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Dynamical friction in the quasi-linear formulation of MOND

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ABSTRACT

Aims. We explore the dynamical friction on a test mass in gravitational systems in the Quasi linear formulation of Modified Newtonian Dynamics (QuMOND).

Methods. Exploiting the quasi linearity of QuMOND we derive a simple expression for the dynamical friction in akin to its Newtonian counterpart in the standard Chandrasekhar derivation. Moreover, adopting a mean field approach based on the Liouville equation we obtain a more rigorous (though in integral form) dynamical friction formula that can be evaluated numerically for a given choice of the QuMOND interpolation function.

Results. Consistently with previous work, we observe that dynamical friction is stronger in MOND with respect to a baryon only Newtonian system with the same mass distribution. This amounts to a correction of the Coulomb logarithmic factor via extra terms proportional to the MOND radius of the system. Morover, with the aid of simple numerical experiments we confirm our theoretical predictions and those of previous work on MOND.

Key words. Galaxies: kinematics and dynamics - Stars: kinematics and dynamics - Gravitation - Methods: analytical

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1. Introduction

Modified Newtonian dynamics (hereafter MOND, Milgrom 1983) is an alternative theory of (classical) gravity originally introduced to solve the missing mass problem on astrophysical and cosmological scales without resorting to dark matter (hereafter DM). In a Lagrangian formulation (see Bekenstein & Milgrom 1984), MOND amends to a substitution of the Poisson equation relating the gravitational potential Φ and mass density ρ, with the non-linear field equation

$$\boldsymbol{\nabla}\cdot\left[\mu\left(\frac{||\boldsymbol{\nabla}\Phi||}{a_0}\right)\boldsymbol{\nabla}\Phi\right]=4\pi G\rho, \quad (1)$$

where the acceleration scale $a_0\approx 1.2\times10^{-2}\mathrm{ms}^{-2}$ is a new universal constant, and the in principle unknown interpolating function μ is monotonic with the asymptotic behavior

$$\mu(x)\to^{x\gg1}1,\quad \mu(x)\sim^{x\ll1}x. \quad (2)$$

A widely adopted form of μ is

$$\mu(x)=\frac{x}{1+x}. \quad (3)$$

According to the equations above, for $\nabla\Phi\gg a_0$ one recovers the Newtonian limit and Eq. (1) reduces to the Poisson equation. Vice versa, for $\nabla\Phi\ll a_0$ the system is in the so-called deep-MOND limit (hereafter dMOND) and Eq. (1) becomes the p-Laplace equation

$$\boldsymbol{\nabla}\cdot(||\boldsymbol{\nabla}\Phi||\boldsymbol{\nabla}\Phi)=4\pi a_0 G\rho. \quad (4)$$

For a given mass density ρ the right-hand-side of the Poisson equation and Eq. (1) are the same. One can therefore eliminate ρ obtaining the relation

$$\mu\left(\frac{||\mathbf{g}_M||}{a_0}\right)\mathbf{g}_M=\mathbf{g}_N+\mathbf{S} \quad (5)$$

between the MOND and Newtonian force fields $\mathbf{g}_M$ and $\mathbf{g}_N$, where $\mathbf{S}\equiv\nabla\times\mathbf{h}(\rho)$ is a density-dependent solenoidal field that zeros-out for systems in spherical, cylindrical or planar symmetry, while it is typically non-zero for more general density profiles.

MOND has been rather successful in reproducing the kinematics of galaxies without the need of DM (Bugg 2015; Zhu et al. 2023), and has proven to be a valid alternative to the ΛCDM paradigm in broad range of gravitational phenomena (Milgrom 1994; Sánchez-Salcedo & Hernandez 2007; McGaugh & Milgrom 2013; Sanders 2021; Bílek et al. 2021; Asencio et al. 2022; Scarpa et al. 2022).

MOND has been extensively investigated numerically

(Brada & Milgrom 1999; Knebe & Gibson 2004; Tiret & Combes 2007; Nipoti et al. 2007a,b,c; Sanders 2008; Wu et al. 2009; Malekjani et al. 2009; Kroupa et al. 2022; Banik et al. 2022; Nagesh et al. 2022; Nagesh et al. 2023; Re & Di Cintio 2023; Sollima et al. 2023), for collisionless mean field processes, such for example violent relaxation and phase-mixing in monolithic collapse, galaxy merging, vertical dynamics of disk galaxies. Unfortunately, due to the non-linearity of the theory (cfr. Eq. 1 and hence the absence of the superposition principle, much less is known for what concerns the collisional processes (see e.g. Bílek et al. 2021). For example, Ciotti & Binney (2004) using an approach based on the fluctuations of uniform field estimated the MOND two-body relaxation (t_{2b}, Chandrasekhar 1941a,b) and dynamical friction (t_{DF}, Chandrasekhar 1943a,b,c) time scales in the dMOND limit. The outcome of this study (see also the numerical simulations in Nipoti et al. 2008) is that a test mass M crossing a purely baryonic system would feel a stronger dynamical friction (hereafter, DF) force due to the encounters with the system's stars in MOND than in the parent Newtonian system without DM, while would undergo an only slightly more efficient DF than in the equivalent Newtonian system (hereafter ENS, i.e. the baryonic plus DM system constructed such that the potential is the same as the purely baryonic MOND model). The Newtonian and MOND DF time scales $t^{\mathrm{N}}_{\mathrm{DF}}$ and $t^{\mathrm{M}}_{\mathrm{DF}}$, are related as

$$t^{\mathrm{M}}_{\mathrm{DF}} = \frac{\sqrt{2}}{1+\mathcal{R}} t^{\mathrm{N}}_{\mathrm{DF}}, \tag{6}$$

where $\mathcal{R}$ is the ratio of the amount of DM to baryons in the ENS.

Since in MOND collisional direct $N-$body simulations are intrinsically unfeasible, results such as Eq. (6) can not be exhaustively explored via numerical experiments; the reason being the absence of a MONDian expression for the force exerted by a point-like particle. Milgrom (1986) proposed the approximated expression for the force exchanged between two masses m_1 and m_2 placed at distance r

$$F_{1,2} \approx \frac{m_1 m_2}{\sqrt{m_1+m_2}} \frac{\sqrt{Ga_0}}{r}, \tag{7}$$

The latter is however valid only in the far field limit, thus making it unusable in a direct $N-$body code, as it lacks a regime bridging to the Newtonian $1/r^2$ limit.

More recently, Milgrom (2010) formulated a quasi-linear MOND theory (hereafter QuMOND) where the governing field equation is

$$\boldsymbol{\nabla} \cdot \left[\nu\left(\frac{\|\boldsymbol{\nabla}\Phi^N\|}{a_0}\right) \boldsymbol{\nabla}\Phi^N \right] = \Delta\Phi^M, \tag{8}$$

formally identical to Equation (1) where now, Φ^N is the Newtonian potential generated by the density ρ through the usual Poisson equation

$$\Delta\Phi^N = -\boldsymbol{\nabla}\cdot\boldsymbol{g}^N = 4\pi G\rho. \tag{9}$$

and the new interpolating function $\nu(y)$ is related to $\mu(x)$ by

$$\begin{cases} x = y\nu(y) \\ y = x\mu(x) \end{cases} \Rightarrow \mu(x) = \frac{y}{x} = \frac{1}{\nu(y)}. \tag{10}$$

The asymptotic behaviours for ν in the Newtonian and in the dMOND limits are

$$\nu(y) \to^{y\gg1} 1, \quad \nu(y) \sim^{y\ll1} y^{-1/2}. \tag{11}$$

We note that, Equation (8) can be rearranged as

$$\tilde{\rho} = -\frac{1}{4\pi G}\nabla\cdot[\nu(g^N/a_0)\mathbf{g}^N]; \quad \mathbf{g}^N = -\nabla\Phi^N, \tag{12}$$

that is, the QuMOND potential satisfies the Poisson equation for the auxiliary density $\tilde{\rho}$. In practice, in QuMOND one has to evaluate the Newtonian potential Φ^N generated by ρ, evaluate its gradient and then via a non-linear algebraic step obtain the auxiliary MOND density $\tilde{\rho}$ via Eq. (12) that acts as source for the potential Φ^M.

At variance with the usual MOND formulation, in its quasi-linear formulation the interpolating function ν has a stronger effect on the form of the gravitational potential than its counterpart μ in Eq. (1). For the usual choice (3), one obtains

$$\nu(y) = \frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{y}}. \tag{13}$$

We note that, QuMOND contains a sort of superposition principle for the auxiliary densities $\tilde{\rho}$. It is therefore tantalizing to try to derive an expression for the DF force in this context, making use of purely kinetic arguments as in the original derivation. In this paper, we first discuss a simple expression of the QuMOND DF summing the contributions of the effective densities for point-particles of mass m interacting with a test mass, with some simple choices of ν. Moreover, we extend the mean field formulation of DF, originally derived in Newtonian gravity by Kandrup (1983) to the case of QuMOND and discuss the results of simple numerical experiments.

2. Dynamical friction: classical approach

2.1. Newtonian case

Before tackling the QuMOND DF formula, let us briefly recall the ingredients of the derivation of its Newtonian expression (for a more detailed review see e.g. Binney & Tremaine 2008 or Ciotti 2021). Let us consider a test mass M travelling at v_M through an infinite background medium of particles with mass m and (by now constant) number density n with velocity distribution function $f(v)$. In each encounter M and a background particle exchange a force $F \approx GMm/b^2$, acting for an interval of time $\Delta t \approx b/v_M$, where b is the impact parameter of the dynamical collision. The associated variation of momentum is $\Delta p \approx F\Delta t \approx GMm/v_M b$. The rate of collisions of M is then

$$\dot{N}_{\mathrm{enc}} = \frac{2\pi nbdbdx}{dt} = 2\pi n v_M b db. \tag{14}$$

From this, the velocity diffusion coefficient along the trajectory of M can be written as

$$D_v = \int \Delta p^2 \frac{dN_{\mathrm{enc}}}{dt} = 2\pi G^2 M^2 \frac{m^2 n}{v_M} \int \frac{db}{b}. \tag{15}$$

In the Equation above, the integral in the impact parameter b diverges for both $b \to 0$ and $b \to \infty$ (infrared and ultraviolet divergence, respectively). It is however possible to impose suitable cut-offs $b_{\min}$ and $b_{\max}$. This factor, after integration takes the form of the Coulomb logarithm (in analogy with the same quantity in plasma physics) $\ln\Lambda$, where $\Lambda = b_{\max}/b_{\min}$.

Let us now assume that $f(v)$, defined such that

$$\int f(\boldsymbol{x},\boldsymbol{v})d^3\boldsymbol{x}d^3\boldsymbol{v} = N \tag{16}$$

is the total number of particles, is position independent and isotropic for all velocities. For such a system, the DF force per unit mass acting on M is given as

$$\frac{d\mathbf{v}_M}{dt} = -16\pi^2 G^2\rho(M+m)\ln\Lambda\frac{\boldsymbol{v}_M}{v_M^3}\int_0^{v_M} v^2 f(v)dv. \quad (17)$$

In the assumption of velocity isotropy, only particles moving slower than M contribute to its slowing down. Moreover for $M \gg m$, in Eq. (17) the factor $m(M+m)n$ becomes $M\rho$, where ρ is the background mass density.

2.2. QuMOND case

We first present a naive general expression of the MONDian DF making use of the fact that in QuMOND

$$F = \nu\left(\frac{Gm}{a_0 b^2}\right)\frac{GMm}{b^2}. \quad (18)$$

For each particle of the background system we define the so-called MOND radius as

$$r_{\rm M} = \sqrt{\frac{Gm}{a_0}} \quad (19)$$

With the same arguments of Sect. 2.1 we now find that the momentum variation of M is

$$\Delta p = \nu(r_{\rm M}^2/b^2)GMm/v_M b. \quad (20)$$

The form of the collision rate factor remains unchanged, as it is a geometric quantity. The diffusion coefficient D_v retains the same definition as in Eq. (15), but with a different integral over the impact parameter

$$I_b = \int_{b_{\min}}^{b_{\max}} \nu(r_{\rm M}^2/b^2)^2 db/b. \quad (21)$$

Once substituting in the DF formula, one obtains

$$\frac{d\mathbf{v}_M}{dt} = -16\pi^2 I_b G^2\rho(M+m)\frac{\boldsymbol{v}_M}{v_M^3}\int_0^{v_M} v^2 f(v)dv. \quad (22)$$

From Equation (22) above we notice immediately that the QuMOND dynamical DF is always larger than its Newtonian expression for the same mass distribution. This happens because $\nu^2 > 1$, thus returning a larger factor within the integral over the impact parameters. In addition, this is also in agreement with the fact that in MOND theories the far field behaviour of a given mass distribution falls off less rapidly than in Newtonian gravity, as implied by the logarithmic trend of the MOND potential.

We now evaluate Equation (22) for a simple choice of ν. Knowing that ν has the asymptotic trends given in Eq. (11) we start with a simplified form

$$\nu(y) = 1 + 1/\sqrt{y}, \quad (23)$$

that is essentially the sum of the Newtonian and the deep MOND régimes. In terms of μ, Eq. (23) corresponds to $\mu(x) = 1 + (1-\sqrt{1+4x})/2x$. For such a choice, the correction factor is $\nu(r_{\rm M}^2/b^2) = 1 + b/r_{\rm M}$, and the integral on the impact parameters becomes

$$I_b = \ln\frac{b_{\max}}{b_{\min}} + 2\frac{b_{\max}-b_{\min}}{r_{\rm M}} + \frac{b_{\max}^2 - b_{\min}^2}{2r_{\rm M}^2} = \cong \ln\Lambda + 2\frac{b_{\max}}{r_{\rm M}} + \frac{b_{\max}^2}{2r_{\rm M}^2}; \quad (24)$$

where we assumed $b_{\min} \ll r_{\rm M}$ in the last equality. We note that, the maximum impact parameter $b_{\max}$ in the context of Newtonian gravity is usually assumed to be some (arbitrary) scale distance[1] of the system at hand. In the original derivation of Chandrasekhar (1943a), where the system is infinite, the definition of $b_{\max}$ remains somewhat unclear. Some authors (see e.g. Van Albada & Szomoru 2020 and references therein) typically take $b_{\max}$ of the order of the average inter-particle distance. Hereby, we follow the same assumption noting that for a star of a solar mass the MOND radius $r_{\rm M}$ is roughly 10^3 AU, much smaller than the typical scale radius of a galaxy, but comparable to the mean inter-particle distance among stars. This established, the QuMOND DF then becomes

$$\left(\frac{d\mathbf{v}_M}{dt}\right) \approx -16\pi^2\left(\ln\Lambda + \frac{b_{\max}^2}{2r_{\rm M}^2} + 2\frac{b_{\max}}{r_{\rm M}}\right)\times \times G^2\rho(M+m)\frac{\boldsymbol{v}_M}{v_M^3}\int_0^{v_M} v^2 f(v)dv. \quad (25)$$

In practice, at first order in QuMOND each particle behaves as a point-like source exerting the usual Newtonian $1/r$ potential plus an infinitely extended "phantom DM" halo whose contribution depends on the specific form of ν. Following this approach, the calculations as shown in Appendix A leads to the same expression of Eq. (25). We note that Equation (25) differs from what one would obtain using the simple derivation of t_{2b} in Sect. 2 of Ciotti & Binney 2004 that is proportional to the crossing time $t_{\rm cross} = 1/\sqrt{G\rho}$ to evaluate the dynamical friction time scale with the Spitzer (1987) relation $t_{\rm DF} = 2t_{2b}m/(m+M)$. This is because their expression of t_{2b} makes use of the force (7), assuming implicitly the dMOND regime.

Unlike the usual MOND interpolating function $\mu(x)$, the form of the QuMOND function $\nu(y)$ affects the behaviour of the MONDian force field. Adopting the commonly used Eq. (13) (see Kroupa et al. 2022), the DF expression becomes

$$\frac{d\mathbf{v}_M}{dt} \approx -16\pi^2\left(\ln\Lambda + \frac{b_{\max}^2}{2r_{\rm M}^2} + \frac{b_{\max}}{r_{\rm M}} - \frac{1}{2}\ln\frac{b_{\max}}{r_{\rm M}} - \frac{2+\ln 2}{4}\right)\times \times G^2\rho(M+m)\frac{\boldsymbol{v}_M}{v_M^3}\int_0^{v_M} v^2 f(v)dv. \quad (26)$$

The explicit derivation is given in Appendix B below.

Noteworthy, we observe in both cases that the MOND correction is proportional to a function of the only $b_{\max}/r_{\rm M}$

$$\left(\frac{d\mathbf{v}_M}{dt}\right)_{\rm QuMOND} - \left(\frac{d\mathbf{v}_M}{dt}\right)_{\rm Newt} \propto \frac{b_{\max}^2}{2r_{\rm M}^2} + 2\frac{b_{\max}}{r_{\rm M}} \quad , \quad \frac{b_{\max}^2}{2r_{\rm M}^2} + \frac{b_{\max}}{r_{\rm M}} - \frac{1}{2}\ln\frac{b_{\max}}{r_{\rm M}} - \frac{1}{2}. \quad (27)$$

The same term $b_{\max}^2/2r_{\rm M}^2$ always dominates the MOND correction of DF, regardless of the exact form of ν, while the smaller terms are affected by the particular choice. Recently some authors (see e.g. Lelli et al. 2017) proposed an expression for ν supported by observational data proportional to the form of radial acceleration relation. However, its functional form is rather complex thus making the explicit evaluation of Eq. (22) particularly cumbersome.

As an example, in Figure 1 we show for $M \approx 2m$ the ratio

[1] Such cutoff length in neutral plasma physics is unequivocally constrained by the Debye length, see Spitzer (1965)

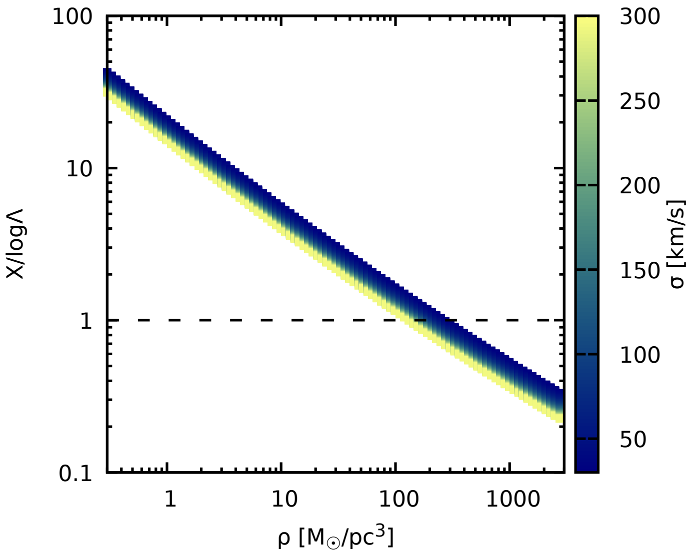

Fig. 1. Ratio of the QuMOND correction factor X to the standard Coulomb logarithm as function of the stellar density. The dependence on the local velocity dispersion σ is color coded.

of the corrective QuMOND term $X = b_{\max}^2/2r_M^2 + b_{\max}/r_M - 1/2\ln(b_{\max}/r_M) - (2+\ln 2)/4$ of Eq. (26) with the standard Coulomb logarithm as function of the stellar density, where we have assumed $b_{\max} = (\rho/M_\odot)^{-1/3}$ and $b_{\min} = G(M+m)/\sigma^2$, where σ (colour-coded in Figure) is the local velocity dispersion (see e.g. Binney & Tremaine 2008). It is evident that for an isolated baryon only system the DF in QuMOND is augmented (up to a factor ≈ 50) at low density, comparable to the typical intra-galactic density of 10^{-1} stars per pc^3, with respect to the Newtonian case. Vice versa and as expected, in denser systems/regions the QuMOND correction is negligible, in particular for large values of the velocity dispersion σ.

3. Mean field approach

A more rigorous evaluation of the DF expression can be carried out using the mean field formalism developed by Kandrup (1983) (see also Gilbert 1968 and Kandrup 1980) in the context of Newtonian gravity that can be applied to any kind of long-range force, not necessarily obeying the superposition principle.

Let us consider the usual system of N equal mass m (field) particles, with coordinates $\mathbf{r}_i$ and momenta $\mathbf{p}_i = m\mathbf{v}_i$ described by the (time-dependent) phase-space distribution function[2] $\mathcal{F}(\mathbf{r},\mathbf{p};t)$; and a test particle M with coordinate $\mathbf{r}_0 = \mathbf{R}$ and momentum $\mathbf{p}_0 = \mathbf{P} = M\mathbf{v}_M$ that perturbs the initial phase-space distribution $\mathcal{F}_0$. In virtue of the Third Law of Dynamics, such perturbation corresponds to the force decelerating the test mass M. We assume the field particles to be statistically uncorrelated in their initial state with Maxwellian velocity distribution. Moreover, we also take the limit of infinite, homogeneous distribution of field particles. In his original work, Kandrup (1983) initially assumes an external potential Φ confining the system; see Appendix C below. We note that, MOND systems with self-consistent gravitational field $g_{\rm in}$ embedded in an external gravitational field $g_{\rm ext}$ are prone to the so-called external field effect (hereafter EFE). The latter implies that for $g_{\rm in} < a_0 < g_{\rm ext}$ the system is purely Newtonian, while for $g_{\rm in} < g_{\rm ext} < a_0$ the system behaves as a Newtonian model with rescaled gravitational constant $G' = Ga_0/g_{\rm ext}$.

Under the assumptions given above, as usual, the average number density of particles $n = N/\mathrm{Vol}$ should be taken constant as $N, \mathrm{Vol} \to \infty$. The dynamical friction force $F_0^{\rm fr}$ can be therefore obtained simply as $\langle F_0^{\rm tot}\rangle_{\mathcal{F}}$.

In the same fashion of the special relativistic extensions of DF (see e.g. Syer 1994 and Chiari & Di Cintio 2023) we shift to the frame of reference of M such that $\hat{\mathbf{r}}, \hat{\mathbf{p}}$. The linear evolution operator for $\mathcal{F}$ takes the form $\dot{\mathcal{F}}(\hat{\mathbf{r}}_i, \hat{\mathbf{p}}_i; t) = -i\mathcal{L}[\mathcal{F}]$. See Eqs. (C.7) and (C.9) in the Appendix below for its explicit expression.

The linear formulation of the evolution equation is allowed since MOND (and therefore QuMOND), although non-linear, is still a Lagrangian and local[3] theory. At this stage, it is not necessary to consider explicitly the Bekenstein & Milgrom (1984) (or Milgrom 2010) Langrangian, since it is enough to consider the particle's energies.

The significant difference with respect to the Newtonian case explored in Kandrup (1983) is that in MOND one cannot substitute the potentials in a form

$$W_i = \sum_{j\neq i} W_{ij}(|\mathbf{r}_i - \mathbf{r}_j|), \tag{28}$$

due to the non-linearity of MOND implied by the absence of a Superposition principle. Such potentials however can be rather calculated in the QuMOND formalism (8).

A general expression for the DF force on the test particle can be written formally as

$$\mathbf{F}_0^{\rm fr} = -\beta \int d\Sigma \mathcal{F}_0 \left(\sum_{j=1}^{N} \mathbf{F}_i^{\rm tot}\right) \int_0^t d\tau G_{\mathcal{L}}(\tau \to t) \left[\sum_{i=1}^{N} \mathbf{v}_i \cdot \mathbf{F}_i^{\rm tot}\right], \tag{29}$$

where $G_{\mathcal{L}}$ is the Greenian of the operator $\mathcal{L}$ and $d\Sigma = d^{3N}\mathbf{r}d^{3N}\mathbf{p}$ is the differential element in phase space. The details of the derivation are discussed in Appendix C.

We now sketch the main simplifying hypotheses needed to perform practically the integrals in Eq. (29). Assuming that the system under consideration has a finite memory (as implied by its ergodicity), we can replace the time integration in τ on the finite interval $[0;t]$ with an integration extended over the semi-infinite interval $[0;\infty)$. Moreover, making use of the standard linear trajectory approximation (see e.g. Ter Haar 1977, see also Syer 1994), allows to simplify $G_{\mathcal{L}}(\tau \to t)[Q] \cong Q(t-\tau)$. The latter is the most delicate step of the present calculation, since in this approximation the effects of the different field particles decouple that is in principle not valid in a non-linear theory.

Under the assumptions listed above, the DF force formula becomes

$$M\frac{d\mathbf{v}_M}{dt} = \mathbf{F}_0^{\rm fr} \cong -\frac{3(GM)^2\rho}{\langle v^2\rangle} \int v_\alpha f(\mathbf{v}) d^3\mathbf{v} \int_0^\infty d\tau \times \int d^3\mathbf{s}\nu\left(\frac{r_{\rm M}^2}{|\mathbf{s}-\tilde{\mathbf{v}}\tau|^2}\right) \frac{s^\alpha - \tilde{v}^\alpha\tau}{|\mathbf{s}-\tilde{\mathbf{v}}_1\tau|^3} \nu\left(\frac{r_{\rm M}^2}{s^2}\right)\frac{\mathbf{s}}{s^3}, \tag{30}$$

where $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{V}$. At this stage, the problem is reduced to solving a three dimensional integral for a given choice of the QuMOND

[2] Note that, $\mathcal{F}$ is not the one particle distribution function $f(r,v)$ entering the collisionless Boltzmann equation, but rather the phase-space distribution of a $6N$ degrees of freedom Hamiltonian system entering the Liouville equation.

[3] MOND is local in the sense that the density at point ξ equals a differential operator acting on the potential in ξ. Vice versa, for the case of $1/r^\alpha$ forces with $\alpha \neq 2$, obeying the superposition principle this is

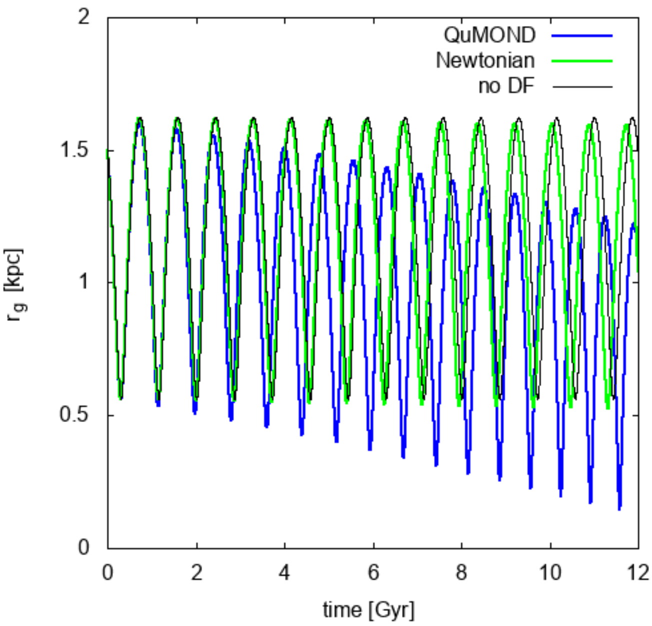

Fig. 2. Evolution of the galactocentric radius r_g for a star cluster orbiting through a dwarf galaxy in Newtonian gravity (green curve) and QuMOND (blue curve). The thin black line marks the case without dynamical friction.

interpolation function ν.

Equation (30) in the Newtonian limit (i.e. $\nu \rightarrow 1$) can be solved by an integration by parts applying

$$\nabla \cdot \frac{\mathbf{s}}{s^3} = -4\pi\delta^3(\mathbf{s}) \tag{31}$$

and

$$\mathbf{v} \cdot \frac{\mathbf{s} - \tilde{\mathbf{v}}\tau}{|\mathbf{s} - \tilde{\mathbf{v}}_1\tau|^3} = \nabla \cdot \frac{\mathbf{v}}{|\mathbf{s} - \tilde{\mathbf{v}}_1\tau|}, \tag{32}$$

so that one has

$$\int \left(\mathbf{v} \cdot \frac{\mathbf{s} - \tilde{\mathbf{v}}\tau}{|\mathbf{s} - \tilde{\mathbf{v}}_1\tau|^3} \right) \frac{\mathbf{s}}{s^3} d^3\mathbf{s} = 4\pi \frac{\mathbf{v}}{\tilde{v}\tau}. \tag{33}$$

Here the term $\tilde{v}\tau = b$ is the standard impact parameter (see again Binney & Tremaine 2008), and thus the integral

$$\int_0^\infty \frac{d\tau}{\tau} = \int_0^\infty \frac{db}{b} \tag{34}$$

returns the usual Coulomb logarithm (see e.g. Kalnajs 1972), while the integral in $d^3\mathbf{v}$ is rewritten as in the Newtonian case, therefore yielding Eq. (17).

The QuMOND case is, even for the simple form of ν given in Eq. (23), is plagued by its intrinsic complexity. Analogously to the naive formula (22) we notice the presence of two factors ν. In this mean field formalism the arguments for the two functions ν in (30) are however slightly different. From such difference $-\tau\tilde{\mathbf{v}}$ one recovers the logarithmic term in τ, as well as any additional QuMOND terms. At variance with the Newtonian case, which requires some cutoff only for the integral over $d\tau$, in the QuMOND DF formula (30) the integral over $d^3\mathbf{s}$ is also diverging. Therefore needing a cutoff at $b_{\max}$. This yields terms of the form $b_{\max}/r_{\rm M}$ and $b^2_{\max}/r^2_{\rm M}$, substantially in agreement with the simplified approach performed in Sect. 2.2.

not true (Stein 1970). In practice, for such general long-range interactions, density and potential are related only via integral relation over the whole domain occupied by the system (Di Cintio & Ciotti 2011; Di Cintio et al. 2013).

4. Numerical experiments

To clarify the different behaviour of DF in the QuMOND and Newtonian gravity model of a given star system, it is instructive perform a simple numerical experiment in a set-up such that in Newtonian gravity with DM the friction on the test object M would be negligible while the gravitational field of the parent system is in the MONDian regime $g \ll a_0$ over a broad interval of radii.

We consider a $M = 10^2 M_\odot$ star cluster moving through a dwarf galaxy with stellar mass $M_* = 2 \times 10^5 M_\odot$ and scale radius $r_s = 0.8$ kpc (parameters compatible to those of the Draco ultrafaint dwarf galaxy, Mashchenko et al. 2006), with stellar distribution given by

$$\rho_*(r) = \frac{3}{4\pi} \frac{M_i r_s}{(r + r_s)^4}, \tag{35}$$

corresponding to a Dehnen (1993) $\gamma-$model for $\gamma = 0$. The associated QuMOND gravitational potential given by Eq. (8) is by construction identical to the potential exerted by a Newtonian system with stellar density (35) plus a DM halo with $\rho_{\rm DM} = \tilde{\rho} - \rho_*$ (cfr. Eq. 12). By contrast, the DF force experienced by M is different in the two paradigms -Newtonian and MOND- being Eq. (17) with $\rho_{DM} + \rho_*$ in lieu of ρ in the first case[4] and Eq. (26) in the second.

Using the numerical approach discussed in Appendix D, see also Pasquato & Di Cintio (2020); Di Cintio et al. (2020); Di Cintio & Casetti (2022), we integrated different orbits under the effect of the same gravitational field and the two different DF expressions, in the assumption of local Maxwellian approximation (i.e. at all radii the velocity distribution is an isotropic Maxwell-Boltzmann with velocity dispersion approximated by $\sigma = \sqrt{2||\Phi||}$). In Figure 2 we show the evolution of the galactocentric distance r_g of M over 12 Gyr for an orbit of initial ellipticity $e = 0.49$. The green and blue curves mark the Newtonian and QuMOND simulations, respectively, while the black curve is the unperturbed orbit in the static potential of the model. We observe that, while the Newtonian DF only minimally alters the orbit of M at around 9 Gyr, in QuMOND, the test particle suffers a rather strong orbital decay with pergalactic radius falling from $\approx$ 1.63 kpc down to $\approx$ 1.2 kpc at 12 Gyr. This is remarkably in agreement with the analytical estimates of Ciotti & Binney (2004), that predicted a MOND inspiral time for an object considerably more massive than a star to the central regions of a Draco-like system of less than a Hubble time, much shorter than its Newtonian analog and argued that this could be the reason behind the absence of star clusters in similar dwarf galaxies.

5. Conclusions

We have investigated the dynamical friction in the Quasi linear formulation of MOND. Using a simple dimensional analysis we find that the expression for the MONDian DF is augmented with respect to its Newtonian counterpart of a factor proportional to the MOND radius of a typical stellar mass $r_{\rm M}$. Such additional term becomes relevant when the maximum impact parameter (proportional to the average interparticle distance, Spitzer 1987) $b_{\max} \gg r_{\rm M}$, cfr. Eqs. (25,26). In practice, in a dense star cluster or in galactic nuclei where the mean stellar density is such that the

[4] We recall that, at equipartition in a multimass system (e.g. stars and elementary particle sized DM), when the test mass M is much larger than the mean mass $\langle m \rangle$ the prefactor of Eq. (17) is dominated by M times the total mass density, see e.g. Ciotti (2021).

typical inter-stellar distance is always smaller than $r_{\rm M}$, the DF is always purely Newtonian. Vice versa, in low density systems, such as ultrafaint Dwarf galaxies (Simon 2019) the $b_{\rm max}/r_{\rm M}$ correction is always of the order of the Coulomb logarithm, thus enhancing the (stellar) DF for a system where in principle it should be negligible. In addition we find that, the explicit form of the QuMOND correction is dependent on the specific choice of the interpolating function ν. However, the strength of the enhancing term does not vary significantly for such different forms of ν.

In additions, we have extended the mean field treatment of DF, pioneered by Kandrup (1983) in the Newtonian case to QuMOND. In this framework, alternative to the fluctuation-based approach by Ciotti & Binney (2004), we recovered an integral expression that contains the usual dependence on ν. Unfortunately, the explicit evaluation of the DF force, becomes rapidly cumbersome (at variance with the simpler Newtonian case) even in the dMOND limit, one is therefore forced to integrate Eq. (30) numerically. Some specific cases will be presented and discussed elsewhere. Working out in QuMOND the two-body relaxation time (and therefore the DF coefficient of Eq. 6 via the relation $t_{DF} = 2t_{2b}m/M$) with the fluctuation approach in Fourier space of Ciotti & Binney (2004), though in principle possible, as it would imply a double application of the classical Poisson equation and a single non-linear algebraic step is hindered by the implicit relation between the Fourier transform of the QuMOND potential and the associated auxiliary density $\tilde{\rho}$.

Interestingly, simple numerical integration of a test particle in a QuMOND and its Newtonian equivalent system evidence that dynamical friction acts in a different way in the two paradigms, being considerably stronger in the first if the model is in a deep MOND regime (such as for DM-dominated dwarf galaxies). We stress the fact however that, in MOND one can not simply add a semi-analytic DF force to a mean field potential obtained solving numerically Eq. (1) for a given density (either imposed extrapolated from particles position), in the same fashion as Alessandrini et al. (2014); Arca-Sedda & Capuzzo-Dolcetta (2014); Di Cintio & Casetti (2022). and therefore our simple numerical estimates could possibly overestimate the MONDian DF. Using the quasi-linear formulation of MOND could in principle allow to incorporate the contribution of DF (and possibly density fluctuations on a scale smaller than the particle-resolution) adding it before evaluating $\tilde{\rho}$ to solve Eq. (8) (see Di Cintio 2023). In conclusion, it is also important to recall that, as a consequence of the EFE, in systems with $g_{\rm int} < g_{\rm ext} < a_0$, the dynamics essentially Newtonian with a rescaled G. In those cases, the DF force would be given by Eq. (17) augmented by the multiplicative factor $(a_0/g_{\rm ext})^2$, and thus a full mean field QuMOND treatment including the DF expression discussed in Sect. 3 is valid only for a isolated system.

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References

Alessandrini, E., Lanzoni, B., Miocchi, P., Ciotti, L., & Ferraro, F. R. 2014, ApJ, 795, 169
Arca-Sedda, M. & Capuzzo-Dolcetta, R. 2014, ApJ, 785, 51
Asencio, E., Banik, I., Mieske, S., et al. 2022, MNRAS, 515, 2981
Banik, I., Thies, I., Truelove, R., et al. 2022, Monthly Notices of the Royal Astronomical Society, 513, 129–158
Bekenstein, J. & Milgrom, M. 1984, ApJ, 286, 7
Binney, J. & Tremaine, S. 2008, Galactic Dynamics: Second Edition (Princeton University Press)
Brada, R. & Milgrom, M. 1999, ApJ, 519, 590
Bugg, D. 2015, Canadian Journal of Physics, 93, 119–125
Bílek, M., Zhao, H., Famaey, B., et al. 2021, Astronomy & Astrophysics, 653, A170
Chandrasekhar, S. 1941a, ApJ, 93, 285
Chandrasekhar, S. 1941b, ApJ, 93, 323
Chandrasekhar, S. 1943a, ApJ, 97, 255
Chandrasekhar, S. 1943b, ApJ, 97, 263
Chandrasekhar, S. 1943c, ApJ, 98, 54
Chiari, C. & Di Cintio, P. 2023, A&A, 677, A140
Ciotti, L. 2021, Introduction to Stellar Dynamics
Ciotti, L. & Binney, J. 2004, MNRAS, 351, 285
Dehnen, W. 1993, MNRAS, 265, 250
Di Cintio, P. 2023, Disruption of star clusters in Newtonian and MOND gravity
Di Cintio, P. & Casetti, L. 2022, in Multi-Scale (Time and Mass) Dynamics of Space Objects, ed. A. Celletti, C. Galeş, C. Beaugé, & A. Lemaître, Vol. 364, 152–157
Di Cintio, P. & Ciotti, L. 2011, International Journal of Bifurcation and Chaos, 21, 2279
Di Cintio, P., Ciotti, L., & Nipoti, C. 2013, MNRAS, 431, 3177
Di Cintio, P., Ciotti, L., & Nipoti, C. 2020, in Proceedings of the IAU Symposia, Vol. 351, Star Clusters: From the Milky Way to the Early Universe, ed. A. Bragaglia, M. Davies, A. Sills, & E. Vesperini, 93–96
Gilbert, I. H. 1968, ApJ, 152, 1043
Kalnajs, A. J. 1972, in Astrophysics and Space Science Library, Vol. 31, IAU Colloq. 10: Gravitational N-Body Problem, ed. M. Lecar, 13
Kandrup, H. E. 1980, Phys. Rep., 63, 1
Kandrup, H. E. 1983, Ap&SS, 97, 435
Kandrup, H. E., Pogorelov, I. V., & Sideris, I. V. 2000, MNRAS, 311, 719
Knebe, A. & Gibson, B. K. 2004, MNRAS, 347, 1055
Kroupa, P., Jerabkova, T., Thies, I., et al. 2022, MNRAS, 517, 3613
Lelli, F., McGaugh, S. S., Schombert, J. M., & Pawlowski, M. S. 2017, ApJ, 836, 152
Malekjani, M., Rahvar, S., & Haghi, H. 2009, ApJ, 694, 1220
Mannella, R. 2004, Phys. Rev. E, 69, 041107
Mashchenko, S., Sills, A., & Couchman, H. M. 2006, ApJ, 640, 252
McGaugh, S. & Milgrom, M. 2013, ApJ, 766, 22
Milgrom, M. 1983, ApJ, 270, 365
Milgrom, M. 1986, ApJ, 302, 617
Milgrom, M. 1994, ApJ, 429, 540
Milgrom, M. 2010, MNRAS, 403, 886
Nagesh, S. T., Kroupa, P., Banik, I., et al. 2022, Monthly Notices of the Royal Astronomical Society, 519, 5128–5148
Nagesh, S. T., Kroupa, P., Banik, I., et al. 2023, MNRAS, 519, 5128
Nipoti, C., Ciotti, L., Binney, J., & Londrillo, P. 2008, MNRAS, 386, 2194
Nipoti, C., Londrillo, P., & Ciotti, L. 2007a, ApJ, 660, 256
Nipoti, C., Londrillo, P., & Ciotti, L. 2007b, MNRAS, 381, L104
Nipoti, C., Londrillo, P., Zhao, H., & Ciotti, L. 2007c, MNRAS, 379, 597
Pasquato, M. & Di Cintio, P. 2020, A&A, 640, A79
Re, F. & Di Cintio, P. 2023, A&A, 678, A110
Sánchez-Salcedo, F. J. & Hernandez, X. 2007, ApJ, 667, 878
Sanders, R. H. 2008, MNRAS, 386, 1588
Sanders, R. H. 2021, Monthly Notices of the Royal Astronomical Society, 507, 803–808
Scarpa, R., Falomo, R., & Treves, A. 2022, MNRAS, 512, 544
Simon, J. D. 2019, ARA&A, 57, 375
Sollima, A., Nipoti, C., Calura, F., Pascale, R., & Baumgardt, H. 2023, MNRAS, 524, 5291
Spitzer, L. 1965, Physics of fully ionized gases (Interscience Publishers)
Spitzer, L. 1987, Dynamical Evolution of Globular Clusters (Princeton University Press)
Stein, E. M. 1970, Singular Integrals and Differentiability Properties of Functions (PMS-30) (Princeton University Press)
Syer, D. 1994, MNRAS, 270, 205
Ter Haar, D. 1977, Lectures on selected topics in statistical mechanics.
Tiret, O. & Combes, F. 2007, A&A, 464, 517
Van Albada, T. S. & Szomoru, A. 2020, Proceedings of the International Astronomical Union, 351, 532
Wu, X., Zhao, H., Wang, Y., Llinares, C., & Knebe, A. 2009, MNRAS, 396, 109
Zhu, Y., Ma, H.-X., Dong, X.-B., et al. 2023, MNRAS, 519, 4479

Appendix A: Effective Dark Matter halos

Combining Equations (9) and (8), we evaluate the equivalent matter distribution from Eq. (12) as

$$\tilde{\rho} = -\frac{1}{4\pi G}\nabla\cdot\boldsymbol{g} = -\frac{1}{4\pi G}\nabla\cdot\left[\nu(|\boldsymbol{g}_N|/a_0)\boldsymbol{g}_N\right]. \tag{A.1}$$

For a point particle, the density becomes $\rho(\boldsymbol{r}) = m\delta^3(\boldsymbol{r})$, so that $\boldsymbol{g}_N(\boldsymbol{r}) = -Gm/r^3\boldsymbol{r}$, and

$$\tilde{\rho} = -\frac{1}{4\pi G}\nabla\cdot\left[-\frac{Gm}{r^3}\boldsymbol{r}\nu\left(\frac{Gm}{a_0 r^2}\right)\right] = \frac{m}{4\pi}\nabla\cdot\left[\frac{\boldsymbol{r}}{r^3}\nu\left(\frac{r_{\rm M}^2}{r^2}\right)\right] \tag{A.2}$$

Substituting the simple interpolation (23), we then find

$$\tilde{\rho} = \frac{m}{4\pi}\nabla\cdot\left(\frac{\boldsymbol{r}}{r^3}\right) + \frac{m}{4\pi r_{\rm M}}\nabla\cdot\left(\frac{\boldsymbol{r}}{r^2}\right) = m\delta^3(\boldsymbol{r}) + \frac{m}{4\pi r_{\rm M} r^2}, \tag{A.3}$$

where the first term here is the real point particle, and the second one represents its phantom dark matter halo $\rho_{pDM} = \tilde{\rho} - \rho$. Its cumulative mass function is

$$m_{DM}(r) = \int \frac{m}{4\pi r_{\rm M} r^2}4\pi r^2 dr = m\frac{r}{r_{\rm M}}. \tag{A.4}$$

Following the approach of Chandrasekhar, the impulsive force generated by the particle and its effective DM halo is then

$$F = \frac{GMm(b)}{b^2} = \frac{GMm}{b^2} + \frac{GMm}{r_{\rm M} b}. \tag{A.5}$$

The associated momentum variation is $\Delta p = Fb/v_M = GMm/v_M\left(b^{-1} + r_{\rm M}^{-1}\right)$ and then the diffusion coefficient becomes

$$D_v = \int \Delta p^2 2\pi n v_M d db = 2\pi G^2 M^2\frac{m^2 n}{v_M}\int\left(\frac{1}{b} + \frac{1}{r_{\rm M}}\right)^2 b db. \tag{A.6}$$

Keeping in mind that $b_m \ll r_{\rm M} < b_{\rm max}$, the integral can be evaluated as $\approx \ln\Lambda + 2b_{\rm max}/r_{\rm M} + b_{\rm max}^2/2r_{\rm M}^2$. The QuMOND DF force is then

$$\begin{aligned}\frac{d\mathbf{v}_M}{dt} &= -2\rho(M+m)D_v\frac{N(|\boldsymbol{v}| < v_M)}{v_M^2}\boldsymbol{v}_M = \\ &= -16\pi^2\left(\ln\Lambda + 2\frac{b_{\rm max}}{r_{\rm M}} + \frac{b_{\rm max}^2}{2r_{\rm M}^2}\right)G^2\rho(M+m)\frac{\boldsymbol{v}_M}{v_M^3}\int_0^{v_M} v^2 f(v)dv\end{aligned} \tag{A.7}$$

Appendix B: Explicit choice of QuMOND interpolating function

If the usual choice (13) of the QuMOND interpolating function ν is used in Eq. (22), we have

$$\nu(r_{\rm M}^2/b^2) = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{b^2}{r_{\rm M}}} \Rightarrow \nu(b)^2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{b^2}{r_{\rm M}^2}} + \frac{b^2}{r_{\rm M}^2}. \tag{B.1}$$

In the Equation above, unfortunately, we can not take the usual approximation for small or large impact parameters b, since the integral runs from $b_{\rm min} < r_{\rm M}$ to $b_{\rm max} > r_{\rm M}$. Vary large values of $b_{\rm max}$ (say of the order of the system size), return an important contribution already in the Newtonian case, in the form of large values of $\log\Lambda$. This must also hold true *a fortiori* in MOND, as in the far-field the forces decay as $1/r$ rather than $1/r^2$. Therefore, we can not expand ν to low order terms in $b/r_{\rm M}$.

We must then solve the exact integral $\int \nu(b)^2 db/b$, where the square root term yields

$$\begin{aligned}\int \sqrt{1 + 4b^2/r_{\rm M}^2}\frac{db}{b} &= \int\frac{\sqrt{1+x^2}}{x}dx = \\ &= \sqrt{1+x^2} - \operatorname{arctanh}(\sqrt{1+x^2}) + \text{const.} = \\ &= \sqrt{1+x^2} - \frac{1}{2}\ln\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}-1} + \left(\text{const} - \frac{\pi}{2}i\right),\end{aligned} \tag{B.2}$$

where $x = 2b/r_{\rm M}$. Evaluating Eq. (B.2) for $x = 2b_{\rm min}/r_{\rm M} \to 0$ returns approximately

$$(1 + x^2/2) - \frac{1}{2}\ln\frac{1+1}{(1+x^2/2)-1} = \ln\frac{b_{\rm min}}{r_{\rm M}} + 1, \tag{B.3}$$

while for $x = 2b_{\rm max}/r_{\rm M} \to \infty$ becomes asymptotically

$$\left(x + \frac{1}{2x}\right) - \frac{1}{2}\ln\frac{x+1}{x-1} = 2\frac{b_{\rm max}}{r_{\rm M}}. \tag{B.4}$$

The integral (B.2) is

$$\int \sqrt{1 + 4b^2/r_{\rm M}^2}\frac{db}{b} \sim 2\frac{b_{\rm max}}{r_{\rm M}} - \ln\frac{b_{\rm min}}{r_{\rm M}} - 1. \tag{B.5}$$

The integral over the impact parameters is then evaluated as

$$\begin{aligned}I_b &= \frac{1}{2}\int\frac{db}{b} + \frac{1}{2}\int\sqrt{1 + 4b^2/r_{\rm M}^2}\frac{db}{b} + \frac{1}{r_{\rm M}^2}\int b db = \\ &\sim \frac{1}{2}\ln\frac{b_{\rm max}}{b_{\rm min}} + \frac{b_{\rm max}}{r_{\rm M}} - \frac{1}{2}\ln\frac{b_{\rm min}}{r_{\rm M}} - \frac{1}{2} + \frac{b_{\rm max}^2}{2r_{\rm M}^2} = \\ &= \ln\Lambda + \frac{b_{\rm max}^2}{2r_{\rm M}^2} + \frac{b_{\rm max}}{r_{\rm M}} - \frac{1}{2}\ln\frac{b_{\rm max}}{r_{\rm M}} - \frac{1}{2}.\end{aligned} \tag{B.6}$$

Appendix C: Further details on the mean field approach

For the system of $N+1$ particles considered in Section 3 described by the distribution function $\mathcal{F}(\mathbf{r}, \mathbf{p}; t)$ the average of a given phase-space observable Q, taken with respect to said distribution function is

$$\langle Q\rangle_{\mathcal{F}}(t) := \int\mathcal{F}(\mathbf{r}, \mathbf{p}; t)Q(\mathbf{r}, \mathbf{p})d\Sigma. \tag{C.1}$$

where $d\Sigma = d^{3N}\mathbf{r}d^{3N}\mathbf{p}$ is the differential phase-space element.

The initial configuration $\mathcal{F}_0$ is perturbed by the passage of the $(N+1)$-th particle, (our usual test particle). In virtue of the Third Law of Dynamics, this corresponds to the force decelerating the test particle once the force due to the unperturbed configuration $\mathcal{F}_0$ is subtracted

$$F_0^{fr} = \langle F_0^{tot}\rangle_{\mathcal{F}} - \langle F_0^{tot}\rangle_{\mathcal{F}_0}, \quad F_0^{tot} = -\sum_{i=1}^{N} F_i^{tot}. \tag{C.2}$$

The field particles are assumed statistically uncorrelated in their initial state $\mathcal{F}_0(\mathbf{r}, \mathbf{p}) = \prod_{i=1}^N f(\mathbf{r}_i, \mathbf{p}_i)$. We also assume their velocity distribution to be a Maxwellian

$$f(\mathbf{r}_i, \mathbf{p}_i) = \mathcal{N} f_1(\mathbf{r}_i)e^{-\beta\mathbf{p}_i^2/2m_i}, \tag{C.3}$$

with temperature β and normalization $\mathcal{N}$. The distribution of the spatial coordinates is chosen such that the field particles are limited within a certain volume, therefore it can be also expressed as $f_1(\mathbf{r}_i) = e^{-\beta m_i \Phi_i(\mathbf{r}_i)}$ with an efficient potential Φ_i.

For our purposes, we will f_1 as the limit of an infinite, homogeneous distribution of field particles. Such homogeneous distribution on a finite volume $Vol \subseteq \mathbb{R}^3$ is given for $f_1(\mathbf{r_i}) = \frac{1}{|Vol|}\chi_{Vol}(\mathbf{r}_i)$, i.e. when Φ_i has the profile of a rigid box. The average number density of particles can be then defined as $n = N/|Vol|$. In the limit of an infinite homogeneous system, this parameters should be taken constant as $N, |Vol| \to \infty$. The unperturbed force $\langle F_0^{tot} \rangle_{\mathcal{F}_0}$ vanishes, so that the dynamical friction F_0^{fr} can be obtained simply as $\langle F_0^{tot} \rangle_{\tilde{\mathcal{F}}}$.

The field particles evolves following the time-dependent Hamiltonian

$$H(\mathbf{r}, \mathbf{p}; t) = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} + \sum_{i=1}^{N} W_i(\mathbf{r}). \tag{C.4}$$

As discussed in Sect. (3) one can not make the usual substitution $W_i = \sum_{j\neq i} W_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$, with $W_{ij}(r) = -Gm_i m_j/r$, as in MOND the Superposition Principle does not hold. The potentials should be rather expressed in using the QuMOND formalism (8), so that

$$-\nabla W_i = m_i \nu\left(\frac{|\mathbf{g}_{Ni}|}{a_0}\right)\mathbf{g}_{Ni}, \quad s.t. \quad -\sum_{j\neq i} \nabla W_{ij} = m_j \mathbf{g}_{Ni}. \tag{C.5}$$

As a consequence, it is useless to define the forces between couples of particles $\mathbf{F}(j \to i) = -\partial_{\mathbf{r}_i} W_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = -\mathbf{F}(i \to j)$, given that the total QuMOND force on the i–th particle is not simply $\mathbf{F}_i^{tot} \neq \sum_{j\neq 0} \mathbf{F}(j \to i)$. The force $\mathbf{F}_i^{tot}$ can be calculated instead, only as

$$\mathbf{F}_i^{tot} = -\partial_{\mathbf{r}_i} W_i(\mathbf{r}), \tag{C.6}$$

where $W_i(\mathbf{r})$ is given by the nonlinear QuMOND formula (C.5). The evolution Equation is therefore expressed as

$$\dot{\mathcal{F}} = -\sum_{i=1}^{N} \frac{\mathbf{p}_i}{m_i} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{r}_i} - \sum_{i=1}^{N} \mathbf{F}_i^{tot} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{p}_i} := -iL[\mathcal{F}]. \tag{C.7}$$

In the equation above the sums arise from those in the Hamiltonian (C.4). In practice, the meaning of Eq. (C.7) is just that the position of each field particle evolves according to the velocity at each instant, as well as the momentum follows the total force.

Following Kandrup (1983), it is useful to change the phase-space coordinates by taking the test particle's reference

$$\hat{\mathbf{r}}_i := \mathbf{r}_i - \mathbf{R}(t) + \mathbf{R}(0), \quad \hat{\mathbf{p}}_i := \mathbf{p}_i - \mathbf{P}(t). \tag{C.8}$$

The evolution operator for $\mathcal{F}$ hence takes the form

$$\dot{\mathcal{F}}(\hat{\mathbf{r}}_i, \hat{\mathbf{p}}_i; t) = -iL[\mathcal{F}] + \sum_{i=1}^{N} \dot{\mathbf{P}}(t) \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{p}_i} := -i\mathcal{L}[\mathcal{F}], \tag{C.9}$$

where $\dot{\mathbf{P}}$ acts as an apparent force on the i-th particle. Except for this, in this frame the forces do not change $\hat{F}_i^{tot} \equiv F_i^{tot}$, since even in MOND they depend only on the respective particles positions. Moreover, it is useful to define the perturbation $\tilde{\mathcal{F}}(t) := \mathcal{F}(t) - \mathcal{F}_0$ of the distribution function so that

$$\langle Q \rangle_{\mathcal{F}} = \langle Q \rangle_{\mathcal{F}_0} + \langle Q \rangle_{\tilde{\mathcal{F}}} \tag{C.10}$$

for any quantity Q. This will be used for evaluating Eq. (C.2), so that $\mathbf{F}_0^{fr} = \langle \mathbf{F}_0^{tot} \rangle_{\tilde{\mathcal{F}}}$. The distribution function $\tilde{\mathcal{F}}$ evolves as

$$\dot{\tilde{\mathcal{F}}} = -i\mathcal{L}[\tilde{\mathcal{F}}] + \beta \mathcal{F}_0(\mathbf{r}, \mathbf{p}) \sum_{i=1}^{N} \left[\frac{\hat{\mathbf{p}}_i}{m_i} + \mathbf{V}(t)\right] \cdot \left[F_i^{tot} + m_i \frac{\partial \hat{\Phi}_i}{\partial \hat{\mathbf{r}}_i}(t)\right], \tag{C.11}$$

where again $\hat{\mathbf{p}}_i/m_i + \mathbf{V}(t) = \mathbf{p}_i/m_i$, and the term $\partial\hat{\Phi}_i/\partial\hat{\mathbf{r}}_i$ vanishes in the infinite, homogeneous limit. In principle, keeping the contribution of $\hat{\Phi}$ would allow one to investigate in a self-consistent fashion the MOND EFE, see Sect. 3. At this point, Equation (C.11) can be rewritten in its simpler form

$$\dot{\tilde{\mathcal{F}}} = -i\mathcal{L}[\tilde{\mathcal{F}}] + \beta \mathcal{F}_0 \sum_{i=1}^{N} \mathbf{p}_i \cdot \mathbf{g}_i^{MOND}. \tag{C.12}$$

Its solution can thus be formally expressed as

$$\tilde{\mathcal{F}}(t) = \beta \mathcal{F}_0 \int_0^t d\tau G_{\mathcal{L}}(\tau \to t) \left[\sum_{i=1}^{N} \mathbf{p}_i \cdot \mathbf{g}_i\right] \tag{C.13}$$

with the Greenian $G_{\mathcal{L}}$ of the operator $\mathcal{L}$. After all the substitution we have a formal, general expression for the dynamical friction on the test particle in the form

$$\begin{aligned}\mathbf{F}_0^{fr} &= \int d\Sigma \mathbf{F}_0^{tot} \tilde{\mathcal{F}} = \int d\Sigma \mathbf{F}_0^{tot} \beta \mathcal{F}_0 \int_0^t d\tau G_{\mathcal{L}}(\tau \to t) \left[\sum_{i=1}^{N} \mathbf{v}_i \cdot \mathbf{F}_i^*\right] = \\ &= -\beta \int d\Sigma \mathcal{F}_0 \left(\sum_{j=1}^{N} \mathbf{F}_i^{tot}\right) \int_0^t d\tau G_{\mathcal{L}}(\tau \to t) \left[\sum_{i=1}^{N} \mathbf{v}_i \cdot \mathbf{F}_i^*\right],\end{aligned} \tag{C.14}$$

where $\mathbf{F}_i^* = \mathbf{F}_i^{tot} + m_i \partial\hat{\Phi}_i/\partial\hat{\mathbf{r}}_i$ becomes $\mathbf{F}_i^{tot}$ in the infinite homogeneous system limit. Here we have exploited the fact that the full set of $N + 1$ particles is a closed, classic dynamical system, so that $\sum_{i=0}^{N} \mathbf{F}_i^{tot} = 0$.

In order to perform the integrals appearing in Eqs. (C.14), we now make some simplifying hypotheses. First of all, we perform the infinite homogeneous limit. Second, if the system has only a finite memory, we can replace the integration of τ on $[0; t]$ with an integration of $[0; \infty)$. Moreover, we make use of the standard linear trajectory approximation, which allows to simplify $G_{\mathcal{L}}(\tau \to t)[Q] \cong Q(t - \tau)$. Under these approximations, the DF force becomes

$$\mathbf{F}_0^{fr} \cong -\beta \int d\Sigma \mathcal{F}_0 \left(\sum_{j=1}^{N} \mathbf{F}_j^{tot}(t)\right) \int_0^{\infty} d\tau \left(\sum_{i=1}^{N} \mathbf{v}_i(t-\tau) \cdot \mathbf{F}_i^{tot}(t-\tau)\right). \tag{C.15}$$

This can be further simplified since $\langle F_i F_j \rangle = 0$ for the off-diagonal combinations $i \neq j$, since the field particles are by definition statistically uncorrelated. We are then left with

$$\begin{aligned}\mathbf{F}_0^{fr} &\cong -\beta \int d\Sigma \mathcal{F}_0 \sum_{i=1}^{N} \mathbf{F}_i^{tot}(t) \int_0^{\infty} d\tau \mathbf{v}_i(t-\tau) \cdot \mathbf{F}_i^{tot}(t-\tau) = \\ &= -\beta \int_0^{\infty} d\tau \left\langle \sum_{i=1}^{N} \mathbf{F}_i^{tot}(t) \left(\mathbf{v}_i(t-\tau) \cdot \mathbf{F}_i^{tot}(t-\tau)\right)\right\rangle_{\mathcal{F}_0} = \\ &= -\beta \int_0^{\infty} d\tau N \left\langle \mathbf{F}_1^{tot}(t) \left(\mathbf{v}_1(t-\tau) \cdot \mathbf{F}_1^{tot}(t-\tau)\right)\right\rangle_{\mathcal{F}_0}.\end{aligned} \tag{C.16}$$

Note that, the last step is justified only if the N field particles have all the same mass m, so that they are statistically equivalent in the initial phase-space distribution. Averaging $\mathcal{F}_0$ for the $i = 1$ particle, practically means that only the $f(\mathbf{r}_1, \mathbf{p}_1)$ distribution function is relevant, hence

$$\begin{aligned}\left\langle \mathbf{F}_1^{tot}\left(\mathbf{v}_1\cdot\mathbf{F}_1^{tot}\right)\right\rangle_{\mathcal{F}_0} &= \int m\mathbf{g}_1^M\left(m\mathbf{v}_1\cdot\mathbf{g}_1^M\right)\prod_{i=1}^N f(\mathbf{r}_i,\mathbf{p}_i)d^{3N}\mathbf{r}d^{3N}\mathbf{p} =\\ &= m^2\int \mathbf{g}_1^M\left(\mathbf{v}_1\cdot\mathbf{g}_1^M\right)f(\mathbf{r}_1,\mathbf{p}_1)d^3\mathbf{r}_1d^3\mathbf{p}_1 =\\ &= m^2\int d^3\mathbf{r}_1\int d^3\mathbf{v}_1\frac{n}{N}f(\mathbf{v}_1)\mathbf{g}_1^M\left(\mathbf{v}_1\cdot\mathbf{g}_1^M\right) =\\ &= m^2\frac{n}{N}\left[\int v_\alpha f(\mathbf{v})d^3\mathbf{v}\right]\left[\int g_1^{\alpha M}\mathbf{g}_1^M d^3\mathbf{r}_1\right],\end{aligned} \tag{C.17}$$

where $f(\mathbf{v}_1) = \mathcal{N}e^{-\beta m v_1^2}$ is again the Maxwellian velocity distribution. The mean square speed is thus given by $\langle v^2\rangle = 3/m\beta$ that can be substituted to obtain

$$\begin{aligned}\mathbf{F}_0^{fr} &\cong -\frac{3n}{m\langle v^2\rangle}m^2\int_0^\infty d\tau\left[\int v_\alpha f(\mathbf{v})d^3\mathbf{v}\right]\left[\int g_1^{\alpha M}(t-\tau)\mathbf{g}_1^M(t)d^3\mathbf{r}_1\right] =\\ &\cong -\frac{3\rho G^2M^2}{\langle v^2\rangle}\int_0^\infty d\tau\int v_\alpha f(\mathbf{v})d^3\mathbf{v}\\ &\quad\times\int d^3\mathbf{s}\nu\left(\frac{r_{\rm M}^2}{|\mathbf{s}-\tilde{\mathbf{v}}\tau|^2}\right)\frac{s^\alpha-\tilde{v}^\alpha\tau}{|\mathbf{s}-\tilde{\mathbf{v}}_1\tau|^3}\nu\left(\frac{r_{\rm M}^2}{s^2}\right)\frac{\mathbf{s}}{s^3},\end{aligned} \tag{C.18}$$

since $\mathbf{g}_1^M(\mathbf{s},t) = GM\nu\left(GM/a_0s^2\right)\mathbf{s}/s^3$ and $\mathbf{s}(t-\tau) \cong \mathbf{s}-\tilde{\mathbf{v}}_1\tau$, being $\tilde{\mathbf{v}} = \mathbf{v}-\mathbf{V}$.

Appendix D: Numerical methods

The dynamics of a particle confined by a static potential Φ under the combined effect of force fluctuations and friction is given by the Langevin type equations

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\nabla\Phi_{\mathbf{r}} - \nu_{\mathbf{r},\mathbf{v}}\mathbf{v} + \delta F;\quad \mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}. \tag{D.1}$$

In the most general case the DF coefficient ν depends explicitly on the position and velocity through the phase-space distribution of the embedding system (also sourcing the potential Φ) and δF is a fluctuating force per unit mass with distribution and amplitude connected to ν via a fluctuation-dissipation relation. For the systems of interest (Newtonian or MOND) we solve Eq. (D.1) using the Mannella (2004) scheme

$$\begin{aligned}x' &= x(t+\Delta t/2) = x(t) + \frac{\Delta t}{2}v(t)\\ v(t+\Delta t) &= c_2\left[c_1v(t) + \Delta t\nabla\Phi(x') + d_1\tilde{F}(x')\right]\\ x(t+\Delta t) &= x' + \frac{\Delta t}{2}v(t+\Delta t),\end{aligned} \tag{D.2}$$

written here for simplicity for a 1D system and fixed time step Δt, where the coefficients c_1, c_2 and d_1 are given by

$$c_1 = 1-\frac{\eta\Delta t}{2};\quad c_2 = \frac{1}{1+\eta\Delta t/2};\quad d_1 = \sqrt{2\zeta\eta\Delta t}. \tag{D.3}$$

In the equations above $\tilde{F}$ is sampled from a norm 1 Gaussian and ζ in the case of a delta correlated noise is fixed by the standard deviation of the distribution of F as

$$\langle F(x,t)F(x,t')\rangle = 2\eta\zeta\delta(t-t'). \tag{D.4}$$

In practice, if such distribution is unknown one takes, following Kandrup et al. (2000) one could assume $\zeta = \mathcal{E}$ where $\mathcal{E}$ is the instantaneous relative (positive) energy per unit mass of the test particle along the orbit. In the simulations discussed in this paper the contribution of the fluctuating term is artificially set to 0 as it would be negligible for the type of gravitational systems under consideration. Note that, if $\nu = \zeta = 0$ Eqs. (D.2) simply become the standard Verlet second order scheme in the drift-kick-drift form.