrm{f}}(k^0, \mathbf{k})}{k^0 - i[k_n - i\mu_a]}, \tag{3.13}$$

where free particle spectral function has the form [51, 53, 54]

$$\rho_a^{\mathrm{f}}(k_0, \mathbf{k}) = \frac{k_0}{|k_0|}\pi\delta(k_0^2 - E_{a\,\mathbf{k}}^2) = \frac{\pi}{2E_{a\,\mathbf{p}}}\left[\delta(k_0 - E_{a\,\mathbf{p}}) - \delta(k_0 + E_{a\,\mathbf{p}})\right], \tag{3.14}$$

with $E_{a\,\mathbf{k}} = \sqrt{\mathbf{k}^2 + m_a^2}$. The expression of the Euclidean free Green'sfunction in Eq. (3.13) can be generalised in the presence of the interaction with the single particle free spectral function (FSF) getting replaced by single particle resummed spectral function (RSF) given as,

$$\mathcal{G}_{a,b}^{E}(K) = \delta_{ab}\int_{-\infty}^\infty \frac{dk^0}{\pi}\frac{\rho_a(k^0, \mathbf{k})}{k^0 - i[k_n - i\mu_a]}. \tag{3.15}$$

Here, $\rho_a(k^0, \mathbf{k})$ is the single particle RSF which contains the information of interaction through the self-energy contribution. From this definition, one can invert the relation to obtain [51, 52, 54]

$$\rho_a(\mathcal{K}) = \frac{1}{2i}\mathrm{Disc}\,\mathcal{G}_{a,a}^{E}\left[i(k_n - i\mu) \to k^0 + i0^+, \mathbf{k}\right] \tag{3.16}$$

$$\equiv \frac{1}{2i}\left[\mathcal{G}_{a,a}^{E}(i(k_n - i\mu) \to k^0 + i0^+, \mathbf{k}) - \mathcal{G}_{a,a}^{E}(i(k_n - i\mu) \to k^0 - i0^+, \mathbf{k})\right], \tag{3.17}$$

---

[5]Additional term $e^{-\mu\tau}$ in the Fourier transformation with respect to τ helps to satisfy the KMS relation $\langle\phi_a(t - i\beta, \mathbf{x})\phi^\dagger(0, \mathbf{0})\rangle = e^{-\mu\beta}\langle\phi_b^\dagger(0, \mathbf{0})\phi_a(t, \mathbf{x})\rangle$ at $t = 0$ for commuting bosonic fields.

[6]Fourier transformation of the commutator of $\phi(\mathcal{X})$ and $\phi^\dagger(0)$ is defined as spectral function: $2\rho(\kappa) = \int_{\mathcal{X}} e^{i\kappa\cdot\mathcal{X}}\left[\phi(\mathcal{X}), \phi^\dagger(0)\right]$.

where, Disc denoted discontinuity of the Green's function.

The relation between the spectral function and the two-point correlation function for single fields ϕ_a can be generalized for composite Hermitian bosonic fields. If $\mathcal{O}_1, \mathcal{O}_2$ are two composite bosonic fields which commute with the charge operator, the two-point correlation functions can be written as (See Eq. (A.23))

$$\mathcal{G}^E_{\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2}(K) = \int_0^\beta d\tau \int_{\mathbf{x}} e^{ik_n\tau - i\mathbf{k}\cdot\mathbf{x}} \mathcal{G}^E_{\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2}(X) = \int_{-\infty}^\infty \frac{dk^0}{\pi} \frac{\rho_{\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2}(k^0, \mathbf{k})}{k^0 - ik_n}, \tag{3.18}$$

where, $\mathcal{G}^E_{\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2}(X) = \langle \mathcal{O}_1(X)\mathcal{O}_2^\dagger(0)\rangle_E$ is the Euclidean coordinate correlation function for the set of composite hermitian operators $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$. The difference between the Fourier representations in Eqs. (3.15) and (3.18) may be noted. The reason for this difference lies in the fact that the single fields do not commute with the total charge operator. This is shown explicitly in Appendix-A.1. Further, similar to Eq. (3.17), the above equation can be inverted to obtain the relation for the composite operators as

$$\rho_{\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2}(\mathcal{K}) = \frac{1}{2i}\text{Disc}\,\mathcal{G}^E_{\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2}(ik_n \to k^0 + i0^+, \mathbf{k}). \tag{3.19}$$

We shall next use this expression to calculate the diffusion coefficient κ_{ab} as given in Eq. (2.46). The diffusion current-current correlation can be computed by plugging this set of two operators $\hat{\mathcal{O}}_1 = \hat{\mathcal{J}}_a^\mu$, $\hat{\mathcal{O}}_2 = \hat{\mathcal{J}}_{b\,\mu}$, into Eq. (3.19).

Now, we can compute the thermodynamic dissipative coefficients with the aid of the Kubo formulas given in Sec-(2). As a toy model, we can take a set of two fields, namely ϕ and ξ with conserved currents under $U_\phi(1) \times U_\xi(1)$ as

$$\hat{J}_a^\mu = -i\left(\partial^\mu\phi_a^\dagger\phi_a - \phi_a^\dagger\partial^\mu\phi_a\right), \tag{3.20}$$

while the conserved charge currents are

$$\hat{J}_A^\mu = \sum_a q_{aA}\hat{J}_a^\mu, \tag{3.21}$$

and the energy-momentum tensor is

$$\hat{T}^{\mu\nu} = \sum_a \partial^\mu\phi_a^\dagger\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L}(x). \tag{3.22}$$

Using Eq. (2.18), one can write down the dissipative heat current and the dissipative number current operators $\hat{\mathcal{Q}}_\mu$ and $\hat{\mathcal{N}}_\mu$ as

$$\hat{\mathcal{Q}}_\mu = u_\alpha\Delta_{\mu\beta}\hat{T}^{\alpha\,\beta}, \tag{3.23}$$

$$\hat{\mathcal{N}}_{a\,\mu} = \Delta_{\mu\alpha}\hat{J}_a^\alpha, \tag{3.24}$$

where the presence of projector $\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$ makes these operators orthogonal to the fluid velocity u^μ.

It is convenient to calculate the correlation function of interest, like Eq. (2.45) which can be evaluated using the thermal equilibrium Green's function within the imaginary-time

Matsubara technique. In order to do so, one needs to convert corresponding operators into their Euclidean counterparts via Wick rotation. In the fluid rest frame $u_\mu = (1, 0, 0, 0)$, $\Delta^{\mu\nu} = \text{diag}(0, -1, -1, -1)$, performing a Wick rotation $t \to -i\tau$, the current operators become

$$\hat{\mathcal{Q}}_i^E = \hat{T}_{\tau i} = i \sum_a \left(\partial_\tau \phi_a^\dagger \partial_i \phi_a + \partial_i \phi_a^\dagger \partial_\tau \phi_a \right), \tag{3.25}$$

$$\hat{\mathcal{N}}_{ai}^E = -i \left(\partial_i \phi_a^\dagger \phi_a - \phi_a^\dagger \partial_i \phi_a \right). \tag{3.26}$$

Now, to calculate the required retarded Green's function in Eq. (2.45), one can perform an analytic continuation process $ik_n \to k_0 + i\epsilon$ to the thermal Green's function of desired operators $\hat{\mathcal{O}}_1 = \hat{\mathcal{J}}_a^i, \hat{\mathcal{O}}_2 = \hat{\mathcal{J}}_b^j$ as given in Eq. (3.18). Therefore, one can write the Euclidean correlation function $\mathcal{G}_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}^E(X)$ for diffusion currents as

$$
\begin{aligned}
\mathcal{G}_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}^E(X) &= \Delta^{ij} \left\langle \hat{\mathcal{J}}_{ai}(X) \hat{\mathcal{J}}_{bj}(0) \right\rangle_l \\
&= - \left[\left\langle \hat{\mathcal{N}}_{ai}(X) \hat{\mathcal{N}}_{bi}(0) \right\rangle_l - \frac{n_b}{h} \left\langle \hat{\mathcal{N}}_{ai}(X) \hat{\mathcal{Q}}_i(0) \right\rangle_l \right. \\
&\quad \left. - \frac{n_a}{h} \left\langle \hat{\mathcal{Q}}_i(X) \hat{\mathcal{N}}_{bj}(0) \right\rangle_l + \frac{n_a n_b}{h^2} \left\langle \hat{\mathcal{Q}}_i(X) \hat{\mathcal{Q}}_j(0) \right\rangle_l \right],
\end{aligned} \tag{3.27}
$$

which will be used to calculate the different contributions to the spectral function given in Eq. (3.18) and eventually for estimating the diffusion coefficient given in Eq. (2.46).

A few comments here are in order. Firstly, it is important to note that while the expression for the dissipative coefficients is estimated using linear response, i.e., restricted to linear order in the gradient for the off-equilibrium correction, the Green's function that appears in Eq. (3.27) is, in general, non-perturbative and not restricted to weak coupling only. We will try to estimate this correlator for the present Lagrangian. The detailed evaluation of these correlators using thermal field theory is given in Appendix-C. The Fourier-transformed correlation function $\mathcal{G}_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}^E(L)$ of Eq. (3.27) is given by

$$\mathcal{G}_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}^E(L) = \delta_{ab} \mathcal{G}_{\hat{\mathcal{N}}_a \hat{\mathcal{N}}_a}^E(L) - \left(\frac{n_a}{h} \mathcal{G}_{\hat{\mathcal{N}}_b \hat{\mathcal{Q}}}^E(L) + \frac{n_b}{h} \mathcal{G}_{\hat{\mathcal{N}}_a \hat{\mathcal{Q}}}^E(L) \right) + \frac{n_a n_b}{h^2} \mathcal{G}_{\hat{\mathcal{Q}} \hat{\mathcal{Q}}}^E(L). \tag{3.28}$$

The different components of the correlation function on the right hand side are given in Eqs. (C.10), (C.13), (C.14) and (C.15) respectively in Appendix-[C]. Let us note that the retarded Green's function are related to the Euclidean Green's function by analytic continuation, as shown in Appendix-[A.1]. The former Green's functions are used to define the transport coefficients as in Eq. (2.45), which are evaluated using the corresponding spectral function Eq. (2.46).

Further, one needs to take appropriate limits of vanishing of three momentum and of frequency for the spectral function $\rho_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}(\omega, \mathbf{l})$. The spectral function in the limit of

momentum $\mathbf{l} \to 0$ in Eq. (C.20) leads to

$$\rho_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}(\omega,\mathbf{0}) = \delta_{ab}\rho_{\hat{\mathcal{N}}_a\hat{\mathcal{N}}_a}(\omega,\mathbf{0}) - \frac{n_b}{h}\rho_{\hat{\mathcal{N}}_a\hat{\mathcal{Q}}}(\omega,\mathbf{0}) - \frac{n_a}{h}\rho_{\hat{\mathcal{N}}_b\hat{\mathcal{Q}}}(\omega,\mathbf{0}) + \frac{n_a n_b}{h^2}\rho_{\hat{\mathcal{Q}}\hat{\mathcal{Q}}}(\omega,\mathbf{0})$$

$$= \int d\mathbf{p} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left[4\delta_{ab}\delta_{ca} - \frac{2}{h}\big(n_a\delta_{cb} + n_b\delta_{ca}\big)(2\omega' + \omega) + \frac{n_a n_b}{h^2}(2\omega' + \omega)^2 \right]$$

$$\times \rho_k(\omega',\mathbf{p})\,\rho_c(\omega' + \omega,\mathbf{p})\Big[f_c(\omega'+\omega) - f_c(\omega')\Big]. \qquad (3.29)$$

Finally, taking the limit $\omega \to 0$ in Eq. (2.46) leads to the diffusion coefficients as [7]

$$\kappa_{ab} = \frac{4}{3}\int d\mathbf{p} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left[\delta_{ab}\delta_{ca} - \frac{\omega'}{h}\big(n_a\delta_{cb} + n_b\delta_{ca}\big) + n_a n_b \Big(\frac{\omega'}{h}\Big)^2 \right]\rho_c(\omega',\mathbf{p})^2$$

$$\times f_c(\omega')\Big[1 + f_c(\omega')\Big]. \qquad (3.30)$$

Hence, the evaluation of the diffusion coefficients reduces to the determination of the one-particle spectral function of the fields.

However, this changes if the spectral function has a finite width. Here, we shall consider the quasi-particle spectral function with a finite width to calculate these diffusion coefficients. The re-summed single-particle spectral function can be calculated from the analytically continued Euclidean propagator's discontinuity as described in Eq. (3.17). In an interacting theory, transport coefficients are finite due to the finite mean free path. This leads to a nonzero broadening of the spectral function around the peak of the quasi-particle excitation. The width of such broadening is inversely proportional to the lifetime of the excitation. In a weakly interacting thermal medium, the spectral function can be approximated by Lorentzian form given as [53]

$$\rho_a(k_0,\mathbf{k}) = \frac{\mathrm{Im}(\Sigma_a(k_0,\mathbf{k}))}{k_0^2 - \mathbf{k}^2 - M_a^2 + |\mathrm{Im}(\Sigma_a(k_0,\mathbf{k}))|^2} \qquad (3.31)$$

The origin of such a form of spectral function can be cast from the re-summed propagator. These resummed propagators carry information of the quasi-particle (of species a) interaction with the medium theough the self-energy function $\Sigma_a = \mathrm{Re}(\Sigma_a(k_0,\mathbf{k})) - i\mathrm{Im}(\Sigma_a(k_0,\mathbf{k}))$. One can show that the real part of the self-energy has the leading order contribution from the tadpole diagram which is momentum independent (See Appendix-[B.1]). This spectral function in the weak coupling limit has a peak at $k_0^2(\mathbf{k}) = \mathbf{k}^2 + m_a^2 + \Sigma_R^a \equiv \mathbf{k}^2 + M_a^2 = E_{a\mathbf{k}}^2$. The width of the spectral function is connected to the imaginary part of the self energy by $\Gamma_{a\mathbf{k}} = \frac{\mathrm{Im}(\Sigma_a(k_0,\mathbf{k}))}{2E_{a\mathbf{k}}}$.

For sufficiently small coupling, the frequency dependence of the self-energy can be neglected over the width of the spectral function. Therefore, the single particle spectral function can be approximated with $|E_{a\mathbf{k}}| \sim \omega$, and we have

$$\rho_a(\omega,\mathbf{p}) \approx \frac{2\omega\Gamma_{ap}}{[\omega^2 - E_p^2]^2 + 4(\omega\Gamma_{ap})^2}. \qquad (3.32)$$

---

[7]To have a non-zero and finite result while taking $\lim_{\omega\to 0}\left(\frac{\rho_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}(\omega,\mathbf{0})}{\omega}\right)$, the numerator should have a term proportion to ω. Such term in this case is $\lim_{\omega\to 0}\left(\frac{f_k(\omega'+\omega)-f_k(\omega')}{\omega}\right) = -\beta f_k(\omega')\big(1 + f_k(\omega')\big)$.

Further in the small width approximation, $\rho_a(\omega, \mathbf{p})$ can be written by a difference of two Lorentzians [50, 52],

$$\rho_a(\omega, \mathbf{p}) \approx \frac{1}{2i} \left[\frac{1}{\left(\omega - i\Gamma_{\mathbf{p}a}\right)^2 - E_{\mathbf{p}a}^2} - \frac{1}{\left(\omega + i\Gamma_{\mathbf{p}a}\right)^2 - E_{\mathbf{p}a}^2} \right] \left[1 + \mathcal{O}\left(\frac{\Gamma_{a\,p}}{E_{a\,p}}\right) \right]$$

$$\equiv \frac{1}{2i} \left[\Delta_{a\,1}(\omega, \mathbf{p}) - \Delta_{a\,2}(\omega, \mathbf{p}) \right]. \tag{3.33}$$

Here, $\Gamma_{\mathbf{p}a}$ is the thermal width, which is the consequence of the scattering with constituents of the thermal bath. The validation of ignoring the $\mathcal{O}\left(\frac{\Gamma_{a\,p}}{E_{a\,p}}\right)$ can be justified at weak coupling limit as to the lowest order, the imaginary part of self-energy Σ_{Ia} arises at the quadratic power of coupling constants. We explicitly compute the self-energy for the Lagrangian (3.1) in the (See Appendix-[B.1]). In passing, we note that the propagators $\Delta_{a\,1}(\omega, \mathbf{p}), \Delta_{a\,2}(\omega, \mathbf{p})$ correspond to the resummed retarded and advanced propagators of Ref [50].

With the form of the single-particle spectral function Eq. (3.33), one can extract the diffusion coefficients using Eq. (3.30). A few comments in this regard are in order. Let us note that in the expression for the diffusion coefficient, Eq. (3.30), the spectral function $\rho_a(k_0, \mathbf{k})$ occurs in quadratic order of the integrand. This will involve squared terms like $\Delta_{a\,1}^2(\omega, \mathbf{p})$, $\Delta_{a\,2}^2(\omega, \mathbf{p})$ and cross-term $\Delta_{a\,1}(\omega, \mathbf{p})\Delta_{a\,2}(\omega, \mathbf{p})$. In each of the squared terms, $\Delta_{a\,1}^2(\omega, \mathbf{p})$, $\Delta_{a\,2}^2(\omega, \mathbf{p})$, the integrand in Eq. (3.30) will have second-order pole on the same side of the real axis, i.e., $\pm E_{a\mathbf{p}} + i\Gamma_{a\mathbf{p}}$, $\pm E_{a\mathbf{p}} - i\Gamma_{a\mathbf{p}}$, respectively. Using the residue theorem for integration over frequency in Eq. (3.30), one can show that the squared terms lead to contributions which are proportional to a polynomial of order $\mathcal{O}\left(\Gamma_{a\,p}/E_{a\,p}\right)^k$ with $k \geq 1$. On the other hand, the dominant contribution of the mixed term $\Delta_{a\,1}(\omega, \mathbf{p})\Delta_{a\,2}(\omega, \mathbf{p})$ lead to a contribution which is of the order $\mathcal{O}\left(1/\Gamma_{a\,p}E_{a\,p}\right)$ and other contribution is suppressed by $\mathcal{O}\left(\Gamma_{a\,p}/E_{a\,p}\right)^k$ with $k \geq 0$. Therefore, the cross terms give a dominant contribution in the small width approximation, which corresponds to the *pinching pole* approximation of Ref. [50]. Hence, by plugging the spectral function of Eq. (3.33) into Eq. (3.30) and performing the integration over ω' by closing the contour in the upper half plane, we have

$$\kappa_{ab} = \frac{1}{6} \int d\mathbf{p} \sum_c \frac{\mathbf{p}^2}{E_{c\mathbf{p}}^2 \Gamma_{c\mathbf{p}}} \left[\delta_{ab}\delta_{ca} - \theta_c \frac{E_{c\mathbf{p}}}{h} \left(n_a \delta_{cb} + n_b \delta_{ca} \right) + n_a n_b \left(\frac{\theta_c E_{c\mathbf{p}}}{h} \right)^2 \right]$$

$$\times \left[f_c(E_{c\mathbf{p}} - \theta_c \mu_c) \left(1 + f_c(E_{c\mathbf{p}} - \theta_c \mu_c) \right) \right]. \tag{3.34}$$

Here $\theta_k = \pm 1$ for particles and antiparticles, respectively, for each given species 'c'. The method of this calculation within the pinching pole approximation is given in Appendix-[D], explicitly using Eq. (D.2).

The different components of the diffusion matrix in the species basis are explicitly given

by

$$\kappa_{\phi\phi} = \frac{1}{3} \int d\mathbf{p}\,\mathbf{p}^2 \left(\frac{1}{2\Gamma_{\phi\mathbf{p}} E_{\phi\mathbf{p}}^2} \left[\left(1 - \frac{n_\phi E_{\phi\mathbf{p}}}{h}\right)^2 f_\phi(E_{\phi\mathbf{p}})\left(1 + f_\phi(E_{\phi\mathbf{p}})\right) + \left(1 + \frac{n_\phi E_{\phi\mathbf{p}}}{h}\right)^2 \right. \right.$$
$$\left. \times \bar{f}_\phi(E_{\phi\mathbf{p}})\left(1 + \bar{f}_\phi(E_{\phi\mathbf{p}})\right) \right] + \frac{1}{2\Gamma_{\xi\mathbf{p}} E_{\xi\mathbf{p}}^2} \left(\frac{n_\xi E_{\xi\mathbf{p}}}{h}\right)^2$$
$$\left. \times \left[f_\xi(E_{\xi\mathbf{p}})\left(1 + f_\xi(E_{\xi\mathbf{p}})\right) + \bar{f}_\xi(E_{\xi\mathbf{p}})\left(1 + \bar{f}_\xi(E_{\xi\mathbf{p}})\right) \right] \right), \qquad (3.35)$$

$$\kappa_{\xi\xi} = \frac{1}{3} \int d\mathbf{p}\,\mathbf{p}^2 \left(\frac{1}{2\Gamma_{\xi\mathbf{p}} E_{\xi\mathbf{p}}^2} \left[\left(1 - \frac{n_\xi E_{\xi\mathbf{p}}}{h}\right)^2 f_\xi(E_{\xi\mathbf{p}})\left(1 + f_\xi(E_{\xi\mathbf{p}})\right) + \left(1 + \frac{n_\xi E_{\xi\mathbf{p}}}{h}\right)^2 \right. \right.$$
$$\left. \times \bar{f}_\xi(E_{\xi\mathbf{p}})\left(1 + \bar{f}_\xi(E_{\xi\mathbf{p}})\right) \right] + \frac{1}{2\Gamma_{\phi\mathbf{p}} E_{\phi\mathbf{p}}^2} \left(\frac{n_\phi E_{\phi\mathbf{p}}}{h}\right)^2$$
$$\left. \times \left[f_\phi(E_{\phi\mathbf{p}})\left(1 + f_\phi(E_{\phi\mathbf{p}})\right) + \bar{f}_\phi(E_{\phi\mathbf{p}})\left(1 + \bar{f}_\phi(E_{\phi\mathbf{p}})\right) \right] \right). \qquad (3.36)$$

The symmetric off-diagonal elements are given by

$$\kappa_{\phi\xi} = \kappa_{\xi\phi} = \frac{1}{3} \int d\mathbf{p}\,\mathbf{p}^2 \left(\frac{1}{2\Gamma_{\phi\mathbf{p}} E_{\phi\mathbf{p}}^2} \left[\frac{n_\xi E_{\phi\mathbf{p}}}{h}\left(\frac{n_\phi E_{\phi\mathbf{p}}}{h} - 1\right) f_\phi(E_{\phi\mathbf{p}})\left(1 + f_\phi(E_{\phi\mathbf{p}})\right) \right. \right.$$
$$\left. + \frac{n_\xi E_{\phi\mathbf{p}}}{h}\left(\frac{n_\phi E_{\phi\mathbf{p}}}{h} + 1\right) \bar{f}_\phi(E_{\phi\mathbf{p}})\left(1 + \bar{f}_\phi(E_{\phi\mathbf{p}})\right) \right] + \frac{1}{2\Gamma_{\xi\mathbf{p}} E_{\xi\mathbf{p}}^2} \left[\frac{n_\phi E_{\xi\mathbf{p}}}{h}\left(\frac{n_\xi E_{\xi\mathbf{p}}}{h} - 1\right) \right.$$
$$\left. \left. f_\xi(E_{\xi\mathbf{p}})\left(1 + f_\xi(E_{\xi\mathbf{p}})\right) + \frac{n_\phi E_{\xi\mathbf{p}}}{h}\left(\frac{n_\phi E_{\xi\mathbf{p}}}{h} + 1\right) \bar{f}_\xi(E_{\xi\mathbf{p}})\left(1 + \bar{f}_\xi(E_{\xi\mathbf{p}})\right) \right] \right). \qquad (3.37)$$

In the above, $f_a(\omega) = (e^{\beta(\omega - \mu_a)} - 1)^{-1}$ and $\bar{f}_a(\omega) = (e^{\beta(\omega + \mu_a)} - 1)^{-1}$ correspond to the particle and anti-particle distribution respectively. We might note here that we also have non-diagonal coefficients in the particle basis. Such a mixing term has its origin from the heat current $\mathcal{Q}^i$ contributing to the diffusion current as in Eq. (2.34) and Eq. (3.23). Both species contribute to the heat flow within the single fluid frame.

To translate the above results for elements of the diffusion matrix into the charge basis, we employ the transformation matrix as given in Eq. (2.44) to obtain

$$\kappa_{AB} = \frac{1}{3} \int d\mathbf{p}\,|\mathbf{p}|^2 \sum_a \left(\frac{1}{2\Gamma_{a\mathbf{p}} E_{a\mathbf{p}}^2} \left[\left(q_{aA} - \frac{n_A E_{a\mathbf{p}}}{h}\right)\left(q_{aB} - \frac{n_B E_{a\mathbf{p}}}{h}\right) f_a(E_{a\mathbf{p}})\left(1 + f_a(E_{a\mathbf{p}})\right) \right] \right.$$
$$\left. + \frac{1}{2\Gamma_{a\mathbf{p}} E_{a\mathbf{p}}^2} \left[\left(q_{aA} + \frac{n_A E_{a\mathbf{p}}}{h}\right)\left(q_{aB} + \frac{n_B E_{a\mathbf{p}}}{h}\right) \bar{f}_a(E_{a\mathbf{p}})\left(1 + \bar{f}_a(E_{a\mathbf{p}})\right) \right] \right). \qquad (3.38)$$

It is interesting to see that in this weak coupling limit, the charge diffusion coefficients are identically the same as derived in Ref. [15] using the kinetic theory approach with the identification of relaxation time $\tau_k = 1/2\Gamma_k$ and including Bose enhancement factor. Such

an expression for diffusion coefficients has the desirable features that κ_{AB} is symmetric w.r.t. $A \leftrightarrow B$ consistent with the Onsangar reciprocal relation. Furthermore, κ_{AA} is manifestly positive and definite. For the sake of completeness, we have also evaluated the thermal width Γ_k to the lowest order, which is explicitly shown in Appendix-[B.2].

4 Summary and outlook

In this paper, we have presented the derivation of the diffusion matrix elements in terms of Kubo relations. For the same, we have used Zubarev's non-equilibrium statistical operator method to derive the Kubo formula for the diffusion coefficients for systems with multiple conserved charges, for e.g., baryon number, electric charge and strangeness in the context of heavy ion collisions. The diffusion matrix elements are given by the zero momentum spectral functions of the diffusive currents in the small frequency limit as in Eq. (2.38) and Eq. (2.45). Such expressions are rather general and can be used for estimating the diffusion coefficients using nonperturbative methods like lattice QCD or using any effective models of strong interactions like Nambu–Jona-Lasinio (NJL) model [55], quasi particle models [14, 56, 57], or sigma models coupled to quarks [58, 59]. This can also be written down in terms of the single particle spectral functions as in Eq. (3.30) along with other thermodynamic quantities like enthlapy density and number densities of the thermodynamic equilibrium systems and can be evaluated within any effective model as above.

As an explicit example, we have applied this correlator method to obtain the charge diffusion coefficients in a toy model of two coupled charged scalar fields having two charges that are conserved. It may be mentioned here that, for scalar fields other transport coefficients like the viscosity coefficients and electrical/thermal conductivity has been derived using Kubo relations while diffusion has not been considered. We have also presented here, in some details, the calculations for the the lowest order thermal decay width that enters in the Kubo formula. Such a width that arises from the sunset diagram in the lowest order had earlier been considered [60] but with vanishing chemical potential. The expressions given here for the diffusion coefficient can be applicable to other models involving scalar fields with multiple conserved charges for hot and dense matter both in QGP as well as in nuclear astrophysics and cosmology [61]. The present work is especially relevant in discussions concerning the physics of compressed baryonic matter at forthcoming FAIR and NICA facilities, or for interpreting recent findings from the isobar run at RHIC.

In the context of neutron stars, it will be interesting to investigate the effects of diffusion on the oscillation of neutron stars as a result of an internal instability or an external perturbation. It may be noted that whether an oscillation mode is excited or not depends upon the interplay of the excitation rate and how efficiently the dissipative mechanisms in the stellar matter counteract this excitation. Much attention has been given on the viscous effects [62, 63] while the effects of diffusion on neutron star oscillations has only been looked into recently for the neutron proton electron matter [64]. In this context, for neutron stars with a hyperon core [65] or a quark matter core [66] the effect of diffusion of particles will be relevant and important to give insight into the structure of stellar matter. Moreover, a

reaction of three distinct species of particles can be viewed as a bulk viscous fluid, which may have important consequences for neutron star physics [67].

It will be interesting to apply this method for diffusion within a three flavor NJL model or different quasi-particle models to discuss the importance of the off-diagonal elements of the diffusion matrix. It will also be interesting to examine, through diagonalisation of the diffusion matrix, which combination of hadrons will diffuse faster giving an insight to the diffusive process that will be relevant for low energy heavy-ion collisions. Further, in the context of off central heavy-ion collisions, where strong electromagnetic fields can be produced [68], the effect of strong electromagnetic field in the diffusion of different charges could also be interesting and relevant. Some of these problems will be investigated in future works.

A   Appendix-A

A.1   Retarded Green's function and spectral some relations.

Our interest here is to estimate the two-point correlation functions of elementary and composite operators to calculate the response function at linear order in the thermal force as in Eq. (2.37) or Eqs. (2.48) and (2.49). This expression is written regarding KMB inner product as in Eq. (2.17). We shall show in the following subsection that KMB inner products are related to the retarded Green's functions and spectral function. Here we shall discuss some useful relations involving this Green's functions.

The retarded Green's function up to arbitrary bosonic operators $\{\hat{\mathcal{O}}_a, \hat{\mathcal{O}}_b^\dagger\}$ in Minkowski space $\mathcal{X} \equiv (\mathbf{x}, t)$ is defined as [42, 51]

$$G_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}^R(\mathcal{X}) = i \left\langle \theta(t) \left[\hat{\mathcal{O}}_a(\mathbf{x}, t), \hat{\mathcal{O}}_b^\dagger(\mathbf{0}, 0) \right] \right\rangle_l. \tag{A.1}$$

In addition, we define Green's function called the spectral function, which is essentially the Fourier transform of the commutator of these operators as

$$\rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathcal{K}) = \frac{1}{2} \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \left\langle \left[\hat{\mathcal{O}}_a(\mathcal{X}), \hat{\mathcal{O}}_b^\dagger(0) \right] \right\rangle_l, \tag{A.2}$$

where $\mathcal{K} = (k^0, \mathbf{k})$. Using the integral representation of the theta function,

$$\theta(t) = i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i0^+}, \tag{A.3}$$

one can show that these two Green's functions are related as

$$G_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}^R(\mathcal{K}) = -i \int_0^\infty dt \int d^3\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \left\langle \theta(t) \left[\hat{\mathcal{O}}_a(\mathbf{x}, t), \hat{\mathcal{O}}_b^\dagger(\mathbf{0}, 0) \right] \right\rangle_l \tag{A.4}$$

$$= \int_{-\infty}^{\infty} \frac{dp^0}{\pi} \frac{\rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathcal{P})}{p^0 - k^0 - i0^+}. \tag{A.5}$$

Therefore, we see that the retarded Green's function $G^R(\mathcal{K})$ can be obtained from the spectral function $\rho(\mathcal{P})$ through analytic continuation in the upper half plane of p^0.

Furthermore, if the operators $\hat{\mathcal{O}}_a$ and $\hat{\mathcal{O}}_b$ are hermitian, one can show that $G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{X})$ is real. Moreover, for the Hermitian operators, the retarded correlation function in the momentum space satisfies the relations

$$\text{Re}\left(G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})\right) = \text{Re}\left(G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(-\mathcal{K})\right), \quad \text{Im}\left(G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})\right) = -\text{Im}\left(G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(-\mathcal{K})\right). \quad (A.6)$$

Similarly, for the spectral function, one can show using the definition Eq. (A.2) that $\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})$ is an odd function of $\mathcal{K}$

$$\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K}) = -\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(-\mathcal{K}). \quad (A.7)$$

Further, if one assumes the spectral function to be real, one can identify the imaginary part of the retarded Green's function as [8]

$$\text{Im}\left(G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})\right) = \rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K}). \quad (A.8)$$

The above equation relates the imaginary part of the retarded Green's function to the spectral function. With the help of Eq. (A.7), one can show using L'Hôpital's rule that

$$\lim_{\omega\to 0}\frac{\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})}{\omega} = \frac{\partial}{\partial\omega}\text{Im}\left(G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})\right)\bigg|_{\omega\to 0}. \quad (A.9)$$

The above relation helps us to express Eq. (2.45) in the form of Eq. (2.46).

Next, we discuss some other relations in the presence of finite chemical potentials in the context of Kubo-Martin-Schwinger (KMS) relations while considering the thermal averages of operators. The commutator of the operators corresponding to the total number operator $\hat{Q}_a$ (spatially integrated number density operator) is given by

$$\left[\hat{\mathcal{O}}_a(\mathcal{X}), \hat{Q}_b\right] = \delta_{ab}\hat{\mathcal{O}}_a(\mathcal{X}). \quad (A.10)$$

The KMS relation for such an operator can be written as [51]

$$\left\langle\hat{\mathcal{O}}_a(t-i\beta,\mathbf{x})\hat{\mathcal{O}}^\dagger_b(0,\mathbf{0})\right\rangle_l = e^{-\beta\mu_a}\left\langle\hat{\mathcal{O}}^\dagger_b(0,\mathbf{0})\hat{\mathcal{O}}_a(t,\mathbf{x})\right\rangle_l, \quad (A.11)$$

where, Eq. (A.10) has been used. Such relations are valid for operators which are odd in powers of fields ϕ_a. In particular, for equal time, Eq. (A.11) reduce to

$$\left\langle\hat{\mathcal{O}}_a(-i\beta,\mathbf{x})\hat{\mathcal{O}}_b(0,\mathbf{0})\right\rangle_l = e^{-\beta\mu_a}\left\langle\hat{\mathcal{O}}_a(0,\mathbf{x})\hat{\mathcal{O}}_b(0,\mathbf{0})\right\rangle_l \quad (A.12)$$

The exponential factor $e^{-\beta\mu_a}$ thus spoils the periodicity of this correlation function. The corresponding Euclidean correlator is defined as

$$\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(K) = \int_0^\beta d\tau \int_{\mathbf{x}} e^{(ik_n+\mu_a)\tau - i\mathbf{k}\cdot\mathbf{x}}\left\langle\hat{\mathcal{O}}_a(X)\hat{\mathcal{O}}_b(0)\right\rangle_l. \quad (A.13)$$

---

[8]This identification can be made by making use of this mnemonic expression $\frac{1}{\Delta\pm i0^+} = P(\frac{1}{\Delta}) \mp i\pi\delta(\Delta)$ and where P denotes the principle value of the integral.

The additional term in the Fourier transform with respect to τ cancels the multiplicative factor $e^{-\beta\mu_a}$ at $\tau = \beta$ so that the τ integrand remains periodic, leading to the bosonic Matsubara frequencies k_n.

Further, for the operators which commute with the total number operator, i.e.,

$$\left[\hat{\mathcal{O}}_a(\mathcal{X}), \hat{Q}_b\right] = 0 \,, \tag{A.14}$$

the KMS relation is given by

$$\left\langle \hat{\mathcal{O}}_a(t - i\beta, \mathbf{x})\hat{\mathcal{O}}_b(0, \mathbf{0})\right\rangle_l = \left\langle \hat{\mathcal{O}}_b(0, \mathbf{0})\hat{\mathcal{O}}_a(t, \mathbf{x})\right\rangle_l \,, \tag{A.15}$$

with no extra chemical potential dependent suppression factor, which may be noted. In particular, for equal time, Eq. (A.15) reduce to

$$\left\langle \hat{\mathcal{O}}_a(-i\beta, \mathbf{x})\hat{\mathcal{O}}_b(0, \mathbf{0})\right\rangle_l = \left\langle \hat{\mathcal{O}}_a(0, \mathbf{x})\hat{\mathcal{O}}_b(0, \mathbf{0})\right\rangle_l \,, \tag{A.16}$$

which is a periodic function with respect to imaginary time. Therefore, in contrast to Eq. (A.13), the Fourier transformation of the Euclidean two-point function can be defined as

$$\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(K) = \int_X e^{iK\cdot X}\left\langle \hat{\mathcal{O}}_a(X)\hat{\mathcal{O}}_b^\dagger(0)\right\rangle_l \,. \tag{A.17}$$

Here $K = (k_n, \mathbf{k})$ is the Euclidean momentum and this correlation function is time-ordered by definition ($0 \le \tau \le \beta$). Since Green's function $\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(X)$ is periodic, we conclude that k_n is a bosonic Matsubara frequency.

Similar to the retarded Green's function $G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})$, the spectral function $\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})$ also has different momentum space representations for the two cases. To evaluate the spectral function representation of the Euclidean Green's function, one needs to make a connection between the Euclidean Green's function $\langle\hat{\mathcal{O}}_a(X)\hat{\mathcal{O}}_b^\dagger(0)\rangle_l$ and the Wightman Green's function $\langle\hat{\mathcal{O}}_a(\mathcal{X})\hat{\mathcal{O}}_b^\dagger(0)\rangle_l$ which are functions of Minkowski coordinates and Euclidean coordinates, respectively. This is possible by performing the analytic continuation $t \to -i\tau$, with $0 \le \tau \le \beta$, on the Wightman function $\langle\hat{\mathcal{O}}_a(\mathcal{X})\hat{\mathcal{O}}_b^\dagger(0)\rangle_l = \int_{\mathcal{K}} e^{i\mathcal{K}\cdot\mathcal{X}}G_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})$. Therefore, the Matsubara Fourier transformed Euclidean Green's function, as given in Eq. (A.13), can be expressed as

$$\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(K) = \int_0^\beta d\tau \int_{\mathbf{x}} e^{(ik_n + \mu_a)\tau - i\mathbf{k}\cdot\mathbf{x}}\left[\int_{\mathcal{P}} e^{-\mathcal{P}\cdot\mathcal{X}}G_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{P})\right]_{it\to\tau} \,. \tag{A.18}$$

Here, the exponential factor $e^{\mu_a\tau}$ appears for those non-commuting operators as given in Eq. (A.10). To explore the relation between Euclidean Green's function and the spectral function, one needs to employ the spectral function representation of the Wightman function as [51]

$$G_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K}) = 2\Big(1 + f_a(k^0 - \mu_a)\Big)\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K}) \,. \tag{A.19}$$

Using this equation into the Eq. (A.18), one can show that

$$
\begin{aligned}
\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(K) &= \int_0^\beta d\tau\, e^{(ik_n+\mu)\tau} \int_{-\infty}^\infty \frac{dp^0}{2\pi} e^{-p^0\tau} 2\Big(1 + f_a(p^0 - \mu_a)\Big)\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(p^0,\mathbf{k}) \\
&= \int_{-\infty}^\infty \frac{dp^0}{2\pi} 2\Big(1 + f_a(p^0 - \mu_a)\Big)\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(p^0,\mathbf{k}) \int_0^\beta d\tau\, e^{(ik_n+\mu-p^0)\tau} \\
&= \int_{-\infty}^\infty \frac{dp^0}{2\pi} 2\Big(1 + f_a(p^0 - \mu_a)\Big)\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(p^0,\mathbf{k})\left[\frac{e^{(ik_n+\mu-p^0)\tau}}{ik_n+\mu-p^0}\right]_0^\beta \\
&= \int_{-\infty}^\infty \frac{dp^0}{2\pi} 2\Big(1 + f_a(p^0 - \mu_a)\Big)\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(p^0,\mathbf{k})\left[\frac{e^{(\mu-p^0)\beta}-1}{ik_n+\mu-p^0}\right] \\
&= \int_{-\infty}^\infty \frac{dp^0}{\pi} \rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(p^0,\mathbf{k})\left[\frac{e^{(p^0-\mu)\beta}}{e^{(p^0-\mu)\beta}-1}\right]\left[\frac{e^{(\mu-p^0)\beta}-1}{ik_n+\mu-p^0}\right] \\
&= \int_{-\infty}^\infty \frac{dk^0}{\pi}\frac{\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(k^0,\mathbf{k})}{k^0 - i[k_n - i\mu]}\,.
\end{aligned}
\tag{A.20}
$$

One can invert the above relation by using the Sokhotski-Plemelj theorem as

$$
\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K}) = \frac{1}{2i}\mathrm{Disc}\,\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(k_n - i\mu \to -ik^0, \mathbf{k}),
\tag{A.21}
$$

where the discontinuity is defined in Eq. (3.17).

Similar to Eq. (A.20), one can also derive the relation between the Euclidean Green's function and spectral function for those bosonic operators, which satisfies the the commutation relation with the total charge operator as given in Eq. (A.14). To do so, one needs to employ a similar relation between the Wightman function and the spectral function as

$$
G_{\mathcal{O}_a\mathcal{O}_b}(\mathcal{K}) = 2\Big(1 + f_B(k^0)\Big)\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K})\,,
\tag{A.22}
$$

where $f_B(k^0) = \frac{1}{\exp(\beta k^0)-1}$ is the bosonic distribution which does not carry the chemical potential, as opposed to Eq. (A.19). Therefore, one can derive the relation between the Euclidean Green's function to the spectral function as follows

$$
\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(K) = \int_{-\infty}^\infty \frac{dk^0}{\pi}\frac{\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(k^0,\mathbf{k})}{k^0 - ik_n}\,.
\tag{A.23}
$$

Here, it is important to note that the absence of the chemical potential in the denominator gives a contrasting representation compared to Eq. (A.20). Moreover, the above relation can be inverted to obtain

$$
\rho_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(\mathcal{K}) = \frac{1}{2i}\mathrm{Disc}\,\mathcal{G}^E_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}(k_n \to -ik^0, \mathbf{k}).
\tag{A.24}
$$

A comparison to the description in Section-(3) can be made if one identifies ϕ and ξ as those operators which satisfy Eq. (A.10), that reflects in the spectral representation of the Euclidean Green's function Eq. (3.15) with a resemblance to Eq. (A.20). Similarly, recognising the operators like particle (charge) diffusion current, $\mathcal{J}^\mu_a$ ($\mathcal{J}^\mu_A$), as those operators which commute with the total charge operator, as given in Eq. (A.14), leads to the resemblance between Eq. (3.18) and Eq. (A.23).

A.2 The KMB inner product and the Retarded Green's function.

In this appendix, we show the relation between the first-order transport coefficients written in terms of KMB inner product, e.g. in Eq. (2.43), and retarded Green's function which is defined in Sec-(2). A similar calculation can also be found in Ref. [38, 44]. The KMB inner product is given by

$$\left(\hat{\mathcal{O}}_a(\mathbf{x},t), \hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1\right)\right) = \frac{1}{\beta}\int_0^\beta d\tau \left\langle \hat{\mathcal{O}}_a(\mathbf{x},t)\left[\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1+i\tau) - \left\langle\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1+i\tau)\right\rangle_l\right]\right\rangle_l .$$
(A.25)

Here, $\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1+i\tau) = e^{-\tau\hat{H}}\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1)e^{\tau\hat{H}}$, with $\hat{H} = u_\mu\hat{P}^\mu - \sum_a \mu_a\hat{n}_a$ being the Hamiltonian.

To extract the transport coefficients from the definition, for e.g. Eq. (2.16), the values of these transport coefficients are computed at the global equilibrium instead of the local equilibrium. Therefore, this global equilibrium distribution is defined with average thermodynamic parameters, like temperature $T = \beta^{-1}$ and the chemical potentials μ_a (μ_A) in particle (charge) basis. In equilibrium, with proper definition of thermodynamic parameters, one can define the global grand-canonical statistical operator $\hat{\varrho}_{eq}$ in fluid rest frame with the time-independent Hamiltonian $\hat{H} = u_\mu\hat{P}^\mu - \sum_a \mu_a\hat{n}_a$ in particle basis or equivalently $\hat{H} = u_\mu\hat{P}^\mu - \sum_A \mu_A\hat{n}_A$ in charge basis, with $\hat{P}^\mu$ being the operator for 4-momentum. A comparison with Eq. (2.8) leads to the recognition that $\hat{\mathcal{A}} = \beta\hat{H}$. Therefore one can re-parameterize Eqs. (2.12) and (2.17) by $\tau = \beta\lambda$ to obtain

$$\left(\hat{\mathcal{O}}_a(\mathbf{x},t), \hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1\right)\right) = \frac{1}{\beta}\int_0^\beta d\tau \left\langle \hat{\mathcal{O}}_a(\mathbf{x},t)\left[\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1+i\tau) - \left\langle\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1+i\tau)\right\rangle_l\right]\right\rangle_l .$$
(A.26)

Here, $\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1+i\tau) = e^{-\tau\hat{H}}\hat{\mathcal{O}}_b(\mathbf{x}_1,t_1)e^{\tau\hat{H}}$, which can be interpreted as the operator $\hat{\mathcal{O}}_b(\mathbf{x},t)$ evolving in Heisenberg picture under a time displacement $\delta t = i\tau$. Further, by performing analytic continuation, one can arrive at the relations

$$\langle\hat{\mathcal{O}}_b(\mathbf{x},t+i\tau)\rangle_l = \langle\hat{\mathcal{O}}_b(\mathbf{x},t)\rangle_l,$$
(A.27)

$$\langle\hat{\mathcal{O}}_a(\mathbf{x},t)\hat{\mathcal{O}}_b(\mathbf{x}',t'+i\beta)\rangle_l = \langle\hat{\mathcal{O}}_b(\mathbf{x}',t')\hat{\mathcal{O}}_a(\mathbf{x},t)\rangle_l.$$
(A.28)

The last relation is known as Kubo-Martin-Schwinger (KMS) relation.

If we consider that the correlation vanishes in the limit $t' \to -\infty$, i.e., [38, 44]

$$\lim_{t'\to-\infty}\langle\hat{\mathcal{O}}_a(\mathbf{x},t)\hat{\mathcal{O}}_b(\mathbf{x}',t'+i\tau)\rangle_l = \lim_{t'\to-\infty}\langle\hat{\mathcal{O}}_a(\mathbf{x},t)\rangle_l\langle\hat{\mathcal{O}}_b(\mathbf{x}',t'+i\tau)\rangle_l,$$
(A.29)

then we can rewrite the Eq. (A.26) by using Eqs. (A.27), (A.28) and (A.29) as

$$
\begin{aligned}
\left(\hat{\mathcal{O}}_a(\mathbf{x},t),\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1\right)\right) = & \frac{1}{\beta}\int_0^\beta d\tau\Bigg[\left\langle\hat{\mathcal{O}}_a(\mathbf{x},t)\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1+i\tau\right)\right\rangle_l \\
& \qquad\qquad - \left\langle\hat{\mathcal{O}}_a(\mathbf{x},t)\right\rangle_l\left\langle\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1+i\tau\right)\right\rangle_l\Bigg] \\
= & \frac{1}{\beta}\int_0^\beta d\tau\int_{-\infty}^{t_1}dt'\frac{d}{dt'}\Bigg[\left\langle\hat{\mathcal{O}}_a(\mathbf{x},t)\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1+i\tau\right)\right\rangle_l \\
& \qquad\qquad - \left\langle\hat{\mathcal{O}}_a(\mathbf{x},t)\right\rangle_l\left\langle\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1+i\tau\right)\right\rangle_l\Bigg].
\end{aligned}
\tag{A.30}
$$

Here, one needs to use the Heisenberg equation of motion of an operator $\hat{\mathcal{O}}(\mathbf{x},t)$.

The Heisenberg equation of motion can be derived if it is evolved using $\hat{\mathcal{O}}_b(\mathbf{x},t+i\tau)=e^{-\tau\hat{H}}\hat{\mathcal{O}}_b(\mathbf{x},t)e^{\tau\hat{H}}$, leading to

$$
\frac{d}{dt}\hat{\mathcal{O}}(\mathbf{x},t+i\tau) = -i\frac{d}{d\tau}\hat{\mathcal{O}}(\mathbf{x},t+i\tau).
\tag{A.31}
$$

Using the above relation, along with Eqs. (A.27) and (A.28), one can turn the KMB inner product of two operators, Eq. (A.26), into a commutator of two operators

$$
\left(\hat{\mathcal{O}}_a(\mathbf{x},t),\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1\right)\right) = \frac{i}{\beta}\int_{-\infty}^{t_1}dt'\left\langle\left[\hat{\mathcal{O}}_a(\mathbf{x},t),\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t'\right)\right]\right\rangle_l
\tag{A.32}
$$

In order to take into account the causality constraint, $t_l\leq t_1\leq t$, one needs to write the last equation in terms of retarded Green's function as,

$$
\left(\hat{\mathcal{O}}_a(\mathbf{x},t),\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t_1\right)\right) = -\frac{1}{\beta}\int_{-\infty}^{t_1}dt'G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}\left(\mathbf{x}-\mathbf{x}_1,t-t'\right),
\tag{A.33}
$$

where two-point Retarded Green's function for uniform medium can be written as

$$
G^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}\left(\mathbf{x}-\mathbf{x}_1,t-t'\right) = -i\theta(t-t')\left\langle\left[\hat{\mathcal{O}}_a(\mathbf{x},t),\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t'\right)\right]\right\rangle_l.
\tag{A.34}
$$

In order to calculate the transport coefficients, one needs to compute the integral such as those appearing in Eq. (2.16). In the following, similar integrals are computed with a set of operators $\{\hat{\mathcal{O}}_a,\hat{\mathcal{O}}_b\}$

$$
\mathcal{I}\left[\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b\right] = \lim_{\epsilon\to 0}\int d^3\mathbf{x}_1\int_{-\infty}^t dt'\,e^{-\epsilon(t-t')}\left(\hat{\mathcal{O}}_a(\mathbf{x},t),\hat{\mathcal{O}}_b\left(\mathbf{x}_1,t'\right)\right),
\tag{A.35}
$$

Now, using Eq. (A.33), one can compute the integral

$$
\mathcal{I}\left[\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b\right] = -\lim_{\epsilon\to 0}\int d^3\mathbf{x}'\int_{-\infty}^t dt'e^{\epsilon(t'-t)}\frac{1}{\beta}\int_{-\infty}^{t'}dsG^R_{\hat{\mathcal{O}}_a\hat{\mathcal{O}}_b}\left(\mathbf{x}-\mathbf{x}',t-s\right).
\tag{A.36}
$$

This can be conveniently done by taking the Fourier transformation $\tilde{G}^R(\omega, \mathbf{k})$ of the retarded Green's function $G^R(\mathbf{x}, t)$, leading to

$$\mathcal{I}\left[\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b\right] = -\lim_{\epsilon \to 0} T \int \frac{d\omega}{2\pi} \frac{1}{i\omega(i\omega + \epsilon)} \lim_{\mathbf{k} \to 0} G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathbf{k}, \omega). \tag{A.37}$$

The right-hand side can be rewritten using partial fraction of the integrand and using the residue theorem. This leads to

$$\mathcal{I}\left[\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b\right] = -\lim_{\epsilon \to 0} \frac{T}{\epsilon} \lim_{\omega \to 0} \lim_{\mathbf{k} \to 0} \left[G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathbf{k}, \omega) - G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathbf{k}, \omega + i\epsilon)\right]. \tag{A.38}$$

Eventually, one needs to take the limit $\epsilon \to 0$ of the integral $\mathcal{I}\left[\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b\right]$, as mentioned in the definition of relevant statistical operator, Eq. (2.2), leading to

$$\mathcal{I}\left[\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b\right] = iT \lim_{\omega \to 0} \partial_\omega G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathbf{0}, \omega). \tag{A.39}$$

Let us consider $G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{k})$ as given in Eq. (A.4). We have,

$$G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{k}) = -i \int_0^\infty dt \int d^3\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \left\langle \theta(t)\left[\hat{\mathcal{O}}_a(\mathbf{x}, t), \hat{\mathcal{O}}_b(\mathbf{0}, 0)\right] \right\rangle_l. \tag{A.40}$$

It is clear from the above equation that

$$\left(G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathbf{0}, \omega)\right)^* = G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\mathbf{0}, -\omega). \tag{A.41}$$

Using this, one can write for the diffusive current-current correlation as

$$\mathcal{I}\left[\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b\right] = -T \lim_{\omega \to 0} \partial_\omega \text{Im}\left(G^R_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}(\mathbf{0}, \omega)\right). \tag{A.42}$$

Using Eq. (A.42) in Eq. (2.43), one obtains Eq. (2.45).

B   Self Energy

B.1   Real part of the self-energy to the lowest order: tadpole diagrams

In this Appendix, we compute the self-energies Σ_a of the fields ϕ and ξ with the Lagrangian given in Eq. (3.1). For this purpose, we note that the Euclidean two point function for these fields can be written in terms of the single particle RSF $\rho_a(k_0, \mathbf{k})$, as given in Eq. (3.15). When the imaginary part of the self-energy vanishes, the spectral functions are given by
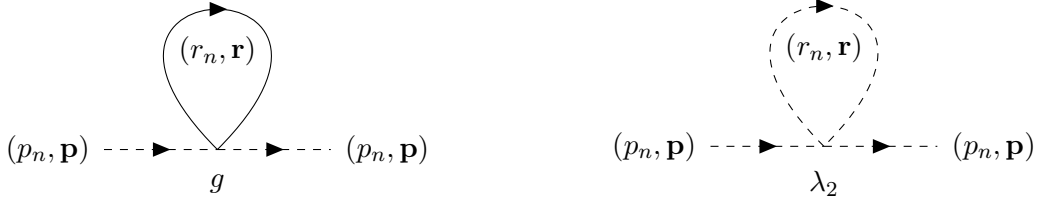
$$\rho_a(k_0, \mathbf{k}) = \frac{\pi}{2\omega_a(\mathbf{k})}\left[\delta(k_0 - \omega_a(\mathbf{k})) - \delta(k_0 + \omega_a(\mathbf{k}))\right], \tag{B.1}$$

where, $\omega_a(\mathbf{k})$ are the quasiparticle energies, given by

$$\omega_a^2(\mathbf{k}) = \mathbf{k}^2 + m_a^2 + \text{Re}(\Sigma_a(\omega_a(\mathbf{k}), \mathbf{k})) \equiv \mathbf{k}^2 + M_a^2. \tag{B.2}$$
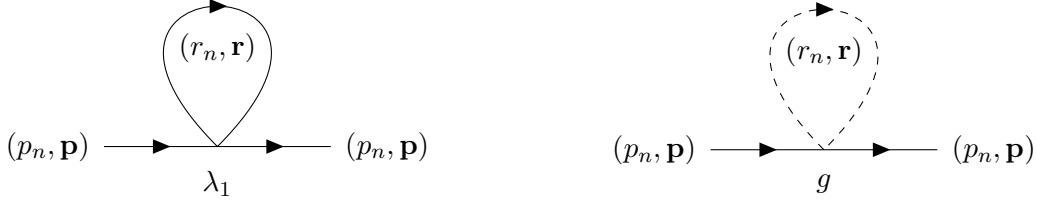
Figure 1. Tadpole diagram for ϕ field with self/mutual-coupling depicts in left/right side.

Figure 2. Tadpole diagram for ϕ field with self/mutual-coupling depicts in left/right side.

On the other hand, when there is non-vanishing imaginary part of the self energy, the spectral functions take the form

$$\rho_a(k_0, \mathbf{k}) = \frac{\text{Im}(\Sigma_a(k_0, \mathbf{k}))}{\omega_a^2 - \mathbf{k}^2 - M_a^2 + |\text{Im}(\Sigma_a(k_0, \mathbf{k}))|^2}. \tag{B.3}$$

The lowest order contribution to the self-energy comes from one-loop tadpole diagrams as shown in Fig. 1 and Fig. 2. These tadpole diagrams do not lead to imaginary part of the self energy and these contributions do not depend upon energy or momentum. From the tadpole diagrams in Fig. 1, contribution to $\text{Re}(\Sigma_\phi)$ is given by,

$$\Sigma_\phi = \frac{\lambda_1}{4} \sum_P \!\!\!\!\!\!\diagdown\!\!\!\!\int \mathcal{G}_\phi^E(P) + \frac{g}{2} \sum_P \!\!\!\!\!\!\diagdown\!\!\!\!\int \mathcal{G}_\xi^E(P). \tag{B.4}$$

Using the single particle RSF in Eq. (3.14) and summing over the Matsubara modes, we have [54, 69]

$$\Sigma_\phi = 4\pi \left(\frac{\lambda_1}{4}\right) \int_0^\infty \frac{dp_0}{\pi} \int_0^\infty \frac{dp}{(2\pi)^3} p^2 \Big[1 + f_\phi(p_0) + \bar{f}_\phi(p_0)\Big] \rho_\phi(p_0, p)$$
$$+ 4\pi \left(\frac{g}{2}\right) \int_0^\infty \frac{dp_0}{\pi} \int_0^\infty \frac{dp}{(2\pi)^3} p^2 \Big[1 + f_\xi(p_0) + \bar{f}_\xi(p_0)\Big] \rho_\xi(p^0, p), \tag{B.5}$$

where, $f_a(p_0) = \left(e^{\beta(p_0 - \mu_a)} + 1\right)^{-1}$ is the equilibrium bosonic particle distribution function with chemical potential μ_a and $\bar{f}_a(p_0) = \left(e^{\beta(p_0 + \mu_a)} + 1\right)^{-1}$ is the corresponding anti-particle distribution function.
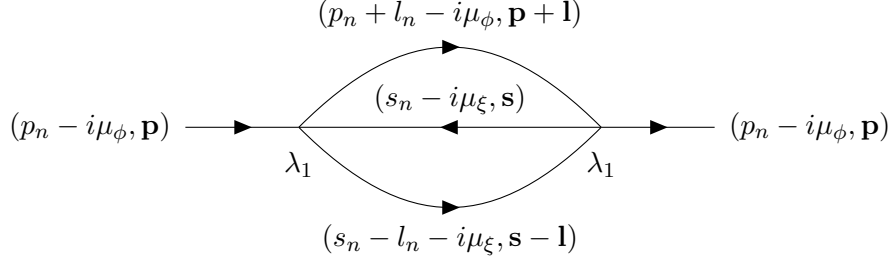
Figure 3. Sunset diagram for the ϕ field with self coupling.
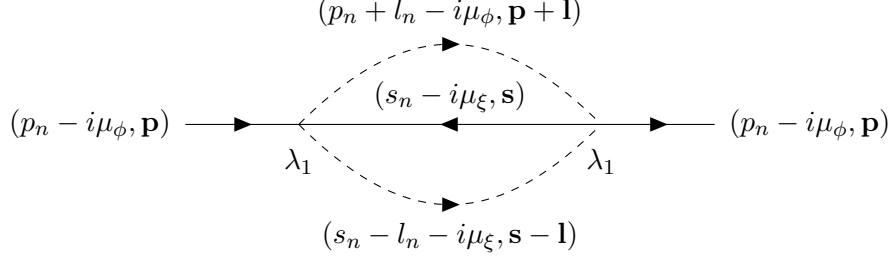
Figure 4. Sunset diagram for the ϕ field with mutual coupling.

Similarly for the ξ fields the contribution to the Σ_ξ is calculated for tadpole diagrams in Fig. 2, and is given by

$$
\begin{aligned}
\Sigma_\xi &= \frac{\lambda_2}{4} \sum_P\!\!\!\!\!\!\!\int \mathcal{G}_\xi^E(P) + \frac{g}{2} \sum_P\!\!\!\!\!\!\!\int \mathcal{G}_\phi^E(P) \\
&= 4\pi \left(\frac{\lambda_2}{4}\right) \int_0^\infty \frac{dp_0}{\pi} \int_0^\infty \frac{dp}{(2\pi)^3} p^2 \Big[1 + f_\xi(p_0) + \bar{f}_\xi(p_0)\Big] \rho_\xi(p_0, p) \\
&\quad + 4\pi \left(\frac{g}{2}\right) \int_0^\infty \frac{dp_0}{\pi} \int_0^\infty \frac{dp}{(2\pi)^3} p^2 \Big[1 + f_\phi(p_0) + \bar{f}_\phi(p_0)\Big] \rho_\phi(p_0, p).
\end{aligned}
\tag{B.6}
$$

Since the real part does not depend upon energy and momentum, they only modify the masses of ϕ and ξ fields. The thermal parts of the self-energies in Eqs. (B.5) and (B.6) $\sim f_a(p_0)$ and are ultraviolet finite. On the other hand, vacuum parts (~ 1) and are divergent which need to be renormalized [51, 54]. Note that these are coupled equation for the self-energies Σ_a which need to be solved self-consistently. For $O(N)$ sigma model these has been investigated in [69].

B.2 Imaginary part of the self-energy to the lowest order; sunset diagram

In the following, we compute the lowest order contribution to the imaginary part of the self-energy that appears at two-loop from the sunset diagrams, given in Figs. 4 and 3. For the ϕ fields this contribution is given by

$$
\Sigma_\phi^{\text{sunset}}(K) = \Sigma_{(a)\phi}^{\text{sunset}}(K) + \Sigma_{(b)\phi}^{\text{sunset}}(K)
\tag{B.7}
$$
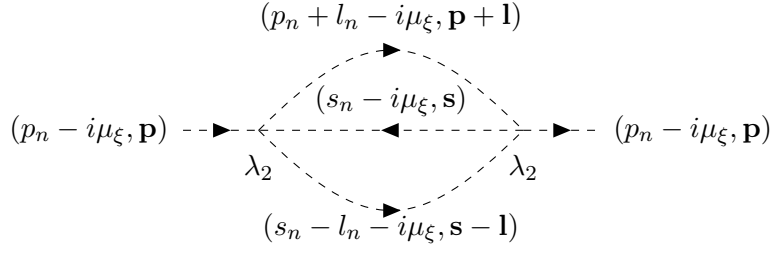
Figure 5. Sunset diagram for the ξ field with self coupling.
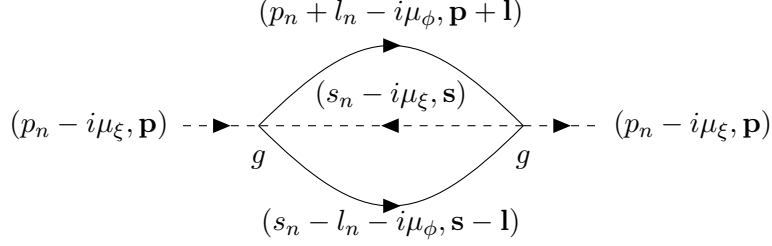
Figure 6. Sunset diagram for the ξ field with mutual coupling.

where, $\Sigma_{(a)\phi}^{\text{sunset}} \sim \lambda_1^2$ arises from Fig. 4

$$\Sigma_{(a)\phi}^{\text{sunset}}(P) = -\frac{\lambda_1^2}{2! \times (4)^2} \sum_{LS}\!\!\!\!\!\!\!\oint \left(8 \times \mathcal{G}_\phi^E(P+L)\mathcal{G}_\phi^E(S)\mathcal{G}_\phi^E(S-L) \right), \qquad (\text{B.8})$$

and $\Sigma_{(b)\phi}^{\text{sunset}} \sim g^2$ arises from the Fig. 3

$$\Sigma_{(b)\phi}^{\text{sunset}}(P) = -\frac{g^2}{2! \times (2)^2} \sum_{LS}\!\!\!\!\!\!\!\oint \left(\mathcal{G}_\xi^E(S)\mathcal{G}_\xi^E(S-L)\mathcal{G}_\phi^E(P+L) \right). \qquad (\text{B.9})$$

Representing these sunset contribution of the self-energy in terms of the single particle RSF can be done using Eq. (3.15), i.e.,

$$\begin{aligned}
\Sigma_\phi^{\text{sunset}}(P) = -\frac{g^2}{2^2 \times 2!} &\left[\sum_{LS}\!\!\!\!\!\!\!\oint\!\!\int \prod_{i=1}^{3} \frac{d\omega_i}{\pi} \left(\frac{\rho_\phi(\omega_1, \mathbf{p}+\mathbf{l})}{\omega_1 - i(p_n + l_n - i\mu_\phi)} \right) \left(\frac{\rho_\xi(\omega_2, \mathbf{s})}{\omega_2 + i(s_n + i\mu_\xi)} \right) \right. \\
&\left(\frac{\rho_\xi(\omega_3, \mathbf{s}-\mathbf{l})}{\omega_1 + i(s_n - l_n + i\mu_\xi)} \right) \right] - 8 \times \frac{\lambda_1^2}{4^2 \times 2!} \left[\sum_{LS}\!\!\!\!\!\!\!\oint\!\!\int \prod_{i=1}^{3} \frac{d\omega_i}{\pi} \left(\frac{\rho_\phi(\omega_1, \mathbf{p}+\mathbf{l})}{\omega_1 - i(p_n + l_n - i\mu_\phi)} \right) \right. \\
&\left. \left(\frac{\rho_\phi(\omega_2, \mathbf{s})}{\omega_2 + i(s_n + i\mu_\phi)} \right) \left(\frac{\rho_\phi(\omega_3, \mathbf{s}-\mathbf{l})}{\omega_1 + i(s_n - l_n + i\mu_\phi)} \right) \right].
\end{aligned} \qquad (\text{B.10})$$

In what follows, without loss of generality, we assume that the self-coupling λ_1 to be much smaller than the cross-coupling g, and consider only the first term of Eq. (B.10).

Before proceeding further, we shall show a general identity that will be useful to perform the Matsubara frequency sums in Eq.(B.10). A double Matsubara sum,

$$
\begin{aligned}
&\mathcal{S}(\omega_1, \omega_2, \omega_3, p_n - i\mu_a) \\
&= T^2 \sum_{s_n, l_n} \frac{1}{\Big(\omega_1 - i(p_n + l_n - i\mu_a)\Big)\Big(\omega_2 + i(s_n + i\mu_b)\Big)\Big(\omega_3 + i(s_n - l_n + i\mu_b)\Big)} \\
&= T \sum_{l_n} \frac{1}{\Big(\omega_1 - i(p_n + l_n - i\mu_a)\Big)} \times \mathcal{I}_n(\omega_2, \omega_3, l_n, \mu_b),
\end{aligned}
\tag{B.11}
$$

can be evaluated by performing two successive summation with respect l_n and s_n respectively. The summation over s_n is performed as

$$
\mathcal{I}_n(\omega_2, \omega_3, l_n, \mu_b) = T \sum_{s_n} \frac{1}{\Big(\omega_2 + i(s_n + i\mu_b)\Big)\Big(\omega_3 + i(s_n - l_n + i\mu_b)\Big)} .
\tag{B.12}
$$

This can be calculated by using $\beta \delta_{j_n, l_n} = \int_0^\beta d\tau\, e^{i\tau(j_n - l_n)}$ and rewriting the above equation as

$$
\begin{aligned}
\mathcal{I}_n(\omega_2, \omega_3, l_n, \mu_b) &= \int_0^\beta d\tau e^{il_n\tau} \left(T \sum_{s_n} \frac{e^{-is_n\tau}}{\omega_2 + i(s_n + i\mu_b)} \right) \left(T \sum_{s_n'} \frac{e^{is_n'\tau}}{\omega_3 + i(s_n' + i\mu_b)} \right) \\
&= \Big(f_b(\omega_2) - f_b(\omega_3) \Big) \times \left(\frac{1}{\omega_3 - \omega_2 - il_n} \right).
\end{aligned}
\tag{B.13}
$$

A similar technique can be repeated while summing over l_n of Eq. (B.11). Doing this, we obtain

$$
\begin{aligned}
&\mathcal{S}(\omega_1, \omega_2, \omega_3, p_n - i\mu_a) \\
&= \frac{\Big(1 + f_a(\omega_1)\Big)\Big(1 + f_b(\omega_3)\Big) f_b(\omega_2)}{\omega_1 - \omega_2 + \omega_3 - i(p_n - i\mu_a)} \Big[1 - \exp\Big(-\beta(\omega_1 - \mu_a + \omega_3 - \omega_2) \Big) \Big].
\end{aligned}
\tag{B.14}
$$

Further, by applying the identities in Eq. (B.14), performing the analytic continuation $i(p_n - i\mu_\phi) \to \omega + i0^+$ in Eq. (B.10), and taking the imaginary part, we obtain

$$
\begin{aligned}
\mathrm{Im}\Big(\Sigma_\phi(\omega, \mathbf{p})\Big) &= \frac{g^2}{2^2 \times 2!} \Big(1 - e^{-\beta(\omega - \mu_\phi)} \Big) \int d\mathbf{l} \int d\mathbf{s} \int \frac{d\omega_2}{\pi} \int \frac{d\omega_3}{\pi} \Bigg[\Big(1 + f_\phi(\omega - \omega_2 - \omega_3) \Big) \\
&\quad \times \Big[1 + \bar{f}_\xi(\omega_2) \Big]\Big[1 + f_\xi(\omega_3) \Big] \rho_\phi(\omega - \omega_2 - \omega_3, \mathbf{p} + \mathbf{l})\, \rho_\xi(\omega_2, \mathbf{s})\, \rho_\xi(\omega_3, \mathbf{s} - \mathbf{l}) \Bigg].
\end{aligned}
\tag{B.15}
$$

Moreover, using the fact that the spectral functions do not depend on the direction of the three momentum, angular integration can be carried out by using the identity

$$
\begin{aligned}
\int d\mathbf{p}\, \mathcal{F}(|\mathbf{k} - \mathbf{p}|)\mathcal{G}(|\mathbf{p}|) &= \frac{1}{(2\pi)^2|\mathbf{k}|} \int_0^\infty dl_1 dl_2\, l_1 l_2 \mathcal{F}(l_1)\mathcal{G}(l_2) \\
&\quad \times \Theta(|l_1 - l_2| \le |\mathbf{k}| \le l_1 + l_2),
\end{aligned}
\tag{B.16}
$$

resulting in the expression for Eq. (B.15) as

$$
\begin{aligned}
\mathrm{Im}\Big[\Sigma_\phi(\omega,\mathbf{p})\Big] &= \frac{g^2\Big(1 - e^{-\beta(\omega-\mu_\phi)}\Big)}{2^2 \times 2! \times (2\pi)^5} \int \frac{d\omega_2}{\pi}\frac{d\omega_3}{\pi} \int_0^\infty dl_1\, dl_2\, dl_3\, \Theta(|l_2 - l_3| \leq l_1 \leq l_2 + l_3) \\
&\quad \times l_1\, l_2\, l_3\, \rho_\phi(\omega - \omega_2 - \omega_3, |\mathbf{p}+\mathbf{l}_1|)\, \rho_\xi(\omega_2, l_2)\, \rho_\xi(\omega_3, l_3) \\
&\quad \times \Big[1 + f_\phi(\omega - \omega_2 - \omega_3)\Big]\Big[1 + \bar{f}_\xi(\omega_2)\Big]\Big[1 + f_\xi(\omega_3)\Big].
\end{aligned}
\tag{B.17}
$$

Here, one can think that l_1, l_2 and l_3 are the magnitude of the three momentums $\mathbf{l}_1, \mathbf{l}_2$ and $\mathbf{l}_3$ respectively. The above form reveals the domain of momentum integration through the step function in this expression. A similar calculation can be done for ξ fields (See Fig. 5 and 6), which leads to the following result

$$
\mathrm{Im}\Big(\Sigma_\xi(\omega,\mathbf{p})\Big) = \mathrm{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)\Big|_{\phi \leftrightarrow \xi}.
\tag{B.18}
$$

The expression as given in Eq. (B.17) and Eq. (B.18) can be written as a sum of different physical processes contributing to imaginary part of the self energy, which we discuss next. For this purpose, we rewrite Eq. (B.15) as with the four vectors $\tilde{l}_i \equiv (\omega_i, \mathbf{l}_i)$ and $\tilde{p} \equiv (\omega, \mathbf{p})$

$$
\begin{aligned}
\mathrm{Im}\Big(\Sigma_\phi(\tilde{p})\Big) &= \frac{g^2}{2!}\Big(1 - e^{-\beta(\omega-\mu_\phi)}\Big)(2\pi)^4 \int \frac{d\tilde{l}_1}{(2\pi)^4}\frac{d\tilde{l}_2}{(2\pi)^4}\frac{d\tilde{l}_3}{(2\pi)^4}\Bigg[\Big(1 + f_\phi(\omega_1)\Big) \\
&\quad \times \Big[1 + \bar{f}_\xi(\omega_2)\Big]\Big[1 + f_\xi(\omega_3)\Big]\rho_\phi(\omega_\phi, \mathbf{l}_1)\,\rho_\xi(\omega_2, \mathbf{l}_2)\,\rho_\xi(\omega_3, \mathbf{l}_3)\delta^4(\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 - \tilde{p})\Bigg].
\end{aligned}
\tag{B.19}
$$

We next note the following relation for the bosonic distribution functions with finite chemical potential

$$
\begin{aligned}
&\Big(1 - e^{-\beta(\omega_1+\omega_2+\omega_3-\mu_\phi)}\Big)\Big(1 + f_\phi(\omega_1)\Big)\Big(1 + \bar{f}_\xi(\omega_2)\Big)\Big(1 + f_\xi(\omega_3)\Big) \\
&= \Bigg[\Big(1 + f_\phi(\omega_1)\Big)\Big(1 + \bar{f}_\xi(\omega_2)\Big)\Big(1 + f_\xi(\omega_3)\Big) - f_\phi(\omega_1)\bar{f}_\xi(\omega_2)f_\xi(\omega_3)\Bigg]
\end{aligned}
\tag{B.20}
$$

We then substitute the expression for the spectral function as given in Eq. (B.1) and use the identity Eq. (B.20) in Eq. (B.19). The two delta functions in each of the three spectral functions occurring in Eq. (B.19) leads to eight different processes that contribute to the imaginary part of the self energy. Therefore we have

$$
\mathrm{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big) = \sum_{a=1}^{8}\mathrm{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_a,
\tag{B.21}
$$

$$
\mathrm{Im}\Big(\Sigma_\xi(\omega,\mathbf{p})\Big) = \sum_{a=1}^{8}\mathrm{Im}\Big(\Sigma_\xi(\omega,\mathbf{p})\Big)_a.
\tag{B.22}
$$

For instance, one can write the first process as

$$\mathrm{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_1 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\,d\mathbf{l}_2\,d\mathbf{l}_3\,\delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p})\times\frac{1}{E_1 E_2 E_3}\Bigg[\Big(1+f_\phi(E_1)\Big)$$

$$\times\Big(1+\bar{f}_\xi(E_2)\Big)\Big(1+f_\xi(E_3)\Big)-f_\phi(E_1)\bar{f}_\xi(E_2)f_\xi(E_3)\Bigg]$$

$$\times\delta(\omega-E_2-E_3-E_1), \tag{B.23}$$

where, $E_1 = \sqrt{\mathbf{l}_1^2+m_\phi^2}$ and E_2, E_3 are $\sqrt{\mathbf{l}_2^2+m_\xi^2}$, $\sqrt{\mathbf{l}_3^2+m_\xi^2}$ respectively. This term can be interpreted as the probability for the $(\phi\rightarrow\phi\xi\bar{\xi})$ with a corresponding bose enhancement factor $(1+f_\phi)(1+f_\xi)(1+\bar{f}_\xi)$ minus the probability for $(\phi\bar{\xi}\xi\rightarrow\phi)$ with the weight $f_\phi\bar{f}_\xi f_\xi$.

Second process can be interpreted as the anti-particle counterpart of the first process and given as,

$$\mathrm{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_2 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\,d\mathbf{l}_2\,d\mathbf{l}_3\,\delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p})\times\frac{1}{E_1 E_2 E_3}\Bigg[\bar{f}_\phi(E_1)f_\xi(E_2)\bar{f}_\xi(E_3)$$

$$-\Big(1+\bar{f}_\phi(E_1)\Big)\Big(1+f_\xi(E_2)\Big)\Big(1+\bar{f}_\xi(E_3)\Big)\Bigg]\times\delta(E_1+E_2+E_3+\omega). \tag{B.24}$$

Both these processes contribute in the zero temperature limit ($T\rightarrow 0$ and $f_a\rightarrow 0$) due to the presence of statistical weight $(1+f_\phi)(1+f_\xi)(1+\bar{f}_\xi)$. Further, in the third process, the contribution arises from the probability of annihilation of a particle-antiparticle pair of ϕ fields and creation of a particle-antiparticle pair of ξ fields ($\phi\bar{\phi}\rightarrow\xi\bar{\xi}$) minus the probability for the reverse process ($\xi\bar{\xi}\rightarrow\phi\bar{\phi}$)

$$\mathrm{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_3 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\,d\mathbf{l}_2\,d\mathbf{l}_3\,\delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p})\times\frac{1}{E_1 E_2 E_3}\Bigg[\bar{f}_\phi(E_1)\Big(1+\bar{f}_\xi(E_2)\Big)$$

$$\times\Big(1+f_\xi(E_3)\Big)-\Big(1+\bar{f}_\phi(E_1)\Big)\bar{f}_\xi(E_2)f_\xi(E_3)\Bigg]\times\delta(E_3+E_2-E_1-\omega). \tag{B.25}$$

Fourth term is due to the scattering process between ϕ and $\bar{\xi}$ particles from different energy states ($\phi\bar{\xi}\rightarrow\phi\bar{\xi}$). The fifth process is different from the fourth one, by changing $\bar{\xi}\rightarrow\xi$ in the same scattering channels ($\phi\xi\rightarrow\phi\xi$). We have

$$\mathrm{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_4 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\,d\mathbf{l}_2\,d\mathbf{l}_3\,\delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p})\times\frac{1}{E_1 E_2 E_3}\Bigg[\Big(1+f_\phi(E_1)\Big)$$

$$\Big(1+\bar{f}_\xi(E_2)\Big)\bar{f}_\xi(E_3)-f_\phi(E_1)\bar{f}_\xi(E_2)\Big(1+\bar{f}_\xi(E_3)\Big)\Bigg]$$

$$\times\delta(E_3-E_3+E_1-\omega), \tag{B.26}$$

and

$$
\text{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_5 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\, d\mathbf{l}_2\, d\mathbf{l}_3\, \delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p}) \times \frac{1}{E_1 E_2 E_3}\Bigg[\Big(1+f_\phi(E_1)\Big)
$$

$$
f_\xi(E_2)\Big(1+f_\xi(E_3)\Big) - f_\phi(E_1)\Big(1+f_\xi(E_2)\Big)f_\xi(E_3)\Bigg]
$$

$$
\times \delta(E_3-E_2-E_1-\omega), \tag{B.27}
$$

Similarly, the sixth term arises from the probability of three body inelastic collision of $\phi\bar{\phi}\bar{\xi} \to \bar{\xi}$ minus the probability $\bar{\xi} \to \bar{\phi}\bar{\xi}\phi$ process, and the seventh one take place with a replacement of particle to anti-particle of ξ field of the same inelastic process.

$$
\text{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_6 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\, d\mathbf{l}_2\, d\mathbf{l}_3\, \delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p}) \times \frac{1}{E_1 E_2 E_3}\Bigg[\bar{f}_\phi(E_1)\Big(1+\bar{f}_\xi(E_2)\Big)
$$

$$
\times \bar{f}_\xi(E_3) - \Big(1+\bar{f}_\phi(E_1)\Big)\bar{f}_\xi(E_2)\Big(1+\bar{f}_\xi(E_3)\Big)\Bigg]
$$

$$
\times \delta(E_2-E_3-E_1-\omega). \tag{B.28}
$$

and

$$
\text{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_7 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\, d\mathbf{l}_2\, d\mathbf{l}_3\, \delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p}) \times \frac{1}{E_1 E_2 E_3}\Bigg[\bar{f}_\phi(E_1)f_\xi(E_2)
$$

$$
\Big(1+f_\xi(E_3)\Big) - \Big(1+\bar{f}_\phi(E_1)\Big)\Big(1+f_\xi(E_2)\Big)f_\xi(E_3)\Bigg]
$$

$$
\times \delta(E_3-E_2-E_1-\omega). \tag{B.29}
$$

The eighth term comes from the difference of the probability of $(\phi\xi\bar{\xi} \to \phi)$ and it's reverse processes $(\phi \to \xi\bar{\xi}\phi)$

$$
\text{Im}\Big(\Sigma_\phi(\omega,\mathbf{p})\Big)_8 = \frac{g^2\pi^4}{8}\int d\mathbf{l}_1\, d\mathbf{l}_2\, d\mathbf{l}_3\, \delta^3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3-\mathbf{p}) \times \frac{1}{E_1 E_2 E_3}\Bigg[\Big(1+f_\phi(E_1)\Big)f_\xi(E_2)
$$

$$
\bar{f}_\xi(E_3) - f_\phi(E_1)\Big(1+f_\xi(E_2)\Big)\Big(1+\bar{f}_\xi(E_3)\Big)\Bigg]
$$

$$
\times \delta(E_1-E_2-E_3-\omega). \tag{B.30}
$$

But this process is different than the first term due to the difference of processes that takes place in internal lines of sunset diagram Fig. 3, which leads to a distinction in statistical weight of internal lines.

Next, we determine the region of $k^2 = \omega^2 - \mathbf{p}^2$ where all the eight physical processes contained in Eq. (B.17) are contingent according to the kinematic constraints, i.e., the integral over l_1 in Eq. (B.17) survives. Kinematically first and second term survives if $k^2 > (m_\phi + M_{\xi\phi})^2$, where $M_{\xi\phi}$ is the invariant mass of the ϕ and ξ. Therefore, the

processes are possible when $k^2 > (2m_\xi + m_\phi)^2$ since $M_{\xi\phi} > m_\xi + m_\phi$. The third and eighth terms are allowed when it satisfies the condition $k^2 > (M_{\xi\xi} - m_\phi)^2 > (2m_\xi - m_\phi)^2$ since $M_{\xi\xi} > 2m_\xi$. The similarities of kinematical conditions between third and eighth processes can be understood by changing external incoming/ougoing ϕ-line to outgoing/incoming $\bar{\phi}$-line in any one of the process. Similarly, fourth, fifth, sixth and seventh term needs to satisfy the same kinematic condition, which is $k^2 < (M_{\xi\phi} - m_\xi)^2$. The similitude of fourth and fifth processes happens because these processes involve particles and anti-particle of ξ fields, respectively. Furthermore, an equivalence with sixth and seventh processes can be made by changing external incoming/outgoing ϕ by its outgoing/incoming $\bar{\phi}$. Therefore, these four scattering processes take place at arbitrary k^2. A similar calculation can be performed for the ξ field by changing each ϕ's propagator to ξ's propagator, and resulting in similar processes can be occurred as described in Eq. (B.23)-(B.30) with an appropriate changes in physical parameters under $\phi \leftrightarrow \xi$.

Thus, the thermal width of the quasi-particle profile of the spectral function can be estimated, which is $\Gamma_a(E_{a\,\mathbf{p}}, \mathbf{p}) = \frac{\text{Im}(\Sigma_a(E_p, \mathbf{p}))}{2E_\mathbf{p}}$ and at the weak coupling limit its contribution start from g^2 order.

C   Euclidean Green's function for the arbitrary bilinear Hermitian bosonic operator.

In this section, we provide some details of the computation of the Euclidean Green's function for two arbitrary bilinear Hermitian bosonic operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$. We consider two such operators,

$$\mathcal{O}_1(X) = \sum_a \phi_a^\dagger \overleftrightarrow{\mathcal{D}}_{1a}(X) \phi_a, \tag{C.1}$$

$$\mathcal{O}_2(X) = \sum_b \phi_b^\dagger \overleftrightarrow{\mathcal{D}}_{2b}(X) \phi_b, \tag{C.2}$$

where, each operator can carry Lorentz and/or particle indices, and that is taken care of by the indices of this $\overleftrightarrow{\mathcal{D}}_1$ and $\overleftrightarrow{\mathcal{D}}_2$. In general, $\overleftrightarrow{\mathcal{D}}_a$'s can be expressed as polynomials of left and right partial derivative, $\overleftarrow{\partial}_X$ and $\overrightarrow{\partial}_X$, which acts on $\phi_a(X)$ and $(\phi_a^\dagger(X)$, respectively. For example, if we consider an operator $\hat{\mathcal{N}}_a^i$ and the operator's structure as given in Eq. (3.26), we obtain $\overleftrightarrow{\mathcal{D}}_a = i \sum_b \delta_{ba}(\overrightarrow{\partial}^i - \overleftarrow{\partial}^i)$. Similar construction can be done with the diffusion current $\hat{\mathcal{J}}_a^i$ and the heat flux $\hat{\mathcal{Q}}^i$ as given in Eq. (2.34) and Eq. (3.25), respectively.

Next, we calculate the Equilibrium Green's function for $\hat{\mathcal{O}}_1 = \hat{\mathcal{J}}_a^i$ and $\hat{\mathcal{O}}_2 = \hat{\mathcal{J}}_b^j$, as required in Eq. (3.27), i.e.,

$$\Delta^{ij} \left\langle \hat{\mathcal{J}}_{a\,i}(X) \hat{\mathcal{J}}_{b\,j}(X') \right\rangle_l = \Delta^{ij} \left[\left\langle \hat{\mathcal{N}}_{a\,i}(X) \hat{\mathcal{N}}_{b\,j}(X') \right\rangle_l - \frac{n_b}{h} \left\langle \hat{\mathcal{N}}_{a\,i}(X) \hat{\mathcal{Q}}_j(X') \right\rangle_l \right.$$

$$\left. - \frac{n_a}{h} \left\langle \hat{\mathcal{Q}}_i(X) \hat{\mathcal{N}}_{b\,j}(X') \right\rangle_l + \frac{n_a n_b}{h^2} \left\langle \hat{\mathcal{Q}}_i(X) \hat{\mathcal{Q}}_j(X') \right\rangle_l \right], \quad \text{(C.3)}$$

where, a, b are the indices for particle types. Nevertheless, we present a calculation with a generic set of composite bosonic and bilinear operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$. If these operators

contain the differential operator as given in Eqs. (C.1) and (C.2), then in the Euclidean momentum space it will be a function of Matsubara frequency and the 3-momentum,[9]. Therefore, the two-point Euclidean Green's function of these generic operators can be written as

$$\left\langle \hat{\mathcal{O}}_1(X)\hat{\mathcal{O}}_2(X') \right\rangle = \sum_{kl} \overleftrightarrow{\mathcal{D}}_{1k}(X) \overleftrightarrow{\mathcal{D}}_{2l}(X') \left\langle \phi_k^\dagger(X)\phi_k(X)\phi_l^\dagger(X')\phi_l(X') \right\rangle$$

$$= \sum_{kl} \overleftrightarrow{\mathcal{D}}_{1k}(X) \overleftrightarrow{\mathcal{D}}_{2l}(X')\mathcal{G}_{lk}^E(X-X')\mathcal{G}_{kl}^E(X'-X). \tag{C.4}$$

Here, one needs to define the action of $\overleftrightarrow{\mathcal{D}}_1(X)$ and $\overleftrightarrow{\mathcal{D}}_2(X')$ on the Euclidean correlation function $\langle \phi_a(X)\phi_b^\dagger(X')\rangle = \mathcal{G}_{ab}^E(X',X) = \mathcal{G}_{ab}^E(X-X')$. This can be obtained from the distinctive action of the differential operator $\overleftarrow{\partial}_X$ and $\overrightarrow{\partial}_X$ on $\phi_a^\dagger$ and ϕ_a. Hence, the left and right partial derivative $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ acts on the left and right variable of the arguments of two-point particle Euclidean correlation function $\mathcal{G}_{ab}^E(X',X)$, respectively.

Functionally, we can show these properties with an example. For instance, a differential operator which contains a linear combination of left and right derivatives, $\overleftrightarrow{\mathcal{D}}(X) = \overrightarrow{\partial}_X - \overleftarrow{\partial}_X$, operates like

$$\overleftrightarrow{\mathcal{D}}(X)\mathcal{G}^E(X',X) = \partial_X\mathcal{G}^E(X',X)\,,$$
$$\overleftrightarrow{\mathcal{D}}(X)\mathcal{G}^E(X,X') = -\partial_X\mathcal{G}^E(X,X')\,,$$
$$\overleftrightarrow{\mathcal{D}}(X')\mathcal{G}^E(X',X) = -\partial_{X'}\mathcal{G}^E(X',X)\,,$$
$$\overleftrightarrow{\mathcal{D}}(X')\mathcal{G}^E(X,X') = \partial_{X'}\mathcal{G}^E(X,X')\,. \tag{C.5}$$

Here, $\partial_X = (\partial_\tau, \partial_{x^i}) = (\partial_\tau, \partial_{\mathbf{x}})$ and $\partial_{X'} = (\partial_{\tau'}, \partial_{x'^i}) = (\partial_{\tau'}, \partial_{\mathbf{x}'})$. Using these properties of left/right towards derivative, one can simplify the following correlation functions.

(a) To evaluate the Euclidean $\hat{\mathcal{N}}_a$-$\hat{\mathcal{N}}_b$ correlation function, which is first term of Eq. (C.3), we can express Eq. (C.4) by using Eqs. (C.1), (C.2). This leads to the identification

$$\overleftrightarrow{\mathcal{D}}_{1a}(X) = i \sum_k \delta_{ka}\left(\overrightarrow{\partial}_{x^i} - \overleftarrow{\partial}_{x^i}\right), \tag{C.6}$$

$$\overleftrightarrow{\mathcal{D}}_{2b}(X') = i \sum_l \delta_{lb}\left(\overrightarrow{\partial}_{x'^i} - \overleftarrow{\partial}_{x'^i}\right). \tag{C.7}$$

Using these differential operators, one can write

$$\Delta^{ij}\left\langle \hat{\mathcal{N}}_{ai}(X)\hat{\mathcal{N}}_{bj}(X') \right\rangle_E$$
$$= -\delta_{ab}\Delta^{ij}\left(\overrightarrow{\partial}_{x^i} - \overleftarrow{\partial}_{x^i}\right)\left(\overrightarrow{\partial}_{x'^j} - \overleftarrow{\partial}_{x'^j}\right)\mathcal{G}_{aa}^E(X',X)\mathcal{G}_{aa}^E(X,X')$$
$$= -\delta_{ab}\Delta^{ij}\left(\partial_{x^i}\mathcal{G}_{aa}^E(X',X)\partial_{x'^j}\mathcal{G}_{aa}^E(X,X') + \partial_{x'^j}\mathcal{G}_{aa}^E(X',X)\partial_{x^i}\mathcal{G}_{aa}^E(X,X')\right.$$
$$\left. - \partial_{x'^j}\partial_{x^i}\mathcal{G}_{aa}^E(X',X)\mathcal{G}_{aa}^E(X,X') - \mathcal{G}_{aa}^E(X',X)\partial_{x'^j}\partial_{x^i}\mathcal{G}_{aa}^E(X,X')\right). \tag{C.8}$$

---

[9]By applying the Euclidean Fourier decomposition of fields as Eq. (3.6) one can show that the differential operator becomes a function of Euclidean momentum, i.e. $\mathcal{O}_1(X) = \sum_a \phi_a^\dagger \overleftrightarrow{\mathcal{D}}_{1a}\phi_a = \sum_a e^{i(Q-P)\cdot X}\mathcal{D}_{1a}(P,Q)\tilde{\phi}_a^\dagger(P)\tilde{\phi}_a(Q)$. If we again take same example $\mathcal{O}_1(X) = \mathcal{N}_a^i(X)$, one can show that $\mathcal{D}_{1a}(P,Q) = (\mathbf{p}^i + \mathbf{q}^i)$.

Moreover, introducing the Fourier decomposition of two-point Eucledian correlation function, Eqs. (3.11) and (3.15), one can further simplify it as

$$\Delta^{ij}\left\langle \hat{\mathcal{N}}_{ai}(X)\hat{\mathcal{N}}_{bj}(X')\right\rangle_E = \delta_{ab}\sum_{PL}e^{iL\cdot(X-X')}\,\Delta_{ij}(2\mathbf{p}^j-\mathbf{l}^j)(2\mathbf{p}^i-\mathbf{l}^i)\int\frac{d\omega_1}{\pi}\frac{d\omega_2}{\pi}$$

$$\times\left[\frac{\rho_a(\omega_1,\mathbf{p})\rho_a(\omega_1,\mathbf{p}-\mathbf{l})}{\left(\omega_1+i\tilde{p}_n\right)\left(\omega_2+i(\tilde{p}_n-l_n)\right)}\right],\quad (C.9)$$

where the chemical potential dependence is hidden in the variable $\tilde{p}_n$ defined as $\tilde{p}_n = p_n+i\mu_a$. We now sum over the Matsubara frequency p_n by using the identities in Eqs. (B.12) and (B.13), leading to the Euclidean correlation function $\mathcal{G}^E_{\mathcal{N}_a\mathcal{N}_a}$ of Eq. (3.28) as

$$\mathcal{G}^E_{\mathcal{N}_a\mathcal{N}_a}(L) = -\int d\mathbf{p}(2\mathbf{p}-\mathbf{l})^2\int\frac{d\omega_1}{\pi}\frac{d\omega_2}{\pi}\rho_a(\omega_1,\mathbf{p})\rho_a(\omega_2,\mathbf{p}-\mathbf{l})\frac{f_a(\omega_1)-f_a(\omega_2)}{\omega_2-\omega_1+il_n}.\quad (C.10)$$

Here, $f_a(\omega) = \left(\exp\beta(\omega-\mu_a)-1\right)^{-1}$ is the thermal distribution function of 'a' type particle.

(b) Similar calculation can be done for the second mixed correlation function of particle and heat current correlation function $\mathcal{N}_a$-$\mathcal{Q}$ of the Eq. (C.3). With the help of the Eqs. (C.1) and (C.2), it is easy to identify

$$\overleftrightarrow{\mathcal{D}}_1(X) = i\left(\overrightarrow{\partial}_{x^i}-\overleftarrow{\partial}_{x^i}\right),\quad (C.11)$$

$$\overleftrightarrow{\mathcal{D}}_2(X') = i\left(\overleftarrow{\partial}_{\tau'}\overrightarrow{\partial}_{x'^i}+\overleftarrow{\partial}_{x'^i}\overrightarrow{\partial}_{\tau'}\right).\quad (C.12)$$

With these differential operators, one can compute Euclidean Green's function in a similar manner as

$$\mathcal{G}^E_{\mathcal{N}_a\mathcal{Q}}(L) = \sum_k\delta_{ak}\Delta^{ij}\int d\mathbf{p}\int\frac{d\omega_1}{\pi}\frac{d\omega_2}{\pi}\left[\omega_1(\mathbf{p}_i-\mathbf{l}_i)+\omega_2\mathbf{p}_i\right]\left(2\mathbf{p}_j-\mathbf{l}_j\right)$$

$$\times\left[\rho_k(\omega_2,\mathbf{p}-\mathbf{l})\,\rho_k(\omega_1,\mathbf{p})\,\frac{f_k(\omega_1)-f_k(\omega_2)}{\omega_2-\omega_1+il_n}\right].\quad (C.13)$$

Likewise, the third mixed correlation function of Eq. (C.3) can be written as

$$\mathcal{G}^E_{\mathcal{Q}\mathcal{N}_a}(L) = \sum_k\delta_{ak}\Delta^{ij}\int d\mathbf{p}\int\frac{d\omega_1}{\pi}\frac{d\omega_2}{\pi}\left[\omega_1(\mathbf{p}_j-\mathbf{l}_j)+\omega_2\mathbf{p}_j\right]\left(2\mathbf{p}_i-\mathbf{l}_i\right)$$

$$\times\left[\rho_k(\omega_2,\mathbf{p}-\mathbf{l})\,\rho_k(\omega_1,\mathbf{p})\,\frac{f_k(\omega_1)-f_k(\omega_2)}{\omega_2-\omega_1+il_n}\right].\quad (C.14)$$

Owing to the symmetry properties of the Δ^{ij}, the second and third $\mathcal{N}_a$-$\mathcal{Q}$ mixed correlation functions are equal.

(c) The last $\mathcal{Q}$-$\mathcal{Q}$ correlation function can be written in a similar manner

$$\mathcal{G}^E_{\mathcal{Q}\mathcal{Q}}(L) = \sum_k\Delta^{ij}\int d\mathbf{p}\int\frac{d\omega_1}{\pi}\frac{d\omega_2}{\pi}\left[\omega_1(\mathbf{p}_i-\mathbf{l}_i)+\omega_2\mathbf{p}_i\right]\left[\omega_1(\mathbf{p}_j-\mathbf{l}_j)+\omega_2\mathbf{p}_j\right]$$

$$\times\left[\rho_k(\omega_2,\mathbf{p}-\mathbf{l})\,\rho_k(\omega_1,\mathbf{p})\,\frac{f_k(\omega_1)-f_k(\omega_2)}{\omega_2-\omega_1+il_n}\right].\quad (C.15)$$

With the help of Eq. (3.19), we evaluate the spectral function of each of the above-mentioned Euclidean Green's functions and the spectral function of $\mathcal{J}_a$-$\mathcal{J}_b$. As mentioned in Eq. (3.19), one needs to employ the Disc function on all of these Green's functions in Eqs. (C.10), (C.13), (C.14) and (C.15). The operation of the Disc function will act on the part which contains a discontinuity of these Green's functions while taking the limit $il_n \to \omega \pm i0^+$. Such part in these Green's function is $\left(\frac{1}{\omega_2 - \omega_1 + il_n}\right)$, which makes them discontinuous across the real ω axis

$$\text{Disc}\left(\frac{1}{\omega_2 - \omega_1 + il_n}\right) = \left(\frac{1}{\omega_2 - \omega_1 + \omega + i0^+} - \frac{1}{\omega_2 - \omega_1 + \omega - i0^+}\right)$$
$$= -2i\pi\delta(\omega_2 - \omega_1 + \omega). \tag{C.16}$$

Applying this identity, we derive the spectral functions for all of these Green's functions, which are given by

$$\rho_{\mathcal{N}_a\mathcal{N}_b}(\omega, \mathbf{l}) = -\delta_{ab}\Delta_{ij}\int d\mathbf{p}(2\mathbf{p}^j - \mathbf{l}^j)(2\mathbf{p}^i - \mathbf{l}^i)\int \frac{d\omega'}{\pi}\rho_a(\omega' + \omega, \mathbf{p})\rho_a(\omega', \mathbf{p} - \mathbf{l})$$
$$\times \left[f_a(\omega' + \omega) - f_a(\omega')\right], \tag{C.17}$$

$$\rho_{\mathcal{N}_a\mathcal{Q}}(\omega, \mathbf{l}) = -\sum_k \delta_{ak}\Delta^{ij}\int d\mathbf{p}\int \frac{d\omega'}{\pi}\left[(\omega' + \omega)(\mathbf{p}_i - \mathbf{l}_i) + \omega'\mathbf{p}_i\right]\left(2\mathbf{p}_j - \mathbf{l}_j\right)$$
$$\times \left[\rho_k(\omega', \mathbf{p} - \mathbf{l})\rho_k(\omega' + \omega, \mathbf{p})\left(f_k(\omega' + \omega) - f_k(\omega')\right)\right], \tag{C.18}$$

$$\rho_{\mathcal{Q}\mathcal{Q}}(\omega, \mathbf{l}) = -\sum_k \Delta^{ij}\int d\mathbf{p}\int \frac{d\omega'}{\pi}\left((\omega' + \omega)(\mathbf{p}_i - \mathbf{l}_i) + \omega'\mathbf{p}_i\right)\left((\omega' + \omega)(\mathbf{p}_j - \mathbf{l}_j) + \omega'\mathbf{p}_j\right)$$
$$\times \left[\rho_k(\omega', \mathbf{p} - \mathbf{l})\rho_k(\omega' + \omega, \mathbf{p})\left(f_k(\omega' + \omega) - f_k(\omega')\right)\right]. \tag{C.19}$$

Adding these spectral functions according to the Eq. (C.3), we obtain the $\mathcal{J}_a$-$\mathcal{J}_b$ spectral function as

$$\rho_{\mathcal{J}_a\mathcal{J}_b}(\omega, \mathbf{l}) = \delta_{ab}\rho_{\mathcal{N}_a\mathcal{N}_a}(\omega, \mathbf{l}) - \frac{n_b}{h}\rho_{\mathcal{N}_a\mathcal{Q}}(\omega, \mathbf{l}) - \frac{n_a}{h}\rho_{\mathcal{N}_b\mathcal{Q}}(\omega, \mathbf{l}) + \frac{n_a n_b}{h^2}\rho_{\mathcal{Q}\mathcal{Q}}(\omega, \mathbf{l}). \tag{C.20}$$

These results are used in Sec-3 to obtain the diffusion coefficient in Eq. (2.46).

D   Pinching Pole Approximation

To extract the transport coefficients, Eq. (3.34), from Eq. (3.33) with the help of representation of the particle RSF as given in Eq. (3.33), we consider the finite thermal width. Physically that corresponds to a transport coefficients arising due to a finite mean free path (or lifetime) in the presence of interactions. On the other hand, transport coefficients are ill-defined in the free theory due to the vanishing lifetime of the quasi-particle excitation. In the weakly coupling limit with non-zero but small (by ignoring the terms of $\mathcal{O}(\Gamma_{a\,\mathbf{p}}/E_{a\,\mathbf{p}})$) thermal width, one can approximate the spectral function as Eq. (3.33) which is written as

a difference of the propagators $\Delta_{a\,1}(k_0, \mathbf{k})$ and $\Delta_{a\,2}(k_0, \mathbf{k})$. Therefore, the dependency on the single-particle spectral function in a quadratic form needs to be evaluated, and finding the dominant contribution is termed as pinching pole approximation. The quadratic term of the spectral function consists of two kind of terms, $\Delta_{a\,1}^2(k_0, \mathbf{k})$ and $\Delta_{a\,2}^2(k_0, \mathbf{k})$ having poles of order two on the same side of the frequency k_0-axis. These lead to contribution smaller than the contribution coming from mixing term $\Delta_{a\,1}(k_0, \mathbf{k})\Delta_{a\,2}(k_0, \mathbf{k})$ which has symmetrically distributed simple poles across the real k_0 axis. In the computation of the transport coefficients in Eq. (3.30), the opposite lying poles with respect to the k_0-axis gives rise to the pinching of the contour and has dominant contribution

$$\int \frac{dk_0}{2\pi} \Delta_{a\,1}(k_0, \mathbf{k})\Delta_{a\,2}(k_0, \mathbf{k})F(k_0, \mathbf{k}) = \tag{D.1}$$

$$\frac{1}{8E_{a\,\mathbf{k}}\Gamma_{a\,\mathbf{k}}}\left[\frac{F(E_{a\,\mathbf{k}} - i\Gamma_{a\,\mathbf{k}}, \mathbf{k})}{E_{a\,\mathbf{k}} - i\Gamma_{a\,\mathbf{k}}} + \frac{F(-E_{a\,\mathbf{k}} - i\Gamma_{a\,\mathbf{k}}, \mathbf{k})}{E_{a\,\mathbf{k}} + i\Gamma_{a\,\mathbf{k}}}\right] - 2\pi i \sum_n \text{Res}(F(k_0, \mathbf{k}), k_0 = \alpha_n).$$

Here, α_n is the position of the pole of the function $F(k_0, \mathbf{k})$ on the complex plane of k_0. Hence, we can expand the last result around small values of thermal width for weakly coupled system and are able to write Eq. (D.1) as

$$\int \frac{dk_0}{2\pi} \Delta_{a\,1}(k_0, \mathbf{k})\Delta_{a\,2}(k_0, \mathbf{k})F(k_0, \mathbf{k}) = \frac{1}{8E_{a\,\mathbf{k}}^2\Gamma_{a\,\mathbf{k}}}\left[F(E_{a\,\mathbf{k}}, \mathbf{k}) + F(-E_{a\,\mathbf{k}}, \mathbf{k})\right]$$
$$+ \sum_{n=0} \mathcal{C}_n(\Gamma_{a\,\mathbf{k}}/E_{a\,\mathbf{k}})^n. \tag{D.2}$$

For weakly coupled theory one can ignore terms of the form $\mathcal{O}(\Gamma_{a\,\mathbf{p}}/E_{a\,\mathbf{p}})^n$ for $n \geq 0$ compared to the first term due to the presence of pinching pole contribution $\mathcal{O}(1/\Gamma_{a\,\mathbf{k}})$. Similar calculation can be performed for Eq. (3.30) and one can show the result as given in Eq. (3.34).

Acknowledgments

References

[1] S. Jeon and U. Heinz, *Introduction to Hydrodynamics*, *Int. J. Mod. Phys. E* **24** (2015) 1530010 [1503.03931].

[2] C. Gale, S. Jeon and B. Schenke, *Hydrodynamic Modeling of Heavy-Ion Collisions*, *Int. J. Mod. Phys. A* **28** (2013) 1340011 [1301.5893].

[3] W. Florkowski, *Phenomenology of Ultra-Relativistic Heavy-Ion Collisions* (3, 2010).

[4] U. Heinz and R. Snellings, *Collective flow and viscosity in relativistic heavy-ion collisions*, *Ann. Rev. Nucl. Part. Sci.* **63** (2013) 123 [`1301.2826`].

[5] P. Kovtun, D.T. Son and A.O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601 [`hep-th/0405231`].

[6] K. Dusling and T. Schäfer, *Bulk viscosity, particle spectra and flow in heavy-ion collisions*, *Phys. Rev. C* **85** (2012) 044909 [`1109.5181`].

[7] P. Bozek, *Bulk and shear viscosities of matter created in relativistic heavy-ion collisions*, *Phys. Rev. C* **81** (2010) 034909 [`0911.2397`].

[8] B. Ling, T. Springer and M. Stephanov, *Hydrodynamics of charge fluctuations and balance functions*, *Phys. Rev. C* **89** (2014) 064901 [`1310.6036`].

[9] K. Tuchin, *Electromagnetic fields in high energy heavy-ion collisions*, *Int. J. Mod. Phys. E* **23** (2014) 1430001.

[10] U. Gursoy, D. Kharzeev and K. Rajagopal, *Magnetohydrodynamics, charged currents and directed flow in heavy ion collisions*, *Phys. Rev. C* **89** (2014) 054905 [`1401.3805`].

[11] Y. Hirono, M. Hongo and T. Hirano, *Estimation of electric conductivity of the quark gluon plasma via asymmetric heavy-ion collisions*, *Phys. Rev. C* **90** (2014) 021903 [`1211.1114`].

[12] M. Greif, J.A. Fotakis, G.S. Denicol and C. Greiner, *Diffusion of conserved charges in relativistic heavy ion collisions*, *Phys. Rev. Lett.* **120** (2018) 242301 [`1711.08680`].

[13] J.A. Fotakis, M. Greif, C. Greiner, G.S. Denicol and H. Niemi, *Diffusion processes involving multiple conserved charges: A study from kinetic theory and implications to the fluid-dynamical modeling of heavy ion collisions*, *Phys. Rev. D* **101** (2020) 076007 [`1912.09103`].

[14] J.A. Fotakis, O. Soloveva, C. Greiner, O. Kaczmarek and E. Bratkovskaya, *Diffusion coefficient matrix of the strongly interacting quark-gluon plasma*, *Phys. Rev. D* **104** (2021) 034014 [`2102.08140`].

[15] A. Das, H. Mishra and R.K. Mohapatra, *Diffusion matrix associated with the diffusion processes of multiple conserved charges in a hot and dense hadronic matter*, *Phys. Rev. D* **106** (2022) 014013 [`2109.01543`].

[16] J.A. Fotakis, E. Molnár, H. Niemi, C. Greiner and D.H. Rischke, *Multicomponent relativistic dissipative fluid dynamics from the Boltzmann equation*, *Phys. Rev. D* **106** (2022) 036009 [`2203.11549`].

[17] M. Asakawa and M. Kitazawa, *Fluctuations of conserved charges in relativistic heavy ion collisions: An introduction*, *Prog. Part. Nucl. Phys.* **90** (2016) 299 [`1512.05038`].

[18] S. Jeon and V. Koch, *Charged particle ratio fluctuation as a signal for QGP*, *Phys. Rev. Lett.* **85** (2000) 2076 [`hep-ph/0003168`].

[19] E.V. Shuryak and M.A. Stephanov, *When can long range charge fluctuations serve as a QGP signal?*, *Phys. Rev. C* **63** (2001) 064903 [`hep-ph/0010100`].

[20] S. Pratt and C. Plumberg, *Determining the Diffusivity for Light Quarks from Experiment*, *Phys. Rev. C* **102** (2020) 044909 [`1904.11459`].

[21] L. Onsager, *Reciprocal Relations in Irreversible Processes. II.*, *Phys. Rev.* **38** (1931) 2265.

[22] P.L. Bhatnagar, E.P. Gross and M. Krook, *A Model for Collision Processes in Gases. 1. Small Amplitude Processes in Charged and Neutral One-Component Systems*, *Phys. Rev.* **94** (1954) 511.

[23] G.S. Rocha, G.S. Denicol and J. Noronha, *Novel Relaxation Time Approximation to the Relativistic Boltzmann Equation*, *Phys. Rev. Lett.* **127** (2021) 042301 [2103.07489].

[24] I. Muller, *Zum Paradoxon der Warmeleitungstheorie*, *Z. Phys.* **198** (1967) 329.

[25] S. Chapman and T.G. Cowling, *The Mathematical Theory of Non- Uniform Gases, 3rd ed*, *Cambridge University Press, Cambridge* (1970) .

[26] W. Israel and J.M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, *Annals Phys.* **118** (1979) 341.

[27] M.A. York and G.D. Moore, *Second order hydrodynamic coefficients from kinetic theory*, *Phys. Rev. D* **79** (2009) 054011 [0811.0729].

[28] B. Betz, D. Henkel and D.H. Rischke, *From kinetic theory to dissipative fluid dynamics*, *Prog. Part. Nucl. Phys.* **62** (2009) 556 [0812.1440].

[29] G.S. Denicol, T. Koide and D.H. Rischke, *Dissipative relativistic fluid dynamics: a new way to derive the equations of motion from kinetic theory*, *Phys. Rev. Lett.* **105** (2010) 162501 [1004.5013].

[30] G.S. Denicol, H. Niemi, E. Molnar and D.H. Rischke, *Derivation of transient relativistic fluid dynamics from the Boltzmann equation*, *Phys. Rev. D* **85** (2012) 114047 [1202.4551].

[31] A. Jaiswal, R. Ryblewski and M. Strickland, *Transport coefficients for bulk viscous evolution in the relaxation time approximation*, *Phys. Rev. C* **90** (2014) 044908 [1407.7231].

[32] A. Jaiswal, B. Friman and K. Redlich, *Relativistic second-order dissipative hydrodynamics at finite chemical potential*, *Phys. Lett. B* **751** (2015) 548 [1507.02849].

[33] A. Gabbana, M. Mendoza, S. Succi and R. Tripiccione, *Kinetic approach to relativistic dissipation*, *Phys. Rev. E* **96** (2017) 023305 [1704.02523].

[34] H.B. Meyer, *Transport Properties of the Quark-Gluon Plasma: A Lattice QCD Perspective*, *Eur. Phys. J. A* **47** (2011) 86 [1104.3708].

[35] G. Aarts, C. Allton, A. Amato, P. Giudice, S. Hands and J.-I. Skullerud, *Electrical conductivity and charge diffusion in thermal QCD from the lattice*, *JHEP* **02** (2015) 186 [1412.6411].

[36] D.N. Zubarev, *Nonequilibrium Statistical Thermodynamics*, Springer, US (1974).

[37] A. Harutyunyan, D.H. Rischke and A. Sedrakian, *Transport coefficients of two-flavor quark matter from the Kubo formalism*, *Phys. Rev. D* **95** (2017) 114021 [1702.04291].

[38] X.-G. Huang, A. Sedrakian and D.H. Rischke, *Kubo formulae for relativistic fluids in strong magnetic fields*, *Annals Phys.* **326** (2011) 3075 [1108.0602].

[39] A. Harutyunyan and A. Sedrakian, *Electrical conductivity of a warm neutron star crust in magnetic fields*, *Phys. Rev. C* **94** (2016) 025805 [1605.07612].

[40] A. Harutyunyan, A. Sedrakian and D.H. Rischke, *Relativistic second-order dissipative hydrodynamics from Zubarev's non-equilibrium statistical operator*, *Annals Phys.* **438** (2022) 168755 [2110.04595].

[41] A. Harutyunyan, *Relativistic hydrodynamics and transport in strongly correlated systems*, Ph.D. thesis, Frankfurt U., 2017.

[42] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, *J. Phys. A* **45** (2012) 473001 [1205.5040].

[43] P. Kovtun, *First-order relativistic hydrodynamics is stable*, *JHEP* **10** (2019) 034 [1907.08191].

[44] A. Hosoya, M.-a. Sakagami and M. Takao, *Nonequilibrium Thermodynamics in Field Theory: Transport Coefficients*, *Annals Phys.* **154** (1984) 229.

[45] K. Fukushima and Y. Hidaka, *Electric conductivity of hot and dense quark matter in a magnetic field with Landau level resummation via kinetic equations*, *Phys. Rev. Lett.* **120** (2018) 162301 [1711.01472].

[46] D.N. Zubarev, A.V. Prozorkevich and S.A. Smolyanskii, *Derivation of nonlinear generalized equations of quantum relativistic hydrodynamics*, *Theor. Math. Phys.* **40** (1979) 821.

[47] L. Gavassino and M. Shokri, *Stability of multicomponent Israel-Stewart-Maxwell theory for charge diffusion*, *Phys. Rev. D* **108** (2023) 096010 [2307.11615].

[48] T. Koide, G.S. Denicol, P. Mota and T. Kodama, *Relativistic dissipative hydrodynamics: A Minimal causal theory*, *Phys. Rev. C* **75** (2007) 034909 [hep-ph/0609117].

[49] T. Koide and T. Kodama, *Transport Coefficients of Non-Newtonian Fluid and Causal Dissipative Hydrodynamics*, *Phys. Rev. E* **78** (2008) 051107 [0806.3725].

[50] A. Czajka and S. Jeon, *Kubo formulas for the shear and bulk viscosity relaxation times and the scalar field theory shear τ_π calculation*, *Phys. Rev. C* **95** (2017) 064906 [1701.07580].

[51] M. Laine and A. Vuorinen, *Basics of Thermal Field Theory*, vol. 925, Springer (2016), 10.1007/978-3-319-31933-9, [1701.01554].

[52] S. Jeon, *Computing spectral densities in finite temperature field theory*, *Phys. Rev. D* **47** (1993) 4586 [hep-ph/9210227].

[53] S. Jeon, *Hydrodynamic transport coefficients in relativistic scalar field theory*, *Phys. Rev. D* **52** (1995) 3591 [hep-ph/9409250].

[54] J.I. Kapusta and C. Gale, *Finite-Temperature Field Theory*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (7, 2023), 10.1017/9781009401968.

[55] M. Buballa, *NJL model analysis of quark matter at large density*, *Phys. Rept.* **407** (2005) 205 [hep-ph/0402234].

[56] A. Peshier and W. Cassing, *The Hot non-perturbative gluon plasma is an almost ideal colored liquid*, *Phys. Rev. Lett.* **94** (2005) 172301 [hep-ph/0502138].

[57] H. Berrehrah, E. Bratkovskaya, T. Steinert and W. Cassing, *A dynamical quasiparticle approach for the QGP bulk and transport properties*, *Int. J. Mod. Phys. E* **25** (2016) 1642003 [1605.02371].

[58] B.-J. Schaefer, M. Wagner and J. Wambach, *Thermodynamics of (2+1)-flavor QCD: Confronting Models with Lattice Studies*, *Phys. Rev. D* **81** (2010) 074013 [0910.5628].

[59] S.K. Rai and V.K. Tiwari, *Phase structure of the on-shell parametrized 2+1 flavor Polyakov quark-meson model*, *Phys. Rev. D* **109** (2024) 034025 [2306.12382].

[60] T. Nishikawa, O. Morimatsu and Y. Hidaka, *On the thermal sunset diagram for scalar field theories*, *Phys. Rev. D* **68** (2003) 076002 [`hep-ph/0302098`].

[61] M. Bastero-Gil, A. Berera and R.O. Ramos, *Dissipation coefficients from scalar and fermion quantum field interactions*, *JCAP* **09** (2011) 033 [`1008.1929`].

[62] D.D. Ofengeim, M.E. Gusakov, P. Haensel and M. Fortin, *Bulk viscosity in neutron stars with hyperon cores*, *Phys. Rev. D* **100** (2019) 103017 [`1911.08407`].

[63] X.-G. Huang, M. Huang, D.H. Rischke and A. Sedrakian, *Anisotropic Hydrodynamics, Bulk Viscosities and R-Modes of Strange Quark Stars with Strong Magnetic Fields*, *Phys. Rev. D* **81** (2010) 045015 [`0910.3633`].

[64] K.Y. Kraav, M.E. Gusakov and E.M. Kantor, *Diffusion as a leading dissipative mechanism in superconducting neutron stars*, *Mon. Not. Roy. Astron. Soc.* **506** (2021) L74 [`2105.08121`].

[65] T.K. Jha, H. Mishra and V. Sreekanth, *Bulk viscosity in hyperonic star and r-mode instability*, *Phys. Rev. C* **82** (2010) 025803 [`1002.4253`].

[66] D. Kumar, H. Mishra and T. Malik, *Non-radial oscillation modes in hybrid stars: consequences of a mixed phase*, *JCAP* **02** (2023) 015 [`2110.00324`].

[67] L. Gavassino, *Relativistic bulk viscous fluids of Burgers type and their presence in neutron stars*, *Class. Quant. Grav.* **40** (2023) 165008 [`2304.05455`].

[68] A.K. Panda, P. Bagchi, H. Mishra and V. Roy, *Electromagnetic fields in low-energy heavy-ion collisions with baryon stopping*, `2404.08431`.

[69] D. Roder, J. Ruppert and D.H. Rischke, *Selfconsistent calculations of spectral densities in the O(N) model: Improving the Hartree-Fock approximation by including nonzero decay widths*, *Nucl. Phys. A* **775** (2006) 127 [`hep-ph/0503042`].