Positive and non-positive measurements in energy extraction from quantum batteries

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We extend the concept of stochastic energy extraction from quantum batteries to the scenario where both positive operator-valued (POV) and physically realizable non-positive operator-valued measurements (NPOVMs) are applied on the auxiliary connected to the battery in presence of noise. The process involves joint evolution of the battery and the auxiliary for a particular time interval, an interaction of the auxiliary with its environment which induces noise in the auxiliary, and performing a POVM or NPOVM on the auxiliary, and finally the selection of a particular measurement outcome. Application of POVM on the auxiliary can be realised by attaching an external system to the auxiliary, which is initially in a product state with the rest of the system, and performing a joint projective measurement on the auxiliary and external. On the other hand, if there are interactions leading to correlations among the auxiliary, environment, and external systems, then performing the projective measurement on the auxiliary-environment-external system can be interpreted as a physically realizable NPOVM operation on the auxiliary. We however utilize interaction between the auxiliary and the environment to implement NPOVMs on the auxiliary. We find the expressions of stochastically extractable energy by performing POVMs and NPOVMs on the auxiliary and show that the latter does not depend on the applied noise. Focusing on a particular model of the governing Hamiltonian of a qubit battery and an auxiliary, and considering the presence of amplitude damping noise affecting the auxiliary, we show that stochastically extractable energy using NPOVMs is greater than or equal to that using POVMs. This advantage of using NPOVMs remains also for bit-flip noise. For dephasing noise, the energies by applying POVMs and NPOVMs are the same. We additionally consider the case when a limited set of measurement operators are allowed, where the limitation is imposed by restricting the projection measurements to those that can be implemented by using only the global unitary generated by the system-auxiliary Hamiltonian.

I. INTRODUCTION

Batteries are one of the most essential devices that are widely used in electrical gadgets to store energy. Large-scale batteries essentially consist of electrochemical cells that store chemical energy and transform it into electric energy whenever needed. Nowadays, the demand for miniaturized technological devices is instigating the development of small-scale batteries. The reduced size of batteries necessitates the consideration of quantum mechanical effects in them, and pottentially led to the development of a battery-like energy-storage device that operates in the quantum domain and possesses quantum mechanical features.

To the best of our knowledge, quantum batteries were first introduced by R. Alicki and M. Fannes in 2013 [1] in an information theoretic context where the concepts of ergotropy and passive states were used [1–3]. Ergotropy refers to the maximum amount of energy that can be extracted from a quantum system using unitary operations. The quantum state having zero ergotropy, i.e., the state from which no energy can be extracted using unitary operations are referred to as passive states [4–11].

The traditional method of charging a quantum battery is to operate a unitary directly on the battery [3, 12–14] or attach another system, which is usually considered as a charger, and operate the unitary on the joint system consisting of the battery and the charger [3, 13, 15, 16]. On the other hand, to extract energy from the battery, conventionally a unitary operator is applied on the battery [1, 17]. The quantities of interest, in this case, that can be used to gauge the efficiency of a quantum battery are ergotropy, power of charging, work capacity, etc [12, 14, 16, 18, 19]. Numerous studies have examined the power of charging of quantum batteries by considering various models for the governing Hamiltonian of the battery, for example the XXZ Heisenberg spin [20], spin cavity [21–28], Bose- and Fermi-Hubbard models [29], non-hermitian systems [30], and so on [31–36]. To explore the role of quantum properties, such as entanglement and quantum coherence, in the charging and discharging of a quantum battery one can go through Refs. [1, 19, 37–46]. Comparative studies between classical and quantum many-body batteries have also been conducted [47]. Experimental studies on quantum batteries have been performed using NMR [48], superconducting qubits [49], quantum dots [50]. A significant number of studies have explored the effect of noise on quantum batteries [51–54]. Various methods to preserve the energy of the battery against noise have also been suggested [55–57].

In Refs. [40] and [58], methods for extraction of energy from quantum batteries were introduced, based on measurements performed on an attached system. The study on measurement-induced energy extraction was further extended in Ref. [59]. The method discussed in Ref. [59] involves connecting an auxiliary system to the quantum battery, jointly evolving the battery and auxiliary for a certain time, performing projective measurement on the auxiliary system, and finally choosing a suitable outcome. To quantify the efficiency of the energy extraction method, a figure of merit, stochastically extracted energy, was defined which is the probability of getting the preferred measurement outcome multiplied by the energy difference between the initial state and the final state of the battery when that outcome is chosen.

In this paper, we introduce the notion of non-positive operator-valued measurements (NPOVM) and utilize it to stochastically extract energy from quantum batteries. Additionally, we compare the performance of positive operator-valued measurements (POVMs) and NPOVMs in the same context. We consider four sub-systems: battery, auxiliary, environment, and external. Initially, all four subsystems are in a product state. The composite system of battery and auxiliary undergoes a unitary evolution under the action of a certain Hamiltonian. Thenceforth, the auxiliary interacts with the environment and becomes entangled with it. On the other hand, the auxiliary and external remain in a product state throughout the process. After that, we perform POVM on the auxiliary by making a projective measurement on the joint product state of the auxiliary and external system. Instead of this, we could have considered an initial four-party state, comprising of battery, auxiliary, environment, and external, where the external is already entangled with the auxiliary and environment. In such a scenario, if we had evolved the system and battery unitarily, subsequently switched on an interaction between the auxiliary and environment, and then performed a projective measurement on the auxiliary, environment and external, then it would also be equivalent to implementing an NPOVM on the auxiliary. We, however, restrict to a specific implementation of NPOVM measurement on the auxiliary. Similar to the POVM case, the battery and auxiliary evolve unitarily and noise acts on the auxiliary via the environment, which gets entangled with the auxiliary in this process. Then a projective measurement is performed on the environment and auxiliary. In this case, the external system acts as the spectator (remains as a product with the rest of the system) throughout the process. The projective measurement appears as a physically realizable NPOVM performed on the auxiliary. We find that even this particular type of NPOVM can provide an advantage in stochastic energy extraction over POVMs. It is important to note that the set of POVMs does not form a subset of the NPOVMs considered here.

After performing POVM or NPOVM on the auxiliary system, we choose a specific outcome. The stochastically extractable energy is defined as the product of the probablity of the chosen outcome and the corresponding energy difference between the initial and final states. We prove that the stochastically extractable energy using NPOVM does not depend on the type or strength of the noise acting on the auxiliary whereas the same using POVM may rely both. Considering a particular model, the dependence of stochastically extractable energy on the strength of the noise acting on the auxiliary and the time duration of the initial evolution of the battery and the auxiliary is explored. By considering an amplitude-damping noise of fixed strength on the auxiliary, we find that NPOVM is always advantageousin comparison to POVM. Moreover, we also examine effects of two types of noise, viz., bit-flip and dephasing, along with amplitude-damping, and show that whatever the amount of noise is, maximum stochastically extractable energy, obtained at a certain time of the auxiliary-battery interaction, by performing NPOVM is always equal to or more than the same using POVM. The optimal time at which we obtain maximum stochastically extractable energy is numerically found to be the same for both POVM and NPOVM operations. We therefore perform both the measurements at this particular time.

In practical scenarios, implementing an arbitrary POVM or NPOVM may not be feasible and therefore in the next part of the paper, we consider stochastically accessible energy as the maximum stochastically extracted energy by applying an optimal POVM or NPOVM on the auxiliary among the set of measurements that are available in the laboratory. The laboratory scenerio is identified as one in which the entangling unitary for generating the global basis on auxiliary and environment space is derived from the entangling Hamiltonian of the battery-auxiliary pair. We determine that even in this restricted scenario, there can be a broad range of noise strength for which stochastically accessible energy is more for NPOVM performed on the auxiliary than POVM applied on the same.

The rest of the paper is organized as follows. In Sec. II, we recapitulate the concept of POVM and extend it further to introduce NPOVMs, and in particular physically realizable ones. In Sec. III, we describe the process of energy extraction from quantum batteries by implementing POVMs and NPOVMs on an attached auxiliary. Considering a particular model of the governing Hamiltonian of the battery and auxiliary duo, the comparison between stochastically extractable energies, using POVM and NPOVM operators, is discussed in Sec. IV. Stochastically accessible energy obtained by performing optimal POVM and NPOVM measurements, where the optimization is performed over a restricted set of measurement operators, is analyzed in Sec. V. Finally, our concluding remarks are presented in Sec. VI.

II. POSITIVE AND NON-POSITIVE OPERATOR-VALUED MEASUREMENTS

In this section, we first briefly recapitulate POVMs and then introduce NPOVMs.

Consider a system, S_1, attached to an environment S_2. Let initially the joint state of S_1 and S_2 are prepared as $\rho_{S_1} \otimes \rho_{S_2}$, i.e., in a product state, where ρ_{S_1} and ρ_{S_2} are states of the system and environment, respectively. If a projective measurement, with projection operators, say $\{|\Psi_i\rangle \langle\Psi_i|\}_i$, is performed on $\rho_{S_1} \otimes \rho_{S_2}$ and $|\Psi_i\rangle \langle\Psi_i|$ gets clicked then the final state of the system, S_1, after the

Process	Spectator	Process	Spectator
(1) **Global evolution** The battery and the auxiliary are globally evolved using U_{BA}.	Environment + External	(1) **Global evolution** The battery and the auxiliary are globally evolved using U_{BA}.	Environment + External
(2) **Noise acted on auxiliary** The auxiliary and the environment interacts with each other.	Battery+ External	(2) **Noise acted on auxiliary** The auxiliary and the environment interacts with each other.	Battery+ External
(3) **POVM measurement** Joint projective measurement is performed on the auxiliary and external qubits.	Battery+Environment	(3) **NPOVM measurement** Joint projective measurement is performed on the auxiliary and the environment.	Battery+External

FIG. 1. Description of the positive and non-positive operator-valued measurement-based energy extraction methods. In the left and right panels, we provide schematic diagrams portraying the process of stochastic extraction of energy by performing, respectively, POVM and NPOVM on an attached auxiliary. The procedure involves four systems: the battery (orange dot), auxiliary (green dot), environment (dark-green dot), and an external system (violet dot). Each solid elliptical shape represents the interaction taking place between the systems contained in the ellipse. The joint projective measurements being performed on the systems are depicted using hollow elliptical shapes. The steps involved in the processes are described below the diagram.

measurement would be

$$\rho^i_{S_1} = \frac{\mathrm{tr}_{S_2}(|\Psi_i\rangle\langle\Psi_i|\,\rho_{S_1}\otimes\rho_{S_2}\,|\Psi_i\rangle\langle\Psi_i|)}{\mathrm{tr}(|\Psi_i\rangle\langle\Psi_i|\,\rho_{S_1}\otimes\rho_{S_2}\,|\Psi_i\rangle\langle\Psi_i|)} = \frac{\chi_i\rho_{S_1}\chi_i^\dagger}{\mathrm{tr}\left(\rho_{S_1}\chi_i^\dagger\chi_i\right)}.$$

Here χ_i is the effective measurement operator acting on the system-state, ρ_{S_1}. The measurement induced on the system, S_1, by performing a projective measurement on the bigger system, S_1S_2, that contains S_1, is popularly known as positive operator-valued measurements or POVM. The reason behind this nomenclature is that the probability, p_i, of getting a particular outcome, say $\rho^i_{S_1}$, can be expressed as $p_i = \mathrm{tr}\left(\rho_{S_1}\chi_i^\dagger\chi_i\right) = \mathrm{tr}(\rho_{S_1}E_i)$, which is a function of the positive semi-definite operator, $E_i = \chi_i^\dagger\chi_i$. Each operator, E_i, is often referred to as the POVM element. The properties that any set of POVM elements, $\{E_i\}$, must satisfy are [60]

- E_i is Hermitian, i.e., $E_i = E_i^\dagger$, $\forall\ i$.
- Each of the POVM elements, E_i, is positive semi-definite, i.e., it has only non-negative eigenvalues.
- The sum of all elements of the set, $\{E_i\}_i$, is equal to the identity operator, i.e., $\sum_i E_i = \mathbb{I}$, where $\mathbb{I}$ is the identity operator which acts on the Hilbert space of the system, S_1.

According to Naimark's dilation theorem [61], any set of operators, $\{E_i\}$, acting on a system, S_1, that satisfy the above-mentioned properties can be thought of as a set of POVM elements, which can be produced by attaching an external system, S_2, to S_1 and performing a projective measurement jointly on S_1 and S_2.

The concept of POVMs depends on the tacit assumption that the initial state, $\rho_{S_1}\otimes\rho_{S_2}$, on which projective measurements would be performed, is a product. This motivates us to introduce a more generalized measurement by relaxing this restriction on the initial state of S_1 and S_2. If we initially prepare the pair of systems, S_1 and S_2, in an initial state $\rho_{S_1S_2}$ which may not necessarily be a product or even separable, perform the projective measurement, $\{|\Psi_i\rangle\langle\Psi_i|\}_i$, on the entire arrangement, and get the output $|\Psi_i\rangle\langle\Psi_i|$, then the final state of the system, S_1, after the application of the measurement would be

$$\rho'^{i}_{S_1} = \frac{\mathrm{tr}_{S_2}\left(|\Psi_i\rangle\langle\Psi_i|\,\rho_{S_1S_2}\,|\Psi_i\rangle\langle\Psi_i|\right)}{\langle\Psi_i|\,\rho_{S_1S_2}\,|\Psi_i\rangle}.$$

In such a scenario, where the initial state, $\rho_{S_1S_2}$, is not necessarily separable, the probability of getting any particular outcome, in general, would not be expressible in terms of positive semidefinite operators. Therefore we refer to the effective measurement performed on S_1 by applying joint projective measurement on an entangled state of the system S_1S_2 as a non-positive operator-valued measurement or NPOVM.

III. STOCHASTIC ENERGY EXTRACTION BY POVM AND NPOVM

In references to POVM and NPOVM s described in Sec. II, here we analyze our scheme for measurement-based energy extraction from quantum batteries. In addition to the quantum battery (B), our set-up involves three more parts, an auxiliary (A), an environment (E), and an external (X). Let the identity operator acting on the Hilbert spaces that describe B, A, E, and X be, respectively, $\mathbb{I}_B$, $\mathbb{I}_A$, $\mathbb{I}_E$, and $\mathbb{I}_X$. Moreover, let us consider the quantum systems, B, A, E, and X, to be initially locally prepared in the states ρ_B, ρ_A, ρ_E, and ρ_X, respectively. To reduce notational complicacy, we will denote the entire system consisting of B, A, E, and X as S. Hence the initial state of S is $\rho^0_S = \rho_B\otimes\rho_A\otimes\rho_E\otimes\rho_X$. The Hamiltonian which describes the energy of the battery can be denoted as H_B. The energy extraction protocol is based on the application of POVM and NPOVM on the auxiliary system, A. For the operations performed on A to affect B, we need B and A to be entangled. As we mentioned before, the joint initial state of B and A are considered as a product, therefore, to entangle them, we perform a unitary, U_{BA}, on the state, $\rho_B\otimes\rho_A$. During this evolution, the environment and the external system act as spectators. Hence the ultimate state of S after the application of U_{BA} is $\rho^1_S = U_{BA}\rho_B\otimes\rho_A U^\dagger_{BA}\otimes\rho_E\otimes\rho_X$. From this moment the auxiliary starts interacting with its environment, E, which results in an evolution of the entire system, S. Because of the interaction, the state of S transforms to $\rho^2_S = (\mathbb{I}_B\otimes U_{AE}\otimes\mathbb{I}_X)\rho^1_S(\mathbb{I}_B\otimes U^\dagger_{AE}\otimes\mathbb{I}_X)$. The reason behind this evolution can just be an uncontrollable effect of the environment on the auxiliary or can also be considered as a manually created interaction by the technician to extract energy from the battery. This interaction between A and E will be utilized to perform NPOVM on A. In the next part, we separately discuss in detail the importance of POVM and NPOVM performed on auxiliary, A, in extracting energy from B.

A. Application of POVM on the auxiliary

One can notice from the expression of ρ^2_S, it is, in general, entangled in the bipartition $B : AEX$ and $BA : EX$ but is separable in the bipartition $BAE : X$. In this situation, a projective measurement, $\{|\Psi_i\rangle_{AX}\langle\Psi_i|\}_i$, is performed on the joint state of A and X, and a particular outcome, say $|\Psi_i\rangle_{AX}\langle\Psi_i|$, is selected. Since, before the measurement the systems, BEA and X, were in a product state, the measurement performed on AX, if we ignore X, reduces to a POVM applied on A. The total change in the energy of the battery-state in the entire process is given by

$$\Delta E = \mathrm{tr}[\rho_B H_B] - \mathrm{tr}\left[H_B\,\mathrm{tr}_{AX}\left(\rho^3_{BAX}\right)\right]/p_i,$$

where $\mathrm{tr}[\rho_B H_B]$ is the initial energy of the battery, ρ^3_{BAX} is the final unnormalised state of BAX after measurement, i.e.,

$$\begin{aligned}\rho^3_{BAX} \;=\; & (\mathbb{I}_B\otimes|\Psi_i\rangle_{AX}\langle\Psi_i|)\,\mathrm{tr}_E\left(\rho^2_S\right)\\ & (\mathbb{I}_B\otimes|\Psi_i\rangle_{AX}\langle\Psi_i|),\end{aligned}$$

and $p_i = \mathrm{tr}\big(\rho^3_{BAX}\big)$. Since the application of measurements is not a deterministic process, the outcome, $|\Psi_i\rangle_{AX}\langle\Psi_i|$, occurs with the probability, p_i. Hence we define this method of extraction of energy as stochastic energy extraction. There can be measurement outcomes corresponding to which ΔE is comparatively larger but the probability of occurrence of that outcome may be very small. To avoid potential misinterpretations, instead of completely focusing on ΔE we consider the following quantity as the figure of merit:

$$S^P = p_i\Delta E,$$

to judge the measurement-based protocol. We refer to S^P as the stochastically extracted energy by performing the POVM on A. The process of energy extraction using this POVM-based method is illustrated in the left panel of Fig. 1. The stochastically extractable energy, S^P_{max}, in this method, can be found by optimizing over the set of all projective measurement operators, $\mathcal{M}$, i.e.,

$$\begin{aligned}S^P_{max} &= \max_{|\Psi_i\rangle_{AX}\in\mathcal{M}}\left(p_i\,\mathrm{tr}[\rho_B H_B] - \mathrm{tr}\left[H_B\,\mathrm{tr}_{AX}\left(\rho^3_{BAX}\right)\right]\right)\\ &= \max_{|\Psi_i\rangle_{AX}\in\mathcal{M}}\mathrm{tr}\big[Z\rho^3_{BAX}\big].\end{aligned}$$

Here ρ^3_{BAX} is the function of $|\Psi_i\rangle_{AX}$ and $Z = \mathrm{tr}[\rho_B H_B]\mathbb{I}_{BAX} - H_B\otimes\mathbb{I}_{AX}$. We can express $|\Psi_i\rangle_{AX}$ as $|\Psi_i\rangle_{AX} = \widetilde{U}_{AX}|0\rangle$ where $|0\rangle$ is any fixed pure state of the system, AX, and can consider the optimization involved in the expression of S^P_{max} as a maximization over the set of all unitaries, $\{\widetilde{U}_{AX}\}$. By further simplification of the expression of S^P_{max}, we finally get the following form:

$$\begin{aligned} S^P_{max} &= \max_{\widetilde{U}_{AX}} \operatorname{tr} \left(\widetilde{U}_{AX} |0\rangle\langle 0| \widetilde{U}^\dagger_{AX} \operatorname{tr}_B \left[\left(\operatorname{tr}_E \left(\rho^2_S\right) Z\right)\right]\right) \\ &= \max_{\widetilde{U}_{AX}} \operatorname{tr}\Big[\widetilde{U}_{AX}\mathcal{A}\widetilde{U}^\dagger_{AX}\mathcal{B}\Big], \end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ denote $|0\rangle\langle 0|$ and $\operatorname{tr}_B\left[\left(\operatorname{tr}_E\left(\rho^2_S\right)Z\right)\right]$, respectively. The maximum would be reached for that $\widetilde{U}_{AX} = \widetilde{U}_{max}$ for which $[\widetilde{U}_{max}\mathcal{A}\widetilde{U}^\dagger_{max}, \mathcal{B}] = 0$ and if the set of eigenvalues, $\{\beta_i\}$, of $\mathcal{B}$ satisfy $\beta_i \leq \beta_j$ then for that order of the eigenstates eigenvalues, $\{\alpha_i\}$, of $\widetilde{U}_{max}\mathcal{A}\widetilde{U}^\dagger_{max}$ will satisfy $\alpha_i \leq \alpha_j$. Thus the expression of S^P_{max} reduces to

$$S^P_{max} = \sum_i \alpha_i \beta_i,$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ follow the same order, as mentioned above. Since $\mathcal{A} = |0\rangle\langle 0|$ is a pure state, it has only one non-zero eigenvalue and that is equal to unity which is clearly the largest among the set because the other eigenvalues are zero. Moreover, we know the action of unitary operation on a state can not change the eigenvalues of the state, hence $\widetilde{U}_{max}\mathcal{A}\widetilde{U}^\dagger_{max}$ will have the same eigenvalues as $\mathcal{A}$. Thus we have

$$S^P_{max} = \beta_{max}, \tag{1}$$

where β_{max} is the largest eigenvalue of $\mathcal{B}$.

B. Implementing NPOVM on the auxiliary

In this case, after getting the state ρ^2_S, instead of performing measurement on the joint state of the auxiliary, A, and the external, X, we perform projective measurement, $\{|\Psi_i\rangle_{AE}\langle\Psi_i|\}$, on the joint state of the auxiliary, A, and environment, E. One can notice that the state, ρ^2_S, may share entanglement in the bipartition $BA : EX$ because of the action of the unitary operator, U_{AE}. Therefore, in this case, in general, the effective measurement performed on A, by applying projective measurement on AE, is a NPOVM. The amount of stochastically extracted energy from B by the NPOVM is given by

$$S^{NP} = p_i \operatorname{tr}[\rho_B H_B] - \operatorname{tr}\big[H_B \operatorname{tr}_{AE}(\rho'^3_{BAE})\big],$$

where $\rho'^3_{BAE} = (\mathbb{I}_B \otimes |\Psi_i\rangle_{AE}\langle\Psi_i|) \operatorname{tr}_X(\rho^2_S)(\mathbb{I}_B \otimes |\Psi_i\rangle_{AE}\langle\Psi_i|)$. Since the external does not play any role in this process, we have simply ignored it and just considered B, A, and E. In the right panel of Fig 1, the entire process of energy extraction from the battery by performing NPOVM on the auxiliary is depicted using a schematic diagram.

Our aim is to find out the stochastically extractable energy, S^{NP}_{max}, in this process, by maximizing over the set of all projective measurements, $\mathcal{M}'$. The expression of S^{NP}_{max} is given by

$$S^{NP}_{max} = \max_{|\Psi_i\rangle_{AE}\in\mathcal{M}'} p_i \operatorname{tr}[\rho_B H_B] - \operatorname{tr}\big[H_B \operatorname{tr}_{AE}(\rho'^3_{BAE})\big].$$

By further simplification of S^{NP}_{max} we get

$$S^{NP}_{max} = \max_{\widetilde{U}_{AE}} \operatorname{tr}\left(\widetilde{U}_{AE}|0'\rangle\langle 0'|\widetilde{U}^\dagger_{AE}\operatorname{tr}_B\left[\operatorname{tr}_X\left(\rho^2_S\right)Z'\right]\right).$$

Here $Z' = \operatorname{tr}[\rho_B H_B]\mathbb{I}_{BAE} - H_B \otimes \mathbb{I}_{AE}$ and $|\Psi_i\rangle_{AE} = \widetilde{U}_{AE}|0'\rangle$ where $|0'\rangle$ is any fixed state of AE. Following the same path of logics as in the case of POVM-based energy extraction, the stochastically extractable energy using NPOVM can be found to have the following form

$$\begin{aligned} S^{NP}_{max} &= \max_{\widetilde{U}_{AE}} \operatorname{tr}\Big[\widetilde{U}_{AE}\mathcal{A}'\widetilde{U}^\dagger_{AE}\mathcal{B}'\Big] \\ &= \beta'_{max}, \end{aligned} \tag{2}$$

where $\mathcal{A}' = |0'\rangle\langle 0'|$, $\mathcal{B}' = \operatorname{tr}_B\left[\operatorname{tr}_X\left(\rho^2_S\right)Z'\right]$, and β'_{max} is the largest eigenvalue of $\mathcal{B}'$. One needs to keep in mind that here the maximization is over all unitaries that can be applied on the state of the system AE. It is interesting to note that since we optimized over all possible unitaries, $\widetilde{U}_{AE}$, the stochastically extractable energy, S^{NP}_{max}, essentially became independent of the applied evolution, U_{AE}.

IV. ENERGY EXTRACTION FROM SINGLE QUBIT BATTERY

Let us now compare between stochastic energy extraction using POVM and NPOVM by focusing on single qubit batteries. To extract energy using POVM following the process that has been discussed in the previous section, we introduce three more qubits, which are, the auxiliary, environment, and the external. Let the Hamiltonian of the battery and the auxiliary be $H_B = h_B\sigma^z$ and $H_A = h_A\sigma^z$, respectively. Here σ^z is the Pauli matrix and h_B and h_A represent the strength of the local fields. We denote the ground and excited states of the battery (auxiliary) as, respectively, $\rho^g_{B(A)} = |g\rangle_{B(A)}\langle g|$ and $\rho^e_{B(A)} = |e\rangle_{B(A)}\langle e|$. Initially, we consider both the battery, B, and the auxiliary, A, to be prepared in the excited states whereas the environment, E, and the external qubits, X are in any arbitrary state, say $|0\rangle_E\langle 0|$ and $|0\rangle_X\langle 0|$ respectively. Hence the joint state of the system, S, is given by $\rho^e_B \otimes \rho^e_A \otimes |0\rangle_E\langle 0| \otimes |0\rangle_X\langle 0|$. At this point, we switch on an interaction for time t between B and A described by the Hamiltonian $H_I = J_{BA}(\sigma^x \otimes \sigma^x)$, where σ^x is the Pauli spin matrix and J_{BA} denotes the strength of the interaction. Hence the total Hamiltonian of BA in between the time $(0, t]$ is given by

$$\begin{aligned} H_{BA} &= H_B + H_A + H_I \\ &= h_B(\sigma^z \otimes \mathbb{I}_A) + h_A(\mathbb{I}_B \otimes \sigma^z) + J_{BA}(\sigma^x \otimes \sigma^x). \end{aligned} \tag{3}$$

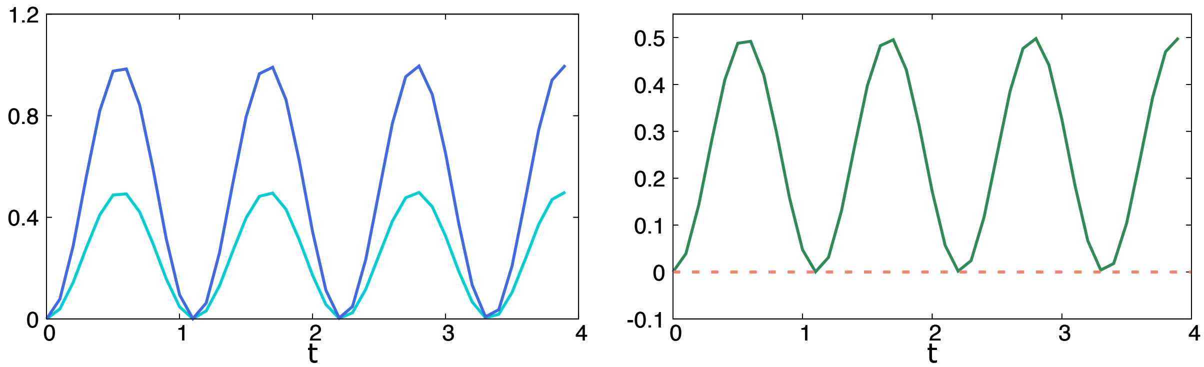

FIG. 2. Comparison between stochastically extractable energy from a battery by performing POVM and NPOVM on a connected auxiliary. Along the vertical axis of the left panel, we plot S^{NP}_{max} (blue curve) and S^{P}_{max} (cyan) with respect to the time duration, t, of the interaction between B and A which occurred in the first step of the measurement-based energy extraction process. The difference between S^{NP}_{max} and S^{P}_{max} is shown in the right panel with respect to t using the green curve. Horizontal axes of both of the panels represent t and are in units of $\hbar/h_A$. The pink dashed line visible in the right panel is used to divide the region into two parts, viz. $S^{NP}_{max} < S^{P}_{max}$ and $S^{NP}_{max} > S^{P}_{max}$. Vertical axes of the panels are in units of h_A. The specific parameter values used for the plots are $J_{BA} = 2h_A$, $h_B = h_A$ and $k = 0.5$.

In all the numerical calculations we will, for specificity, consider $J_{BA} = 2h_A$, and $h_B = h_A$. This evolution will not affect E and X which is initially prepared in the state $|0\rangle_E$ and $|0\rangle_X$, respectively. Therefore, the final state after this interaction is $U_{BA}(\rho^e_B \otimes \rho^e_A)U^\dagger_{BA} \otimes |0\rangle_E \langle 0| \otimes |0\rangle_X \langle 0|$, where in this case $U_{BA} = \exp(-iH_{BA}t/\hbar)$. After time t we evolve the joint state of AE using a unitary $U^1_{AE}(k)$ which transforms states in the following way

$$\begin{aligned} U^1_{AE}(k)\,|e\rangle_A\,|0\rangle_E &= \sqrt{1-k}\,|e\rangle_A\,|0\rangle_E + \sqrt{k}\,|g\rangle_A\,|1\rangle_E\,,\\ U^1_{AE}(k)\,|g\rangle_A\,|0\rangle_E &= |g\rangle_A\,|0\rangle_E\,. \end{aligned}$$

It is noticeable that such unitary acts as an amplitude-damping noise on A where k is the strength of that noise, i.e., the interaction between A and E. Finally, we perform POVM on A by applying the optimal projective measurement on the joint state of AX to get the maximum stochastically extracted energy, S^P_{max} [see Eq. (1)]. The nature of S^P_{max} is depicted in the left panel of Fig. 2 using a cyan curve with respect to t. To get the plot we have considered a fixed interaction between A and E described by the unitary, $U^1_{AE}(0.5)$.

Due to the action of the unitary, $U^1_{AE}(k)$, the state of A and E may become entangled, therefore, if instead of performing the projective measurement on the joint state of AX, measurement is performed on the joint state of AE, it will effectively appear as an application of NPOVM on the auxiliary, A. By measuring the system, AE, in the optimal projective-measurement basis, we can get the stochastically extractable energy, S^{NP}_{max} [see Eq. (2)]. The characteristic of S^{NP}_{max} for the considered model is illustrated in the left panel of the same figure, Fig. 2, using a blue curve for the same noise strength, i.e., $k = 0.5$. The right panel of the figure shows the behaviour of the difference between S^{NP}_{max} and S^P_{max} with time for the same interaction, $U^1_{AE}(0.5)$, between A and E. The plots of the figure demonstrate that S^{NP}_{max} is always greater or equal to S^P_{max} for any fixed time, t. Moreover, the peaks of the oscillation of S^{NP}_{max} can be seen in the plot to be much higher than the same of S^P_{max} demonstrating that if the time duration, t, is suitably chosen, S^{NP}_{max} would be much larger than the maximum S^P_{max} optimised over t.

Taking a fixed noise strength, $k = 0.5$, we graphically examined that there exists a clear hierarchy between the effectiveness of POVMs and NPOVMs, with NPOVMs always being equally or more beneficial than POVMs in stochastic energy extraction. Let us now examine if the advantage of NPOVM over POVM still holds if we change the strength of the interaction, k, between A and E. In this regard, we numerically maximize S^{NP}_{max} and S^P_{max} over time within the time range, $t = 0$ to $t = 3600\hbar/h_A$, and plot these optimal values in Fig. 2, with respect to k, using dark-blue and red curves, respectively. It is visible from the figure that there is a finite gap between S^{NP}_{max} and S^P_{max} for all values of k, except $k = 0$ and $k = 1$. The reason behind this behaviour is that for $k = 0$ and 1 the final state after the action of the unitary, U^1_{AE}, is separable in the bipartition $BA : EX$ and therefore even if we perform projective measurement on AE it results as an application of POVM on A.

To examine how the stochastically extractable energies using POVM or NPOVM depend on the nature of the applied unitary on the joint state of AE, we also consider the following two types of unitaries:

$$\begin{aligned} U^2_{AE}(k)\,|e\rangle_A\,|0\rangle_E &= \sqrt{1-k}\,|e\rangle_A\,|0\rangle_E + \sqrt{k}\,|g\rangle_A\,|1\rangle_E\,,\\ U^2_{AE}(k)\,|g\rangle_A\,|0\rangle_E &= \sqrt{1-k}\,|g\rangle_A\,|1\rangle_E + \sqrt{k}\,|e\rangle_A\,|0\rangle_E\,, \end{aligned}$$

and

$$\begin{aligned} U^3_{AE}(k)\,|e\rangle_A\,|0\rangle_E &= \sqrt{1-k}\,|e\rangle_A\,|0\rangle_E + \sqrt{k}\,|g\rangle_A\,|0\rangle_E\,,\\ U^3_{AE}(k)\,|g\rangle_A\,|0\rangle_E &= |g\rangle_A\,|0\rangle_E\,. \end{aligned}$$

One can notice that the unitaries, $U^2_{AE}(k)$ and $U^3_{AE}(k)$, affect A like a bit flip and dephasing noise having strength

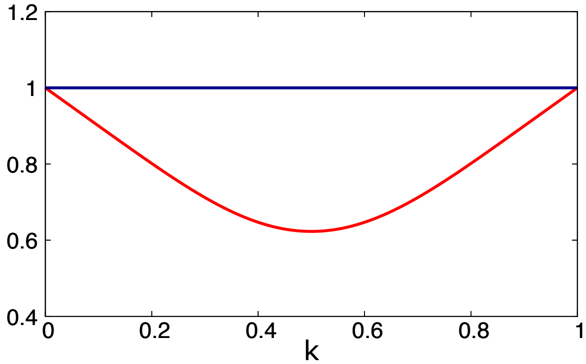

FIG. 3. Nature of stochastically extractable energy from a qubit battery by performing POVM and NPOVM on an attached auxiliary. We individually considered the effects of three types of noise, viz. amplitude damping, dephasing, and bit-flip on the auxiliary qubit before performing the measurement. S_{max}^{NP} and S_{max}^{P} are plotted along the vertical axis with respect to the strength of the noise, k, represented along the horizontal axis. The red curve denotes S_{max}^{P} when the applied noise is either amplitude damping or bit-flip. On the other hand, the dark blue curve depicts S_{max}^{NP} for the same pair of noise. When the system is exposed to dephasing noise, S_{max}^{NP} and S_{max}^{P} become equal for all k and therefore have been shown in the figure using the same colour, dark blue. The values of the other parameters are taken to be $J_{BA} = 2h_A$, $h_B = h_A$ and $t = 0.3h_A/J_{BA}$. The vertical axis is in units of h_A and the horizontal axis is dimensionless.

k. In this case, after the evolution of BA for time, t, we separately apply $U_{AE}^2(k)$ and $U_{AE}^3(k)$ on the joint state of AE, instead of applying $U_{AE}^1(k)$, and finally implement POVM and NPOVM on A by performing a joint projective measurement on AX and AE respectively. The behaviour of the maximum stochastically extractable energy using POVMs and NPOVMs maximized within the time range, $t = 0$ to $t = 3600\hbar/h_A$, in case of bit flip noise acting on A, is shown in Fig. 3 for varying noise strength, k, using, respectively, red and dark-blue lines. In the case of dephasing noise acting upon the auxiliary, the stochastically extractable energy by performing POVMs and NPOVMs on the auxiliary qubit is always equal and therefore only the maximum value of stochastically extractable energy using NPOVM is shown in the figure using the dark-blue curve. It is visible from Fig. 3, that in the case of POVM performed on the auxiliary affected by amplitude damping or bit-flip noise, the stochastically extractable energy at first reduces with the increase in noise strength, k, reaching a minimum value at $k = 0.5$, after which, it starts to increase with k. But in the case of NPOVM-based energy extraction, the maximum stochastically extractable energy is not only always greater or equal to the same for POVM-based energy extraction, but it also does not depend on the noise parameter, k, or even on the noise type, i.e., it is the same for all considered noise types: amplitude damping, bit flip, and dephasing as we expected from Eq. (2).

A. POVMs do not form a subset of the considered NPOVMs

Since all the examples discussed in the previous section prove NPOVMs are always equally or more advantageous than POVMs, one might wonder if the set of POVMs applied on the auxiliary is a subset of the NPOVMs applied on the same. Here, we demonstrate that this is not the case; specifically, the set of all POVMs is not contained within the set of NPOVMs considered.

In the considered scenario, we took the initial state of the total system as $\rho_B^e \otimes \rho_A^e \otimes |0\rangle_E \langle 0| \otimes |0\rangle_X \langle 0|$, where B, A, E and X denote the battery, auxiliary, environment and external respectively. Each of ρ_B^e, ρ_A^e, $|0\rangle_E \langle 0|$ and $|0\rangle_X \langle 0|$ are rank-one states, and the dimension of each of the systems is considered to be 2. Initially, the battery and auxiliary undergo a unitary evolution which, in general, generates entanglement between the battery and auxiliary. As a result, the rank of the local state of the auxiliary gets increased. Let us consider the eigenvalues of the state of the auxiliary at the end of its interaction with the battery to be λ_1 and λ_2. Since, at this instant, the auxiliary and environment are in a product state and the environment is in a pure state, the set of eigenvalues of the composite system consisting of only the auxiliary and environment is given by, $\{\lambda_1, \lambda_2, 0, 0\}$. After the interaction between the battery and the auxiliary, noise acts on the auxiliary by turning on of an interaction between the auxiliary and the environment. At the end of the interaction, the auxiliary-environment state is $\rho_{AE}^2 = \text{tr}_{BX}(\rho_S^2)$. As the eigenspectrum of an operator remains invariant under unitary operation, the set of eigenvalues of the joint state of the auxiliary and environment will remain unchanged in this evolution, and only the eigenspectrums of the individual subsystems, i.e., the auxiliary and environment, may change. Hence the eigenvalues of $\rho_{AE}^2 = \text{tr}_{BX}(\rho_S^2)$ are elements of $\{\lambda_1, \lambda_2, 0, 0\}$. Let the set of eigenvalues of the auxiliary system, $\rho_A^2 = \text{tr}_{EBX}(\rho_S^2)$, after the operation of the noise be $\{x_1, x_2\}$, which in general is not equal to the set $\{\lambda_1, \lambda_2\}$. At this moment, we applied the POVM on the auxiliary qubit by ignoring the environment and performing a joint projective measurement on the auxiliary-external system, whose pre-measurement state is $\rho_A^2 \otimes |0\rangle_X \langle 0|$, whereas in case the considered NPOVM is applied, it is performed by acting with a projective measurement on the joint state, ρ_{AE}^2, of the auxiliary and environment.

A rank-one projective measurement on an arbitrary basis of a bipartite system can be implemented by operating a joint unitary, U, on the entire bipartite system and then measuring it in the computational basis. Therefore the performance of the POVM and NPOVM on the auxiliary are equivalent to measuring the states $U_1 \rho_A^2 \otimes |0\rangle_X \langle 0| U_1^\dagger$ and $U_2 \rho_{AE}^2 U_2^\dagger$ in the computational basis, respectively, where U_1 and U_2 are unitaries respectively acting on AX and AE, and specifies the basis of the corresponding projective measurements. For the NPOVM applied on the auxiliary to reduce into a POVM, we need the auxiliary to be in a product state with the environment at the moment when the projective measurement is being performed. This implies that there should exist a unitary, U_2, such that $U_2\rho_{AE}^2 U_2^\dagger = \rho_A^2 \otimes \chi$, where χ is an arbitrary state of the environment. But it can be shown that no such unitary can be constructed. This can be argued as follows: The eigenspectrum of ρ_{AE}^2 is $\{\lambda_1, \lambda_2, 0, 0\}$, while that of ρ_A^2 is $\{x_1, x_2\}$. Let the eigenspectrum of χ be $\{y_1, y_2\}$. Hence the eigenspectrum of $\rho_A^2 \otimes \mathcal{X}$ is $\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$. Since unitaries do not change eigenvalues, for the condition, $U_2\rho_{AE}^2 U_2^\dagger = \rho_A^2 \otimes \mathcal{X}$, to hold, we must have $\{\lambda_1, \lambda_2, 0, 0\} = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ which implies either $\{x_1, x_2\} = \{\lambda_1, \lambda_2\}$ and $\{y_1, y_2\} = \{0, 1\}$ or $\{y_1, y_2\} = \{\lambda_1, \lambda_2\}$ and $\{x_1, x_2\} = \{0, 1\}$. But this is not true in general, and therefore we can conclude that the NPOVMs considered here do not form a superset of all POVMs.

V. STOCHASTICALLY ACCESSIBLE ENERGY USING POVM AND NPOVM

In the previous section, we have depicted the situation where an optimal NPOVM, performed within a particular range of time, on an auxiliary qubit attached to a quantum battery, can stochastically extract equal or more energy than any POVM applied on the same. However in practical scenarios, preparing and performing the optimal measurement on the auxiliary qubit, A, i.e., the best projective measurement on AE or AX may be expensive. The experimentalists, in such scenarios, may economically have access to a restricted set of measurements. To examine such situations, in this section, we will perform the optimization involved in the definitions of S_{max}^{NP} and S_{max}^{P} over a limited set of measurement operators. The projective measurements applied on AX (in case of POVM) and AE (in case of NPOVM) can be implemented by operating unitaries on, respectively, AX and AE, and then measuring the systems, AX and AE, in the computational basis. Since we are already using a unitary, $U_{BA} = \exp(-iH_{BA}t/\hbar)$, to entangle B and A in the first step, we can assume that such unitary, U_{BA}, is available in the laboratory. Therefore we will use the same unitary, $U_{AE(AX)} = \exp\big(-iH_{AE(AX)}t/\hbar\big)$, on AE (AX) to perform measurement on AE (AX) and finally apply a local projective measurement, $\{\left|\Psi_i\right\rangle_A \left|\Psi_j\right\rangle_{E(X)}\}$ on AE (AX). Here $H_{AE(AX)}$ is given by

$$H_{AE(AX)} = h_A(\sigma^z \otimes \mathbb{I}_{E(X)}) + h_{E(X)}(\mathbb{I}_A \otimes \sigma^z) + J_{AE(AX)}(\sigma^x \otimes \sigma^x),$$

which is the same Hamiltonian as H_{BA} with the only difference that it is defined to act on the system, AE (AX). Since the form of the unitary, $U_{AE(AX)}$ is the same with U_{BA}, we expect that if U_{BA} can be produced in the laboratory then $U_{AE(AX)}$ can also be constructed. Furthermore, since performing measurement in a local basis, $\{\left|\Psi_i\right\rangle_A \left|\Psi_j\right\rangle_{E(X)}\}$, does not require any additional non-local resources we believe measurements in the basis, $\{\left|\Psi_i\right\rangle_A \left|\Psi_j\right\rangle_{E(X)}\}$, can also be easily implemented. Therefore, instead of optimizing S^{NP} (S^P) over all unitaries, $\widetilde{U}_{AE(AX)}$, to determine the stochastically extractable energy, S^{NP} (S^P), we optimize S^{NP} (S^P) over a restricted set of unitaries, $\{U_{AE(AX)}\widetilde{U}_A \otimes \widetilde{U}_{E(X)}\}$, production of which involves lesser cost, where the set runs over all local unitaries, $\widetilde{U}_A \otimes \widetilde{U}_{E(X)}$. The stochastically extracted energy optimized over this smaller set of unitaries is named stochastically accessible energy, in the sense, that it is the amount of energy of the battery that can be accessed in the laboratory. We denote them as S_A^P and S_A^{NP}, for the case when the auxiliary undergoes a POVM and NPOVM respectively. Formally, the expressions of S_A^P and S_A^{NP} are given by, respectively,

$$\begin{aligned} S_A^P &= \max_{\widetilde{U}_A, \widetilde{U}_X} \operatorname{tr}\Big(\widetilde{U}_A \otimes \widetilde{U}_X \left|\psi\right\rangle\!\left\langle\psi\right| \widetilde{U}_A^\dagger \otimes \widetilde{U}_X^\dagger \\ &\qquad \operatorname{tr}_B\left[\left(\operatorname{tr}_E\left(\rho_S^2\right) Z\right)\right]\Big), \\ S_A^{NP} &= \max_{\widetilde{U}_A, \widetilde{U}_E} \operatorname{tr}\Big(\widetilde{U}_A \otimes \widetilde{U}_E \left|\psi'\right\rangle\!\left\langle\psi'\right| \widetilde{U}_A^\dagger \otimes \widetilde{U}_E^\dagger \\ &\qquad \operatorname{tr}_B\left[\operatorname{tr}_X\left(\rho_S^2\right) Z'\right]\Big), \end{aligned}$$

where $\left|\psi\right\rangle = \exp(-iH_{AX}t/\hbar)\left|0\right\rangle$ and $\left|\psi'\right\rangle = \exp(-iH_{AE}t/\hbar)\left|0'\right\rangle$. Similar to Sec. IV, here also we started with the state $\rho_B^e \otimes \rho_A^e \otimes \left|0\right\rangle_E \left\langle 0\right|_E \otimes \left|0\right\rangle_X \left\langle 0\right|_X$, applied the unitary $U_{BA} = \exp(-iH_{BA}t/\hbar)$ [see Eq. (3) for expression of H_{BA}] on BA, operated $U_{AE}^i(k)$ on AE, where i=1, 2, and 3, and finally performed the optimal NPOVM or POVM on A to extract S_A^{NP} or S_A^P energy from the battery. To find the optimal $\widetilde{U}_A$ and $\widetilde{U}_E$ we have used a non-linear optimization package. The behaviour of S_A^{NP} or S_A^P are presented in Fig. 4. The left, middle, and right panels of the figure correspond to the action of unitary $U_{AE}^1(k)$, $U_{AE}^2(k)$, and $U_{AE}^3(k)$, on the joint system of AE, which, affects A as an amplitude damping, a bit-flip, and a dephasing noise, respectively. The dark blue and red lines represent stochastically accessible energies as a function of k for, respectively, NPOVM- and POVM-based energy extraction. From Fig. 4, it is clear that in the presence of amplitude damping and bit-flip noise, the application of NPOVM provides a benefit over POVM within the for $k \in [0, 0.626]$ (for amplitude damping) and $k \in [0, 0.828]$ (for bit-flip), i.e., for higher noise strength POVM outperforms NPOVM providing a greater amount of stochastically accessible energy. In case of dephasing noise, the stochastically accessible energies using POVM and NPOVM are the same and therefore represented using the same colour, dark blue, in the right panel of Fig. 4. If we compare the three types of unitaries, U_{AE}^1, U_{AE}^2, and U_{AE}^3, from the figure we can notice U_{AE}^2 provides the most stochastically accessible energy if NPOVM is performed. It should be noted that, since we optimize over a limited set of unitaries, the stochastically accessible energy is, as expected, less

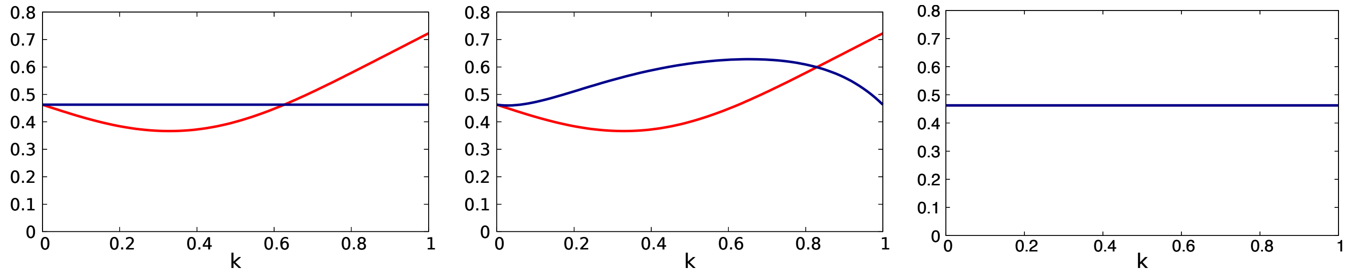

FIG. 4. Behaviour of stochastically accessible energy with respect to noise strength. The dark blue curves depict the nature of S_A^{NP} when the noise applied on the auxiliary qubit are amplitude damping (left panel), bit-flip (middle panel), and dephasing (right panel). The red curves of the left and middle panels and the blue curve of the right panel represent characteristics of S_A^P when the considered noise are, respectively, amplitude damping, bit-flip, and dephasing. In all the panels, the horizontal axes represent the strength of the applied noise while the vertical axes represent S_A^P and S_A^{NP}. The parameter values which are considered to obtain the plots are $J_{BA} = 2h_A$, $h_B = h_A$ and $t = 0.75h_A/J_{BA}$. The horizontal axes are dimensionless whereas the vertical axes are in the units of h_A.

than the stochastically extractable energy for each of the three noise models.

VI. CONCLUSION

A measurement-based method of energy extraction from quantum batteries was introduced in Ref. [40], which involved measurements on an auxiliary attached to the battery. The energy extraction method was examined in Ref. [59] focusing on two qubit batteries and considering projective measurements on the auxiliary.

Here we considered measurement-based energy extraction from quantum batteries and began by using POVMs. Subsequently we identified the most general quantum mechanically allowed measurements, which could be NPOVMs, and use them in the energy extraction. The energy extraction method considered here included first attaching an auxiliary system to the battery, unitarily evolving the joint battery-auxiliary state for a certain time, connecting the auxiliary with an environment which introduces noise in the auxiliary and finally performing a POVM or a physically realizable NPOVM on the auxiliary and selecting a preferred outcome. A POVM measurement is performed on the auxiliary by attaching an external to it, which is in a product state with the rest of the system, and making a projective measurement on the auxiliary and external system. On the other hand, if there is an initial entanglement among the auxiliary, environment, and external system, then joint projective measurement on the auxiliary-environment-system state can be interpreted as a physically realizable NPOVM operation on the auxiliary. In this paper, however, we performed NPOVM operation on the auxiliary by making a projective measurement on the joint state of the auxiliary and environment, which had become entangled due to the consideration of the interaction between them.

One should, therefore, keep in mind that the set of POVM measurements that has been used here is not a subset of the set of NPOVM operators that has been implemented.

The stochastically extractable energies using POVMs and NPOVMs were defined as the maximum energy that can be extracted in the corresponding method multiplied by the probability of getting the particular outcome which results in this amount of energy extraction, optimized over the relevant measurements. We first derived the expressions for the stochastically extractable energies using POVMs and NPOVMs and proved that stochastically extractable energy using NPOVMs does not depend on the applied noise on the auxiliary.

Subsequently, we focused on qubit batteries with a particular Hamiltonian for the battery-auxiliary pair. We compared the stochastically extractable energy using POVMs and NPOVMs in such a model. First, the effect of amplitude damping channel on the auxiliary, induced by an environment was examined. By fixing the strength of interaction between the auxiliary and the environment, we showed that the stochastically extractable energy with NPOVMs is always equal to or more than the stochastically extractable energy using POVMs. Additionally, by separately considering amplitude damping, bit-flip, and dephasing noise, we maximized the stochastically extractable energy over the time of initial battery-auxiliary interaction and proved that the optimized stochastically extractable energy using NPOVMs is again always greater than or equal to the same for POVMs for all considered noise strengths.

Keeping in mind the fact that in reality, it may not be possible to perform arbitrary measurements. In the next part of the paper, we dealt with situations where access to only a restricted set of POVM and NPOVM is provided. We referred to the energy that can be extracted multiplied by the probability of getting the corresponding outcome maximized over the set of available measurements as stochastically accessible energy. By fixing the time interval of the interaction between the auxiliary and the battery, we found that, for amplitude damping and bit-flip noise, the stochastically accessible energy by

NPOVM is still larger than the POVMs over a wide range of noise parameters, though not for the entire range, whereas, in the case of dephasing noise, the stochastically accessible energies by both POVM and NPOVM remain equal to each other and constant over the entire range of noise strength.

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