Bifurcations for Lagrangian systems and geodesics

Guangcun Lu*

Preliminary version

Abstract In this paper we shall use the abstract bifurcation theorems developed by the author in previous papers to study bifurcations of solutions for Lagrangian systems on manifolds linearly or nonlinearly dependent on parameters under various boundary value conditions. As applications, many bifurcation results for geodesics on Finsler and Riemannian manifolds are derived.

Contents

2020 *Mathematics Subject Classification.* Primary 37J20, 34C23, 53C22.
Key words and phrases. Bifurcation, Lagrangian system, geodesic.
*E-mail: gclu@bnu.edu.cn/April 29, 2024.

Introduction

Basic assumptions and conventions. Let M be a n-dimensional, connected C^7 submanifold of $\mathbb{R}^N$. Its tangent bundle TM is a C^6-smooth manifold of dimension $2n$, whose points are denoted by (x, v), with $x \in M$ and $v \in T_xM$. The bundle projection $\pi : TM \to M$, $(x, v) \mapsto x$ is C^6. Let g be a C^6 Riemannian metric and $\mathbb{I}_g$ a C^7 isometry on (M, g), i.e., $\mathbb{I}_g : M \to M$ is C^7 and satisfies $g((\mathbb{I}_g)_*(u), (\mathbb{I}_g)_*(v)) = g(u, v)$ for all $u, v \in TM$. (Thus the Christoffiel symbols Γ^i_{jk} and the exponential map $\exp : TM \to M$ are C^5.) Without special statements, Λ denotes a topological space.

This paper is a continuation of our program on variational bifurcations beginning at [33, 34, 35]. Using the abstract bifurcation theory developed in [34, 36] we studied bifurcations for solutions of several types of Hamiltonian boundary value problems [37]. The current manuscript focuses on bifurcations research for the following Lagrangian boundary value problem:

$$\left.\begin{aligned}&\tfrac{d}{dt}\Big(\partial_v L_\lambda(t,\gamma(t),\dot\gamma(t))\Big)-\partial_q L_\lambda(t,\gamma(t),\dot\gamma(t))=0\ \ \forall t\in[0,\tau],\\&(\gamma(0),\gamma(\tau))\in\mathbf{N}\quad\text{and}\\&\partial_v L_\lambda(0,\gamma(0),\dot\gamma(0))[v_0]=\partial_v L_\lambda(\tau,\gamma(\tau),\dot\gamma(\tau))[v_1]\ \ \forall (v_0,v_1)\in T_{(\gamma(0),\gamma(\tau))}\mathbf{N}\end{aligned}\right\}\tag{0.1}$$

with respect to a continuous family $\{\gamma_\lambda\,|\,\lambda\in\Lambda\}$ of solutions of this problem, where $\mathbf{N}$ is a submanifold $M\times M$, (precisely $\mathbf{N}$ is either a product of two submanifolds in M or the graph of an Riemannian isometry on (M,g)), and $L:\Lambda\times[0,\tau]\times TM\to\mathbb{R}$ is as in Assumption 1.1. If every neighborhood of (μ,γ_μ) in $\Lambda\times C^1([0,\tau];\mathbb{R}^{2n})$ contains a point $(\lambda,\alpha_\lambda)\notin\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$ satisfying (0.1) we say (μ,γ_μ) to be bifurcation point of (0.1) in $\Lambda\times C^1([0,\tau];\mathbb{R}^{2n})$ with respect to the trivial branch $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$. Using the Morse index $m^-(\mathcal{E}_\lambda,\gamma_\gamma)$ and nullity $m^0(\mathcal{E}_\lambda,\gamma_\lambda)$ at γ_λ of C^2 functionals

$$C^1_{\mathbf{N}}([0,\tau];M)\to\mathbb{R},\ \gamma\mapsto\mathcal{E}_\lambda(\gamma)=\int_0^\tau L_\lambda(t,\gamma(t),\dot\gamma(t))dt$$

on $C^1_{\mathbf{N}}([0,\tau];M)=\{\gamma\in C^1([0,\tau];M)\,|\,(\gamma(0),\gamma(\tau))\in\mathbf{N}\}$ we shall characterize the following questions:

(1) Under what conditions (μ,γ_μ) is a bifurcation point in the above sense?

(2) What are the necessary (resp. sufficient) condition for a given point (μ,γ_μ) to be a bifurcation point in the above sense?

(3) How is the solutions of (0.1) distributed near a bifurcation point (μ,γ_μ) as above ?

Let $\Delta(\mathcal{E}_\lambda,\gamma_\lambda):=[m^-(\mathcal{E}_\lambda,\gamma_\lambda),m^-(\mathcal{E}_\lambda,\gamma_\lambda)+m^0(\mathcal{E}_\lambda,\gamma_\lambda)]$. Roughly speaking, our answers are:

(a) If Λ is path-connected and there exist two points $\lambda^+,\lambda^-\in\Lambda$ such that $\Delta(\mathcal{E}_{\lambda^-},\gamma_{\lambda^-})\cap\Delta(\mathcal{E}_{\lambda^+},\gamma_{\lambda^+})=\emptyset$, and either $m^0(\mathcal{E}_{\lambda^+},\gamma_{\lambda^+})=0$ or $m^0(\mathcal{E}_{\lambda^-},\gamma_{\lambda^-})=0$, then there exists a bifurcation point (μ,γ_μ) as above.

(b) If Λ is first countable and there exist two points $\lambda^+,\lambda^-\in\Lambda$ in any neighborhood of some $\mu\in\Lambda$ satisfying the properties as in (a), (μ,γ_μ) is a bifurcation point. Conversely, for a bifurcation point (μ,γ_μ) it must hold that $m^0(\mathcal{E}_\mu,\gamma_\mu)>0$. possesses the above properties.

(c) If Λ is a real interval, $\mu\in\mathrm{Int}(\Lambda)$, then the solutions of (0.1) near a bifurcation point (μ,γ_μ) have alternative bifurcations of Rabinowitz's type (as in [51]) provided that $m^0(\mathcal{E}_\lambda,\gamma_\lambda)=0$ for each $\lambda\in\Lambda\setminus\{\mu\}$ near μ, and $m^-(\mathcal{E}_\lambda,\gamma_\lambda)$ take, respectively, values $m^-(\mathcal{E}_\mu,\gamma_\mu)$ and $m^-(\mathcal{E}_\mu,\gamma_\mu)+m^0(\mathcal{E}_\mu,\gamma_\mu)$ as $\lambda\in\Lambda$ varies in two deleted half neighborhoods of μ.

More results and assumptions are precisely stated in Section 1. Proofs are based on the abstract theory in [34, 36] and Appendix in [37]. These constitute Part I of this paper.

Clearly, many bifurcation theorems of geodesics on Riemannian manifolds can be immediately obtained as direct consequences of results in Part I. For geodesics on Finsler manifolds there are questions corresponding to (1)-(3) above. Using a technique by the author [31] we may derive similar answers from the above (a)-(c). These are completed in Part II.

To the author's knowledge there only are a few results on geodesic bifurcations in the literature. The following classical result was first proved by Morse-Littauer [45] for analytic Finsler spaces, and then was generalized to the C^∞ Finsler space by Savage [53]. See Warner [55] for a new proof.

Theorem 0.1. *The exponential map* $\exp^F$ *of a* C^∞ *Finsler space* (M, F) *is not locally injective near any critical point.*

Precisely speaking, if $v \in T_pM \setminus \{0\}$ is a critical point of $\exp_p^F$ then there exist two sequences $(v_k^1), (v_k^2) \subset T_pM \setminus \{v\}$ converging to v such that $v_k^1 \neq v_k^2$ and $\exp_p^F(v_k^1) = \exp_p^F(v_k^2)$ for each $k \in \mathbb{N}$. That is, we have always at least two distinct geodesics from p to some point of any neighborhood of $\exp_p^F(v)$ near the geodesic $[0,1] \ni t \mapsto \exp_p^F(tv)$. Theorem 0.1 is absolutely not trivial because the C^∞ map $\mathbb{R}^2 \ni (x, y) \to (x^3, y) \in \mathbb{R}^2$ is a bijection and has singularity at each point of the y axis.

In the case of a two-dimensional Riemannian manifold Klingenberg used geometrical techniques to study bifurcation at a conjugate point on a geodesic ([22, complement 2.1.13]) and geodesic bifurcations in the case of a smoothly varying family of Riemannian metrics ([22, section 3.4]). There are several generalizations of these to bifurcation of geodesics in semi-Riemannian manifolds and Lorentzian manifolds; see [14, 15, 19, 49] and the references therein. For example, if there exists a nondegenerate conjugate instant $t_0 \in (0, 1)$ along a geodesic $\gamma : [0, 1] \to M$ in a semi-Riemannian manifold (M, g) with $\mathrm{sgn}(t_0) \neq 0$, Piccione, Portaluri and Tausk [49, Corollaries 5.5 and 5.7] concluded that $\gamma(t_0)$ is a bifurcation point along γ and that the exponential map $\exp_{\gamma(0)}$ is not injective on any neighborhood of $t_0\dot{\gamma}(0)$. For a lightlike geodesic $z : [0, 1] \to M$ in a Lorentzian manifold (M, g), Javaloyes and Piccione [19, Corollary 11] showed that $z(t_0)$ with $t_0 \in (0, 1)$ is conjugate to $z(0)$ along z if and only if the exponential map $\exp : \mathcal{A} \cap (\cup_{s\in[0,1]}\{v \in T_{z(s)}M \,|\, g(v, v) \neq 0\}) \to M$ is not locally injective around $t_0\dot{z}(0)$.

Using a bifurcation result (Theorem 1.9) about Euler-Lagrange curves of Lagrangian systems we derive a bifurcation theorem about Finsler geodesics, Theorem 10.5, whose following special form greatly improved Theorem 0.1.

Theorem 0.2. *Let* $M \subset \mathbb{R}^N$ *be a* C^7 *manifold and let* $F : TM \to \mathbb{R}$ *be a* C^6 *Finsler metric. If* v *is a critical point of the restriction* $\exp_p^F$ *of the exponential map* $TM \supseteq \mathcal{D} \ni u \mapsto \exp^F(u) \in M$ *to* $\mathcal{D}_p := \mathcal{D} \cap T_pM$*, then one of the following alternatives occurs:*

(i) *There exists a sequence* (v_k) *of distinct points in* $\mathcal{D}_p \setminus \{v\}$ *converging to* v*, such that* $\exp_p^F(v_k) = \exp_p^F(v)$ *for each* $k = 1, 2, \cdots$.

(ii) *For every* $\lambda \in \mathbb{R} \setminus \{1\}$ *near* 1 *there exists* $v_\lambda \in \mathcal{D}_p \setminus \{v\}$ *such that* $\exp_p^F(\lambda v_\lambda) = \exp_p^F(\lambda v)$ *and* $v_\lambda \to v$ *as* $\lambda \to 1$.

(iii) *Given a small neighborhood* $\mathcal{O}$ *of* v *in* $\mathcal{D}_p$ *there is an one-sided neighborhood* Λ^* *of* 1 *in* $\mathbb{R}$ *such that for any* $\lambda \in \Lambda^* \setminus \{1\}$*, there exist at least two points* v_λ^1 *and* v_λ^2 *in* $\mathcal{O} \setminus \{v\}$ *such that* $\exp_p^F(\lambda v_\lambda^k) = \exp_p^F(\lambda v)$ *for each* $k = 1, 2$*. Moreover the points* v_λ^1 *and* v_λ^2 *above can also be chosen to satisfy* $F(v_\lambda^1) \neq F(v_\lambda^2)$ *if* $\dim \mathrm{Ker}\left(D\exp_p^F(v)\right) > 1$ *and* $\mathcal{O} \setminus \{v\}$ *only contains finitely many points,* $v_1, \cdots, v_m$*, such that* $\exp_p^F(\lambda v_i) = \exp_p^F(\lambda v)$*,* $i = 1, \cdots, m$.

There exist examples to show that the latter two cases of Theorem 0.2 cannot appear. For example, if $M = \mathbb{S}^n$ is the n-sphere with the round metric, then the geodesics are great circles, and the cut locus of the south pole is the north pole. Suppose that p is the south pole and the norm of $v \in T_p\mathbb{S}^n$ is equal to π (the length of semi-great circle). Then $\exp_p(v)$ is the north pole. It is easily seen that only (i) in Theorem 0.2 occurs.

As a continuation of this article, Part I will be generalized to Lagrangian systems of higher order in [39]. We shall also prove similar results to those of Part II for other geometrical variational problems such as minimal submanifolds, harmonic maps, and so on in [40].

Part I
Bifurcations of Lagrangian systems

1 Statement of main results

1.1 Bifurcations for Lagrangian trajectories connecting submanifolds

Assumption 1.1. Let (M, g) be as in "Basic assumptions and conventions" in Introduction. For a real $\tau > 0$ and a topological space Λ, let $L : \Lambda \times [0, \tau] \times TM \to \mathbb{R}$ be a continuous function such that for each C^3 chart $\alpha : U_\alpha \to \alpha(U_\alpha) \subset \mathbb{R}^n$ and the induced bundle chart $T\alpha : TM|_{U_\alpha} \to \alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ the function

$$L^\alpha : \Lambda \times [0, \tau] \times \alpha(U_\alpha) \times \mathbb{R}^n \to \mathbb{R},\ (\lambda, t, q, v) \mapsto L(\lambda, t, (T\alpha)^{-1}(q, v))$$

is C^2 with respect to (t, q, v) and strictly convex with respect to v, and all its partial derivatives also depend continuously on (λ, t, q, v). Let S_0 and S_1 be two boundaryless and connected submanifolds of M of dimensions less than $\dim M$.

By [48, Theorem 4.2], for each integer $1 \le i \le 4$, $C^i([0, \tau]; M)$ is a C^{5-i} Banach manifold modeled on the Banach space $C^i([0, \tau]; \mathbb{R}^n)$ with the tangent space

$$T_\gamma C^i([0, \tau]; M) = C^i(\gamma^* TM) = \{\xi \in C^i([0, \tau]; \mathbb{R}^N) \,|\, \xi(t) \in T_{\gamma(t)}M\ \forall t\}$$

at $\gamma \in C^i([0, \tau]; M)$. Thus

$$C^1_{S_0 \times S_1}([0, \tau]; M) := \left\{\gamma \in C^1([0, \tau]; M) \,|\, (\gamma(0), \gamma(\tau)) \in S_0 \times S_1\right\} \tag{1.1}$$

is a C^4 Banach submanifold of $C^1([0, \tau]; M)$. Its tangent space at $\gamma \in C^1_{S_0 \times S_1}([0, \tau]; M)$ is

$$C^1_{S_0 \times S_1}(\gamma^* TM) := \left\{\xi \in C^1(\gamma^* TM) \,|\, (\xi(0), \xi(\tau)) \in T_{(\gamma(0), \gamma(1))}(S_0 \times S_1)\right\},$$

which is dense in the Hilbert subspace

$$W^{1,2}_{S_0 \times S_1}(\gamma^* TM) := \left\{\xi \in W^{1,2}(\gamma^* TM) \,|\, (\xi(0), \xi(\tau)) \in T_{(\gamma(0), \gamma(1))}(S_0 \times S_1)\right\} \tag{1.2}$$

of $W^{1,2}(\gamma^* TM)$ (consisting of all $W^{1,2}$-sections of the pull-back bundle $\gamma^* TM \to [0, \tau]$) with inner product given by

$$\langle \xi, \eta \rangle_1 = \int_0^\tau \langle \xi(t), \eta(t) \rangle dt + \int_0^\tau \langle \nabla^g_{\dot\gamma} \xi(t), \nabla^g_{\dot\gamma} \xi(t) \rangle dt \tag{1.3}$$

(using the L^2 covariant derivative along γ associated to the Levi-Civita connection ∇^g of the metric g). Hereafter $\langle u, v \rangle = g(u, v)$ for $u, v \in TM$.

For each $\lambda \in \Lambda$, as in the proof of the first claim in [35, Proposition 4.2] we get that

$$\mathcal{E}_\lambda : C^1_{S_0 \times S_1}([0, \tau]; M) \to \mathbb{R},\ \gamma \mapsto \int_0^\tau L_\lambda(t, \gamma(t), \dot\gamma(t)) dt \tag{1.4}$$

is a C^2 functional. A path $\gamma_0 \in C^1_{S_0 \times S_1}([0, \tau]; M)$ is a critical point of $\mathcal{E}_\lambda$ if and only if it belongs to $C^2([0, \tau]; M)$ and satisfies the Euler-Lagrange equation

$$\frac{d}{dt}\Big(\partial_v L_\lambda(t, \gamma(t), \dot\gamma(t))\Big) - \partial_q L_\lambda(t, \gamma(t), \dot\gamma(t)) = 0\ \forall t \in [0, \tau] \tag{1.5}$$

and the boundary condition

$$\left.\begin{array}{l}(\gamma(0),\gamma(\tau))\in S_0\times S_1 \quad \text{and}\\ \partial_v L_\lambda(0,\gamma(0),\dot\gamma(0))[v_0]=0 \quad \forall v_0\in T_{\gamma(0)}S_0,\\ \partial_v L_\lambda(\tau,\gamma(\tau),\dot\gamma(\tau))[v_1]=0 \quad \forall v_1\in T_{\gamma(\tau)}S_1.\end{array}\right\} \tag{1.6}$$

By [11], the second-order differential $D^2\mathcal{E}_\lambda(\gamma_0)$ of $\mathcal{E}_\lambda$ at such a critical point γ_0 can be extended into a continuous symmetric bilinear form on $W^{1,2}_{S_0\times S_1}(\gamma^*TM)$ with finite Morse index and nullity

$$m^-(\mathcal{E}_\lambda,\gamma_0) \quad \text{and} \quad m^0(\mathcal{E}_\lambda,\gamma_0).$$

Assumption 1.2. Under Assumption 1.1, for each $\lambda\in\Lambda$ let $\gamma_\lambda\in C^2([0,\tau];M)$ satisfy (1.5)-(1.6). It is also assumed that $\Lambda\times[0,\tau]\ni(\lambda,t)\mapsto\gamma_\lambda(t)\in M$ and $\Lambda\times[0,\tau]\ni(\lambda,t)\mapsto\dot\gamma_\lambda(t)\in TM$ are continuous, that is, for any C^2 coordinate chart $\phi:W\to\phi(W)\subset\mathbb{R}^n$, maps

$$(\lambda,t)\mapsto(\phi\circ\gamma_\lambda)(t),\quad (\lambda,t)\mapsto\frac{d}{dt}(\phi\circ\gamma_\lambda)(t)$$

are continuous.

Definition 1.3. Let $X=W^{1,2}_{S_0\times S_1}([0,\tau];M)$ (or $C^1_{S_0\times S_1}([0,\tau];M)$, or $C^2_{S_0\times S_1}([0,\tau];M)$). For $\mu\in\Lambda$, we call (μ,γ_μ) a bifurcation point of the problem (1.5)–(1.6) in $\Lambda\times X$ with respect to the branch $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$ if there exists a point (λ_0,γ_0) in any neighborhood of (μ,γ_μ) in $\Lambda\times X$ such that $\gamma_0\ne\gamma_{\lambda_0}$ is a solution of (1.5)–(1.6) with $\lambda=\lambda_0$. Moreover, (μ,γ_μ) is said to be a bifurcation point along sequences of the problem (1.5)–(1.6) in $\Lambda\times X$ with respect to the branch $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$ if there exists a sequence $\{(\lambda_k,\gamma^k)\}_{k\ge1}$ in $\Lambda\times X$, converging to (μ,γ_μ) such that each $\gamma^k\ne\gamma_{\lambda_k}$ is a solution of (1.5)–(1.6) with $\lambda=\lambda_k$, $k=1,2,\cdots$. (These two notions are equivalent if Λ is first countable.)

Recall that an isolated critical point p of a C^1-functional f on a Banach manifold $\mathcal{M}$ is said to be homological visible if there exists a nonzero critical group $C_m(f,p;\mathbf{K})$ for some Abel group $\mathbf{K}$.

Theorem 1.4. *Let Assumptions 1.1, 1.2 be satisfied, and $\mu\in\Lambda$ be such that* [1] $\gamma_\mu(0)\ne\gamma_\mu(\tau)$ *in the case* $\dim S_0>0$ *and* $\dim S_1>0$.

(I) (Necessary condition): *Suppose that (μ,γ_μ) is a bifurcation point along sequences of the problem (1.5)–(1.6) with respect to the branch $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$ in $\Lambda\times C^1_{S_0\times S_1}([0,\tau];M)$. Then $m^0(\mathcal{E}_\mu,\gamma_\mu)>0$.*

(II) (Sufficient condition): *Suppose that Λ is first countable and that there exist two sequences in Λ converging to μ, (λ_k^-) and (λ_k^+), such that one of the following conditions is satisfied:*

(II.1) *For each $k\in\mathbb{N}$, either $\gamma_{\lambda_k^+}$ is not an isolated critical point of $\mathcal{E}_{\lambda_k^+}$, or $\gamma_{\lambda_k^-}$ is not an isolated critical point of $\mathcal{E}_{\lambda_k^-}$, or $\gamma_{\lambda_k^+}$ (resp. $\gamma_{\lambda_k^-}$) is an isolated critical point of $\mathcal{E}_{\lambda_k^+}$ (resp. $\mathcal{E}_{\lambda_k^-}$) and $C_m(\mathcal{E}_{\lambda_k^+},\gamma_{\lambda_k^+};\mathbf{K})$ and $C_m(\mathcal{E}_{\lambda_k^-},\gamma_{\lambda_k^-};\mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m\in\mathbb{Z}$.*

[1] This assumption is to guarantee the existence of a Riemannian metric g on M such that S_0 (resp. S_1) is totally geodesic near $\gamma_\mu(0)$ (resp. $\gamma_\mu(\tau)$) when we reduce the problems to Euclidean spaces in Section 3.1.1. Therefore it is not needed if M is an open subset in Euclidean spaces and S_0 and S_1 are linear subspaces. Actually, when $\gamma_\mu(0)=\gamma_\mu(\tau)$ we only need a weaker condition that there exists a coordinate chart (U,φ) around this point on M such that $\varphi(S_0\cap S_1\cap U)$ is the intersection $\varphi(U)$ of the union of two linear subspaces in $\mathbb{R}^n$.

(II.2) *For each* $k \in \mathbb{N}$, *there exists* $\lambda \in \{\lambda_k^+, \lambda_k^-\}$ *such that* γ_λ *is an either nonisolated or homological visible critical point of* $\mathcal{E}_\lambda$, *and*

$$\left.\begin{array}{l}[m^-(\mathcal{E}_{\lambda_k^-}, \gamma_{\lambda_k^-}), m^-(\mathcal{E}_{\lambda_k^-}, \gamma_{\lambda_k^-}) + m^0(\mathcal{E}_{\lambda_k^-}, \gamma_{\lambda_k^-})] \\ \cap[m^-(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+}), m^-(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+}) + m^0(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+})] = \emptyset.\end{array}\right\} \qquad (*_k)$$

(II.3) *For each* $k \in \mathbb{N}$, $(*_k)$ *holds true, and either* $m^0(\mathcal{E}_{\lambda_k^-}, \gamma_{\lambda_k^-}) = 0$ *or* $m^0(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+}) = 0$.

Then there exists a sequence $\{(\lambda_k, \gamma^k)\}_{k\geq 1}$ *in* $\hat{\Lambda} \times C^2_{S_0\times S_1}([0, \tau]; M)$ *converging to* (μ, γ_μ) *such that each* $\gamma^k \neq \gamma_{\lambda_k}$ *is a solution of the problem (1.5)–(1.6) with* $\lambda = \lambda_k$, $k = 1, 2, \cdots$, *where* $\hat{\Lambda} = \{\mu, \lambda_k^+, \lambda_k^- \,|\, k \in \mathbb{N}\}$. *In particular,* (μ, γ_μ) *is a bifurcation point of the problem (1.5)–(1.6) in* $\hat{\Lambda} \times C^2_{S_0\times S_1}([0, \tau]; M)$ *respect to the branch* $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \hat{\Lambda}\}$ *(and so* $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \Lambda\}$*).*

Theorem 1.5 (Existence for bifurcations)**.** *Let Assumptions 1.1, 1.2 be satisfied, and let* Λ *be path-connected. Suppose that there exist two points* $\lambda^+, \lambda^- \in \Lambda$ *such that one of the following conditions is satisfied:*

(i) *Either* γ_{λ^+} *is not an isolated critical point of* $\mathcal{E}_{\lambda^+}$, *or* γ_{λ^-} *is not an isolated critical point of* $\mathcal{E}_{\lambda^-}$, *or* γ_{λ^+} *(resp.* γ_{λ^-}*) is an isolated critical point of* $\mathcal{E}_{\lambda^+}$ *(resp.* $\mathcal{E}_{\lambda^-}$*) and* $C_m(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}; \mathbf{K})$ *and* $C_m(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}; \mathbf{K})$ *are not isomorphic for some Abel group* $\mathbf{K}$ *and some* $m \in \mathbb{Z}$.

(ii) $[m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}), m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-})+m^0(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-})]\cap[m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}), m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+})+m^0(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+})] = \emptyset$, *and there exists* $\lambda \in \{\lambda^+, \lambda^-\}$ *such that* γ_λ *is an either non-isolated or homological visible critical point of* $\mathcal{E}_\lambda$.

(iii) $[m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}), m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-})+m^0(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-})]\cap[m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}), m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+})+m^0(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+})] = \emptyset$, *and either* $m^0(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}) = 0$ *or* $m^0(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}) = 0$.

Then for any path $\alpha : [0, 1] \to \Lambda$ *connecting* λ^+ *to* λ^- *such that* $\gamma_{\alpha(s)}(0) \neq \gamma_{\alpha(s)}(\tau)$ *for any* $s \in [0, 1]$ *in the case* $\dim S_0 > 0$ *and* $\dim S_1 > 0$, *there exists a sequence* $(\lambda_k) \subset \alpha([0, 1])$ *converging to some* $\mu \in \alpha([0, 1])$, *and solutions* $\gamma^k \neq \gamma_{\lambda_k}$ *of the problem (1.5)–(1.6) with* $\lambda = \lambda_k$, $k = 1, 2, \cdots$, *such that* $\|\gamma^k - \gamma_{\lambda_k}\|_{C^2([0,\tau];\mathbb{R}^N)} \to 0$ *as* $k \to \infty$. *(In particular,* (μ, γ_μ) *is a bifurcation point along sequences of the problem (1.5)–(1.6) in* $\Lambda\times C^2_{S_0\times S_1}([0, \tau]; M)$ *with respect to the branch* $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \Lambda\}$*.) Moreover,* μ *is not equal to* λ^+ *(resp.* λ^-*) if* $m^0_\tau(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}) = 0$ *(resp.* $m^0_\tau(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}) = 0$*).*

Theorem 1.6 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumptions 1.1, 1.2 with* Λ *being a real interval, let* $\mu \in \mathrm{Int}(\Lambda)$ *satisfy* $\gamma_\mu(0) \neq \gamma_\mu(\tau)$ *(if* $\dim S_0 > 0$ *and* $\dim S_1 > 0$*) and* $m^0(\mathcal{E}_\mu, \gamma_\mu) > 0$. *If* $m^0(\mathcal{E}_\lambda, \gamma_\lambda) = 0$ *for each* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ, *and* $m^-(\mathcal{E}_\lambda, \gamma_\lambda)$ *take, respectively, values* $m^-(\mathcal{E}_\mu, \gamma_\mu)$ *and* $m^-(\mathcal{E}_\mu, \gamma_\mu) + m^0(\mathcal{E}_\mu, \gamma_\mu)$ *as* $\lambda \in \Lambda$ *varies in two deleted half neighborhoods of* μ, *then one of the following alternatives occurs:*

(i) *The problem (1.5)–(1.6) with* $\lambda = \mu$ *has a sequence of solutions,* $\gamma_k \neq \gamma_\mu$, $k = 1, 2, \cdots$, *which converges to* γ_μ *in* $C^2([0, \tau], M)$.

(ii) *For every* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ *there is a solution* $\alpha_\lambda \neq \gamma_\lambda$ *of (1.5)–(1.6) with parameter value* λ, *such that* $\alpha_\lambda - \gamma_\lambda$ *converges to zero in* $C^2([0, \tau], \mathbb{R}^N)$ *as* $\lambda \to \mu$. *(Recall that we have assumed* $M \subset \mathbb{R}^N$*.)*

(iii) *For a given neighborhood $\mathcal{W}$ of γ_μ in $C^1([0,\tau],M)$, there is an one-sided neighborhood Λ^0 of μ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (1.5)–(1.6) with parameter value λ has at least two distinct solutions in $\mathcal{W}$, $\gamma^1_\lambda \neq \gamma_\lambda$ and $\gamma^2_\lambda \neq \gamma_\lambda$, which can also be chosen to satisfies $\mathcal{E}_\lambda(\gamma^1_\lambda) \neq \mathcal{E}_\lambda(\gamma^2_\lambda)$ provided that $m^0(\mathcal{E}_\mu, \gamma_\mu) > 1$ and (1.5)–(1.6) with parameter value λ has only finitely many distinct solutions in $\mathcal{W}$.*

When M is an open subset in $\mathbb{R}^n$, the conditions in Assumptions 1.1, 1.2 in theorems above can be weakened, see Theorems 3.5, 3.6, 3.7.

Assumption 1.7. Let S_0 be a boundaryless submanifold of M of dimension $\dim S_0 < \dim M$, and let $L : [0,\tau] \times TM \to \mathbb{R}$ be C^3 and fiberwise strictly convex, that is, for each $(t,q,v) \in [0,\tau] \times TM$ the bilinear form $\partial_{vv}L(t,q,v)$ is positive definite.

Under Assumption 1.7, a C^2 curve $\gamma : [0,\lambda] \to M$ with $\lambda \in (0,\tau]$ is called a Euler-Lagrange curve of L emanating perpendicularly from S_0 if it solves the following boundary problem

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\partial_v L(t,\gamma(t),\dot\gamma(t))\Big) - \partial_q L(t,\gamma(t),\dot\gamma(t)) = 0,\ 0 \le t \le \lambda, \\ \gamma(0) \in S_0 \quad \text{and} \quad \partial_v L(0,\gamma(0),\dot\gamma(0))[v] = 0\ \forall v \in T_{\gamma(0)}S_0. \end{array}\right\} \tag{1.7}$$

In particular, if S_0 consists of a point p we call a Euler-Lagrange curve of L starting at p. Since L is C^3, with a local coordinate chart it may follow from [6, Proposition 4.3] that the Euler-Lagrange curves of L are C^3. Clearly, for each $s \in (0,\lambda]$ the Euler-Lagrange curve $\gamma_s := \gamma|_{[0,s]}$ of L emanating perpendicularly from S_0 is a critical point of the C^2 functional

$$\mathcal{L}_{S_0,s}(\alpha) := \int_0^s L(t,\alpha(t),\dot\alpha(t))dt \tag{1.8}$$

on the C^4 Banach manifold

$$C^1_{S_0\times\{\gamma(s)\}}([0,s];M) = \left\{\alpha \in C^1([0,s];M)\,\middle|\,\alpha(0) \in S_0, \alpha(s) = \gamma(s))\right\}. \tag{1.9}$$

We say $s \in (0,\lambda]$ to be a S_0-focal point along γ if the linearization of (1.7) on $[0,s]$ (called the the Jacobi equation of the functional $\mathcal{L}_{S_0,s}$) has nonzero solutions, i.e., the second order differential $D^2\mathcal{L}_{S_0,s}(\gamma_s)$ of $\mathcal{L}_{S_0,s}$ at γ_s is degenerate; moreover $\dim \mathrm{Ker}(D^2\mathcal{L}_{S_0,s}(\gamma_s))$ is called the multiplicity of s, denoted by $\nu^{S_0}_\gamma(s)$ or $m^0(\mathcal{L}_{S_0,s},\gamma_s)$.

As done in [49, Definition 6.1] for geodesics, similar to Jacobi's original definition of conjugate points along an extremal of quadratic functionals (cf. [13, Definition 4, page 114]) we introduce:

Definition 1.8. Under Assumption 1.7, $\mu \in (0,\lambda)$ is called a bifurcation instant for (S_0,γ) if there exists a sequence $(t_k) \subset (0,\lambda]$ converging to μ and a sequence of Euler-Lagrange curves of L emanating perpendicularly from S_0, $\gamma_k : [0,\lambda] \to M$, such that

$$\gamma^k(t_k) = \gamma(t_k) \text{ for all } k \in \mathbb{N}, \tag{1.10}$$

$$0 < \|\gamma^k - \gamma\|_{C^1([0,\lambda],\mathbb{R}^N)} \to 0 \ \text{ as } k \to \infty. \tag{1.11}$$

As proved in Lemma 2.6(ii), using local coordinate charts we can derive from the basic existence, uniqueness and smoothness theorem of ODE solutions that the limit of (1.11) is equivalent to any one of the following two conditions:

- $\|\gamma^k - \gamma\|_{C^2([0,\lambda],\mathbb{R}^N)} \to 0$ as $k \to \infty$.
- $\gamma_k(0) \to \gamma(0)$ and $\dot\gamma_k(0) \to \dot\gamma(0)$.

Theorem 1.9. *Under Assumption 1.7, let $\gamma : [0, \tau] \to M$ be a Euler-Lagrange curve of L emanating perpendicularly from S_0. Then:*

(i) *There exists only finitely many S_0-focal points along γ.*

(ii) *If $\mu \in (0, \tau]$ is a bifurcation instant for (S_0, γ), then it is a S_0-focal point along γ.*

(iii) *If $\mu \in (0, \tau)$ is a S_0-focal point along γ, then it is a bifurcation instant for (S_0, γ), and one of the following alternatives occurs:*

(iii-1) *There exists a sequence distinct C^3 Euler-Lagrange curves of L emanating perpendicularly from S_0 and ending at $\gamma(\mu)$, $\alpha_k : [0, \mu] \to M$, $\alpha_k \neq \gamma|_{[0,\mu]}$, $k = 1, 2, \cdots$, such that $\alpha_k \to \gamma|_{[0,\mu]}$ in $C^2([0, \mu], \mathbb{R}^N)$ as $k \to \infty$.*

(iii-2) *For every $\lambda \in (0, \tau) \setminus \{\mu\}$ near μ there exists a C^3 Euler-Lagrange curves of L emanating perpendicularly from S_0 and ending at $\gamma(\lambda)$, $\alpha_\lambda : [0, \lambda] \to M$, $\alpha_\lambda \neq \gamma|_{[0,\lambda]}$, such that $\|\alpha_\lambda - \gamma|_{[0,\lambda]}\|_{C^2([0,\lambda],\mathbb{R}^N)} \to 0$ as $\lambda \to \mu$.*

(iii-3) *For a given small $\epsilon > 0$ there is an one-sided neighborhood Λ^* of μ such that for any $\lambda \in \Lambda^* \setminus \{\mu\}$, there exist at least two distinct C^3 Euler-Lagrange curves of L emanating perpendicularly from S_0 and ending at $\gamma(\lambda)$, $\beta^i_\lambda : [0, \lambda] \to M$, $\beta^i_\lambda \neq \gamma|_{[0,\lambda]}$, $i = 1, 2$, to satisfy the condition that $\|\beta^i_\lambda - \gamma|_{[0,\lambda]}\|_{C^1([0,\lambda],\mathbb{R}^N)} < \epsilon$, $i = 1, 2$. Moreover, if the multiplicity of $\gamma(\mu)$ as a S_0-focal point along γ is greater than one and there exist only finitely many distinct C^3 Euler-Lagrange curves of L emanating perpendicularly from S_0 and ending at $\gamma(\lambda)$, $\alpha_1, \cdots, \alpha_m$, such that $\|\alpha_i - \gamma|_{[0,\lambda]}\|_{C^1([0,\lambda],\mathbb{R}^N)} < \epsilon$, $i = 1, \cdots, m$, then the above two distinct C^3 L-curves $\beta^i_\lambda \neq \gamma|_{[0,\lambda]}$, $i = 1, 2$, can also be chosen to satisfy*

$$\int_0^\lambda L(t, \beta^1_\lambda(t), \dot{\beta}^1_\lambda(t))dt \neq \int_0^\lambda L(t, \beta^2_\lambda(t), \dot{\beta}^2_\lambda(t))dt. \tag{1.12}$$

Remark 1.10. If M is an open subset in the Euclidean space, we may assume that L in Assumption 1.7 and Theorem 1.9 is C^2, see Theorems 3.5, 3.7.

1.2 Bifurcations for generalized periodic solutions of time dependent Lagrangian systems

Assumption 1.11. Let (M, g) be as in "Basic assumptions and conventions" in Introduction, let $\mathbb{I}_g$ be a C^7 isometry on (M, g), and let (τ, Λ, L) be as in Assumption 1.1. For each $\lambda \in \Lambda$ let $\gamma_\lambda \in C^2([0, \tau]; M)$ satisfy the following boundary problem

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\partial_v L(t, \gamma(t), \dot{\gamma}(t))\Big) - \partial_q L(t, \gamma(t), \dot{\gamma}(t)) = 0, \ 0 \le t \le \tau, \\ \mathbb{I}_g(\gamma(0)) = \gamma(\tau) \quad \text{and} \quad d\mathbb{I}_g(\gamma(0))\Big[\frac{\partial L_\lambda}{\partial v}(0, \gamma(0), \dot{\gamma}(0))\Big] = \frac{\partial L_\lambda}{\partial v}(\tau, \gamma(\tau), \dot{\gamma}(\tau)). \end{array}\right\} \tag{1.13}$$

Suppose also that $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto \gamma_\lambda(t) \in M$ and $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto \dot{\gamma}_\lambda(t) \in TM$ are continuous,

Consider the following C^4 Banach submanifold of $C^1([0, \tau]; M)$ of codimension n,

$$C^1_{\mathbb{I}_g}([0, \tau]; M) := \{\gamma \in C^i([0, \tau]; M) \,|\, \mathbb{I}_g(\gamma(0)) = \gamma(\tau)\}. \tag{1.14}$$

Its tangent space at $\gamma \in C^1_{\mathbb{I}_g}([0, \tau]; M)$ is

$$C^1_{\mathbb{I}_g}(\gamma^* TM) := T_\gamma C^1_{\mathbb{I}_g}([0, \tau]; M) = \{\xi \in C^1(\gamma^* TM) \,|\, d\mathbb{I}_g(\gamma(0))[\xi(0)] = \xi(\tau)\},$$

which is dense in the Hilbert subspace

$$W^{1,2}_{\mathbb{I}_g}(\gamma^*TM) := \{\xi \in H^1(\gamma^*TM)\,|\, d\mathbb{I}_g(\gamma(0))[\xi(0)] = \xi(\tau)\}$$

of $W^{1,2}(\gamma^*TM)$.

For each $\lambda \in \Lambda$, as above we have a C^2 functional

$$\mathcal{E}_\lambda : C^1_{\mathbb{I}_g}([0,\tau]; M) \to \mathbb{R},\ \gamma \mapsto \int_0^\tau L_\lambda(t, \gamma(t), \dot\gamma(t))dt. \tag{1.15}$$

By [6, Proposition 4.2] $\gamma \in C^1_{\mathbb{I}_g}([0,\tau]; M)$ is a critical point of $\mathcal{E}_\lambda$ if and only if it belongs to $C^3([0,\tau]; M)$ and satisfy (1.13). By [11], the second-order differential $D^2\mathcal{E}_\lambda(\gamma)$ of $\mathcal{E}_\lambda$ at such a critical point γ can be extended into a continuous symmetric bilinear form on $W^{1,2}_{\mathbb{I}_g}(\gamma^*TM)$ with finite Morse index and nullity

$$m^-_\tau(\mathcal{E}_\lambda, \gamma) \quad\text{and}\quad m^0_\tau(\mathcal{E}_\lambda, \gamma). \tag{1.16}$$

Definition 1.12. In Definition 1.3, "$W^{1,2}_{S_0\times S_1}([0,\tau]; M)$ (or $C^1_{S_0\times S_1}([0,\tau]; M)$, or $C^2_{S_0\times S_1}([0,\tau]; M)$)" is replaced by "$W^{1,2}_{\mathbb{I}_g}([0,\tau]; M)$ (or $C^1_{\mathbb{I}_g}([0,\tau]; M)$, or $C^2_{\mathbb{I}_g}([0,\tau]; M)$)", and "(1.5)–(1.6)" is replaced by "(1.13)".

Theorem 1.13. *Let Assumption 1.11 be satisfied, and $\mu \in \Lambda$.*

(I) (Necessary condition): *Suppose that (μ, γ_μ) is a bifurcation point along sequences of (1.13) with respect to the branch $\{(\lambda, \gamma_\lambda)\,|\,\lambda \in \Lambda\}$ in $C^1_{\mathbb{I}_g}([0,\tau]; M)$. Then $m^0(\mathcal{E}_\mu, \gamma_\mu) > 0$.*

(II) (Sufficient condition): *Suppose that Λ is first countable and that there exist two sequences in Λ converging to μ, (λ^-_k) and (λ^+_k), such that one of the following conditions is satisfied:*

(II.1) *For each $k \in \mathbb{N}$, either $\gamma_{\lambda^+_k}$ is not an isolated critical point of $\mathcal{E}_{\lambda^+_k}$, or $\gamma_{\lambda^-_k}$ is not an isolated critical point of $\mathcal{E}_{\lambda^-_k}$, or $\gamma_{\lambda^+_k}$ (resp. $\gamma_{\lambda^-_k}$) is an isolated critical point of $\mathcal{E}_{\lambda^+_k}$ (resp. $\mathcal{E}_{\lambda^-_k}$) and $C_m(\mathcal{E}_{\lambda^+_k}, \gamma_{\lambda^+_k}; \mathbf{K})$ and $C_m(\mathcal{E}_{\lambda^-_k}, \gamma_{\lambda^-_k}; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$.*

(II.2) *For each $k \in \mathbb{N}$, there exists $\lambda \in \{\lambda^+_k, \lambda^-_k\}$ such that γ_λ is an either nonisolated or homological visible critical point of $\mathcal{E}_\lambda$, and*

$$\left.\begin{array}{l}[m^-(\mathcal{E}_{\lambda^-_k}, \gamma_{\lambda^-_k}), m^-(\mathcal{E}_{\lambda^-_k}, \gamma_{\lambda^-_k}) + m^0(\mathcal{E}_{\lambda^-_k}, \gamma_{\lambda^-_k})]\\ \cap[m^-(\mathcal{E}_{\lambda^+_k}, \gamma_{\lambda^+_k}), m^-(\mathcal{E}_{\lambda^+_k}, \gamma_{\lambda^+_k}) + m^0(\mathcal{E}_{\lambda^+_k}, \gamma_{\lambda^+_k})] = \emptyset.\end{array}\right\} \tag{$2*_k$}$$

(II.3) *For each $k \in \mathbb{N}$, ($2*_k$) holds true, and either $m^0(\mathcal{E}_{\lambda^-_k}, \gamma_{\lambda^-_k}) = 0$ or $m^0(\mathcal{E}_{\lambda^+_k}, \gamma_{\lambda^+_k}) = 0$.*

Then there exists a sequence $\{(\lambda_k, \gamma^k)\}_{k\ge1}$ in $\hat\Lambda \times C^2_{\mathbb{I}_g}([0,\tau]; M)$ converging to (μ, γ_μ) such that each $\gamma^k \ne \gamma_{\lambda_k}$ is a solution of the problem (1.13) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, where $\hat\Lambda = \{\mu, \lambda^+_k, \lambda^-_k\,|\,k \in \mathbb{N}\}$. In particular, (μ, γ_μ) is a bifurcation point of the problem (1.13) in $\hat\Lambda \times C^2_{\mathbb{I}_g}([0,\tau]; M)$ respect to the branch $\{(\lambda, \gamma_\lambda)\,|\,\lambda \in \hat\Lambda\}$ (and so $\{(\lambda, \gamma_\lambda)\,|\,\lambda \in \Lambda\}$).

Theorem 1.14 (Existence for bifurcations)**.** *Let Assumption 1.11 be satisfied, and let Λ be path-connected. Suppose that there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that one of the following conditions is satisfied:*

(i) *Either* γ_{λ^+} *is not an isolated critical point of* $\mathcal{E}_{\lambda^+}$*, or* γ_{λ^-} *is not an isolated critical point of* $\mathcal{E}_{\lambda^-}$*, or* γ_{λ^+} *(resp.* γ_{λ^-}*) is an isolated critical point of* $\mathcal{E}_{\lambda^+}$ *(resp.* $\mathcal{E}_{\lambda^-}$*) and* $C_m(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}; \mathbf{K})$ *and* $C_m(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}; \mathbf{K})$ *are not isomorphic for some Abel group* $\mathbf{K}$ *and some* $m \in \mathbb{Z}$*.*

(ii) $[m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}), m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}) + m^0(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-})] \cap [m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}), m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}) + m^0(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+})] = \emptyset$*, and there exists* $\lambda \in \{\lambda^+, \lambda^-\}$ *such that* γ_λ *is an either non-isolated or homological visible critical point of* $\mathcal{E}_\lambda$*.*

(iii) $[m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}), m^-(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}) + m^0(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-})] \cap [m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}), m^-(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}) + m^0(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+})] = \emptyset$*, and either* $m^0(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}) = 0$ *or* $m^0(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}) = 0$*.*

Then for any path $\alpha : [0,1] \to \Lambda$ *connecting* λ^+ *to* λ^- *there exists a sequence* $(\lambda_k) \subset \alpha([0,1])$ *converging to some* $\mu \in \alpha([0,1])$*, and solutions* $\gamma^k \neq \gamma_{\lambda_k}$ *of the problem (1.13) with* $\lambda = \lambda_k$*,* $k = 1, 2, \cdots$*, such that* $\|\gamma^k - \gamma_{\lambda_k}\|_{C^2([0,\tau];\mathbb{R}^N)} \to 0$ *as* $k \to \infty$*. (In particular,* (μ, γ_μ) *is a bifurcation point along sequences of the problem (1.13) in* $\Lambda \times C^2_{\mathbb{I}_g}([0,\tau]; M)$ *with respect to the branch* $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \Lambda\}$*.) Moreover,* μ *is not equal to* λ^+ *(resp.* λ^-*) if* $m^0_\tau(\mathcal{E}_{\lambda^+}, \gamma_{\lambda^+}) = 0$ *(resp.* $m^0_\tau(\mathcal{E}_{\lambda^-}, \gamma_{\lambda^-}) = 0$*).*

Theorem 1.15 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumption 1.11 with* Λ *being a real interval, let* $\mu \in \mathrm{Int}(\Lambda)$ *satisfy* $m^0(\mathcal{E}_\mu, \gamma_\mu) > 0$*. If* $m^0(\mathcal{E}_\lambda, \gamma_\lambda) = 0$ *for each* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ*, and* $m^-(\mathcal{E}_\lambda, \gamma_\lambda)$ *take, respectively, values* $m^-(\mathcal{E}_\mu, \gamma_\mu)$ *and* $m^-(\mathcal{E}_\mu, \gamma_\mu) + m^0(\mathcal{E}_\mu, \gamma_\mu)$ *as* $\lambda \in \Lambda$ *varies in two deleted half neighborhoods of* μ*, then one of the following alternatives occurs:*

(i) *The problem (1.13) with* $\lambda = \mu$ *has a sequence of solutions,* $\gamma_k \neq \gamma_\mu$*,* $k = 1, 2, \cdots$*, which converges to* γ_μ *in* $C^2([0,\tau], M)$*.*

(ii) *For every* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ *there is a solution* $\alpha_\lambda \neq \gamma_\lambda$ *of (1.13) with parameter value* λ*, such that* $\alpha_\lambda - \gamma_\lambda$ *converges to zero in* $C^2([0,\tau], \mathbb{R}^N)$ *as* $\lambda \to \mu$*. (Recall that we have assumed* $M \subset \mathbb{R}^N$*.)*

(iii) *For a given neighborhood* $\mathcal{W}$ *of* γ_μ *in* $C^1_{\mathbb{I}_g}([0,\tau]; M)$*, there is an one-sided neighborhood* Λ^0 *of* μ *such that for any* $\lambda \in \Lambda^0 \setminus \{\mu\}$*, (1.13) with parameter value* λ *has at least two distinct solutions in* $\mathcal{W}$*,* $\gamma^1_\lambda \neq \gamma_\lambda$ *and* $\gamma^2_\lambda \neq \gamma_\lambda$*, which can also be chosen to satisfies* $\mathcal{E}_\lambda(\gamma^1_\lambda) \neq \mathcal{E}_\lambda(\gamma^2_\lambda)$ *provided that* $m^0(\mathcal{E}_\mu, \gamma_\mu) > 1$ *and (1.13) with parameter value* λ *has only finitely many distinct solutions in* $\mathcal{W}$*.*

Assumption 1.16. In Assumption 1.11, the interval $[0,\tau]$ is replaced by $\mathbb{R}$, and L is also required to be $\mathbb{I}_g$-invariant in the following sense:

$$L(\lambda, t+\tau, \mathbb{I}_g(x), d\mathbb{I}_g(x)[v]) = L(\lambda, t, x, v) \quad \forall (t,x,v) \in \mathbb{R} \times TM. \tag{1.17}$$

The problem (1.13) is replaced by

$$\left.\begin{aligned} &\tfrac{d}{dt}\Big(\partial_v L(t, \gamma(t), \dot\gamma(t))\Big) - \partial_q L(t, \gamma(t), \dot\gamma(t)) = 0 \ \forall t \in \mathbb{R}, \\ &\mathbb{I}_g(\gamma(t)) = \gamma(t+\tau) \quad \forall t \in \mathbb{R}. \end{aligned}\right\} \tag{1.18}$$

Solutions of (1.18) are also called $\mathbb{I}_g$-periodic trajectories with period τ ([9]). When $\mathbb{I}_g$ generates a cyclic group, that is, it is of finite order $p \in \mathbb{N}$, every $\mathbb{I}_g$-periodic trajectory is $p\tau$-periodic.

A C^2 curve $\gamma:\mathbb{R}\to M$ satisfies (1.18) if and only if it is a critical point of the C^2 functional defined by

$$\mathfrak{E}_\lambda(\gamma)=\int_0^\tau L_\lambda(t,\gamma(t),\dot\gamma(t))dt \tag{1.19}$$

on a C^4 Banach manifold $\mathcal{X}^1_\tau(M,\mathbb{I}_g)$, where

$$\mathcal{X}^i_\tau(M,\mathbb{I}_g):=\{\gamma\in C^i(\mathbb{R},M)\,|\,\mathbb{I}_g(\gamma(t))=\gamma(t+\tau)\ \forall t\},\quad i=0,1,2,\cdots. \tag{1.20}$$

Clearly, a solution γ of the problem (1.18) restricts to a solution $\gamma|_{[0,\tau]}$ of (1.13). Conversely, any solution γ^* of (1.13) may extend into that of (1.18), $\gamma:\mathbb{R}\to M$, via

$$\gamma(t)=(\mathbb{I}_g)^k(\gamma^*(t-k\tau))\text{ if }k\tau<t\le(k+1)\tau\text{ with }\pm k\in\mathbb{N}. \tag{1.21}$$

Moreover, for a solution γ of (1.18) we call

$$m^-_\tau(\mathfrak{E}_\lambda,\gamma):=m^-_\tau(\mathcal{E}_\lambda,\gamma|_{[0,\tau]})\quad\text{and}\quad m^0_\tau(\mathfrak{E}_\lambda,\gamma):=m^0_\tau(\mathcal{E}_\lambda,\gamma|_{[0,\tau]}) \tag{1.22}$$

the Morse index and nullity of $\mathfrak{E}_\lambda$ at γ, respectively, where $m^-_\tau(\mathcal{E}_\lambda,\gamma|_{[0,\tau]})$ and $m^0_\tau(\mathcal{E}_\lambda,\gamma|_{[0,\tau]})$ are as in (1.16). These are well-defined by [9, §4].

Theorem 1.13, 1.14, 1.15 immediately leads to the following two results, respectively.

Theorem 1.17. *Let Assumption 1.16 be satisfied, and $\mu\in\Lambda$.*

(I) (Necessary condition): *Suppose that (μ,γ_μ) is a bifurcation point along sequences of the problem (1.18) in $\Lambda\times\mathcal{X}^1_\tau(M,\mathbb{I}_g)$ with respect to the branch $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$. Then $m^0_\tau(\mathfrak{E}_\mu,\gamma_\mu)\ne0$.*

(II) (Sufficient condition): *Suppose that Λ is first countable and that there exist two sequences in Λ converging to μ, (λ^-_k) and (λ^+_k), such that one of the following conditions is satisfied:*

(II.1) *For each $k\in\mathbb{N}$, either $\gamma_{\lambda^+_k}$ is not an isolated critical point of $\mathfrak{E}_{\lambda^+_k}$, or $\gamma_{\lambda^-_k}$ is not an isolated critical point of $\mathfrak{E}_{\lambda^-_k}$, or $\gamma_{\lambda^+_k}$ (resp. $\gamma_{\lambda^-_k}$) is an isolated critical point of $\mathfrak{E}_{\lambda^+_k}$ (resp. $\mathfrak{E}_{\lambda^-_k}$) and $C_m(\mathfrak{E}_{\lambda^+_k},\gamma_{\lambda^+_k};\mathbf{K})$ and $C_m(\mathfrak{E}_{\lambda^-_k},\gamma_{\lambda^-_k};\mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m\in\mathbb{Z}$.*

(II.2) *For each $k\in\mathbb{N}$, there exists $\lambda\in\{\lambda^+_k,\lambda^-_k\}$ such that γ_λ is an either nonisolated or homological visible critical point of $\mathfrak{E}_\lambda$, and*

$$\left.\begin{aligned}&[m^-(\mathfrak{E}_{\lambda^-_k},\gamma_{\lambda^-_k}),m^-(\mathfrak{E}_{\lambda^-_k},\gamma_{\lambda^-_k})+m^0(\mathfrak{E}_{\lambda^-_k},\gamma_{\lambda^-_k})]\\&\cap[m^-(\mathfrak{E}_{\lambda^+_k},\gamma_{\lambda^+_k}),m^-(\mathfrak{E}_{\lambda^+_k},\gamma_{\lambda^+_k})+m^0(\mathfrak{E}_{\lambda^+_k},\gamma_{\lambda^+_k})]=\emptyset.\end{aligned}\right\}\tag{$3*_k$}$$

(II.3) *For each $k\in\mathbb{N}$, ($3*_k$) holds true, and either $m^0(\mathfrak{E}_{\lambda^-_k},\gamma_{\lambda^-_k})=0$ or $m^0(\mathfrak{E}_{\lambda^+_k},\gamma_{\lambda^+_k})=0$.*

Then there exists a sequence $\{(\lambda_k,\gamma^k)\}_{k\ge1}$ in $\hat\Lambda\times\mathcal{X}^2_\tau(M,\mathbb{I}_g)$ converging to (μ,γ_μ) such that each $\gamma^k\ne\gamma_{\lambda_k}$ is a solution of the problem (1.18) with $\lambda=\lambda_k$, $k=1,2,\cdots$, where $\hat\Lambda=\{\mu,\lambda^+_k,\lambda^-_k\,|\,k\in\mathbb{N}\}$. In particular, (μ,γ_μ) is a bifurcation point of the problem (1.18) in $\hat\Lambda\times\mathcal{X}^2_\tau(M,\mathbb{I}_g)$ respect to the branch $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\hat\Lambda\}$ (and so $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$).

Theorem 1.18 (Existence for bifurcations)**.** *Let Assumption 1.16 be satisfied, and let Λ be path-connected. Suppose that there exist two points $\lambda^+,\lambda^-\in\Lambda$ such that one of the following conditions is satisfied:*

(i) *Either* γ_{λ^+} *is not an isolated critical point of* $\mathfrak{E}_{\lambda^+}$*, or* γ_{λ^-} *is not an isolated critical point of* $\mathfrak{E}_{\lambda^-}$*, or* γ_{λ^+} *(resp.* γ_{λ^-}*) is an isolated critical point of* $\mathfrak{E}_{\lambda^+}$ *(resp.* $\mathfrak{E}_{\lambda^-}$*) and* $C_m(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+};\mathbf{K})$ *and* $C_m(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-};\mathbf{K})$ *are not isomorphic for some Abel group* $\mathbf{K}$ *and some* $m\in\mathbb{Z}$*.*

(ii) $[m^-(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-}), m^-(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-})+m^0(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-})]\cap[m^-(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+}), m^-(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+})+m^0(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+})] = \emptyset$*, and there exists* $\lambda\in\{\lambda^+,\lambda^-\}$ *such that* γ_λ *is an either non-isolated or homological visible critical point of* $\mathfrak{E}_\lambda$*.*

(iii) $[m^-(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-}), m^-(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-})+m^0(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-})]\cap[m^-(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+}), m^-(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+})+m^0(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+})] = \emptyset$*, and either* $m^0(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+})=0$ *or* $m^0(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-})=0$*.*

Then for any path $\alpha:[0,1]\to\Lambda$ *connecting* λ^+ *to* λ^- *there exists a sequence* $(\lambda_k)\subset\alpha([0,1])$ *converging to* $\mu\in\alpha([0,1])$*, and solutions* $\gamma^k\neq\gamma_{\lambda_k}$ *of the problem (1.18) with* $\lambda=\lambda_k$*,* $k=1,2,\cdots$*, such that* $(\gamma^k-\gamma_{\lambda_k})|_{[0,\tau]}\to 0$ *in* $C^2([0,\tau];\mathbb{R}^N)$ *as* $k\to\infty$*. (In particular,* (μ,γ_μ) *is a bifurcation point along sequences of the problem (1.18) in* $\Lambda\times\mathcal{X}^2_\tau(M,\mathbb{I}_g)$ *with respect to the branch* $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$*.) Moreover,* μ *is not equal to* λ^+ *(resp.* λ^-*) if* $m^0_\tau(\mathfrak{E}_{\lambda^+},\gamma_{\lambda^+})=0$ *(resp.* $m^0_\tau(\mathfrak{E}_{\lambda^-},\gamma_{\lambda^-})=0$*).*

Theorem 1.19 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumption 1.16 with* Λ *being a real interval, let* $\mu\in\mathrm{Int}(\Lambda)$ *satisfy* $m^0_\tau(\mathfrak{E}_\mu,\gamma_\mu)\neq 0$*. Suppose that* $m^0_\tau(\mathfrak{E}_\lambda,\gamma_\lambda)=0$ *for each* $\lambda\in\Lambda\setminus\{\mu\}$ *near* μ*, and that* $m^-_\tau(\mathfrak{E}_\lambda,\gamma_\lambda)$ *take, respectively, values* $m^-_\tau(\mathfrak{E}_\mu,\gamma_\mu)$ *and* $m^-_\tau(\mathfrak{E}_\mu,\gamma_\mu)+m^0_\tau(\mathfrak{E}_\mu,\gamma_\mu)$ *as* $\lambda\in\Lambda$ *varies in two deleted half neighborhoods of* μ*. Then one of the following alternatives occurs:*

(i) *The problem (1.18) with* $\lambda=\mu$ *has a sequence of solutions,* $\gamma_k\neq\gamma_\mu$*,* $k=1,2,\cdots$*, such that* $\gamma_k\to\gamma_\mu$ *in* $\mathcal{X}^2_\tau(M,\mathbb{I}_g)$ *(or equivaliently* $\gamma_k|_{[0,\tau]}-\gamma_\mu|_{[0,\tau]}$ *converges to zero in* $C^2([0,\tau],\mathbb{R}^N)$*).*

(ii) *For every* $\lambda\in\Lambda\setminus\{\mu\}$ *near* μ *there exists a solution* $\alpha_\lambda\neq\gamma_\lambda$ *of (1.18) with parameter value* λ*, such that* $\|(\alpha_\lambda-\gamma_\lambda)|_{[0,\tau]}\|_{C^2([0,\tau],\mathbb{R}^N)}\to 0$ *as* $\lambda\to\mu$*.*

(iii) *For a given neighborhood* $\mathcal{W}$ *of* γ_μ *in* $\mathcal{X}^1_\tau(M,\mathbb{I}_g)$*, there exists an one-sided neighborhood* Λ^0 *of* μ *such that for any* $\lambda\in\Lambda^0\setminus\{\mu\}$*, the problem (1.18) with parameter value* λ *has at least two distinct solutions in* $\mathcal{W}$*,* $\gamma^1_\lambda\neq\gamma_\lambda$ *and* $\gamma^2_\lambda\neq\gamma_\lambda$*, which can also be chosen to satisfy* $\mathfrak{E}_\lambda(\gamma^1_\lambda)\neq\mathfrak{E}_\lambda(\gamma^2_\lambda)$ *provided that if* $m^0_\tau(\mathfrak{E}_\mu,\gamma_\mu)>1$ *and the problem (1.18) with parameter value* $\lambda\in\Lambda^0\setminus\{\mu\}$ *has only finitely many distinct solutions in* $\mathcal{W}$*.*

1.3 Bifurcations for generalized periodic solutions of autonomous Lagrangian systems

When L_λ in (1.18) is independent of time t, the problem (1.18) becomes

$$\left.\begin{array}{l}\frac{d}{dt}\Big(\partial_vL(\gamma(t),\dot\gamma(t))\Big)-\partial_qL(\gamma(t),\dot\gamma(t))=0\;\forall t\in\mathbb{R},\\ \mathbb{I}_g(\gamma(t))=\gamma(t+\tau)\quad\forall t\in\mathbb{R}.\end{array}\right\}\tag{1.23}$$

Two solutions γ_1 and γ_2 of (1.23) are said $\mathbb{R}$-distinct if $\gamma_1(\theta+\cdot)\neq\gamma_2$ for any $\theta\in\mathbb{R}$. The corresponding functionals $\mathfrak{E}_\lambda$, Morse indexes $m^-_\tau(\mathfrak{E}_\lambda,\gamma)$ and nullities $m^0_\tau(\mathfrak{E}_\lambda,\gamma)$ are also defined by (1.22).

1.3.1 Bifurcations of (1.23) starting at constant solutions

Theorem 1.20 (Alternative bifurcations of Fadell-Rabinowitz's type). *Under Assumption 1.16 with Λ being a real interval, suppose also that L is independent of t and that $\mathbb{I}_g$ satisfies $\mathbb{I}_g^l = id_M$ for some $l \in \mathbb{N}$. Let*

$$\left.\begin{array}{l}\Lambda \ni \lambda \to \gamma_\lambda \in \mathrm{Fix}(\mathbb{I}_g) \subset M \textit{ be continuous and} \\ \partial_x L_\lambda(\gamma_\lambda, 0) = 0\ \forall \lambda \in \Lambda.\end{array}\right\} \tag{1.24}$$

(Therefore γ_λ is a constant solution of (1.23). Hereafter the points γ_λ are also understood as constant value maps from $\mathbb{R}$ to M without special statements.) Suppose that for some $\mu \in \mathrm{Int}(\Lambda)$ and $\tau > 0$,

(a) *$\partial_{vv} L_\mu(\gamma_\mu, 0)$ is positive definite;*

(b) *$\partial_{xx} L_\mu(\gamma_\mu, 0)u = 0$ and $d\mathbb{I}_g(\gamma_\mu)u = u$ have only the zero solution in $T_{\gamma_\mu}M$;*

(c) *$m^0_\tau(\mathfrak{E}_\mu, \gamma_\mu) \neq 0$, $m^0_\tau(\mathfrak{E}_\lambda, \gamma_\mu) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near μ, and $m^-_\tau(\mathfrak{E}_\lambda, \gamma_\mu)$ take, respectively, values $m^-_\tau(\mathfrak{E}_\mu, \gamma_\mu)$ and $m^-_\tau(\mathfrak{E}_\mu, \gamma_\mu) + m^0_\tau(\mathfrak{E}_\mu, \gamma_\mu)$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of μ.*

Then one of the following alternatives occurs:

(i) *The problem (1.23) with $\lambda = \mu$ has a sequence of $\mathbb{R}$-distinct solutions, γ^k, $k = 1, 2, \cdots$, which are $\mathbb{R}$-distinct with γ_μ and converges to γ_μ on any compact interval $I \subset \mathbb{R}$ in C^2-topology.*

(ii) *There exist left and right neighborhoods Λ^- and Λ^+ of μ in Λ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq m^0_\tau(\mathfrak{E}_\mu, \gamma_\mu)/2$, and for $\lambda \in \Lambda^- \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^+ \setminus \{\mu\}$), (1.23) with parameter value λ has at least n^- (resp. n^+) $\mathbb{R}$-distinct solutions solutions, γ^i_λ, $i = 1, \cdots, n^-$ (resp. n^+), which are $\mathbb{R}$-distinct with γ_μ and converge to γ_μ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $\lambda \to \mu$.*

Moreover, if $m^0_\tau(\mathfrak{E}_\mu, \gamma_\mu) \geq 3$, then (ii) may be replaced by the following alternatives:

(iii) *For every $\lambda \in \Lambda \setminus \{\mu\}$ near μ there is a solution $\alpha_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$ of (1.23) with parameter value λ, such that $\alpha_\lambda - \gamma_\lambda$ converges to zero on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $\lambda \to \mu$.*

(iv) *For a given small $\epsilon > 0$ there is an one-sided neighborhood Λ^0 of μ in Λ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (1.23) with parameter value λ has either infinitely many $\mathbb{R}$-distinct solutions $\bar\alpha^k_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$ such that $\|\bar\alpha^k_\lambda|_{[0,\tau]} - \gamma_\lambda|_{[0,\tau]}\|_{C^2([0,\tau];\mathbb{R}^N)} < \epsilon$, $k = 1, 2, \cdots$, or at least two $\mathbb{R}$-distinct solutions $\beta^1_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$ and $\beta^2_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$ such that $\|\beta^i_\lambda|_{[0,\tau]} - \gamma_\lambda|_{[0,\tau]}\|_{C^2([0,\tau];\mathbb{R}^N)} < \epsilon$, $i = 1, 2$, and that $\mathfrak{E}_\lambda(\beta^1_\lambda) \neq \mathfrak{E}_\lambda(\beta^2_\lambda)$. (Recall that we have assumed $M \subset \mathbb{R}^N$.)*

1.3.2 Bifurcations of (1.23) starting at nonconstant solutions

We need make stronger:

Assumption 1.21. Let (M, g) be as in "Basic assumptions and conventions" in Introduction and let $\mathbb{I}_g$ be a C^7 isometry on (M, g). For a real $\tau > 0$ and a topological space Λ, let $L : \Lambda \times TM \to \mathbb{R}$ be a continuous function such that each $L(\lambda, \cdot) : TM \to \mathbb{R}$, $\lambda \in \Lambda$, is C^6 and all its partial derivatives of order no more than two depend continuously on $(\lambda, x, v) \in \Lambda \times TM$.

Each $L_\lambda(\cdot) = L(\lambda, \cdot)$ is fiberwise strictly convex, and $\mathbb{I}_g$-invariant (i.e., $L(\lambda, \mathbb{I}_g(x), d\mathbb{I}_g(x)[v]) = L(\lambda, x, v)$ for all $(x, v) \in TM$). Let $\bar\gamma : \mathbb{R} \to M$ be a nonconstant C^2 map satisfying (1.23) with this L for all $\lambda \in \Lambda$. ($\bar\gamma$ is actually C^6 by [6, Proposition 4.3].)

Under this assumption, each element in $\mathbb{R}\cdot\bar\gamma := \{\bar\gamma(\theta+\cdot) \,|\, \theta \in \mathbb{R}\}$ ($\mathbb{R}$-orbit) also satisfies (1.23) with this L for all $\lambda \in \Lambda$. It follows that $m^0_\tau(\mathcal{E}_\lambda, \bar\gamma) \neq 0$ for all $\lambda \in \Lambda$. Thus each point $(\lambda, \bar\gamma)$ in $\Lambda \times \{\bar\gamma\}$ is a bifurcation point of (1.23) in the sense of Definition 1.12. In order to give an exact description for bifurcation pictures of solutions of (1.23) near $\mathbb{R}\cdot\bar\gamma$, we introduce:

Definition 1.22. $\mathbb{R}$-orbits of solutions of the problem (1.23) with a parameter $\lambda \in \Lambda$ is said sequently bifurcating at μ with respect to the $\mathbb{R}$-orbit $\mathbb{R}\cdot\bar\gamma$ if there exists a sequence $(\lambda_k) \subset \Lambda$ converging to μ, and a solution γ^k of (1.23) with parameter value λ_k for each k, such that: (i) $\gamma^k \notin \mathbb{R}\cdot\bar\gamma$ $\forall k$, (ii) all γ^k are $\mathbb{R}$-distinct, (iii) $\gamma^k|_{[0,\tau]} \to \bar\gamma|_{[0,\tau]}$ in $C^1([0,\tau]; M)$. [Passing to a subsequence (i) is implied in (ii).]

Theorem 1.23 (Necessary condition)**.** *Under Assumption 1.21, suppose that $\mathbb{R}$-orbits of solutions of the problem (1.23) with a parameter $\lambda \in \Lambda$ sequently bifurcate at $\mu \in \Lambda$ with respect to the $\mathbb{R}$-orbit $\mathbb{R}\cdot\bar\gamma$. Then $m^0_\tau(\mathfrak{E}_\mu, \bar\gamma) \geq 2$.*

Theorem 1.24 (Sufficient condition)**.** *Under Assumption 1.21, suppose that Λ is first countable, $\mu \in \Lambda$ and:*

(a) *$\bar\gamma$ is periodic, and $m^0_\tau(\mathfrak{E}_\mu, \bar\gamma) \geq 2$;*

(b) *there exist two sequences in Λ converging to μ, (λ^-_k) and (λ^+_k), such that for each $k \in \mathbb{N}$,*

$$[m^-_\tau(\mathfrak{E}_{\lambda^-_k}, \bar\gamma), m^-_\tau(\mathfrak{E}_{\lambda^-_k}, \bar\gamma) + m^0_\tau(\mathfrak{E}_{\lambda^-_k}, \bar\gamma) - 1] \cap [m^-_\tau(\mathfrak{E}_{\lambda^+_k}, \bar\gamma), m^-_\tau(\mathfrak{E}_{\lambda^+_k}, \bar\gamma) + m^0_\tau(\mathfrak{E}_{\lambda^+_k}, \bar\gamma) - 1] = \emptyset$$

and either $m^0_\tau(\mathfrak{E}_{\lambda^-_k}, \bar\gamma) = 1$ or $m^0_\tau(\mathfrak{E}_{\lambda^+_k}, \bar\gamma) = 1$;

(c) *for any solution γ of (1.23) with $\lambda = \mu$, if there exists a sequence (s_k) of reals such that $s_k \cdot \gamma$ converges to $\bar\gamma$ on any compact interval $I \subset \mathbb{R}$ in C^1-topology, then γ is periodic. (Clearly, this holds if $(\mathbb{I}_g)^l = id_M$ for some $l \in \mathbb{N}$.)*

Then there exists a sequence $(\lambda_k) \subset \hat\Lambda = \{\mu, \lambda^+_k, \lambda^-_k \,|\, k \in \mathbb{N}\}$ converging to μ and C^6 solutions γ_k of the corresponding problem (1.23) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, such that any two of these γ_k are $\mathbb{R}$-distinct and that (γ_k) converges to $\bar\gamma$ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $k \to \infty$.

Theorem 1.25 (Existence for bifurcations)**.** *Under Assumption 1.21, suppose that Λ is path-connected, $(\mathbb{I}_g)^l = id_M$ for some $l \in \mathbb{N}$, and the following is satisfied:*

(d) *There exist two points $\lambda^+, \lambda^- \in \Lambda$ such that*

$$[m^-_\tau(\mathfrak{E}_{\lambda^-}, \bar\gamma), m^-_\tau(\mathfrak{E}_{\lambda^-}, \bar\gamma) + m^0_\tau(\mathfrak{E}_{\lambda^-}, \bar\gamma) - 1] \cap [m^-_\tau(\mathfrak{E}_{\lambda^+}, \bar\gamma), m^-_\tau(\mathfrak{E}_{\lambda^+}, \bar\gamma) + m^0_\tau(\mathfrak{E}_{\lambda^+}, \bar\gamma) - 1] = \emptyset$$

and either $m^0_\tau(\mathfrak{E}_{\lambda^-}, \bar\gamma) = 1$ or $m^0_\tau(\mathfrak{E}_{\lambda^+}, \bar\gamma) = 1$.

Then for any path $\alpha : [0,1] \to \Lambda$ connecting λ^+ to λ^- there exists a sequence (λ_k) in $\alpha([0,1])$ converging to $\mu \in \alpha([0,1]) \subset \Lambda$, and C^6 solutions γ_k of the corresponding problem (1.23) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, such that any two of these γ_k are $\mathbb{R}$-distinct and that (γ_k) converges to $\bar\gamma$ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $k \to \infty$. Moreover, this μ is not equal to λ^+ (resp. λ^-) if $m^0_\tau(\mathfrak{E}_{\lambda^+}, \bar\gamma) = 1$ (resp. $m^0_\tau(\mathfrak{E}_{\lambda^-}, \bar\gamma) = 1$).

Theorem 1.26 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumption 1.21 with Λ being a real interval, let $\mu \in \mathrm{Int}(\Lambda)$, $\mathbb{I}_g = id_M$ and $\bar{\gamma}$ have least period τ. Suppose that*

$$m^0_\tau(\mathfrak{E}_\mu, \bar{\gamma}) \geq 2, \quad m^0_\tau(\mathfrak{E}_\lambda, \bar{\gamma}) = 1 \text{ for each } \lambda \in \Lambda \setminus \{\mu\} \text{ near } 0,$$

and that $m^-_\tau(\mathfrak{E}_\lambda, \bar{\gamma})$ take, respectively, values $m^-_\tau(\mathfrak{E}_\mu, \bar{\gamma})$ and $m^-_\tau(\mathfrak{E}_\mu, \bar{\gamma}) + m^0_\tau(\mathfrak{E}_\mu, \bar{\gamma}) - 1$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of 0. Then one of the following alternatives occurs:

(i) *The corresponding problem (1.23) with $\lambda = \mu$ has a sequence of $\mathbb{R}$-distinct C^6 solutions, γ_k, $k = 1, 2, \cdots$, such that (γ_k) converges to $\bar{\gamma}$ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $k \to \infty$.*

(ii) *For every $\lambda \in \Lambda \setminus \{\mu\}$ near μ there is a C^6 solution γ_λ of (1.23) with parameter value λ, which is $\mathbb{R}$-distinct with $\bar{\gamma}$ and converges to $\bar{\gamma}$ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $\lambda \to \mu$.*

(iii) *For a given neighborhood $\mathcal{W}$ of $\bar{\gamma}$ in $\mathcal{X}^1_\tau(M, \mathbb{I}_g)$, there exists an one-sided neighborhood Λ^0 of μ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (1.23) with parameter value λ has at least two $\mathbb{R}$-distinct C^6 solutions in $\mathcal{W}$, $\gamma^1_\lambda \notin \mathbb{R} \cdot \bar{\gamma}$ and $\gamma^2_\lambda \notin \mathbb{R} \cdot \bar{\gamma}$, which can also be chosen to satisfy $\mathfrak{E}_\lambda(\gamma^1_\lambda) \neq \mathfrak{E}_\lambda(\gamma^2_\lambda)$ provided that $m^0_\tau(\mathfrak{E}_\mu, \bar{\gamma}) \geq 3$ and (1.23) with parameter value λ has only finitely many $\mathbb{R}$-distinct solutions in $\mathcal{W}$ which are $\mathbb{R}$-distinct from $\bar{\gamma}$.*

In the above Theorems 1.23, 1.24, 1.25, 1.26, if M is an open subset U of $\mathbb{R}^n$ and $\mathbb{I}_g$ is an orthogonal matrix E of order n which maintain U invariant, Assumption 1.21 can be replaced by a weaker Assumption 6.12, see Section 6.5.

1.4 Bifurcations for brake orbits of Lagrangian systems

Assumption 1.27. Let (M, g) be as in "Basic assumptions and conventions" in Introduction. For a real $\tau > 0$ and a topological space Λ, let $L : \Lambda \times \mathbb{R} \times TM \to \mathbb{R}$ be a continuous function satisfying

$$L(\lambda, -t, q, -v) = L(\lambda, t, q, v) = L(\lambda, t + \tau, q, v) \quad \forall (t, q, v) \in \Lambda \times \mathbb{R} \times TM. \tag{1.25}$$

Suppose that for each C^3 chart $\alpha : U_\alpha \to \alpha(U_\alpha) \subset \mathbb{R}^n$ and the induced bundle chart $T\alpha : TM|_{U_\alpha} \to \alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ the function

$$L^\alpha : \Lambda \times \mathbb{R} \times \alpha(U_\alpha) \times \mathbb{R}^n \to \mathbb{R}, \ (\lambda, t, q, v) \mapsto L(\lambda, t, (T\alpha)^{-1}(q, v))$$

is C^2 with respect to (t, q, v) and strictly convex with respect to v, and all its partial derivatives also depend continuously on (λ, t, q, v).

Consider the following problem

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\partial_v L_\lambda(t, \gamma(t), \dot{\gamma}(t))\Big) - \partial_x L_\lambda(t, \gamma(t), \dot{\gamma}(t)) = 0 \ \forall t \in \mathbb{R} \\ \gamma(-t) = \gamma(t) = \gamma(t + \tau) \quad \forall t \in \mathbb{R} \end{array}\right\} \tag{1.26}$$

and C^4 Banach manifolds

$$EC^k(S_\tau; M) := \{\gamma \in C^1(\mathbb{R}; M) \,|\, \gamma(t + \tau) = \gamma(t) \,\&\, \gamma(-t) = \gamma(t) \ \forall t \in \mathbb{R}\}, \quad k \in \mathbb{N}. \tag{1.27}$$

Solutions of (1.26) are called brake orbits. Assumption 1.27 assures that the solutions of (1.26) are critical points of the C^2 functionals

$$\mathcal{L}_\lambda^E : EC^1(S_\tau; M) \to \mathbb{R},\ \gamma \mapsto \int_0^\tau L_\lambda(t, \gamma(t), \dot{\gamma}(t))dt \in \mathbb{R}, \quad \lambda \in \Lambda.$$

For a critical point γ of $\mathcal{L}_\lambda^E$, the second-order differential $D^2\mathcal{L}_\lambda^E(\gamma)$ can be extended into a continuous symmetric bilinear form on $W^{1,2}(\gamma^*TM)$ with finite Morse index and nullity

$$m_\tau^-(\mathcal{L}_\lambda^E, \gamma) \quad \text{and} \quad m_\tau^0(\mathcal{L}_\lambda^E, \gamma). \tag{1.28}$$

Assumption 1.28. For each $\lambda \in \Lambda$ let $\gamma_\lambda \in EC^1(S_\tau; M) \cap C^2(S_\tau; M)$ satisfy (1.26) and the maps $\Lambda \times \mathbb{R} \ni (\lambda, t) \to \gamma_\lambda(t) \in M$ and $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto \dot{\gamma}_\lambda(t) \in TM$ are continuous.

For $\mu \in \Lambda$ we call (μ, γ_μ) a bifurcation point along sequences of the problem (1.26) in $\Lambda \times EC^1(S_\tau; M)$ with respect to the branch $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \Lambda\}$ if there exists a sequence $\{(\lambda_k, \gamma^k)\}_{k\ge 1}$ in $\Lambda \times EC^1(S_\tau; M)$ converging to (μ, γ_μ), such that each $\gamma^k \ne \gamma_{\lambda_k}$ is a solution of (1.26) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$.

Theorem 1.29. *Let Assumptions 1.27,1.28 be satisfied.*

(I) (Necessary condition): *Suppose that (μ, γ_μ) is a bifurcation point along sequences of the problem (1.26). Then $m_\tau^0(\mathcal{L}_\mu^E, \gamma_\mu) > 0$.*

(II) (Sufficient condition): *Let Λ be first countable. Suppose that there exist two sequences in Λ converging to μ, (λ_k^-) and (λ_k^+), such that one of the following conditions is satisfied:*

(II.1) *For each $k \in \mathbb{N}$, either $\gamma_{\lambda_k^+}$ is not an isolated critical point of $\mathcal{L}_{\lambda_k^+}^E$, or $\gamma_{\lambda_k^-}$ is not an isolated critical point of $\mathcal{L}_{\lambda_k^-}^E$, or $\gamma_{\lambda_k^+}$ (resp. $\gamma_{\lambda_k^-}$) is an isolated critical point of $\mathcal{L}_{\lambda_k^+}^E$ (resp. $\mathcal{L}_{\lambda_k^-}^E$) and $C_m(\mathcal{L}_{\lambda_k^+}^E, \gamma_{\lambda_k^+}; \mathbf{K})$ and $C_m(\mathcal{L}_{\lambda_k^-}^E, \gamma_{\lambda_k^-}; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$.*

(II.2) *For each $k \in \mathbb{N}$, there exists $\lambda \in \{\lambda_k^+, \lambda_k^-\}$ such that γ_λ is an either nonisolated or homological visible critical point of $\mathcal{L}_\lambda^E$, and*

$$\left.\begin{array}{l} [m^-(\mathcal{L}_{\lambda_k^-}^E, \gamma_{\lambda_k^-}), m^-(\mathcal{L}_{\lambda_k^-}^E, \gamma_{\lambda_k^-}) + m^0(\mathcal{L}_{\lambda_k^-}^E, \gamma_{\lambda_k^-})] \\ \cap [m^-(\mathcal{L}_{\lambda_k^+}^E, \gamma_{\lambda_k^+}), m^-(\mathcal{L}_{\lambda_k^+}^E, \gamma_{\lambda_k^+}) + m^0(\mathcal{L}_{\lambda_k^+}^E, \gamma_{\lambda_k^+})] = \emptyset. \end{array}\right\} \tag{4$*_k$}$$

(II.3) *For each $k \in \mathbb{N}$, (4$*_k$) holds true, and either $m^0(\mathcal{L}_{\lambda_k^-}^E, \gamma_{\lambda_k^-}) = 0$ or $m^0(\mathcal{L}_{\lambda_k^+}^E, \gamma_{\lambda_k^+}) = 0$.*

Then there exists a sequence $\{(\lambda_k, \gamma^k)\}_{k\ge1}$ in $\hat{\Lambda} \times EC^1(S_\tau; M)$ converging to (μ, γ_μ) such that each $\gamma^k \ne \gamma_{\lambda_k}$ is a solution of the problem (1.26) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, where $\hat{\Lambda} = \{\mu, \lambda_k^+, \lambda_k^- \,|\, k \in \mathbb{N}\}$. In particular, (μ, γ_μ) is a bifurcation point of the problem (1.26) in $\hat{\Lambda} \times EC^1(S_\tau; M)$ respect to the branch $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \hat{\Lambda}\}$ (and so $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \Lambda\}$).

Theorem 1.30 (Existence for bifurcations)**.** *Let Assumptions 1.27,1.28 be satisfied, and let Λ be path-connected. Suppose that there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that one of the following conditions is satisfied:*

(i) *Either γ_{λ^+} is not an isolated critical point of $\mathcal{L}_{\lambda^+}^E$, or γ_{λ^-} is not an isolated critical point of $\mathcal{L}_{\lambda^-}^E$, or γ_{λ^+} (resp. γ_{λ^-}) is an isolated critical point of $\mathcal{L}_{\lambda^+}^E$ (resp. $\mathcal{L}_{\lambda^-}^E$) and $C_m(\mathcal{L}_{\lambda^+}^E, \gamma_{\lambda^+}; \mathbf{K})$ and $C_m(\mathcal{L}_{\lambda^-}^E, \gamma_{\lambda^-}; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$.*

(ii) $[m^-(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-}), m^-(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-})+m^0(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-})]\cap[m^-(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+}), m^-(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+})+m^0(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+})]$ $=\emptyset$, *and there exists* $\lambda\in\{\lambda^+,\lambda^-\}$ *such that* γ_λ *is an either non-isolated or homological visible critical point of* $\mathcal{L}^E_\lambda$.

(iii) $[m^-(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-}), m^-(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-})+m^0(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-})]\cap[m^-(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+}), m^-(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+})+m^0(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+})]$ $=\emptyset$, *and either* $m^0(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+})=0$ *or* $m^0(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-})=0$.

Then for any path $\alpha:[0,1]\to\Lambda$ *connecting* λ^+ *to* λ^- *there exists a sequence* $(\lambda_k)\subset\alpha([0,1])$ *converging to some* $\mu\in\alpha([0,1])$, *and solutions* $\gamma^k\neq\gamma_{\lambda_k}$ *of the problem (1.26) with* $\lambda=\lambda_k$, $k=1,2,\cdots$, *such that* $\|\gamma^k-\gamma_{\lambda_k}\|_{C^2(S_\tau;\mathbb{R}^N)}\to 0$ *as* $k\to\infty$. *(In particular,* (μ,λ_μ) *is a bifurcation point along sequences of the problem (1.26) in* $\Lambda\times EC^1(S_\tau;M)$ *with respect to the branch* $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\Lambda\}$.*) Moreover,* μ *is not equal to* λ^+ *(resp.* λ^-*) if* $m^0_\tau(\mathcal{L}^E_{\lambda^+},\gamma_{\lambda^+})=0$ *(resp.* $m^0_\tau(\mathcal{L}^E_{\lambda^-},\gamma_{\lambda^-})=0$*).*

Theorem 1.31 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumptions 1.27,1.28 with* Λ *being a real interval, let* $\mu\in\mathrm{Int}(\Lambda)$ *satisfy* $m^0_\tau(\mathcal{L}^E_\mu,\gamma_\mu)>0$. *Suppose that* $m^0_\tau(\mathcal{L}^E_\lambda,\gamma_\lambda)=0$ *for each* $\lambda\in\Lambda\setminus\{\mu\}$ *near* μ, *and that* $m^-_\tau(\mathcal{L}^E_\lambda,\gamma_\lambda)$ *take, respectively, values* $m^-_\tau(\mathcal{L}^E_\mu,\gamma_\mu)$ *and* $m^-_\tau(\mathcal{L}^E_\mu,\gamma_\mu)+m^0_\tau(\mathcal{L}^E_\mu,\gamma_\mu)$ *as* $\lambda\in\Lambda$ *varies in two deleted half neighborhoods of* μ. *Then one of the following alternatives occurs:*

(i) *The problem (1.26) with* $\lambda=\mu$ *has a sequence of solutions,* $\gamma_k\neq\gamma_\mu$, $k=1,2,\cdots$, *which converges to* γ_μ *in* $C^2(S_\tau,M)$.

(ii) *For every* $\lambda\in\Lambda\setminus\{\mu\}$ *near* μ *there exists a solution* $\alpha_\lambda\neq\gamma_\lambda$ *of (1.26) with parameter value* λ, *such that* $\alpha_\lambda-\gamma_\lambda$ *converges to zero in* $C^2(S_\tau,\mathbb{R}^N)$ *as* $\lambda\to\mu$. *(Recall that* $M\subset\mathbb{R}^N$.*)*

(iii) *For a given neighborhood* $\mathcal{W}$ *of* γ_μ *in* $C^2(S_\tau,M)$, *there exists an one-sided neighborhood* Λ^0 *of* μ *such that for any* $\lambda\in\Lambda^0\setminus\{\mu\}$, *(1.26) with parameter value* λ *has at least two distinct solutions in* $\mathcal{W}$, $\gamma^1_\lambda\neq\gamma_\lambda$ *and* $\gamma^2_\lambda\neq\gamma_\lambda$, *which can also be chosen to satisfy* $\mathcal{L}^E_\lambda(\gamma^1_\lambda)\neq\mathcal{L}^E_\lambda(\gamma^2_\lambda)$ *provided that* $m^0_\tau(\mathcal{L}^E_\mu,\gamma_\mu)>1$ *and (1.26) with parameter value* λ *has only finitely many solutions in* $\mathcal{W}$.

As noted in Remark 2.8, when M is an open subset in $\mathbb{R}^n$ the conditions in the above theorems may be weakened suitably.

Remark 1.32. Clearly, if the Lagrangian L in Assumption 1.27 comes from a family of C^6 Riemannian metrics $\{h_\lambda\,|\,\lambda\in\Lambda\}$ on M, i.e., $L(\lambda,t,x,v)=(h_\lambda)_x(v,v)$ for all (λ,t,x,v), as direct consequences of the above results we immediately obtain many bifurcation theorems of geodesics on Riemannian manifolds. See Section 14 for an outline.

Further researches. As natural continuations to this work the following can be considered.

(i) Because of [2, 3] and [50] we may also consider the case of free period (resp. free time) for those in Sections 1.2,1.3 (resp. in Section 1.1).

(ii) We may also study the case where reflections are allowed as in [56].

2 Preparations and some technical lemmas

In this section we collect a few preliminaries which will be used throughout the proof.

2.1 Technical lemmas

Notations and conventions Following [37], all vectors in $\mathbb{R}^m$ will be understand as column vectors. The transpose of a matrix $M \in \mathbb{R}^{m\times m}$ is denoted by M^T. We denote $(\cdot,\cdot)_{\mathbb{R}^m}$ by the standard Euclidean inner product in $\mathbb{R}^m$ and by $|\cdot|$ the corresponding norm. Let $\mathcal{L}_s(\mathbb{R}^m)$ be the set of all real symmetric matrices of order m, and $\mathrm{Sp}(2n) := \{M \in \mathrm{GL}(2n) \,|\, M^TJM = J\}$, where J is the standard complex structure on $\mathbb{R}^{2n}$ given by

$$(q_1,\cdots,q_n,p_1,\cdots,p_n)^T \mapsto J(q_1,\cdots,q_n,p_1,\cdots,p_n)^T = (-p_1,\cdots,-p_n,q_1,\cdots,q_n)^T.$$

For a map f from X to Y, $Df(x)$ (resp. $df(x)$ or $f'(x)$) denotes the Gâteaux (resp. Fréchet) derivative of f at $x \in X$, which is an element in $\mathcal{L}(X,Y)$. Of course, we also use $f'(x)$ to denote $Df(x)$ without occurring of confusions. When $Y = \mathbb{R}$, $f'(x) \in \mathcal{L}(X,\mathbb{R}) = X^*$, and if $X = H$ we call the Riesz representation of $f'(x)$ in H gradient of f at x, denoted by $\nabla f(x)$. The the Fréchet (or Gâteaux) derivative of ∇f at $x \in H$ is denoted by $f''(x)$, which is an element in $\mathcal{L}_s(H)$. (Precisely, $f''(x) = (f')'(x) \in \mathcal{L}(H;\mathcal{L}(H;\mathbb{R})$ is a symmetric bilinear form on H, and is identified with $D(\nabla f)(x)$ after $\mathcal{L}(H,\mathbb{R}) = H^*$ is identified with H via the Riesz representation theorem.)

Let $L^2([0,\tau];\mathbb{R}^n) = (L^2([0,\tau];\mathbb{R}))^n$ and $W^{1,2}([0,\tau];\mathbb{R}^n) = (W^{1,2}([0,\tau];\mathbb{R}))^n$ be the Hilbert spaces equipped with L^2-inner product and $W^{1,2}$-inner product

$$\begin{aligned}(u,v)_2 &= \int_0^\tau (u(t),v(t))_{\mathbb{R}^n}dt, &(2.1)\\ (u,v)_{1,2} &= \int_0^\tau [(u(t),v(t))_{\mathbb{R}^{2n}} + (\dot u,\dot v)_{\mathbb{R}^n}]dt, &(2.2)\end{aligned}$$

respectively. The corresponding norms are denoted by $\|\cdot\|_2$ and $\|\cdot\|_{1,2}$, respectively. (As usual each $u \in L^2([0,\tau];\mathbb{R}^n)$ will be identified with any fixed representative of it; in particular, for $k \in \mathbb{N}$ we do not distinguish $u \in W^{1,k}([0,\tau];\mathbb{R}^n)$ with its unique continuous representation.) Then $W^{1,k}([0,\tau];\mathbb{R}^n) \hookrightarrow C^0([0,\tau];\mathbb{R}^n)$ and

$$\left.\begin{array}{l}\|u\|_{C^0} \le (\tau+1)\|u\|_{1,1}, \quad \forall u \in W^{1,1}([0,\tau];\mathbb{R}^n),\\ \|u\|_{C^0} \le (\sqrt{\tau}+1/\sqrt{\tau})\|u\|_{1,2}, \quad \forall u \in W^{1,2}([0,\tau];\mathbb{R}^n).\end{array}\right\} \tag{2.3}$$

Lemma 2.1 ([31, Lemma 2.1])**.** *Given positive numbers $c > 0$ and $C_1 \ge 1$, choose positive parameters $0 < \varepsilon < \delta < \frac{2c}{3C_1}$. Then:*

(i) *There exists a C^∞ function $\psi_{\varepsilon,\delta} : [0,\infty) \to \mathbb{R}$ such that: $\psi'_{\varepsilon,\delta} > 0$ and $\psi_{\varepsilon,\delta}$ is convex on (ε,∞), $\psi_{\varepsilon,\delta}$ vanishes in $[0,\varepsilon)$ and is equal to the affine function $\kappa t + \varrho_0$ on $[\delta,\infty)$, where $\kappa > 0$ and $\varrho_0 < 0$ are suitable constants.*

(ii) *There exists a C^∞ function $\phi_{\mu,b} : [0,\infty) \to \mathbb{R}$ depending on parameters $\mu > 0$ and $b > 0$, such that: $\phi_{\mu,b}$ is nondecreasing and concave (and hence $\phi''_{\mu,b} \le 0$), and equal to the affine function $\mu t - \mu\delta$ on $[0,\delta]$, and equal to constant $b > 0$ on $[\frac{2c}{3C_1},\infty)$.*

(iii) *Under the above assumptions, $\psi_{\varepsilon,\delta}(t)+\phi_{\mu,b}(t)-b = \kappa t+\varrho_0$ for any $t \ge \frac{2c}{3C_1}$ (and hence for $t \ge \frac{2c}{3}$). Moreover, $\psi_{\varepsilon,\delta}(t)+\phi_{\mu,b}(t)-b \ge -\mu\delta-b$ $\forall t \ge 0$, and $\psi_{\varepsilon,\delta}(t)+\phi_{\mu,b}(t)-b = -\mu\delta-b$ if and only if $t = 0$.*

(iv) *Under the assumptions (i)-(ii), suppose that the constant $\mu > 0$ satisfies*

$$\mu + \frac{\varrho_0}{\delta-\varepsilon} > 0 \quad \textit{and} \quad \mu\delta + b + \varrho_0 > 0. \tag{2.4}$$

Then $\psi_{\varepsilon,\delta}(t)+\phi_{\mu,b}(t)-b\le\kappa t+\varrho_0\ \forall t\ge\varepsilon$, *and* $\psi_{\varepsilon,\delta}(t)+\phi_{\mu,b}(t)-b\le\kappa t+\varrho_0\ \forall t\in[0,\varepsilon]$ *if* $\kappa\ge\mu$.

Assumption 2.2. For a real $\tau>0$, a topology space Λ, let $L:\Lambda\times[0,\tau]\times U\times\mathbb{R}^n\to\mathbb{R}$ be a continuous function such that the following partial derivatives

$$\partial_t L(\cdot),\ \partial_q L(\cdot),\ \partial_v L(\cdot),\ \partial_{tv} L(\cdot),\ \partial_{vt} L(\cdot),\ \partial_{qv} L(\cdot),\ \partial_{vq} L(\cdot),\ \partial_{qq} L(\cdot),\ \partial_{vv} L(\cdot)$$

exist and depend continuously on $(\lambda,t,q,v)\in\Lambda\times[0,\tau]\times U\times\mathbb{R}^n$. Moreover, $\Lambda\times[0,\tau]\times U\times\mathbb{R}^n\ni(\lambda,t,q,v)\mapsto L(\lambda,t,q,v)$ is convex with respect to v, that is, the second partial derivative $\partial_{vv}L(\lambda,t,q,v)$ is positive semi-definite as a quadratic form.

Under Assumption 2.2 let E be a real orthogonal matrix of order n such that $(EU)\cap U\ne\emptyset$. Consider the Lagrangian boundary value problem on U:

$$\tfrac{d}{dt}\left(\partial_v L_\lambda(t,x(t),\dot x(t))\right)-\partial_q L_\lambda(t,x(t),\dot x(t))=0, \tag{2.5}$$

$$E(x(0))=x(\tau)\quad\text{and}\quad (E^T)^{-1}\left[\partial_v L_\lambda(0,x(0),\dot x(0))\right]=\partial_v L_\lambda(\tau,x(\tau),\dot x(\tau)). \tag{2.6}$$

Assumption 2.3. For a real $\tau>0$, a topological space Λ, a real orthogonal matrix E of order n, and an E-invariant path-connected open subset $U\subset\mathbb{R}^n$ let $L:\Lambda\times\mathbb{R}\times U\times\mathbb{R}^n\to\mathbb{R}$ be a continuous function such that the following partial derivatives

$$\partial_t L(\cdot),\ \partial_q L(\cdot),\ \partial_v L(\cdot),\ \partial_{tv} L(\cdot),\ \partial_{vt} L(\cdot),\ \partial_{qv} L(\cdot),\ \partial_{vq} L(\cdot),\ \partial_{qq} L(\cdot),\ \partial_{vv} L(\cdot)$$

exist and depend continuously on $(\lambda,t,q,v)\in\Lambda\times[0,\tau]\times U\times\mathbb{R}^n$. Moreover, for each $(\lambda,t,q)\in\Lambda\times\mathbb{R}\times U$, $L(\lambda,t,q,v)$ is convex in v, and satisfies

$$L(\lambda,t+\tau,Eq,Ev)=L(\lambda,t,q,v)\quad\forall(\lambda,t,q,v)\in\Lambda\times\mathbb{R}\times U\times\mathbb{R}^n. \tag{2.7}$$

Lemma 2.4. *Under Assumption 2.2, let the topological space* Λ *be either* compact *or* sequentially compact. *Suppose that for some real* $\rho>0$ *the function* $B^n_\rho(0)\ni v\mapsto L_\lambda(t,q,v)$ *is strictly convex for each* $(\lambda,t,q)\in\Lambda\times[0,\tau]\times U$. *Then for any given real* $0<\rho_0<\rho$ *there exists a continuous function* $\tilde L:\Lambda\times[0,\tau]\times U\times\mathbb{R}^n\to\mathbb{R}$ *and a constant* $\kappa>0$ *satisfying the following properties:*

(i) $\tilde L$ *is equal to* L *on* $\Lambda\times[0,\tau]\times U\times B^n_{\rho_0}(0)$.

(ii) *The partial derivatives*

$$\partial_t\tilde L(\cdot),\ \partial_q\tilde L(\cdot),\ \partial_v\tilde L(\cdot),\ \partial_{tv}\tilde L(\cdot),\ \partial_{vt}\tilde L(\cdot),\ \partial_{qv}\tilde L(\cdot),\ \partial_{vq}\tilde L(\cdot),\ \partial_{qq}\tilde L(\cdot),\ \partial_{vv}\tilde L(\cdot)$$

exist and depend continuously on (λ,t,q,v).

(iii) $\mathbb{R}^n\ni v\mapsto\tilde L_\lambda(t,q,v)$ *is strictly convex for each* $(\lambda,t,q)\in\Lambda\times[0,\tau]\times U$, *that is, the second partial derivative* $\partial_{vv}\tilde L_\lambda(t,x,v)$ *is positive definite as a quadratic form.*

(iv) *For any given compact subset* $S\subset U$, *there exists a constant* $C>0$ *such that*

$$\tilde L_\lambda(t,q,v)\ge\kappa|v|^2-C,\quad\forall(\lambda,t,q,v)\in\Lambda\times[0,\tau]\times S\times\mathbb{R}^n.$$

(v) *For each* (λ,t,q), *if* $L(\lambda,t,q,v)$ *is even in* v *then* $\tilde L_\lambda(t,q,v)$ *can be required to be even in* v.

(vi) *If* U *is a symmetric open neighborhood of the origin in* $\mathbb{R}^n$, *and for each* (λ,t) *the function* $L(\lambda,t,q,v)$ *is even in* (q,v) *then* $\tilde L(\lambda,t,q,v)$ *can be also required to be even in* (q,v).

(vii) *For each (λ, q), if $L(\lambda, t, q, v)$ is even in (t, v), then $\tilde{L}(\lambda, t, q, v)$ can be chosen to be even in (t, v).*

(viii) *If L is independent of time t, so is $\tilde{L}$.*

(ix) *If Assumption 2.2 is replaced by Assumption 2.3, the function $\tilde{L}$ given by (2.8) may be replaced by*

$$\tilde{L} : \Lambda \times \mathbb{R} \times U \times \mathbb{R}^n \to \mathbb{R},\ (\lambda, t, q, v) \mapsto L(\lambda, t, q, v) + \psi_{\rho_0,\rho_1}(|v|^2),$$

which also satisfies (2.7) because E is a real orthogonal matrix.

Proof. Fix a positive real $\rho_1 \in (\rho_0, \rho)$. By Lemma 2.1 ([31, Lemma 2.1]) we have a C^∞ convex function $\psi_{\rho_0,\rho_1} : [0, \infty) \to \mathbb{R}$ such that $\psi'_{\rho_0,\rho_1}(t) > 0$ for $t \in (\rho_0^2, \infty)$, $\psi_{\rho_0,\rho_1}(t) = 0$ for $t \in [0, \rho_0^2)$ and $\psi_{\rho_0,\rho_1}(t) = \kappa t + \varrho_0$ for $t \in [\rho_1^2, \infty)$, where $\kappa > 0$ and $\varrho_0 < 0$ are suitable constants. We conclude

$$\tilde{L} : \Lambda \times [0, \tau] \times U \times \mathbb{R}^n \to \mathbb{R},\ (\lambda, t, q, v) \mapsto L(\lambda, t, q, v) + \psi_{\rho_0,\rho_1}(|v|^2) \tag{2.8}$$

to satisfy the desired requirements. By Assumption 2.2 and the choice of ψ_{ρ_0,ρ_1} it is clear that $\tilde{L}$ satisfies (i)-(ii).

In order to see that $\tilde{L}$ satisfies (iii), note that

$$\frac{\partial^2}{\partial t \partial s}\psi_{\rho_0,\rho_1}(|v + su + tu|^2)\Big|_{s=0,t=0} = 2\psi'_{\rho_0,\rho_1}(|v|^2)|u|^2 + 4\psi''_{\rho_0,\rho_1}(|v|^2)\big((v, u)_{\mathbb{R}^n}\big)^2$$

(cf. the proof of [31, Lemma 2.1]) and therefore

$$\partial_{vv}\tilde{L}(\lambda, t, q, v)[u, u] = \partial_{vv}L(\lambda, t, q, v)[u, u] + 2\psi'_{\rho_0,\rho_1}(|v|^2)|u|^2 + 4\psi''_{\rho_0,\rho_1}(|v|^2)\big((v, u)_{\mathbb{R}^n}\big)^2.$$

Since $\psi'_{\rho_0,\rho_1} \geq 0$ and $\psi''_{\rho_0,\rho_1}(t) \geq 0$, by Assumption 2.2 we deduce

$$\partial_{vv}\tilde{L}(\lambda, t, q, v)[u, u] \geq \partial_{vv}L(\lambda, t, q, v)[u, u] > 0 \quad \text{for } |v| < \rho \text{ and } u \neq 0.$$

Moreover, if $|v| > \rho_1$ and $u \neq 0$ we obtain $\partial_{vv}\tilde{L}(\lambda, t, q, v)[u, u] \geq 2\kappa|u|^2$ because $\partial_{vv}L(\lambda, t, q, v)[u, u] \geq 0$. Hence $\tilde{L}(\lambda, t, q, v)$ is strictly convex in v.

Let us prove (iv). Fixing $v_0 \in B^n_\rho(0) \setminus B^n_{\rho_1}(0)$, by [12, Proposition 1.2.10] we get

$$\begin{aligned}
\tilde{L}(\lambda, t, q, v) \;&\geq\; \tilde{L}(\lambda, t, q, v_0) + \partial_v\tilde{L}(\lambda, t, q, v_0)[v - v_0] \\
&\geq\; L(\lambda, t, q, v_0) + \partial_v L(\lambda, t, q, v_0)[v - v_0] + 2\psi'_{\rho_0,\rho_1}(|v_0|^2)(v, v - v_0)_{\mathbb{R}^n} \\
&= L(\lambda, t, q, v_0) - \partial_v L(\lambda, t, q, v_0)[v_0] + \partial_v L(\lambda, t, q, v_0)[v] - 2\kappa(v, v_0)_{\mathbb{R}^n} + 2\kappa|v|^2
\end{aligned}$$

for all $v \in \mathbb{R}^n$. Since $2\kappa|(v, v_0)_{\mathbb{R}^n}| \leq \kappa|v|^2/2 + 2\kappa|v_0|^2$ and

$$|\partial_v L(\lambda, t, q, v_0)[v]| = \left|\sum_{j=1}^n \frac{\partial L}{\partial v_j}(\lambda, t, q, v_0)v_j\right| \leq \frac{1}{\kappa}\sum_{j=1}^n \left|\frac{\partial L}{\partial v_j}(\lambda, t, q, v_0)\right|^2 + \frac{\kappa}{4}|v|^2,$$

we derive

$$\tilde{L}(\lambda, t, q, v) \geq L(\lambda, t, q, v_0) - \partial_v L(\lambda, t, q, v_0)[v_0] - \frac{1}{\kappa}\sum_{j=1}^n \left|\frac{\partial L}{\partial v_j}(\lambda, t, q, v_0)\right|^2 - 2\kappa|v_0|^2 + \frac{5\kappa}{4}|v|^2$$

for all $v \in \mathbb{R}^n$. Since Λ is either compact or sequential compact, so is $\Lambda\times[0,\tau]\times S$, in either case we can always derive that both $L(\lambda,t,q,v_0)$ and $\partial_v L(\lambda,t,q,v_0)$ are bounded on $\Lambda\times[0,\tau]\times S$. Therefore there exists a constant $C>0$ such that

$$\tilde{L}(\lambda,t,q,v)\ge \frac{5\kappa}{4}|v|^2-C,\quad \forall(\lambda,t,q,v)\in\Lambda\times[0,\tau]\times S\times\mathbb{R}^n.$$

Other conclusions are clear by the above construction. $\square$

Assumption 2.5. Under Assumption 2.2, for each $\lambda\in\Lambda$, let $x_\lambda:[0,\tau]\to U$ be a C^2 path satisfying (2.5). Suppose: (i) $\Lambda\times[0,\tau]\ni(\lambda,t)\to x_\lambda(t)\in U$ and $\Lambda\times[0,\tau]\ni(\lambda,t)\mapsto \dot{x}_\lambda(t)\in\mathbb{R}^n$ are continuous; (ii) for any compact or sequential compact subset $\hat\Lambda\subset\Lambda$ there exists $\rho>0$ such that

$$\sup\left\{|\dot{x}_\lambda(t)|\,\middle|\,(\lambda,t)\in\hat\Lambda\times[0,\tau]\right\}<\rho$$

and that $\hat\Lambda\times[0,\tau]\times U\times B^n_\rho(0)\ni(\lambda,t,q,v)\mapsto L_\lambda(t,q,v)$ is strictly convex with respect to v.

Lemma 2.6. *Under Assumption 2.5, the following holds:*

(i) $\Lambda\times[0,\tau]\ni(\lambda,t)\mapsto\ddot{x}_\lambda(t)\in\mathbb{R}^n$ *is continuous.*

(ii) *If there exists a sequence $(\lambda_k)\subset\Lambda$ converging to $\mu\in\Lambda$ and solutions $x_k\in C^2([0,\tau],U)\setminus\{x_{\lambda_k}\}$ of (2.5) with $\lambda=\lambda_k\in\Lambda$, $k=1,2,\cdots$, such that $\|x_k-x_{\lambda_k}\|_{C^1}\to 0$, then $\|x_k-x_{\lambda_k}\|_{C^2}\to 0$ as $k\to\infty$.*

Proof. *Step 1*[Prove (i)]. Since x_λ is C^2, we have

$$\begin{aligned}&\partial_{vt}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))+\partial_{vq}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))\dot{x}_\lambda(t)\\&+\partial_{vv}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))\ddot{x}_\lambda(t)-\partial_q L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))=0\end{aligned}$$

and therefore

$$\begin{aligned}\ddot{x}_\lambda(t)&=[\partial_{vv}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))]^{-1}\partial_q L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))\\&\quad-[\partial_{vv}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))]^{-1}\partial_{vt}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))\\&\quad-[\partial_{vv}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))]^{-1}\partial_{vq}L_\lambda(t,x_\lambda(t),\dot{x}_\lambda(t))\dot{x}_\lambda(t)\end{aligned}$$

because (ii) in Assumption 2.5 implies that the matrixes $\partial_{vv}L_\lambda(t,x(t),\dot{x}(t))$ are positive definite and therefore invertible. Moreover, by Assumption 2.2 maps

$$\begin{aligned}&(\lambda,t,q,v)\mapsto\partial_{vt}L_\lambda(t,q,v),&&(\lambda,t,q,v)\mapsto\partial_{vq}L_\lambda(t,q,v),\\&(\lambda,t,q,v)\mapsto\partial_{vv}L_\lambda(t,q,v),&&(\lambda,t,q,v)\mapsto\partial_q L_\lambda(t,q,v)\end{aligned}$$

are continuous. The desired conclusion may follow from these, (i) in Assumption 2.5 and the above equality directly.

Step 2[Prove (ii)]. Let $\hat\Lambda=\{\mu,\lambda_k\,|\,k\in\mathbb{N}\}$. It is a sequential compact subset of Λ. By the assumption (ii) in Assumption 2.5 there exists $\rho>0$ such that $\sup\{|\dot{x}_\lambda(t)|\,|\,(\lambda,t)\in\hat\Lambda\times[0,\tau]\}<\rho$ and that $\hat\Lambda\times[0,\tau]\times U\times B^n_\rho(0)\ni(\lambda,t,q,v)\mapsto L_\lambda(t,q,v)$ is strictly convex with respect to v. In particular, we may obtain $0<M_1<M_2<\infty$ such that

$$M_1I_n\le\partial_{vv}L_\mu(t,x_\mu(t),\dot{x}_\mu(t))\le M_2I_n,\quad\forall t\in[0,\tau].$$

Suppose that there exists a sequence $(t_i)\subset[0,\tau]$ such that for each $i=1,2,\cdots$,

$$\partial_{vv}L_{\lambda_{k_i}}(t_i,x_{\lambda_{k_i}}(t_i),\dot{x}_{\lambda_{k_i}}(t_i))<\frac{1}{2}M_1I_n\quad\text{or}\quad\partial_{vv}L_{\lambda_{k_i}}(t_i,x_{\lambda_{k_i}}(t_i),\dot{x}_{\lambda_{k_i}}(t_i))>\frac{1}{2}M_2I_n.$$

We can assume $t_i \to t_0 \in [0,\tau]$. By (i) in Assumption 2.5 and the continuity of the map

$$\Lambda \times [0,\tau] \times U \times \mathbb{R}^n \ni (\lambda, t, q, v) \mapsto \partial_{vv}L(\lambda, t, q, v) \in \mathbb{R}^{n\times n},$$

we derive

$$\partial_{vv}L_\mu(t_0, x_\mu(t_0), \dot{x}_\mu(t_0)) \le \frac{1}{2}M_1 I_n \quad \text{or} \quad \partial_{vv}L_\mu(t_0, x_\mu(t_0), \dot{x}_\mu(t_0)) \ge \frac{1}{2}M_2 I_n.$$

This contradiction shows that if k is large enough then

$$\frac{1}{2}M_1 I_n \le \partial_{vv}L_{\lambda_k}(t, x_{\lambda_k}(t), \dot{x}_{\lambda_k}(t)) \le \frac{1}{2}M_2 I_n, \quad \forall t\in[0,\tau]. \tag{2.9}$$

Similarly, since $\|x_k - x_{\lambda_k}\|_{C^1} \to 0$, for sufficiently large k we have

$$\frac{1}{2}M_1 I_n \le \partial_{vv}L_{\lambda_k}(t, x_k(t), \dot{x}_k(t)) \le \frac{1}{2}M_2 I_n, \quad \forall t\in[0,\tau]. \tag{2.10}$$

Note that for each $k\in\mathbb{N}$, both (λ_k, x_k) and $(\lambda_k, x_{\lambda_k})$ satisfy (2.5), that is,

$$\begin{aligned}&\partial_{vv}L_{\lambda_k}(t, x_{\lambda_k}(t), \dot{x}_{\lambda_k}(t))\ddot{x}_{\lambda_k}(t) + \partial_{tv}L_{\lambda_k}(t, x_{\lambda_k}(t), \dot{x}_{\lambda_k}(t))\\ &+\partial_{qv}L_{\lambda_k}(t, x_{\lambda_k}(t), \dot{x}_{\lambda_k}(t))\dot{x}_{\lambda_k}(t) - \partial_q L_{\lambda_k}(t, x_{\lambda_k}(t), \dot{x}_{\lambda_k}(t)) = 0, \qquad (2.11)\\ &\partial_{vv}L_{\lambda_k}(t, x_k(t), \dot{x}_k(t))\ddot{x}_k(t) + \partial_{tv}L_{\lambda_k}(t, x_k(t), \dot{x}_k(t))\\ &+\partial_{qv}L_{\lambda_k}(t, x_k(t), \dot{x}_k(t))\dot{x}_k(t) - \partial_q L_{\lambda_k}(t, x_k(t), \dot{x}_k(t)) = 0. \qquad (2.12)\end{aligned}$$

By contradiction, passing to subsequences (if necessary) suppose that there exists $\varepsilon>0$ and a sequence $(t_k)\subset[0,\tau]$ converging to t_0 such that $|\ddot{x}_{\lambda_k}(t_k) - \ddot{x}_k(t_k)| \ge \varepsilon$ for all $k\in\mathbb{N}$. Then for each large k, (2.9) and (2.10) imply that $(\partial_{vv}L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k)))$ and $(\partial_{vv}L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k)))$ are invertible. It follows from (2.11) and (2.12) that for each large k,

$$\begin{aligned}\varepsilon &\le |\ddot{x}_{\lambda_k}(t_k) - \ddot{x}_k(t_k)|\\ &= \big|(\partial_{vv}L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k)))^{-1}\partial_{tv}L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k))\\ &\quad - (\partial_{vv}L_{\lambda_k}(t_k, x_k(t_k), \dot{x}_k(t_k)))^{-1}\partial_{tv}L_{\lambda_k}(t_k, x_k(t_k), \dot{x}_k(t_k))\\ &\quad + (\partial_{vv}L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k)))^{-1}\partial_{qv}L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k))\dot{x}_{\lambda_k}(t_k)\\ &\quad - (\partial_{vv}L_{\lambda_k}(t_k, x_k(t_k), \dot{x}_k(t_k)))^{-1}\partial_{qv}L_{\lambda_k}(t_k, x_k(t_k), \dot{x}_k(t_k))\dot{x}_k(t_k)\\ &\quad - (\partial_{vv}L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k)))^{-1}\partial_q L_{\lambda_k}(t_k, x_{\lambda_k}(t_k), \dot{x}_{\lambda_k}(t_k))\\ &\quad + (\partial_{vv}L_{\lambda_k}(t_k, x_k(t_k), \dot{x}_k(t_k)))^{-1}\partial_q L_{\lambda_k}(t_k, x_k(t_k), \dot{x}_k(t_k))\big|. \qquad (2.13)\end{aligned}$$

By (i) in Assumption 2.5, $|x_{\lambda_k}(t_k) - x_\mu(t_0)| \to 0$ and $|\dot{x}_{\lambda_k}(t_k) - \dot{x}_\mu(t_0)| \to 0$. Moreover

$$\begin{aligned}|x_k(t_k) - x_\mu(t_0)| &\le |x_{\lambda_k}(t_k) - x_\mu(t_0)| + \|x_{\lambda_k} - x_k\|_{C^0} \to 0,\\ |\dot{x}_k(t_k) - \dot{x}_\mu(t_0)| &\le |\dot{x}_{\lambda_k}(t_k) - \dot{x}_\mu(t_0)| + \|\dot{x}_{\lambda_k} - \dot{x}_k\|_{C^0} \to 0.\end{aligned}$$

Letting $k\to\infty$ in (2.13), by Assumption 2.2 we get $\varepsilon\le 0$. This contradiction shows $\|\ddot{x}_{\lambda_k} - \ddot{x}_k\|_{C^0} \to 0$. Combing the condition $\|x_k - x_{\lambda_k}\|_{C^1}\to 0$, we arrive at $\|x_k - x_{\lambda_k}\|_{C^2} \to 0$ as $k\to\infty$. $\square$

Note: the continuity of $\partial_{tv}L$ is used in the proof of Lemma 2.6.

Under Assumption 2.5, for a given compact or sequential compact subset $\hat{\Lambda}\subset\Lambda$ there exist positive numbers $0<\rho_0<\rho$ such that

$$\rho_{00} := \sup\left\{|\dot{x}_\lambda(t)| \,\big|\, (\lambda, t)\in\hat{\Lambda}\times[0,\tau]\right\} < \rho_0 < \rho$$

and that $\hat{\Lambda}\times[0,\tau]\times U\times B^n_\rho(0)\ni(\lambda,t,q,v)\mapsto L_\lambda(t,q,v)$ is strictly convex with respect to v. By Lemma 2.4 we have an associated continuous function $\tilde{L}:\hat{\Lambda}\times[0,\tau]\times U\times\mathbb{R}^n\to\mathbb{R}$. Since a subset of $\mathbb{R}^n$ is compact if and only if it is sequential compact, whether $\hat{\Lambda}$ is compact or sequential compact the continuous map $\hat{\Lambda}\times[0,\tau]\ni(\lambda,t)\mapsto x_\lambda(t)\in\mathbb{R}^n$ has a compact image set and therefore we can choose $\delta>0$ so small that the compact subset

$$S:=Cl\big(\cup_{\lambda\in\hat{\Lambda}}\cup_{t\in[0,\tau]}(x_\lambda(t)+B^n_\delta(0))\big)$$

is contained in U. Lemma 2.4(iv) yields a constant $C>0$ such that

$$\tilde{L}_\lambda(t,q,v)\ge\kappa|v|^2-C,\quad\forall(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times S\times\mathbb{R}^n.$$

Define

$$\hat{L}:\hat{\Lambda}\times[0,\tau]\times B^n_\delta(0)\times\mathbb{R}^n\to\mathbb{R},\ (\lambda,t,q,v)\mapsto\tilde{L}(\lambda,t,q+x_\lambda(t),v+\dot{x}_\lambda(t))\tag{2.14}$$

and $\hat{L}_\lambda(\cdot)=\hat{L}(\lambda,\cdot)$ for $\lambda\in\hat{\Lambda}$. By Lemmas 2.4, 2.6, we obtain:

Lemma 2.7. (A) *The function $\hat{L}$ is continuous; partial derivatives*

$$\partial_t\hat{L}(\cdot),\ \partial_q\hat{L}(\cdot),\ \partial_v\hat{L}(\cdot),\ \partial_{tv}\hat{L}(\cdot),\ \partial_{vt}\hat{L}(\cdot),\ \partial_{qv}\hat{L}(\cdot),\ \partial_{vq}\hat{L}(\cdot),\ \partial_{qq}\hat{L}(\cdot),\ \partial_{vv}\hat{L}(\cdot)$$

exist and depend continuously on (λ,t,q,v).

(B) *For each $(\lambda,t,q)\in\hat{\Lambda}\times[0,\tau]\times B^n_\delta(0)$, $\hat{L}_\lambda(t,q,v)$ is strictly convex in v, and*

$$\hat{L}_\lambda(t,q,v)\ge\kappa|v+\dot{x}_\lambda(t)|^2-C\ge\frac{\kappa}{2}|v|^2-\kappa\rho^2-C\tag{2.15}$$

for all $(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times B^n_\delta(0)\times\mathbb{R}^n$.

(C) *For each (λ,t,q), if each x_λ is constant and $L(\lambda,t,q,v)$ is even in v then $\hat{L}_\lambda(t,q,v)$ can be required to be even in v.*

(D) *If U is a symmetric open neighborhood of the origin in $\mathbb{R}^n$, $x_\lambda\equiv0\ \forall\lambda$, and for each (λ,t) the function $L(\lambda,t,q,v)$ is even in (q,v), then $\hat{L}(\lambda,t,q,v)$ can be also required to be even in (q,v).*

(E) *For each (λ,q), if $L(\lambda,t,q,v)$ is even in (t,v), and $x_\lambda\equiv0\ \forall\lambda$, then $\hat{L}(\lambda,t,q,v)$ can be chosen to be even in (t,v).*

(F) *If L is independent of time t, so is $\hat{L}$.*

(G) *If Assumption 2.2 is replaced by Assumption 2.3, and $Ex_\lambda(t)=x_\lambda(t)\ \forall t\in\hat{\Lambda}$, then the function $\hat{L}$ given by (2.14) may be replaced by*

$$\hat{L}:\hat{\Lambda}\times\mathbb{R}\times U\times\mathbb{R}^n\to\mathbb{R},\ (\lambda,t,q,v)\mapsto L(\lambda,t,q+x_\lambda(t),v+\dot{x}_\lambda(t))+\psi_{\rho_0,\rho_1}(|v+\dot{x}_\lambda(t)|^2),$$

which also satisfies (2.7) because E is a real orthogonal matrix.

Remark 2.8. For a given positive number $\rho_0>0$, replacing $\tilde{L}^*$ and ι by $\hat{L}$ and δ in the proof of Lemma 3.8 we may obtain a continuous function $\check{L}:\hat{\Lambda}\times[0,\tau]\times B^n_{3\delta/4}(0)\times\mathbb{R}^n\to\mathbb{R}$ satisfying (L1)-(L6) in Lemma 3.8 with $\tilde{L}^*=\hat{L}$ and $\iota=\delta$, and Lemma 3.8(L0) without $\partial_t\check{L}(\cdot)$. Because of Remark 3.12, as in the proofs of Theorems 1.4, 1.5, 1.6 in Section 3 we may obtain the corresponding versions of these theorems under weaker Assumptions 2.2, 2.5. Similarly, when M is an open subset in $\mathbb{R}^n$ the conditions in Theorem 1.9 may be weakened suitably.

3 Proofs of Theorems 1.4, 1.5, 1.6 and 1.9

3.1 Proofs of Theorems 1.4, 1.5, 1.6

3.1.1 Reduction to Euclidean spaces

Since $\gamma_\mu(0)\ne\gamma_\mu(\tau)$, we may choose the C^6 Riemannian metric g on M so that S_0 (resp. S_1) is totally geodesic near $\gamma_\mu(0)$ (resp. $\gamma_\mu(\tau)$). (Indeed, by the definition of submanifolds there exists a coordinate chart (U,φ) around $\gamma_\mu(0)$ (resp. $\gamma_\mu(\tau)$) on M such that $\varphi(S_0\cap U)=\varphi(U)\cap V_0$ (resp. $\varphi(S_1\cap U)=\varphi(U)\cap V_1$) for some linear subspace V_0 (resp. V_1) in $\mathbb{R}^n$. extending the pullback of the standard metric on $\mathbb{R}^n$ to U yields a required metric.) There exists a fibrewise convex open neighborhood $\mathcal{U}(0_{TM})$ of the zero section of TM such that the exponential map of g gives rise to C^5 immersion

$$\mathbb{F}:\mathcal{U}(0_{TM})\to M\times M,\ (q,v)\mapsto(q,\exp_q(v)),\tag{3.1}$$

(cf. Appendix A). By (A.3), $d\mathbb{F}(q,0_q):T_{(q,0_q)}\to T_{(q,q)}(M\times M)=T_qM\times T_qM$ is an isomorphism for each $q\in M$. Since $\mathbb{F}$ is injective on the closed subset $0_{TM}\subset TM$, it follows from Exercise 7 in [16, page 41] that $\mathbb{F}|_{\mathcal{W}(0_{TM})}$ is a C^5 embedding of some smaller open neighborhood $\mathcal{W}(0_{TM})\subset\mathcal{U}(0_{TM})$ of 0_{TM}. Note that $\mathbb{F}(0_{TM})$ is equal to the diagonal Δ_M in $M\times M$, and that $\gamma_\mu([0,\tau])$ is compact. We may choose a number $\iota>0$ such that

(♣) the closure $\bar{\mathbf{U}}_{3\iota}(\gamma_\mu([0,\tau]))$ of $\mathbf{U}_{3\iota}(\gamma_\mu([0,\tau])):=\{p\in M\,|\,d_g(p,\gamma_\mu([0,\tau]))<3\iota\}$ is a compact neighborhood of $\gamma_\mu([0,\tau])$ in M, and $\bar{\mathbf{U}}_{3\iota}(\gamma_\mu([0,\tau]))\times\bar{\mathbf{U}}_{3\iota}(\gamma_\mu([0,\tau]))$ is contained in the image of $\mathbb{F}|_{\mathcal{W}(0_{TM})}$;

(♠) $\{(q,v)\in TM\,|\,q\in\bar{\mathbf{U}}_{3\iota}(\gamma_\mu([0,\tau])),\ |v|_q\le 3\iota\}\subset\mathcal{W}(0_{TM})$.

Then 3ι is less than the injectivity radius of g at each point on $\bar{\mathbf{U}}_{3\iota}(\gamma_\mu([0,\tau]))$. Let us take a path $\overline{\gamma}\in C^7([0,\tau];M)$ such that

$$\overline{\gamma}(0)=\gamma_\mu(0),\quad \overline{\gamma}(\tau)=\gamma_\mu(\tau),\quad\text{and}\quad \mathrm{dist}_g(\gamma_\mu(t),\overline{\gamma}(t))<\iota\ \forall t\in[0,\tau].\tag{3.2}$$

We first assume:

$$d_g(\gamma_\lambda(t),\overline{\gamma}(t))<\iota,\quad \forall(\lambda,t)\in\Lambda\times[0,\tau].\tag{3.3}$$

(For cases of Theorems 1.4, 1.6, by contradiction we may use nets to prove that (3.3) is satisfied after shrinking Λ toward μ.) Then (3.2) and (3.3) imply

$$d_g(\gamma_\lambda(t),\gamma_\mu([0,\tau]))\le d_g(\gamma_\lambda(t),\gamma_\mu(t))<2\iota,\quad \forall(\lambda,t)\in\Lambda\times[0,\tau].\tag{3.4}$$

Using a unit orthogonal parallel C^5 frame field along $\overline{\gamma}$, $[0,\tau]\ni t\mapsto(e_1(t),\cdots,e_n(t))$, we get a C^5 map

$$\phi_{\overline{\gamma}}:[0,\tau]\times B^n_{2\iota}(0)\to M,\ (t,x)\mapsto\exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n x_ie_i(t)\right)\tag{3.5}$$

(Note that the tangent map $d\phi_{\overline{\gamma}}:T([0,\tau]\times B^n_{2\iota}(0))\to TM$ is C^4.) By Step 1 in [31, §4] there exist two linear subspaces of $\mathbb{R}^n$, V_0 and V_1, such that $v\in V_0$ (resp. $v\in V_1$) if and only if $\sum_{k=1}^n v_ke_k(0)\in T_{\gamma(0)}S_0$ (resp. $\sum_{k=1}^n v_ke_k(\tau)\in T_{\gamma(0)}S_1$). By [48, Theorem 4.2], $C^1_{S_0\times S_1}([0,\tau];M)$ is a C^4 Banach manifold; and it follows from [48, Theorem 4.3] that the map

$$\Phi_{\overline{\gamma}}:C^1_{V_0\times V_1}([0,\tau];B^n_{2\iota}(0))\to C^1_{S_0\times S_1}([0,\tau];M)\tag{3.6}$$

defined by $\Phi_{\overline{\gamma}}(\xi)(t) = \phi_{\overline{\gamma}}(t, \xi(t))$ gives a C^2 coordinate chart around $\overline{\gamma}$ on $C^1_{S_0\times S_1}([0,\tau]; M)$, where

$$C^k_{V_0\times V_1}([0,\tau]; B^n_{2\iota}(0)) = \{\xi \in C^k([0,\tau]; B^n_{2\iota}(0)) \,|\, \xi(0) \in V_0,\ \xi(\tau) \in V_1\} \quad \text{with } k \in \mathbb{N} \cup \{0\}.$$

Moreover, it is clear that

$$\Phi_{\overline{\gamma}}\left(C^1_{V_0\times V_1}([0,\tau]; B^n_{2\iota}(0))\right) = \left\{\gamma \in C^1_{S_0\times S_1}([0,\tau], M) \,\middle|\, \sup_t \mathrm{dist}_g(\gamma(t), \overline{\gamma}(t)) < 2\iota\right\}.$$

(**Note**: $\Phi_{\overline{\gamma}}$ also defines an at least C^1 map from $C^2_{V_0\times V_1}([0,\tau]; B^n_{2\iota}(0))$ to $C^2_{S_0\times S_1}([0,\tau]; M)$.)

By (3.3), for each $\lambda \in \Lambda$ there exists a unique map $\mathbf{u}_\lambda : [0,\tau] \to B^n_\iota(0)$ such that

$$\gamma_\lambda(t) = \phi_{\overline{\gamma}}(t, \mathbf{u}_\lambda(t)) = \exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n \mathbf{u}^i_\lambda(t) e_i(t)\right), \quad t \in [0,\tau].$$

Clearly, $\mathbf{u}_\lambda$ satisfies the first assertion of the following lemma, whose proof will be given in Appendix A.

Lemma 3.1. *$\mathbf{u}_\mu(0) = 0 = \mathbf{u}_\mu(\tau)$, $\mathbf{u}_\lambda \in C^2_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$, and*

$$(\lambda, t) \mapsto \mathbf{u}_\lambda(t) \quad \text{and} \quad (\lambda, t) \mapsto \dot{\mathbf{u}}_\lambda(t)$$

are continuous as maps from $\Lambda \times [0,\tau]$ to $\mathbb{R}^n$.

Define $L^* : \Lambda \times [0,\tau] \times B^n_{2\iota}(0) \times \mathbb{R}^n \to \mathbb{R}$ by

$$L^*(\lambda, t, q, v) = L^*_\lambda(t, q, v) = L_\lambda\left(t, \phi_{\overline{\gamma}}(t, q), D_t\phi_{\overline{\gamma}}(t, q) + D_q\phi_{\overline{\gamma}}(t, q)[v]\right) \tag{3.7}$$

Since $\phi_{\overline{\gamma}}$ is C^5, by Assumption 1.1, L^* is C^2 with respect to (t, q, v) and strictly convex with respect to v, and all its partial derivatives also depend continuously on (λ, t, q, v). Moreover, $\mathbf{u}_\lambda$ solves the following boundary problem:

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\partial_v L^*_\lambda(t, x(t), \dot{x}(t))\Big) - \partial_q L^*_\lambda(t, x(t), \dot{x}(t)) = 0, \\ x \in C^2([0,\tau]; B^n_\iota(0)),\ (x(0), x(\tau)) \in V_0 \times V_1 \quad \text{and} \\ \partial_v L^*_\lambda(0, x(0), \dot{x}(0))[v_0] = \partial_v L^*_\lambda(\tau, x(\tau), \dot{x}(\tau))[v_1] \\ \qquad \forall (v_0, v_1) \in V_0 \times V_1. \end{array}\right\}$$

By Lemmas 2.6, 3.1 we directly obtain:

Lemma 3.2. *$\Lambda \times [0,\tau] \ni (\lambda, t) \mapsto \ddot{\mathbf{u}}_\lambda(t) \in \mathbb{R}^n$ is continuous.*

(This is necessary for us to derive that $\partial_{vt}\tilde{L}^*$ is continuous in Proposition 3.3.)

Define $\tilde{L}^* : \Lambda \times [0,\tau] \times B^n_\iota(0) \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{L}^*(\lambda, t, q, v) = \tilde{L}^*_\lambda(t, q, v) = L^*(\lambda, t, q + \mathbf{u}_\lambda(t), v + \dot{\mathbf{u}}_\lambda(t)). \tag{3.8}$$

Then

$$\begin{aligned} \partial_t\tilde{L}^*(\lambda, t, q, v) &= \partial_t L^*(\lambda, t, q + \mathbf{u}_\lambda(t), v + \dot{\mathbf{u}}_\lambda(t)) \\ &\quad + \partial_q L^*(\lambda, t, q + \mathbf{u}_\lambda(t), v + \dot{\mathbf{u}}_\lambda(t))\dot{\mathbf{u}}_\lambda(t) \\ &\quad + \partial_v L^*(\lambda, t, q + \mathbf{u}_\lambda(t), v + \dot{\mathbf{u}}_\lambda(t))\ddot{\mathbf{u}}_\lambda(t), \\ \partial_q\tilde{L}^*(\lambda, t, q, v) &= \partial_q L^*(\lambda, t, q + \mathbf{u}_\lambda(t), v + \dot{\mathbf{u}}_\lambda(t)) \\ \partial_v\tilde{L}^*(\lambda, t, q, v) &= \partial_v L^*(\lambda, t, q + \mathbf{u}_\lambda(t), v + \dot{\mathbf{u}}_\lambda(t)). \end{aligned}$$

By these and Lemmas 3.1, 3.2 it is not hard to see that $\tilde{L}^*$ satisfies the following:

Proposition 3.3. (a) *$\tilde{L}^*$ is continuous, and the following partial derivatives*

$$\partial_t\tilde{L}^*(\cdot),\ \partial_q\tilde{L}^*(\cdot),\ \partial_v\tilde{L}^*(\cdot),\ \partial_{tv}\tilde{L}^*(\cdot),\ \partial_{vt}\tilde{L}^*(\cdot),\ \partial_{qv}\tilde{L}^*(\cdot),\ \partial_{vq}\tilde{L}^*(\cdot),\ \partial_{qq}\tilde{L}^*(\cdot),\ \partial_{vv}\tilde{L}^*(\cdot)$$

exist and depend continuously on (λ, t, q, v).

(b) *For each $(\lambda, t, q) \in \Lambda \times [0,\tau] \times B^n_\iota(0)$, $\tilde{L}^*_\lambda(t, q, v)$ is strictly convex in v.*

Clearly, $\tilde{L}^*$ satisfies Assumption 2.2, and Assumption 2.5 with $x_\lambda \equiv 0\ \forall\lambda$.

Remark 3.4. Actually, for our next arguments in this section it is suffices that $\tilde{L}^*$ satisfies (a) and the following weaker condition:

(b') $\tilde{L}^*(\lambda, t, q, v)$ is convex in v, and for any compact or sequential compact subset $\hat{\Lambda} \subset \Lambda$ there exists $\rho > 0$ such that $\hat{\Lambda} \times [0,\tau] \times B^n_\iota(0) \times B^n_\rho(0) \ni (\lambda, t, q, v) \mapsto \tilde{L}^*(\lambda, t, q, v)$ is strictly convex with respect to v.

This means: In Assumption 1.1 we may only require that L is fiberwise convex; but in Assumption 1.2 we need to add the condition: for any compact or sequential compact subset $\hat{\Lambda} \subset \Lambda$ there exist $0 < \rho_0 < \rho$ such that $\sup\{|\dot{\gamma}_\lambda(t)|_g \,|\, (\lambda, t) \in \hat{\Lambda} \times [0,\tau]\} < \rho_0$ and L is fiberwise strictly convex in $(\lambda, t, q, v) \in \hat{\Lambda} \times [0,\tau] \times TM \,|\, |v|_g < \rho\}$.

The condition (a) in Proposition 3.3 assure that each functional

$$\tilde{\mathcal{E}}^*_\lambda : C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0)) \to \mathbb{R},\ x \mapsto \int_0^1 \tilde{L}^*_\lambda(t, x(t), \dot{x}(t))dt \tag{3.9}$$

is C^2, and satisfies

$$\tilde{\mathcal{E}}^*_\lambda(x) = \mathcal{E}_\lambda\left(\Phi_{\overline{\gamma}}(x + \mathbf{u}_\lambda)\right)\ \forall x \in C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0)) \quad \text{and} \quad d\tilde{\mathcal{E}}^*_\lambda(0) = 0. \tag{3.10}$$

Hence for each $\lambda \in \Lambda$, $x \in C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$ satisfies $d\tilde{\mathcal{E}}^*_\lambda(x) = 0$ if and only if $\gamma := \Phi_{\overline{\gamma}}(x+\mathbf{u}_\lambda)$ satisfies $d\mathcal{E}_\lambda(\gamma) = 0$; and in this case $m^-(\tilde{\mathcal{E}}^*_\lambda, x) = m^-(\mathcal{E}_\lambda, \gamma)$ and $m^0(\tilde{\mathcal{E}}^*_\lambda, x) = m^0(\mathcal{E}_\lambda, \gamma)$. In particular, we have

$$m^-(\tilde{\mathcal{E}}^*_\lambda, 0) = m^-(\mathcal{E}_\lambda, \gamma_\lambda) \quad \text{and} \quad m^0(\tilde{\mathcal{E}}^*_\lambda, 0) = m^0(\mathcal{E}_\lambda, \gamma_\lambda). \tag{3.11}$$

By [6, Proposition 4.2] the critical points of $\tilde{\mathcal{E}}^*_\lambda$ correspond to the solutions of the following boundary problem:

$$\frac{d}{dt}\Big(\partial_v\tilde{L}^*_\lambda(t, x(t), \dot{x}(t))\Big) - \partial_q\tilde{L}^*_\lambda(t, x(t), \dot{x}(t)) = 0, \tag{3.12}$$

$$\left.\begin{array}{l} x \in C^2([0,\tau]; B^n_\iota(0)),\ (x(0), x(\tau)) \in V_0 \times V_1 \quad \text{and} \\ \partial_v\tilde{L}^*_\lambda(0, x(0), \dot{x}(0))[v_0] = 0 \quad \forall v_0 \in V_0, \\ \partial_v\tilde{L}^*_\lambda(\tau, x(\tau), \dot{x}(\tau))[v_1] = 0 \quad \forall v_1 \in V_1. \end{array}\right\} \tag{3.13}$$

Let $W^{1,2}_{V_0\times V_1}([0,\tau]; B^n_\iota(0)) = \{\xi \in W^{1,2}([0,\tau]; B^n_\iota(0)) \,|\, (\xi(0), \xi(\tau)) \in V_0 \times V_1\}$. The following three theorems may be, respectively, viewed as corresponding results of Theorems 1.4, 1.5, 1.6 provided that $\tilde{L}^*$ satisfies (a) and (b) in Proposition 3.3.

Theorem 3.5. (I) (Necessary condition): *Suppose that $(\mu, 0) \in \Lambda \times C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$ is a bifurcation point along sequences of the problem (3.12)–(3.13) with respect to the trivial branch $\{(\lambda, 0) \,|\, \lambda \in \Lambda\}$ in $\Lambda \times C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$. Then $m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 0$.*

(II) (Sufficient condition): *Suppose that Λ is first countable and that there exist two sequences in Λ converging to μ, (λ_k^-) and (λ_k^+), such that one of the following conditions is satisfied:*

(II.1) *For each $k \in \mathbb{N}$, either 0 is not an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda_k^+}$, or 0 is not an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda_k^-}$, or 0 is an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda_k^+}$ and $\tilde{\mathcal{E}}^*_{\lambda_k^-}$ and $C_m(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0; \mathbf{K})$ and $C_m(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$.*

(II.2) *For each $k \in \mathbb{N}$, there exists $\lambda \in \{\lambda_k^+, \lambda_k^-\}$ such that 0 is an either nonisolated or homological visible critical point of $\tilde{\mathcal{E}}^*_\lambda$, and*

$$[m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0)] = \emptyset.$$

(II.3) $[m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0)] = \emptyset$, *and either $m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) = 0$ or $m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) = 0$ for each $k \in \mathbb{N}$.*

Then there exists a sequence $\{(\lambda_k, x_k)\}_{k\ge1}$ in $\hat{\Lambda} \times C^2([0,\tau], \mathbb{R}^n)$ converging to $(\mu, 0)$ such that each $x_k \ne 0$ is a solution of the problem (3.12)–(3.13) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, where $\hat{\Lambda} = \{\mu, \lambda_k^+, \lambda_k^- \mid k \in \mathbb{N}\}$.

Theorem 3.6 (Existence for bifurcations)**.** *Let Λ be connected. For $\lambda^-, \lambda^+ \in \Lambda$ suppose that one of the following conditions is satisfied:*

(i) *Either 0 is not an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda^+}$, or 0 is not an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda^-}$, or 0 is an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda^+}$ and $\tilde{\mathcal{E}}^*_{\lambda^-}$ and $C_m(\tilde{\mathcal{E}}^*_{\lambda^+}, 0; \mathbf{K})$ and $C_m(\tilde{\mathcal{E}}^*_{\lambda^-}, 0; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$.*

(ii) $[m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^+}, 0)] = \emptyset$, *and there exists $\lambda \in \{\lambda^+, \lambda^-\}$ such that 0 is an either nonisolated or homological visible critical point of $\mathcal{E}^*_\lambda$.*

(iii) $[m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^+}, 0)] = \emptyset$, *and either $m^0(\mathcal{E}^*_{\lambda^+}, 0) = 0$ or $m^0(\mathcal{E}^*_{\lambda^-}, 0) = 0$.*

Then for any path $\alpha : [0,1] \to \Lambda$ connecting λ^+ to λ^- there exists a sequence $(t_k) \subset [0,1]$ converging to some $\bar{t} \in [0,1]$, and a nonzero solution x_k of the problem (1.5)–(1.6) with $\lambda = \alpha(t_k)$ for each $k \in \mathbb{N}$ such that $\|x_k\|_{C^2([0,\tau];\mathbb{R}^n)} \to 0$ as $k \to \infty$. Moreover, $\alpha(\bar{t})$ is not equal to λ^+ (resp. λ^-) if $m^0(\tilde{\mathcal{E}}^*_{\lambda^+}, 0) = 0$ (resp. $m^0(\tilde{\mathcal{E}}^*_{\lambda^-}, 0) = 0$).

Theorem 3.7 (Alternative bifurcations of Rabinowitz's type)**.** *Let Λ be a real interval and $\mu \in \mathrm{Int}(\Lambda)$. Suppose that $m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 0$, and that $m^0(\tilde{\mathcal{E}}^*_\lambda, 0) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near μ, and $m^-(\tilde{\mathcal{E}}^*_\lambda, 0)$ take, respectively, values $m^-(\tilde{\mathcal{E}}^*_\mu, 0)$ and $m^-(\tilde{\mathcal{E}}^*_\mu, 0) + m^0(\tilde{\mathcal{E}}^*_\mu, 0)$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of μ. Then one of the following alternatives occurs:*

(i) *The problem (3.12)–(3.13) with $\lambda = \mu$ has a sequence of solutions, $x_k \ne 0$, $k = 1, 2, \cdots$, which converges to 0 in $C^2([0,\tau], \mathbb{R}^n)$.*

(ii) *For every $\lambda \in \Lambda \setminus \{\mu\}$ near μ there is a solution $y_\lambda \ne 0$ of (3.12)–(3.13) with parameter value λ, such that y_λ converges to zero in $C^2([0,\tau], \mathbb{R}^n)$ as $\lambda \to \mu$.*

(iii) *For a given neighborhood $\mathfrak{W}$ of $0 \in C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$, there is an one-sided neighborhood Λ^0 of μ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, the problem (3.12)–(3.13) with parameter value λ has at least two distinct solutions in $\mathfrak{W}$, $y^1_\lambda \neq 0$ and $y^2_\lambda \neq 0$, which can also be required to satisfy $\tilde{\mathcal{E}}^*_\lambda(y^1_\lambda) \neq \tilde{\mathcal{E}}^*_\lambda(y^2_\lambda)$ provided that $m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 1$ and the problem (3.12)–(3.13) with parameter value λ has only finitely many solutions in $\mathfrak{W}$.*

Theorems 1.4, 1.6 are derived from Theorems 3.5, 3.7, respectively. We first admit them and postpone their proof to Section 3.1.3. Theorem 3.6 can only lead to Theorem 1.5 under the assumption (3.3). We shall directly prove Theorem 1.5 in Section 3.1.4.

3.1.2 Proofs of Theorems 1.4, 1.6

Proof of Theorem 1.4. (I) By the assumption there exists a sequence in $\Lambda\times C^1_{S_0\times S_1}([0,\tau]; M)$ converging to (μ,γ_μ), $\{(\lambda_k,\gamma^k)\}_{k\ge 1}$, such that each $\gamma^k \neq \gamma_{\lambda_k}$ is a solution of (1.5)–(1.6) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$. After removing the finite terms (if necessary) we may assume that all γ^k are contained in the image of the chart $\Phi_{\overline{\gamma}}$ in (3.6). Then for each $k \in \mathbb{N}$ there exists a unique $\mathbf{u}^k \in C^2_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$ such that $\Phi_{\overline{\gamma}}(\mathbf{u}^k) = \gamma^k$. Since $\gamma_{\lambda_k} \neq \gamma^k$, $d\mathcal{E}_{\lambda_k}(\gamma_{\lambda_k}) = 0$ and $d\mathcal{E}_{\lambda_k}(\gamma^k) = 0$, we obtain $\mathbf{u}^k \neq \mathbf{u}_{\lambda_k}$, and $d\tilde{\mathcal{E}}_{\lambda_k}(\mathbf{u}_{\lambda_k}) = 0$ and $d\tilde{\mathcal{E}}_{\lambda_k}(\mathbf{u}^k) = 0$. Recall that we have assumed $M \subset \mathbb{R}^N$. Assumption 1.2 implies that $\gamma_\lambda - \gamma_\mu \to 0$ in $C^1([0,\tau]; \mathbb{R}^N)$ as $\lambda \to \mu$. Moreover, $\gamma^k \to \gamma_\mu$ in $C^1_{S_0\times S_1}([0,\tau]; M) \subset C^1([0,\tau]; \mathbb{R}^N)$ as $k \to \infty$. Therefore $\|\gamma^k - \gamma_{\lambda_k}\|_{C^1([0,\tau];\mathbb{R}^N)} \to 0$ as $k \to \infty$. This implies that $\|\mathbf{u}^k - \mathbf{u}_{\lambda_k}\|_{C^1([0,\tau];\mathbb{R}^n)} \to 0$ as $k \to \infty$. In particular, there exists an integer $k_0 > 0$ such that $\|\mathbf{u}^k - \mathbf{u}_{\lambda_k}\|_{C^1([0,\tau];\mathbb{R}^n)} < \iota$ for all $k \ge k_0$. Since $\mathbf{u}^k = (\mathbf{u}^k - \mathbf{u}_{\lambda_k}) + \mathbf{u}_{\lambda_k}$, by the arguments below (3.9) we get $d\tilde{\mathcal{E}}^*_{\lambda_k}(\mathbf{u}^k - \mathbf{u}_{\lambda_k}) = 0$ for all $k \ge k_0$. These show that $(\mu, 0) \in \Lambda \times C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$ is a bifurcation point along sequences of the problem (3.12)–(3.13) in $\Lambda\times C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$ with respect to the trivial branch $\{(\lambda,0)\,|\,\lambda\in\Lambda\}$. Then Theorem 3.5(I) concludes $m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 0$, and therefore $m^0(\mathcal{E}_\mu,\gamma_\mu) > 0$ by (3.11).

(II) Follow the above notations. By the assumption, (3.11) we get that for all $k \in \mathbb{N}$,

$$\begin{aligned}
&[m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0)] \\
&= [m^-_\tau(\mathcal{E}_{\lambda_k^-}, \gamma_{\lambda_k^-}), m^-_\tau(\mathcal{E}_{\lambda_k^-}, \gamma_{\lambda_k^-}) + m^0_\tau(\mathcal{E}_{\lambda_k^-}, \gamma_{\lambda_k^-})] \\
&\qquad\qquad \cap[m^-_\tau(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+}), m^-_\tau(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+}) + m^0_\tau(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+})] = \emptyset
\end{aligned}$$

and either $m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) = m^0_\tau(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^-}) = 0$ or $m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) = m^0_\tau(\mathcal{E}_{\lambda_k^+}, \gamma_{\lambda_k^+}) = 0$. By Theorem 3.5(II) we have a sequence $\{(\lambda_k, \mathbf{v}^k)\}_{k\ge1} \subset \{\mu, \lambda^+_k, \lambda^-_k\,|\,k\in\mathbb{N}\} \times C^2_{V_0\times V_1}([0,\tau]; B^n_\iota(0))$ such that $\lambda_k \to \mu$ and $0 < \|\mathbf{v}^k\|_{C^2} \to 0$, and that each $\mathbf{v}^k$ is a solution of (3.12)–(3.13) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$. Therefore for k large enough, $\gamma^k := \Phi_{\overline{\gamma}}(\mathbf{v^k} + \mathbf{u}_{\lambda_k})$ defined by $\Phi_{\overline{\gamma}}(\mathbf{v^k} + \mathbf{u}_{\lambda_k})(t) = \phi_{\overline{\gamma}}(t, \mathbf{v^k}(t) + \mathbf{u}_{\lambda_k}(t))$ is a solution of (1.5)–(1.6) with $\lambda = \lambda_k$, $\gamma^k \neq \gamma_{\lambda_k}$, and as $k \to \infty$ we have $\gamma^k \to \gamma_\mu$ in $C^2_{S_0\times S_1}([0,\tau]; \mathbb{R}^N)$ because $\Phi_{\overline{\gamma}}$ is also a C^1 map from $C^2_{V_0\times V_1}([0,\tau]; B^n_{2\iota}(0))$ to $C^2_{S_0\times S_1}([0,\tau]; M)$ as noted below (3.6). Theorem 1.4(II) is proved. $\square$

Proof of Theorem 1.6. Follow the above notations. By the assumption, (3.11) we obtain that $m^0(\tilde{\mathcal{E}}^*_\mu, 0) \neq 0$, and that $m^0(\tilde{\mathcal{E}}^*_\lambda, 0) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near μ, and $m^-(\tilde{\mathcal{E}}^*_\lambda, 0)$ take, respectively, values $m^-(\tilde{\mathcal{E}}^*_\mu, 0)$ and $m^-(\tilde{\mathcal{E}}^*_\mu, 0) + m^0(\tilde{\mathcal{E}}^*_\mu, 0)$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of μ. Therefore one of the conclusions (i)-(iii) in Theorem 3.7 occurs.

Let (x_k) be as in (i) in Theorem 3.7. Since $\|x_k\|_{C^2} \to 0$, we can choose $k_0 > 0$ such that $\|x_k\|_{C^2} < \iota$ for $k \ge k_0$. Then for each $k \ge k_0$, $\gamma^k := \Phi_{\overline{\gamma}}(x_k + \mathbf{u}_\mu) \ne \gamma_\mu$ is a solution of (1.5)–(1.6) with $\lambda = \mu$, and as above we may deduce that $\gamma^k \to \gamma_\mu$ in $C^2_{S_0\times S_1}([0,\tau];\mathbb{R}^N)$ as $k \to \infty$. This is, (i) of Theorem 1.6 occurs.

For $y_\lambda \ne 0$ in (ii) in Theorem 3.7, we can shrink Λ toward μ so that $\|y_\lambda\|_{C^2} < \iota$ for all $\lambda \in \Lambda$. Then $\alpha_\lambda := \Phi_{\overline{\gamma}}(y_\lambda + \mathbf{u}_\lambda) \ne \gamma_\lambda$ is a solution of (1.5)–(1.6) with parameter value λ, and $\alpha_\lambda - \gamma_\lambda$ converges to zero in $C^2([0,\tau],\mathbb{R}^N)$ as $\lambda \to \mu$. Namely, (ii) of Theorem 1.6 occurs.

For a given neighborhood $\mathcal{W}$ of γ_μ in $C^1([0,\tau],M)$, let us choose a neighborhood $\mathfrak{W}$ of $0 \in C^1_{V_0\times V_1}([0,\tau];B^n_\iota(0))$ such that $\Phi_{\overline{\gamma}}(\mathbf{u}_\mu + \mathfrak{W}) \subset \mathcal{W}$. Let Λ^0, $y^1_\lambda \ne 0$ and $y^2_\lambda \ne 0$ be as in (iii) in Theorem 3.7. Put $\gamma^i_\lambda := \Phi_{\overline{\gamma}}(y^i_\lambda + \mathbf{u}_\lambda) \ne \gamma_\lambda$, $i = 1,2$. Both sit in $\mathcal{W}$ and are distinct solutions of (1.5)–(1.6) with parameter value λ. Suppose that $m^0(\mathcal{E}_\mu,\gamma_\mu) = m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 1$ and (1.5)–(1.6) with parameter value λ has only finitely many distinct solutions in $\mathcal{W}$. Then the problem (3.12)–(3.13) with parameter value λ has only finitely many solutions in $\mathfrak{W}$ as well. In this case (iii) in Theorem 3.7 concludes that the above $y^1_\lambda \ne 0$ and $y^2_\lambda \ne 0$ are chosen to satisfies $\tilde{\mathcal{E}}^*_\lambda(y^1_\lambda) \ne \tilde{\mathcal{E}}^*_\lambda(y^2_\lambda)$, which implies $\mathcal{E}_\lambda(\gamma^1_\lambda) \ne \mathcal{E}_\lambda(\gamma^2_\lambda)$. Hence (iii) in Theorem 3.7 occurs. $\square$

3.1.3 Proofs of Theorems 3.5, 3.6, 3.7

We need to make modifications for the Lagrangian $\tilde{L}^*$ in (7.9).

Lemma 3.8. *Given a positive number $\rho_0 > 0$ and a subset $\hat{\Lambda} \subset \Lambda$ which is either compact or sequential compact, there exists a continuous function $\check{L} : \hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties for some constants $\check{\kappa} > 0$ and $0 < \check{c} < \check{C}$:*

(L0) *The following partial derivatives*

$$\partial_t\check{L}(\cdot),\ \partial_q\check{L}(\cdot),\ \partial_v\check{L}(\cdot),\ \partial_{tv}\check{L}(\cdot),\ \partial_{vt}\check{L}(\cdot),\ \partial_{qv}\check{L}(\cdot),\ \partial_{vq}\check{L}(\cdot),\ \partial_{qq}\check{L}(\cdot),\ \partial_{vv}\check{L}(\cdot)$$

exist and depend continuously on (λ,t,q,v). (These are all used in the proof of Proposition 3.11.)

(L1) *$\check{L}$ and $\tilde{L}^*$ are equal in $\hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times B^n_{\rho_0}(0)$.*

(L2) $\partial_{vv}\check{L}_\lambda(t,q,v) \ge \check{c}I_n, \quad \forall(\lambda,t,q,v) \in \hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

(L3) $\left|\frac{\partial^2}{\partial q_i\partial q_j}\check{L}_\lambda(t,q,v)\right| \le \check{C}(1+|v|^2), \quad \left|\frac{\partial^2}{\partial q_i\partial v_j}\check{L}_\lambda(t,q,v)\right| \le \check{C}(1+|v|),$ *and*
$\left|\frac{\partial^2}{\partial v_i\partial v_j}\check{L}_\lambda(t,q,v)\right| \le \check{C}, \quad \forall(\lambda,t,q,v) \in \hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

(L4) $\check{L}(\lambda,t,q,v) \ge \check{\kappa}|v|^2 - \check{C}, \quad \forall(\lambda,t,q,v) \in \hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

(L5) $|\partial_q\check{L}(\lambda,t,q,v)| \le \check{C}(1+|v|^2)$ *and* $|\partial_v\check{L}(\lambda,t,q,v)| \le \check{C}(1+|v|)$ *for all* $(\lambda,t,q,v) \in \hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

(L6) $|\check{L}_\lambda(t,q,v)| \le \check{C}(1+|v|^2), \quad \forall(\lambda,t,q,v) \in \hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

Proof. Step 1. Fix a positive number $\rho_1 > \rho_0$. As in the proof of Lemma 2.4, we may choose a C^∞ convex function $\psi_{\rho_0,\rho_1} : [0,\infty) \to \mathbb{R}$ such that $\psi'_{\rho_0,\rho_1}(t) > 0$ for $t \in (\rho_0^2,\infty)$, $\psi_{\rho_0,\rho_1}(t) = 0$ for $t \in [0,\rho_0^2)$ and $\psi_{\rho_0,\rho_1}(t) = \kappa t + \varrho_0$ for $t \in [\rho_1^2,\infty)$, where $\kappa > 0$ and $\varrho_0 < 0$ are suitable constants. Define $\tilde{L}^{**} : \Lambda \times [0,\tau] \times B^n_\iota(0) \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{L}^{**}(\lambda,t,q,v) = \tilde{L}^*(\lambda,t,q,v) + \psi_{\rho_0,\rho_1}(|v|^2). \tag{3.14}$$

It possess the same properties as $\check{L}$ in (L0) and also satisfies

$$\tilde{L}^{**}(\lambda,t,q,v)=\tilde{L}^{*}(\lambda,t,q,v),\quad \forall(\lambda,t,q,v)\in\Lambda\times[0,\tau]\times B^n_\iota(0)\times B^n_{\rho_0}(0). \tag{3.15}$$

Since the closure of $B^n_{3\iota/4}(0)$ is a compact subset in $B^n_\iota(0)$, and $\hat{\Lambda}$ is either compact or sequential compact, by Lemma 2.4 (or the proof of Lemma 2.4(iv)) there exists a constant $C'>0$ such that

$$\tilde{L}^{**}(\lambda,t,q,v)\ge\kappa|v|^2-C',\quad \forall(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\mathbb{R}^n. \tag{3.16}$$

Step 2. Take a smooth function $\Gamma:\mathbb{R}\to\mathbb{R}$ such that $\Gamma(s)=s$ for $s\le 1$ and that $\Gamma(s)$ is constant for $s\ge 2$. Fix positive numbers $\rho>\rho_1$ and ϑ such that

$$\vartheta\ge\max\{\tilde{L}^{**}_\lambda(q,v)\,|\,(\lambda,q,v)\in\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\bar{B}^n_\rho(0)\}.$$

Define $\tilde{L}^{***}:\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\mathbb{R}^n\to\mathbb{R}$ by $\tilde{L}^{***}=\vartheta\Gamma(\tilde{L}^{**}/\vartheta)$. Then the choice of ϑ implies

$$\tilde{L}^{***}(\lambda,t,q,v)=\tilde{L}^{**}(\lambda,t,q,v),\quad \forall(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times B^n_\rho(0). \tag{3.17}$$

By (3.16), $\tilde{L}^{***}$ is equal to a constant C'' outside $\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times B^n_R(0)$ for a large $R>\rho$. Because of this fact and

$$\partial_{vv}\tilde{L}^{**}(\lambda,t,q,v)[u,u]=\partial_{vv}\tilde{L}^{*}(\lambda,t,q,v)[u,u]+2\psi'_{\rho_0,\rho_1}(|v|^2)|u|^2+4\psi''_{\rho_0,\rho_1}(|v|^2)\big((v,u)_{\mathbb{R}^n}\big)^2 \tag{3.18}$$

for each $(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times B^n_\iota(0)\times\mathbb{R}^n$, there exist positive constants Υ and C'_0 such that

$$\partial_{vv}\tilde{L}^{***}_\lambda(t,q,v)[u,u]\ge-\Upsilon|u|^2\quad \forall(\lambda,t,q,v,u)\in\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\mathbb{R}^n, \tag{3.19}$$

$$\tilde{L}^{***}(\lambda,t,q,v)\ge\kappa|v|^2-C'_0,\quad \forall(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\mathbb{R}^n. \tag{3.20}$$

Choose a smooth function $\Xi:[0,\infty)\to\mathbb{R}$ such that:

$\Xi'\ge0$, Ξ is convex on $[\rho_0^2,\infty)$, vanishes in $[0,\rho_0^2)$, and is equal to
the affine function $\Upsilon s+\Theta$ on $[\rho^2,\infty)$, where $\Theta<0$ is a suitable constant.

(See [31, Lemma 2.1] or [1, §5]). Define $\check{L}:\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\mathbb{R}^n\to\mathbb{R}$ by

$$\check{L}:\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\mathbb{R}^n\to\mathbb{R},\ (\lambda,t,q,v)\mapsto\tilde{L}^{***}(\lambda,t,q,v)+\Xi(|v|^2). \tag{3.21}$$

Since $\tilde{L}^{***}=\vartheta\Gamma(\tilde{L}^{**}/\vartheta)$, it clearly satisfies (L0) by (3.14). (L1) may follow from (3.15), (3.17) and the fact that Ξ vanishes in $[0,\rho_0^2)$. (3.20) leads to (L4) because $\Xi\ge0$.

Let us prove that $\check{L}$ satisfies (L2) and (L3).

- If $|v|<\rho$, by (3.17) $\check{L}(\lambda,t,q,v)=\tilde{L}^{***}(\lambda,t,q,v)+\Xi(|v|^2)=\tilde{L}^{**}(\lambda,t,q,v)+\Xi(|v|^2)$ and so

$$\begin{aligned}\partial_{vv}\check{L}(\lambda,t,q,v)[u,u]&=\partial_{vv}\tilde{L}^{**}(\lambda,t,q,v)[u,u]+\partial_{vv}(\Xi(|v|^2))[u,u]\\&=\partial_{vv}\tilde{L}^{*}(\lambda,t,q,v)[u,u]+2\psi'_{\rho_0,\rho_1}(|v|^2)|u|^2\\&\quad+4\psi''_{\rho_0,\rho_1}(|v|^2)\big((v,u)_{\mathbb{R}^n}\big)^2+2\psi'_{\rho_0,\rho_1}(|v|^2)|u|^2\\&\quad+2\Xi'(|v|^2)|u|^2+4\Xi''(|v|^2)\big((v,u)_{\mathbb{R}^n}\big)^2\end{aligned}$$

because of (3.18) and the equality $\partial_{vv}(\Xi(|v|^2))[u,u]=2\Xi'(|v|^2)|u|^2+4\Xi''(|v|^2)\big((v,u)_{\mathbb{R}^n}\big)^2$ for all $u\in\mathbb{R}^n$. Recall that $\psi'_{\rho_0,\rho_1}\ge0$, $\psi''_{\rho_0,\rho_1}\ge0$, $\Xi'\ge0$ and $\Xi''\ge0$. Since both $\partial_{vv}\check{L}(\lambda,t,q,v)$ and $\partial_{vv}\tilde{L}^{**}(\lambda,t,q,v)$ depend continuously on (λ,t,q,v) we deduce

$$\partial_{vv}\check{L}(\lambda,t,q,v)\ge\partial_{vv}\tilde{L}^{*}(\lambda,t,q,v),\quad \forall(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times\overline{B^n_{3\iota/4}(0)}\times\overline{B^n_\rho(0)}. \tag{3.22}$$

- If $|v| \geq \rho$ then (3.19) may lead to

$$\begin{aligned}\partial_{vv}\check{L}(\lambda,t,q,v)[u,u] &= \partial_{vv}\tilde{L}^{***}(\lambda,t,q,v)[u,u] + 2\Xi'(|v|^2)|u|^2 \\ &\geq \Upsilon|u|^2, \quad \forall u \in \mathbb{R}^n. \end{aligned} \tag{3.23}$$

By Proposition 3.3(c), $\tilde{L}^*_\lambda(t,q,v)$ is strictly convex in v. (L2) may follow from (3.23) and (3.22) because $\hat{\Lambda} \times [0,\tau] \times \overline{B^n_{3\iota/4}(0)} \times \overline{B^n_\rho(0)}$ is either compact or sequential compact. Using the same reason we obtain that $\check{L}$ satisfies (L3) because

$$\check{L}(\lambda,t,q,v) = \tilde{L}^{***}(\lambda,t,q,v) + \Xi(|v|^2) = C'' + \Upsilon|v|^2 + \Theta \quad \forall |v| > R.$$

Finally, since $\partial_q\check{L}(\lambda,t,0,0)$ and $\partial_v\check{L}(\lambda,t,0,0)$ are bounded using the Taylor formula (L5) and (L6) easily follows from (L3). $\square$

Consider the Banach subspace

$$\mathbf{X}_{V_0\times V_1} := \{\xi \in C^1([0,\tau];\mathbb{R}^n)\,|\,(\xi(0),\xi(\tau)) \in V_0 \times V_1\}$$

of $C^1([0,\tau],\mathbb{R}^n)$, and the Hilbert subspace

$$\mathbf{H}_{V_0\times V_1} := \{\xi \in W^{1,2}([0,\tau];\mathbb{R}^n)\,|\,(\xi(0),\xi(\tau)) \in V_0 \times V_1\}$$

of $W^{1,2}([0,\tau];\mathbb{R}^n)$. The spaces $\mathbf{H}_{V_0\times V_1}$ and $\mathbf{X}_{V_0\times V_1}$ have the following open subsets

$$\begin{aligned}\mathcal{U} : &= W^{1,2}_{V_0\times V_1}\left([0,\tau];B^n_{\iota/2}(0)\right) = \left\{\xi \in W^{1,2}\left([0,\tau];B^n_{\iota/2}(0)\right)\,|\,(\xi(0),\xi(\tau)) \in V_0 \times V_1\right\}, \\ \mathcal{U}^X : &= \mathcal{U} \cap \mathbf{X}_{V_0\times V_1} = C^1_{V_0\times V_1}([0,\tau];B^n_{\iota/2}(0))\end{aligned}$$

respectively. Define a family of functionals $\check{\mathcal{E}}_\lambda : \mathcal{U} \to \mathbb{R}$ given by

$$\check{\mathcal{E}}_\lambda(x) = \int_0^\tau \check{L}_\lambda(t,x(t),\dot{x}(t))dt, \quad \lambda \in \hat{\Lambda}. \tag{3.24}$$

Since $\tilde{\mathcal{E}}^*_\lambda$ is defined on $C^1_{V_0\times V_1}([0,\tau];B^n_\iota(0))$ and $\mathcal{U}^X = C^1_{V_0\times V_1}([0,\tau];B^n_{\iota/2}(0))$ is an open neighborhood of $0 \in C^1_{V_0\times V_1}([0,\tau];B^n_\iota(0))$, by (L1) in Lemma 3.8 we obtain

$$\tilde{\mathcal{E}}^*_\lambda = \check{\mathcal{E}}_\lambda|_{\mathcal{U}^X} \quad \text{in} \quad \{x \in \mathcal{U}^X\,|\,\|x\|_{C^1} < \rho_0\} \subset \mathcal{U}^X, \tag{3.25}$$

and therefore the following (3.26).

Proposition 3.9. (i) *Each $\check{\mathcal{E}}_\lambda$ is C^{2-0} and twice Gâteaux-differentiable, and $d\check{\mathcal{E}}_\lambda(0) = 0$ and*

$$m^\star(\tilde{\mathcal{E}}^*_\lambda,0) = m^\star(\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X},0) = m^\star(\check{\mathcal{E}}_\lambda,0), \quad \star = -,0. \tag{3.26}$$

(ii) *Each critical point of $\check{\mathcal{E}}_\lambda$ sits in $C^2\left([0,\tau];B^n_{\iota/2}(0)\right) \cap \mathcal{U}^X$, and satisfies the boundary problem:*

$$\left.\begin{aligned}&\tfrac{d}{dt}\Big(\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t))\Big) - \partial_q\check{L}_\lambda(t,x(t),\dot{x}(t)) = 0, \\ &\partial_v\check{L}_\lambda(0,x(0),\dot{x}(0))[v_0] = 0 \quad \forall v_0 \in V_0, \\ &\partial_v\check{L}_\lambda(\tau,x(\tau),\dot{x}(\tau))[v_1] = 0 \quad \forall v_1 \in V_1.\end{aligned}\right\} \tag{3.27}$$

(iii) *The gradient of* $\check{\mathcal{E}}_\lambda$ *at* $x \in \mathcal{U}$*, denoted by* $\nabla\check{\mathcal{E}}_\lambda(x)$*, is given by*

$$\begin{aligned}\nabla\check{\mathcal{E}}_\lambda(x)(t) &= e^t \int_0^t \left[e^{-2s}\int_0^s e^r f_{\lambda,x}(r)dr\right] ds + c_1(\lambda,x)e^t + c_2(\lambda,x)e^{-t} \\ &\quad + \int_0^t \partial_v \check{L}_\lambda(s,x(s),\dot{x}(s))ds, \end{aligned} \tag{3.28}$$

where $c_1(\lambda,x), c_2(\lambda,x) \in \mathbb{R}^n$ *are suitable constant vectors and*

$$f_{\lambda,x}(t) = -\partial_q \check{L}_\lambda(t,x(t),\dot{x}(t)) + \int_0^t \partial_v \check{L}_\lambda(s,x(s),\dot{x}(s))ds. \tag{3.29}$$

(iv) $\nabla\check{\mathcal{E}}_\lambda$ *restricts to a* C^1 *map* A_λ *from* $\mathcal{U}^X$ *to* $\mathbf{X}_{V_0\times V_1}$.

(v) $\nabla\check{\mathcal{E}}_\lambda$ *has the Gâteaux derivative* $B_\lambda(\zeta) \in \mathcal{L}_s(\mathbf{H}_{V_0\times V_1})$ *at* $\zeta \in \mathcal{U}$ *given by*

$$\begin{aligned}(B_\lambda(\zeta)\xi,\eta) = \int_0^\tau \Big(&\partial_{vv}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\dot{\xi}(t),\dot{\eta}(t)\big] + \partial_{qv}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\xi(t),\dot{\eta}(t)\big] \\ &+\partial_{vq}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\dot{\xi}(t),\eta(t)\big] \\ &+\partial_{qq}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\xi(t),\eta(t)\big]\Big)\, dt \end{aligned} \tag{3.30}$$

for any $\xi,\eta \in \mathbf{H}_{V_0\times V_1}$. $B_\lambda(\zeta)$ *is a self-adjoint Fredholm operator and has a decomposition* $B_\lambda(\zeta) = P_\lambda(\zeta)+Q_\lambda(\zeta)$*, where* $P_\lambda(\zeta) \in \mathcal{L}_s(H)$ *is a positive definitive linear operator defined by*

$$(P_\lambda(\zeta)\xi,\eta)_{1,2} = \int_0^\tau \Big(\partial_{vv}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)[\dot{\xi}(t),\dot{\eta}(t)] + \big(\xi(t),\eta(t)\big)_{\mathbb{R}^n}\Big)\, dt, \tag{3.31}$$

and $Q_\lambda(\zeta) \in \check{\mathcal{L}}_s(H)$ *is a compact self-adjoint linear operator. Moreover, (L2) Lemma 3.8 implies that* $(P_\lambda(\zeta)\xi,\xi)_{1,2} \geq \min\{c,1\}\|\xi\|^2_{1,2}$ *for all* $x \in \mathcal{U}$ *and* $\xi \in \mathbf{H}_{V_0\times V_1}$.

(vi) *If* $(\lambda_k) \subset \hat{\Lambda}$ *and* $(\zeta_k) \subset \mathcal{U}$ *converge to* $\mu \in \hat{\Lambda}$ *and* 0*, respectively, then* $\|P_{\lambda_k}(\zeta_k)\xi - P_\mu(0)\xi\|_{1,2} \to 0$ *for each* $\xi \in \mathbf{H}_{V_0\times V_1}$.

(vii) $\mathcal{U} \ni \zeta \mapsto Q_\lambda(\zeta) \in \mathcal{L}_s(\mathbf{H}_{V_0\times V_1})$ *is uniformly continuous at* $0 \in \mathcal{U}$ *with respect to* $\lambda \in \hat{\Lambda}$ *and* $\|Q_{\lambda_k}(0) - Q_\mu(0)\| \to 0$ *as* $(\lambda_k) \subset \Lambda$ *converges to* $\mu \in \hat{\Lambda}$.

Proof. (i) is obtained by [31, §4] or [32, §3] and [30]. (ii) follows from [6, Theorem 4.5] because of conditions (L0), (L2) and (L4)-(L6) in Lemma 3.8. (iii) is obtained by (4.13) and (4.14) in [31]. (iv) and (v) are proved in [31, §4].

Proof of (vi). By (6.25) we have

$$\|[P_{\lambda_k}(\zeta_k) - P_\mu(0)]\xi\|^2_{1,2} \leq \int_0^\tau \left|[\partial_{vv}\check{L}_{\lambda_k}\big(t,\zeta_k(t),\dot{\zeta}_k(t)\big) - \partial_{vv}\check{L}_\mu\big(t,0,0\big)]\dot{\xi}(t)\right|^2_{\mathbb{R}^n} dt.$$

Note that $\|\zeta_k\|_{1,2} \to 0$ implies $\|\zeta_k\|_{C^0} \to 0$. Since $(\lambda,t,x,v) \mapsto \partial_{vv}\check{L}_\lambda(t,x,v)$ is continuous, by the third inequality in (L3) in Lemma 3.8 we may apply [37, Prop. B.9] ([35, Prop. C.1]) to

$$f(t,\eta;\lambda) = \partial_{vv}\check{L}(\lambda,t,\zeta_k(t),\dot{\zeta}_k(t))\eta$$

to get that

$$\int_0^\tau \left|[\partial_{vv}\check{L}_{\lambda_k}\big(t,\zeta_k(t),\dot{\zeta}_k(t)\big) - \partial_{vv}\check{L}_\mu\big(t,0,0\big)]\dot{\xi}(t)\right|^2_{\mathbb{R}^n} dt \to 0.$$

Moreover, the Lebesgue dominated convergence theorem also leads to

$$\int_0^\tau \left|[\partial_{vv}\check{L}_{\lambda_k}(t,0,0) - \partial_{vv}\check{L}_\mu(t,0,0)]\dot{\xi}(t)\right|^2_{\mathbb{R}^n} dt \to 0.$$

Hence $\|[P_{\lambda_k}(\zeta_k) - P_\mu(0)]\xi\|_{1,2} \to 0$.

Proof of (vii). Write $Q_\lambda(\zeta) := Q_{\lambda,1}(\zeta) + Q_{\lambda,2}(\zeta) + Q_{\lambda,3}(\zeta)$, where

$$\begin{aligned}
(Q_{\lambda,1}(\zeta)\xi,\eta)_{1,2} &= \int_0^\tau \partial_{vq}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)[\dot{\xi}(t),\eta(t)]dt,\\
(Q_{\lambda,2}(\zeta)\xi,\eta)_{1,2} &= \int_0^\tau \partial_{qv}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)[\xi(t),\dot{\eta}(t)]dt,\\
(Q_{\lambda,3}(\zeta)\xi,\eta)_{1,2} &= \int_0^\tau \Big(\partial_{qq}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)[\xi(t),\eta(t)] - \big(\xi(t),\eta(t)\big)_{\mathbb{R}^n}\Big)\, dt.
\end{aligned}$$

As above the first claim follows from (L3) in Lemma 3.8 and [37, Prop. B.9] ([35, Prop. C.1]) directly.

In order to prove the second claim, as in the proof of [30, page 571] we have

$$\begin{aligned}
&\|Q_{\lambda_k,1}(0) - Q_{\mu,1}(0)\|_{\mathcal{L}(\mathbf{H})}\\
\le\;& 2(e^\tau+1)\left(\int_0^\tau \left|\partial_{vq}\check{L}_{\lambda_k}(s,0,0) - \partial_{vq}\check{L}_\mu(s,0,0)\right|^2 ds\right)^{1/2}.
\end{aligned}$$

Because of the second inequality in (**L2**), it follows from the Lebesgue dominated convergence theorem that $\|Q_{\lambda_k,1}(0) - Q_{\mu,1}(0)\|_{\mathcal{L}(\mathbf{H})} \to 0$. Observe that $(Q_{\lambda,2}(\zeta)\xi,\eta)_{1,2} = \big(\xi,(Q_{\lambda,1}(\zeta))^*\eta\big)_{1,2}$. Hence $\|Q_{\lambda_k,2}(0) - Q_{\mu,2}(0)\|_{\mathcal{L}(\mathbf{H})} \to 0$. Finally, it is easy to deduce that

$$\|Q_{\lambda_k,3}(0) - Q_{\mu,3}(0)\|^2_{\mathcal{L}(\mathbf{H})} \le \int_0^\tau \left|\partial_{qq}\check{L}_{\lambda_k}\big(t,0,0\big) - \partial_{qq}\check{L}_\mu\big(t,0,0\big)\right|^2 dt.$$

By the Lebesgue dominated convergence theorem the right side converges to zero. Then $\|Q_{\lambda_k,3}(0) - Q_{\mu,3}(0)\|_{\mathcal{L}(\mathbf{H})} \to 0$ and therefore $\|Q_{\lambda_k}(0) - Q_\mu(0)\| \to 0$.

□

In order to apply our abstract theory in [34, 36, 37] to the family of functionals in (3.24) we also need two results.

Proposition 3.10. *Both maps $\hat{\Lambda}\times\mathcal{U}^X \ni (\lambda,x) \mapsto \check{\mathcal{E}}_\lambda(x) \in \mathbb{R}$ and $\hat{\Lambda}\times\mathcal{U}^X \ni (\lambda,x) \mapsto A_\lambda(x) \in \mathbf{X}_{V_0\times V_1}$ are continuous.*

Proof. Step 1(*Prove that $\hat{\Lambda}\times\mathcal{U} \ni (\lambda,x) \mapsto \check{\mathcal{L}}_\lambda(x) \in \mathbb{R}$ is continuous, and therefore obtain the first claim*). Indeed, for any two points (λ,x) and (λ_0,x_0) in $\hat{\Lambda}\times\mathcal{U}$ we can write

$$\begin{aligned}
\check{\mathcal{E}}_\lambda(x) - \check{\mathcal{E}}_{\lambda_0}(x_0) \;=\;& \left[\int_0^\tau \check{L}_\lambda(t,x(t),\dot{x}(t))dt - \int_0^\tau \check{L}_\lambda(t,x_0(t),\dot{x}_0(t))dt\right]\\
&+\left[\int_0^\tau \check{L}_\lambda(t,x_0(t),\dot{x}_0(t))dt - \int_0^\tau \check{L}_{\lambda_0}(t,x_0(t),\dot{x}_0(t))dt\right].
\end{aligned}$$

As $(\lambda,x)\to(\lambda_0,x_0)$, we derive from (L6) in Lemma 3.8 and [37, Prop.B.9] or [35, Proposition C.1] (resp. (L6) in Lemma 3.8 and the Lebesgue dominated convergence theorem) that the first (resp. second) bracket on the right side converges to the zero.

Step 2(*Prove that* $\hat\Lambda\times\mathcal{U}^X\ni(\lambda,x)\mapsto A_\lambda(x)\in\mathbf{X}_{V_0\times V_1}$ *is continuous*). By [31, (4.14)] we have

$$\begin{aligned}\frac{d}{dt}\nabla\check{\mathcal{E}}_\lambda(x)(t) &= e^t\int_0^t\left[e^{-2s}\int_0^s e^r f_{\lambda,x}(r)dr\right]ds+e^{-t}\int_0^t e^r f_{\lambda,x}(r)dr\\ &\quad +c_1(\lambda,x)e^t-c_2(\lambda,x)e^{-t}+\partial_v\check{L}_\lambda\left(t,x(t),\dot{x}(t)\right).\end{aligned}\tag{3.32}$$

This and (3.28) lead to

$$\begin{aligned}2c_1(\lambda,x)e^t &= -2e^t\int_0^t\left[e^{-2s}\int_0^s e^r f_{\lambda,x}(r)dr\right]ds-e^{-t}\int_0^t e^r f_{\lambda,x}(r)dr\\ &\quad -\int_0^t\partial_v\check{L}_\lambda(s,x(s),\dot{x}(s))ds-\partial_v\check{L}_\lambda\left(t,x(t),\dot{x}(t)\right)\\ &\quad +\nabla\check{\mathcal{E}}_\lambda(x)(t)+\frac{d}{dt}\nabla\check{\mathcal{E}}_\lambda(x)(t)\end{aligned}\tag{3.33}$$

and

$$\begin{aligned}2c_2(\lambda,x)e^{-t} &= e^{-t}\int_0^t e^r f_{\lambda,x}(r)dr-\int_0^t\partial_v\check{L}_\lambda(s,x(s),\dot{x}(s))ds\\ &\quad +\partial_v\check{L}_\lambda\left(t,x(t),\dot{x}(t)\right)+\nabla\check{\mathcal{E}}_\lambda(x)(t)-\frac{d}{dt}\nabla\check{\mathcal{E}}_\lambda(x)(t).\end{aligned}\tag{3.34}$$

Moreover, since

$$\begin{aligned}&d\check{\mathcal{E}}_{\lambda_1}(x)[\xi]-d\check{\mathcal{E}}_{\lambda_2}(y)[\xi]\\ =\ &\int_0^\tau\left(\partial_q\check{L}(\lambda_1,t,x(t),\dot{x}(t))-\partial_q\check{L}(\lambda_2,t,y(t),\dot{y}(t))\right)\cdot\xi(t)dt\\ &+\int_0^\tau\Big(\partial_v\check{L}\left(\lambda_1,t,x(t),\dot{x}(t)\right)-\partial_v\check{L}(\lambda_2,t,y(t),\dot{y}(t))\big)\cdot\dot{\xi}(t)\Big)\,dt,\end{aligned}$$

we have

$$\begin{aligned}&\|\nabla\check{\mathcal{E}}_{\lambda_1}(x)-\nabla\check{\mathcal{E}}_{\lambda_2}(y)\|_{1,2}\\ \le\ &\left(\int_0^\tau\left|\partial_q\check{L}(\lambda_1,t,x(t),\dot{x}(t))-\partial_q\check{L}(\lambda_2,t,y(t),\dot{y}(t))\right|^2dt\right)^{1/2}\\ &+\left(\int_0^\tau\left|\partial_v\check{L}(\lambda_1,t,x(t),\dot{x}(t))-\partial_v\check{L}(\lambda_2,t,y(t),\dot{y}(t))\right|^2dt\right)^{1/2}.\end{aligned}$$

Fix a point $(\lambda_1,x)\in\hat\Lambda\times\mathcal{U}^X$. Then $\{(\lambda_1,t,x(t),\dot{x}(t))\,|\,t\in[0,\tau]\}$ is a compact subset of $\hat\Lambda\times[0,\tau]\times B^n_{\iota/2}(0)\times\mathbb{R}^n$. Since $\partial_q\check{L}$ and $\partial_v\check{L}$ are uniformly continuous in any compact neighborhood of this compact subset we deduce: If $(\lambda_2,y)\in\hat\Lambda\times\mathcal{U}^X$ converges to (λ_1,x) in $\hat\Lambda\times\mathcal{U}^X$, then

$$\|\nabla\check{\mathcal{E}}_{\lambda_1}(x)-\nabla\check{\mathcal{E}}_{\lambda_2}(y)\|_{1,2}\to0\quad\text{and so}\quad\|\nabla\check{\mathcal{E}}_{\lambda_1}(x)-\nabla\check{\mathcal{E}}_{\lambda_2}(y)\|_{C^0}\to0.$$

This fact, (3.29) and (3.33)–(3.34) imply that $c_1(\lambda,x)$ and $c_2(\lambda,x)$ are continuous in $\hat\Lambda\times\mathcal{U}^X$. From the latter claim, (3.28)–(3.29) and (3.32), it easily follows that as $(\lambda_2,y)\in\hat\Lambda\times\mathcal{U}^X$ converges to (λ_1,x) in $\hat\Lambda\times\mathcal{U}^X$,

$$\left\|\frac{d}{dt}\nabla\check{\mathcal{E}}_{\lambda_1}(x)-\frac{d}{dt}\nabla\check{\mathcal{E}}_{\lambda_2}(y)\right\|_{C^0}\to0$$

and hence $\|\nabla\check{\mathcal{E}}_{\lambda_1}(x)-\nabla\check{\mathcal{E}}_{\lambda_2}(y)\|_{C^1}\to0$. $\square$

Proposition 3.11. *For any given $\epsilon > 0$ there exists $\varepsilon > 0$ such that if a critical point x of $\check{\mathcal{E}}_\lambda$ satisfies $\|x\|_{1,2} < \varepsilon$ then $\|x\|_{C^2} < \epsilon$. (***Note***: ε is independent of $\lambda \in \hat{\Lambda}$.) Consequently, if $0 \in \mathcal{U}^X$ is an isolated critical point of $\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}$ then $0 \in \mathcal{U}$ is also an isolated critical point of $\check{\mathcal{E}}_\lambda$.*

By Proposition 3.9(ii), $\mathrm{Crit}(\check{\mathcal{E}}) := \{(\lambda, x) \in \hat{\Lambda} \times \mathcal{U} \,|\, d\check{\mathcal{E}}_\lambda(x) = 0\} \subset \hat{\Lambda} \times C^2\left([0,\tau]; B^n_{\iota/2}(0)\right) \cap \mathcal{U}^X$. Proposition 3.11 claims that $\hat{\Lambda} \times \mathcal{U}$ and $\hat{\Lambda} \times C^2\left([0,\tau]; B^n_{\iota/2}(0)\right) \cap \mathcal{U}^X$ induce the equivalence topologies.

Proof of Proposition 3.11. The second claim may follow from the first one by contradiction. Let us prove the first one. By Proposition 3.9(ii), x is C^2. Let $c_1(\lambda, x)$ and $c_2(\lambda, x)$ be given by (3.33) and (3.34), respectively.

Step 1 (*Prove that both $|c_1(\lambda, x) - c_1(\lambda, 0)|$ and $|c_2(\lambda, x) - c_2(\lambda, 0)|$ uniformly converge to zero in $\lambda \in \hat{\Lambda}$ as $\|x\|_{1,2} \to 0$*).

Since $\nabla\check{\mathcal{E}}_\lambda(x)(t) \equiv 0$ and $\frac{d}{dt}\nabla\check{\mathcal{E}}_\lambda(x)(t) \equiv 0$, by (3.33) for any $t \in [0,\tau]$ we have

$$\begin{aligned}
&2|c_1(\lambda, x) - c_1(\lambda, 0)| \le 2|c_1(\lambda, x)e^t - c_1(\lambda, 0)e^t| \\
\le\ & 2e^t \int_0^t \left[e^{-2s}\int_0^s e^r |f_{\lambda,x}(r) - f_{\lambda,0}(r)|dr\right] ds + e^{-t}\int_0^t e^r|f_{\lambda,x}(r) - f_{\lambda,0}(r)|dr \\
&+ \int_0^t |\partial_v \check{L}_\lambda(s, x(s), \dot{x}(s)) - \check{L}_\lambda(s,0,0)|ds + |\partial_v\check{L}_\lambda(t, x(t), \dot{x}(t)) - \partial_v\check{L}_\lambda(t,0,0)| \\
\le\ & 2e^{2\tau}\tau \int_0^\tau |f_{\lambda,x}(r) - f_{\lambda,0}(r)|dr + \int_0^\tau |f_{\lambda,x}(r) - f_{\lambda,0}(r)|dr \\
&+ \int_0^\tau |\partial_v \check{L}_\lambda(s, x(s), \dot{x}(s)) - \check{L}_\lambda(s,0,0)|ds + |\partial_v\check{L}_\lambda(t, x(t), \dot{x}(t)) - \partial_v\check{L}_\lambda(t,0,0)|
\end{aligned}$$

and (by integrating this inequality over $[0,\tau]$) hence

$$\begin{aligned}
&2\tau|c_1(\lambda, x) - c_1(\lambda, 0)| \le 2\int_0^\tau |c_1(\lambda, x)e^t - c_1(\lambda, 0)e^t|dt \\
\le\ & 2e^{2\tau}\tau^2 \int_0^\tau |f_{\lambda,x}(r) - f_{\lambda,0}(r)|dr + \tau\int_0^\tau |f_{\lambda,x}(r) - f_{\lambda,0}(r)|dr \\
&+(\tau+1)\int_0^\tau |\partial_v \check{L}_\lambda(s, x(s), \dot{x}(s)) - \partial_v\check{L}_\lambda(s,0,0)|ds,
\end{aligned}$$

that is,

$$\begin{aligned}
|c_1(\lambda, x) - c_1(\lambda, 0)| \ \le\ & (e^{2\tau}\tau + 1)\int_0^\tau |f_{\lambda,x}(r) - f_{\lambda,0}(r)|dr \\
&+\frac{(\tau+1)}{2\tau}\int_0^\tau |\partial_v \check{L}_\lambda(s, x(s), \dot{x}(s)) - \partial_v\check{L}_\lambda(s,0,0)|ds.
\end{aligned} \tag{3.35}$$

Moreover, (3.29) leads to

$$\begin{aligned}
\int_0^\tau |f_{\lambda,x}(t) - f_{\lambda,0}(t)|dt \ \le\ & \int_0^\tau |\partial_q \check{L}_\lambda(t, x(t), \dot{x}(t)) - \partial_q\check{L}_\lambda(t,0,0)|dt \\
&+\tau\int_0^\tau |\partial_v \check{L}_\lambda(t, x(t), \dot{x}(t)) - \partial_v\check{L}_\lambda(t,0,0)|dt
\end{aligned} \tag{3.36}$$

From this and (3.35) we derive

$$|c_1(\lambda, x) - c_1(\lambda, 0)| \le (e^{2\tau}\tau + 1)\int_0^\tau |\partial_q \check{L}_\lambda(t, x(t), \dot{x}(t)) - \partial_q\check{L}_\lambda(t,0,0)|dt$$

$$+\left((e^{2\tau}\tau+1)\tau+\frac{\tau+1}{2\tau}\right)\int_0^\tau|\partial_v\check{L}_\lambda(s,x(s),\dot{x}(s))-\check{L}_\lambda(s,0,0)|ds. \tag{3.37}$$

Similarly, by (3.34) we obtain

$$\begin{aligned}2|c_2(\lambda,x)-c_2(\lambda,0)|\le&\int_0^t e^r|f_{\lambda,x}(r)-f_{\lambda,0}(r)|dr\\&+e^t\int_0^t|\partial_v\check{L}_\lambda(s,x(s),\dot{x}(s))-\partial_v\check{L}_\lambda(s,0,0)|ds+e^t|\partial_v\check{L}_\lambda\left(t,x(t),\dot{x}(t)\right)-\partial_v\check{L}_\lambda(t,0,0)|\end{aligned}$$

and (by integrating this inequality over $[0,\tau]$) so

$$\begin{aligned}2\tau|c_2(\lambda,x)-c_2(\lambda,0)|\le&\ e^\tau\int_0^\tau|\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_q\check{L}_\lambda(t,0,0)|dt\\&+e^\tau(2\tau+1)\int_0^\tau|\partial_v\check{L}_\lambda\left(t,x(t),\dot{x}(t)\right)-\partial_v\check{L}_\lambda(t,0,0)|dt\end{aligned}\tag{3.38}$$

by (3.36).

Note that (L5) of Lemma 3.8 and [37, Prop.B.9] ([35, Proposition C.1]) imply that

$$\begin{aligned}&\int_0^\tau\left|\left[\partial_q\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_q\check{L}_\lambda\big(s,0,0\big)\right]\right|ds\to 0\quad\text{and}\\&\int_0^\tau\left|\left[\partial_v\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_v\check{L}_\lambda\big(s,0,0\big)\right]\right|ds\\&\le\sqrt{\tau}\left(\int_0^\tau\left|\left[\partial_v\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_v\check{L}_\lambda\big(s,0,0\big)\right]\right|^2ds\right)^{1/2}\to 0\end{aligned}$$

uniformly in $\lambda\in\hat{\Lambda}$ as $\|x\|_{1,2}\to 0$. The required claim follows from this and (3.37)-(3.38).

Step 2(*Prove that for any given $\nu'>0$ there exists $\varepsilon'>0$ such that $\nabla\check{\mathcal{E}}_\lambda(x)=0$ and $\|x\|_{1,2}\le\varepsilon'$ imply $\|x\|_{C^1}<\nu'$*).

Since $\nabla\check{\mathcal{E}}_\lambda(x)=0$ and $\nabla\check{\mathcal{E}}_\lambda(0)=0$, by (3.32) we have

$$\begin{aligned}0\ =\ & e^t\int_0^t\left[e^{-2s}\int_0^s e^rf_{\lambda,x}(r)dr\right]ds+e^{-t}\int_0^t e^rf_{\lambda,x}(r)dr\\&+c_1(\lambda,x)e^t-c_2(\lambda,x)e^{-t}+\partial_v\check{L}_\lambda\left(t,x(t),\dot{x}(t)\right).\end{aligned}$$

and

$$\begin{aligned}0\ =\ & e^t\int_0^t\left[e^{-2s}\int_0^s e^rf_{\lambda,0}(r)dr\right]ds+e^{-t}\int_0^t e^rf_{\lambda,0}(r)dr\\&+c_1(\lambda,0)e^t-c_2(\lambda,0)e^{-t}+\partial_v\check{L}_\lambda\left(t,0,0\right).\end{aligned}$$

For each $0\le t\le\tau$, the former minus the latter gives rise to

$$\begin{aligned}&|\partial_v\check{L}_\lambda\left(t,x(t),\dot{x}(t)\right)-\partial_v\check{L}_\lambda\left(t,0,0\right)|\le e^t|c_1(\lambda,x)-c_1(\lambda,0)|+e^{-t}|c_2(\lambda,x)-c_2(\lambda,0)|\\&\quad+e^t\int_0^t\left[e^{-2s}\int_0^s e^r|f_{\lambda,x}(r)-f_{\lambda,0}(r)|dr\right]ds+e^{-t}\int_0^t e^r|f_{\lambda,x}(r)-f_{\lambda,0}(r)|dr\\&\le e^\tau|c_1(\lambda,x)-c_1(\lambda,0)|+|c_2(\lambda,x)-c_2(\lambda,0)|+e^\tau(\tau+1)\int_0^\tau|f_{\lambda,x}(r)-f_{\lambda,0}(r)|dr\end{aligned}$$

and (by (3.36)) hence

$$
\begin{aligned}
|\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_v\check{L}_\lambda(t,0,0)\,|&\le e^\tau|c_1(\lambda,x)-c_1(\lambda,0)|+|c_2(\lambda,x)-c_2(\lambda,0)|\\
&+e^\tau(\tau+1)\int_0^\tau|\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_q\check{L}_\lambda(t,0,0)|dt\\
&+e^\tau(\tau+1)\tau\int_0^\tau|\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_v\check{L}_\lambda(t,0,0)|dt.
\end{aligned}\tag{3.39}
$$

Note that (L3) in Lemma 3.8 and the mean value theorem of integrals may lead to

$$
\check{c}|v|^2\le\int_0^1\big(\partial_{vv}\check{L}_\lambda(t,q,sv)[v],v\big)_{\mathbb{R}^n}\,ds=\big(\partial_v\check{L}_\lambda(t,q,v)-\partial_v\check{L}_\lambda(t,q,0),v\big)_{\mathbb{R}^n}
$$

and so $c|v|\le\big|\partial_v\check{L}_\lambda(t,q,v)-\partial_v\check{L}_\lambda(t,q,0)\big|$ for any $(\lambda,t,q,v)\in\hat{\Lambda}\times[0,\tau]\times B^n_{3\iota/4}(0)\times\mathbb{R}^n$. In particular, for all $t\in[0,\tau]$ we have

$$
\begin{aligned}
\check{c}|\dot{x}(t)|&\le\big|\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_v\check{L}_\lambda(t,x(t),0)\big|\\
&\le\big|\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_v\check{L}_\lambda(t,0,0)\big|+\big|\partial_v\check{L}_\lambda(t,x(t),0)-\partial_v\check{L}_\lambda(t,0,0)\big|.
\end{aligned}
$$

By this and (3.39) we arrive at

$$
\begin{aligned}
\check{c}|\dot{x}(t)|&\le e^\tau|c_1(\lambda,x)-c_1(\lambda,0)|+|c_2(\lambda,x)-c_2(\lambda,0)|+\big|\partial_v\check{L}_\lambda(t,x(t),0)-\partial_v\check{L}_\lambda(t,0,0)\big|\\
&+e^\tau(\tau+1)\int_0^\tau|\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_q\check{L}_\lambda(t,0,0)|dt\\
&+e^\tau(\tau+1)\tau\int_0^\tau|\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_v\check{L}_\lambda(t,0,0)|dt.
\end{aligned}\tag{3.40}
$$

Since $\|x\|_{C^0}\le(\sqrt{\tau}+1/\sqrt{\tau})\|x\|_{1,2}$, as in the final proof of Step 1, the required claim may follow from (3.40) and the conclusion in Step 1.

Step 3(*Complete the proof for the first claim*). Note that x satisfies

$$
\begin{aligned}
0&=\frac{d}{dt}\Big(\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t))\Big)-\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t))\\
&=\partial_{vv}\check{L}_\lambda(t,x(t),\dot{x}(t))\ddot{x}(t)+\partial_{vq}\check{L}_\lambda(t,x(t),\dot{x}(t))\dot{x}(t)+\partial_{vt}\check{L}_\lambda(t,x(t),\dot{x}(t))\\
&\quad-\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t)).
\end{aligned}\tag{3.41}
$$

In particular, taking $x=0$ we get

$$
0=\partial_{vt}\check{L}_\lambda(t,0,0)-\partial_q\check{L}_\lambda(t,0,0).\tag{3.42}
$$

(6.62) minus (6.63) gives rise to

$$
\begin{aligned}
0&=\partial_{vv}\check{L}_\lambda(t,x(t),\dot{x}(t))\ddot{x}(t)+\partial_{vq}\check{L}_\lambda(t,x(t),\dot{x}(t))\dot{x}(t)\\
&\quad+\partial_{vt}\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_{vt}\check{L}_\lambda(t,0,0)\\
&\quad-\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t))+\partial_q\check{L}_\lambda(t,0,0).
\end{aligned}\tag{3.43}
$$

Since (L2) of Lemma 3.8 implies $|[\partial_{vv}\check{L}_\lambda(t,x(t),\dot{x}(t))]^{-1}\xi|\le\frac{1}{\check{c}}|\xi|\ \forall\xi\in\mathbb{R}^n$, (6.64) and (L3) in Lemma 3.8 lead to

$$
|\ddot{x}(t)|\le\frac{1}{\check{c}}|\partial_{vq}\check{L}_\lambda(t,x(t),\dot{x}(t))|\cdot|\dot{x}(t)|+\frac{1}{\check{c}}|\partial_{vt}\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_{vt}\check{L}_\lambda(t,0,0)|
$$

$$
\begin{aligned}
&+\frac{1}{\check{c}}|\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_q\check{L}_\lambda(t,0,0)|\\
\le\ & \frac{\check{C}}{\check{c}}(1+|\dot{x}(t)|)|\cdot|\dot{x}(t)|+\frac{1}{\check{c}}|\partial_{vt}\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_{vt}\check{L}_\lambda(t,0,0)|\\
&+\frac{1}{\check{c}}|\partial_q\check{L}_\lambda(t,x(t),\dot{x}(t))-\partial_q\check{L}_\lambda(t,0,0)|. \qquad (3.44)
\end{aligned}
$$

Recall that $\hat{\Lambda}$ is either compact or sequential compact and that $\partial_q\check{L}_\lambda(t,q,v)$ and $\partial_{vt}\check{L}_\lambda(t,q,v)$ is continuous in (λ,t,q,v) by Lemma 3.8(L0). The desired claim easily follows from (6.65) and the result in Step 2. □

Remark 3.12. (i) The existence and continuity of the partial derivative $\partial_t\check{L}(\cdot)$ in Lemma 3.8(L0) are not used in the proofs of Propositions 3.9, 3.10, 3.11.

(ii) The existence and continuity of the partial derivatives $\partial_{tv}\check{L}(\cdot)$ and $\partial_{vt}\check{L}(\cdot)$ in Lemma 3.8(L0) are not used in the proofs of Propositions 3.9, 3.10; but they are necessary for the proof of Proposition 3.11.

Proof of Theorem 3.5(I). Since there exist a sequence $\{(\lambda_k,x_k)\}_{k\ge1}\subset\Lambda\times C^1_{V_0\times V_1}([0,\tau];B^n_\iota(0))$ converging to $(\mu,0)$ such that each x_k is a nonzero solution of the problem (3.12)–(3.13) with $\lambda=\lambda_k$, i.e., $\nabla\tilde{\mathcal{E}}^*_{\lambda_k}(x_k)=0$, $k=1,2,\cdots$, by (L1) in Lemma 3.8 with $\hat{\Lambda}=\{\mu,\lambda_k\,|\,k\in\mathbb{N}\}$ we deduce that $\nabla\check{\mathcal{E}}_{\lambda_k}(x_k)=0$ for k large enough. Therefore $(\mu,0)$ is a bifurcation point along sequence of $\nabla\check{\mathcal{E}}_\lambda(x)=0$ in $\hat{\Lambda}\times\mathcal{U}$. By (i) and (v)-(vii) of Proposition 3.9 we see that the conditions of [34, Theorem 3.1] ([37, Theorem C.6]) are satisfied with $\mathcal{F}_\lambda=\check{\mathcal{E}}_\lambda$ and $H=X=\mathbf{H}_{V_0\times V_1}$, $U=\mathcal{U}$ and $\lambda^*=\mu$. Then $m^0_\tau(\check{\mathcal{E}}_\mu,0)>0$ and so $m^0(\tilde{\mathcal{E}}^*_\lambda,0)>0$ by (3.26). □

Proof of Theorem 3.7. Since Λ is a real interval and $\mu\in\mathrm{Int}(\Lambda)$ we can take a small $\varepsilon>0$ so that $\hat{\Lambda}:=[\mu-\varepsilon,\mu+\varepsilon]\subset\Lambda$. Propositions 3.9, 3.10 shows that $(\mathcal{U},\mathcal{U}^X,\{\check{\mathcal{E}}_\lambda\,|\,\lambda\in\hat{\Lambda}\})$ satisfies the conditions in [35, Theorem 3.6] with $\lambda^*=\mu$ except for the condition (f). The latter may follow from (3.26) and the assumption, that is, $m^0(\check{\mathcal{E}}_\mu,0)\ne0$ and $m^0(\check{\mathcal{E}}_\lambda,0)=0$ for all $\lambda\in\hat{\Lambda}\setminus\{\mu\}$ near μ, and $m^-(\check{\mathcal{E}}_\lambda,0)$ take, respectively, values $m^-(\check{\mathcal{E}}_\mu,0)$ and $m^-(\check{\mathcal{E}}_\mu,0)+m^0(\check{\mathcal{E}}_\mu,0)$ as $\lambda\in\hat{\Lambda}$ varies in two deleted half neighborhoods of μ. Therefore from [37, Theorem C.7] ([36, Theorem 3.6]) we deduce that one of the following occurs:

(i) $\nabla\check{\mathcal{E}}_\mu$ has a sequence of nontrivial zero points converging to 0 in $\mathcal{U}$.

(ii) For every $\lambda\in\hat{\Lambda}\setminus\{\mu\}$ near μ, A_λ has a zero point $y_\lambda\ne0$, which converge to zero in $\mathcal{U}^X$ as $\lambda\to\mu$.

(iii) For a given neighborhood $\mathfrak{M}$ of $0\in\mathcal{U}^X$, there is an one-sided neighborhood Λ^0 of μ in $\hat{\Lambda}$ (therefore in Λ) such that for any $\lambda\in\Lambda^0\setminus\{\mu\}$, A_λ has at least two distinct nontrivial zero points in $\mathfrak{M}$, y^1_λ and y^2_λ, which can also be required to satisfy $\check{\mathcal{E}}_\lambda(y^1_\lambda)\ne\check{\mathcal{E}}_\lambda(y^2_\lambda)$ provided that $m^0(\check{\mathcal{E}}_\mu,0)>1$ and A_λ has only finitely many nontrivial zero points in $\mathfrak{M}$.

As above the required results may follow from Proposition 3.11 and (L0) in Lemma 3.8. □

In order to prove Theorem 3.5(II) and Theorem 3.6, noting that because of Propositions 3.9, 3.10, (specially Proposition 3.9(iv) implies that $\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}\in C^2(\mathcal{U}^X,\mathbb{R})$ and B_λ is continuous as a map from $\mathcal{U}^X$ to $\mathcal{L}_s(\mathbf{H}_{V_0\times V_1})$ because $A'_\lambda=B_\lambda$), we may, respectively, apply [37, Theorem C.4] and [37, Theorem C.5] to $(\mathcal{U},\mathcal{U}^X,\{\check{\mathcal{E}}_\lambda\,|\,\lambda\in\hat{\Lambda}\})$ to obtain (I) and (II) of the following theorem.

Theorem 3.13. (I) (Sufficient condition): *Suppose that $\hat{\Lambda}$ is first countable and that there exist two sequences in Λ converging to μ, (λ_k^-) and (λ_k^+), such that one of the following conditions is satisfied:*

(I.1) *For each $k \in \mathbb{N}$, either 0 is not an isolated critical point of $\check{\mathcal{E}}_{\lambda_k^+}$, or 0 is not an isolated critical point of $\check{\mathcal{E}}_{\lambda_k^-}$, or 0 is an isolated critical point of $\check{\mathcal{E}}_{\lambda_k^+}$ and $\check{\mathcal{E}}_{\lambda_k^-}$ and $C_m(\check{\mathcal{E}}_{\lambda_k^+}, 0; \mathbf{K})$ and $C_m(\check{\mathcal{E}}_{\lambda_k^-}, 0; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$. Moreover, in the third case, "$C_m(\check{\mathcal{E}}_{\lambda_k^+}, 0; \mathbf{K})$ and $C_m(\check{\mathcal{E}}_{\lambda_k^-}, 0; \mathbf{K})$" may be replaced by "$C_*(\check{\mathcal{E}}_{\lambda_k^+}|_{\mathcal{U}^X}, 0; \mathbf{K})$ and $C_*(\check{\mathcal{E}}_{\lambda_k^-}|_{\mathcal{U}^X}, 0; \mathbf{K})$".*

(I.2) *For each $k \in \mathbb{N}$, $[m^-(\check{\mathcal{E}}_{\lambda_k^+}, 0), m^-(\check{\mathcal{E}}_{\lambda_k^+}, 0) + m^0(\check{\mathcal{E}}_{\lambda_k^+}, 0)] \cap [m^-(\check{\mathcal{E}}_{\lambda_k^-}, 0), m^-(\check{\mathcal{E}}_{\lambda_k^-}, 0) + m^0(\check{\mathcal{E}}_{\lambda_k^-}, 0)] = \emptyset$, and there exists $\lambda \in \{\lambda_k^+, \lambda_k^-\}$ such that 0 is an either nonisolated or homological visible critical point of $\check{\mathcal{E}}_\lambda$. Moreover, $\check{\mathcal{E}}_\lambda$ can be replaced by $\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}$ in the second condition.*

(I.3) *For each $k \in \mathbb{N}$, $[m^-(\check{\mathcal{E}}_{\lambda_k^+}, 0), m^-(\check{\mathcal{E}}_{\lambda_k^+}, 0) + m^0(\check{\mathcal{E}}_{\lambda_k^+}, 0)] \cap [m^-(\check{\mathcal{E}}_{\lambda_k^-}, 0), m^-(\check{\mathcal{E}}_{\lambda_k^-}, 0) + m^0(\check{\mathcal{E}}_{\lambda_k^-}, 0)] = \emptyset$, and either $m^0(\check{\mathcal{E}}_{\lambda_k^+}, 0) = 0$ or $m^0(\check{\mathcal{E}}_{\lambda_k^-}, 0) = 0$.*

Then there exists a sequence $\{(\lambda_k, x_k)\}_{k\ge1}$ in $\check{\Lambda} \times (\mathcal{U} \cap C^2([0,\tau], \mathbb{R}^n))$ such that $\lambda_k \to \mu$, $0 < \|x_k\|_{C^2} \to 0$ and $\nabla\check{\mathcal{E}}_\lambda(x_k) = 0$ for $k = 1, 2, \cdots$, where $\check{\Lambda} = \{\mu, \lambda_k^+, \lambda_k^- \,|\, k \in \mathbb{N}\}$. In particular, $(\mu, 0)$ is a bifurcation point of the problem $\nabla\check{\mathcal{E}}_\lambda(x) = 0$ in $\check{\Lambda} \times C^2([0,\tau], \mathbb{R}^n)$ with respect to the branch $\{(\lambda, 0) \,|\, \lambda \in \check{\Lambda}\}$ (and so $\{(\lambda, 0) \,|\, \lambda \in \hat{\Lambda}\}$).

(II) (Existence for bifurcations): *For λ^-, λ^+ in a path-connected component of $\hat{\Lambda}$ suppose that one of the following conditions is satisfied:*

(II.1) *Either 0 is not an isolated critical point of $\check{\mathcal{E}}_{\lambda^+}$, or 0 is not an isolated critical point of $\check{\mathcal{E}}_{\lambda^-}$, or 0 is an isolated critical point of $\check{\mathcal{E}}_{\lambda^+}$ and $\check{\mathcal{E}}_{\lambda^-}$ and $C_m(\check{\mathcal{E}}_{\lambda^+}, 0; \mathbf{K})$ and $C_m(\check{\mathcal{E}}_{\lambda^-}, 0; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$. Moreover, in the final case, "$C_m(\check{\mathcal{E}}_{\lambda^+}, 0; \mathbf{K})$ and $C_m(\check{\mathcal{E}}_{\lambda^-}, 0; \mathbf{K})$" may be replaced by "$C_*(\check{\mathcal{E}}_{\lambda^+}|_{\mathcal{U}^X}, 0; \mathbf{K})$ and $C_*(\check{\mathcal{E}}_{\lambda^-}|_{\mathcal{U}^X}, 0; \mathbf{K})$".*

(II.2) *$[m^-(\check{\mathcal{E}}_{\lambda^+}, 0), m^-(\check{\mathcal{E}}_{\lambda^+}, 0) + m^0(\check{\mathcal{E}}_{\lambda^+}, 0)] \cap [m^-(\check{\mathcal{E}}_{\lambda^-}, 0), m^-(\check{\mathcal{E}}_{\lambda^-}, 0) + m^0(\check{\mathcal{E}}_{\lambda^-}, 0)] = \emptyset$, and there exists $\lambda \in \{\lambda^+, \lambda^-\}$ such that 0 is an either nonisolated or homological visible critical point of $\check{\mathcal{E}}_\lambda$. Moreover, in the second condition, $\check{\mathcal{E}}_\lambda$ can be replaced by $\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}$.*

(II.3) *$[m^-(\check{\mathcal{E}}_{\lambda^+}, 0), m^-(\check{\mathcal{E}}_{\lambda^+}, 0) + m^0(\check{\mathcal{E}}_{\lambda^+}, 0)] \cap [m^-(\check{\mathcal{E}}_{\lambda^-}, 0), m^-(\check{\mathcal{E}}_{\lambda^-}, 0) + m^0(\check{\mathcal{E}}_{\lambda^-}, 0)] = \emptyset$, and either $m^0(\check{\mathcal{E}}_{\lambda^+}, 0) = 0$ or $m^0(\check{\mathcal{E}}_{\lambda^-}, 0) = 0$.*

Then for any path $\alpha : [0,1] \to \hat{\Lambda}$ connecting λ^+ to λ^- there exists a sequence $\{(t_k, x_k)\}_{k\ge1}$ in $[0,1] \times \mathcal{U}$ converging to $(\bar{t}, 0)$ for some $\bar{t} \in [0,1]$, such that each x_k is a nonzero solution of $\nabla\check{\mathcal{E}}_{\alpha(t_k)}(x) = 0$, $k = 1, 2, \cdots$. (In fact, $\|x_k\|_{C^2} \to 0$ by Proposition 3.11.) Moreover, $\alpha(\bar{t})$ is not equal to λ^+ (resp. λ^-) if $m^0(\check{\mathcal{E}}_{\lambda^+}, 0) = 0$ (resp. $m^0(\check{\mathcal{E}}_{\lambda^-}, 0) = 0$).

Proof. Step 1(*Prove (I)*). Because of Propositions 3.9, 3.10, (specially Proposition 3.9(iv) implies that $\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X} \in C^2(\mathcal{U}^X, \mathbb{R})$ and B_λ is continuous as a map from $\mathcal{U}^X$ to $\mathcal{L}_s(\mathbf{H}_{V_0\times V_1})$ since $A'_\lambda = B_\lambda$), for the case (I.2) [resp. (I.3)] we apply [37, Theorem C.4(B.1),(B.2)] (resp. [37, Theorem C.4(B.3)]) to obtain:

($*$) There exists a sequence $\{(\lambda_k, x_k)\}_{k\ge1} \subset \hat{\Lambda} \times \mathcal{U} \setminus \{(\mu, 0)\}$ converging to $(\mu, 0)$ such that $x_k \neq 0$ and $\nabla\check{\mathcal{E}}_\lambda(x_k) = 0$ for $k = 1, 2, \cdots$.

For the case (I.1), if $\{\check{\mathcal{E}}_{\lambda_k^+}\}_{k\ge1}$ or $\{\check{\mathcal{E}}_{\lambda_k^-}\}_{k\ge1}$ has a subsequence such that each term of it has 0 as a non-isolated critical point, then we have $(*)$ naturally. Otherwise, for each sufficiently large integer l, 0 is an isolated critical point of $\check{\mathcal{E}}_{\lambda_l^+}$ and $\check{\mathcal{E}}_{\lambda_l^-}$ and $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^+}, 0; \mathbf{K}_l)$ and $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^-}, 0; \mathbf{K}_l)$ are not isomorphic for some Abel group $\mathbf{K}_l$ and some $m_l \in \mathbb{Z}$, which implies that for each such an integer l, 0 is an isolated critical point of $\check{\mathcal{E}}_{\lambda_l^+}|_{\mathcal{U}^X}$ and $\check{\mathcal{E}}_{\lambda_l^-}|_{\mathcal{U}^X}$ and $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^+}|_{\mathcal{U}^X}, 0; \mathbf{K}_l)$ and $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^-}|_{\mathcal{U}^X}, 0; \mathbf{K}_l)$ are not isomorphic because by Proposition 3.9 we may use [21, Corollary 2.8] to deduce that $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^+}|_{\mathcal{U}^X}, 0; \mathbf{K}_l) \cong C_{m_l}(\check{\mathcal{E}}_{\lambda_l^+}, 0; \mathbf{K}_l)$ and $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^-}|_{\mathcal{U}^X}, 0; \mathbf{K}_l) \cong C_{m_l}(\check{\mathcal{E}}_{\lambda_l^-}, 0; \mathbf{K}_l)$. Then we may use [37, Theorem C.4(A)] to get a contradiction provided that $(*)$ is not true.

In summary, we get $(*)$ in this case, and therefore the desired statements by Proposition 3.11.

Step 2(*Prove (II)*). Applying [37, Theorem C.5] to $(\mathcal{U}, \mathcal{U}^X, \{\check{\mathcal{E}}_\lambda \,|\, \lambda \in \hat{\Lambda}\})$ a similar proof to that of Step 1 yields the required results. $\square$

Proof of Theorem 3.5(II). Suppose that (II.1) is satisfied. If $\{\tilde{\mathcal{E}}^*_{\lambda_k^+}\}_{k\ge1}$ or $\{\tilde{\mathcal{E}}^*_{\lambda_k^-}\}_{k\ge1}$ has a subsequence such that each term of it has 0 as a non-isolated critical point, then the required result may follow from (3.25) and Proposition 3.11. Otherwise, for each sufficiently large integer l, 0 is an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda_l^+}$ and $\tilde{\mathcal{E}}^*_{\lambda_l^-}$ and $C_{m_l}(\tilde{\mathcal{E}}^*_{\lambda_l^+}, 0; \mathbf{K}_l)$ and $C_{m_l}(\tilde{\mathcal{E}}^*_{\lambda_l^-}, 0; \mathbf{K}_l)$ are not isomorphic for some Abel group $\mathbf{K}_l$ and some $m_l \in \mathbb{Z}$, which implies by (3.25) that 0 is an isolated critical point of $\check{\mathcal{E}}_{\lambda_l^+}|_{\mathcal{U}^X}$ and $\check{\mathcal{E}}_{\lambda_l^-}|_{\mathcal{U}^X}$ and $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^+}|_{\mathcal{U}^X}, 0; \mathbf{K}_l)$ and $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^-}|_{\mathcal{U}^X}, 0; \mathbf{K}_l)$ are not isomorphic. That is, the condition (I.1) in Theorem 3.13 is satisfied. Hence (3.25) and the conclusion in Theorem 3.13(I) lead to the required result.

Next, let (II.2) be satisfied. If the statements in the second sentence in last paragraph are true we are done. Otherwise, for each sufficiently large integer l, 0 is an isolated critical point of $\tilde{\mathcal{E}}^*_{\lambda_l^+}$ and $\tilde{\mathcal{E}}^*_{\lambda_l^-}$ and either $C_{m_l}(\tilde{\mathcal{E}}^*_{\lambda^+}, 0; \mathbf{K}_l) \ne 0$ for some Abel group $\mathbf{K}_l$ and some $m_l \in \mathbb{Z}$ or and $C_{n_l}(\tilde{\mathcal{E}}^*_{\lambda_l^-}, 0; \mathbf{K}'_l) \ne 0$ for some Abel group $\mathbf{K}'_l$ and some $n_l \in \mathbb{Z}$. Therefore by (3.25) 0 is an isolated critical point of $\check{\mathcal{E}}_{\lambda_l^+}|_{\mathcal{U}^X}$ and $\check{\mathcal{E}}_{\lambda_l^-}|_{\mathcal{U}^X}$ and either $C_{m_l}(\check{\mathcal{E}}_{\lambda_l^+}|_{\mathcal{U}^X}, 0; \mathbf{K}_l) \ne 0$ or $C_{n_l}(\check{\mathcal{E}}_{\lambda_l^-}|_{\mathcal{U}^X}, 0; \mathbf{K}'_l) \ne 0$. These mean that the condition (I.2) in Theorem 3.13 is satisfied. As above, Theorem 3.13(I) and (3.25) yield the desired conclusions.

For the case (I.3), by (3.26) we see that the condition (I.3) in Theorem 3.13 is satisfied. The required statements are derived as above. $\square$

Proof of Theorem 3.6. For the case (i) in Theorem 3.6. The first two cases easily follow from Proposition 3.11 and (3.25). For the third case, 0 is also an isolated critical point of $\check{\mathcal{E}}_{\lambda^+}$ and $\check{\mathcal{E}}_{\lambda^-}$ by Proposition 3.11 and (3.25), and $C_m(\check{\mathcal{E}}_{\lambda^+}|_{\mathcal{U}^X}, 0; \mathbf{K})$ and $C_m(\check{\mathcal{E}}_{\lambda^-}|_{\mathcal{U}^X}, 0; \mathbf{K})$ are not isomorphic. Theorem 3.13(II) leads to the required results.

For the case (ii) in Theorem 3.6, by (3.25) "0 is an either nonisolated or homological visible critical point of $\mathcal{E}^*_\lambda$" is equivalent to "0 is an either nonisolated or homological visible critical point of $\check{\mathcal{E}}_{\lambda^+}|_{\mathcal{U}^X}$". Because of these and (3.26), Theorem 3.13(II) yields the required results.

The case (iii) in Theorem 3.6 follows from (3.26) and Theorem 3.13(II). $\square$

3.1.4 Proof of Theorem 1.5

By contradiction suppose that there exists a path $\alpha : [0,1] \to \Lambda$ connecting λ^+ to λ^- such that each point $(\alpha(s), \gamma_{\alpha(s)})$, $s \in [0,1]$, is not a bifurcation point of the problem (1.5)–(1.6) in $\alpha([0,1]) \times C^2_{S_0\times S_1}([0,\tau]; M)$ with respect to the branch $\{(\lambda, \gamma_\lambda) \,|\, \lambda \in \alpha([0,1])\}$. Then for some small $\epsilon > 0$ we have:

$$\left.\begin{array}{l}\gamma \in C^2_{\mathbb{I}_g}([0,\tau];M) \text{ satisfies } \|\gamma - \gamma_{\alpha(s)}\|_{C^2([0,\tau];\mathbb{R}^N)} \le \epsilon \\ \text{and (1.5)–(1.6) with } \lambda = \alpha(s) \text{ for some } s \in [0,1] \\ \Longrightarrow\ \gamma = \gamma_\lambda \text{ for some } \lambda \in \alpha([0,1]).\end{array}\right\} \tag{3.45}$$

Fix a point $\mu \in \alpha([0,1])$. Let $\overline{\gamma}$ be as in (3.2). We have a compact neighborhood $\hat{\Lambda}$ of μ in $\alpha([0,1])$ such that (3.3) is satisfied for all $(\lambda,t) \in \hat{\Lambda} \times [0,\tau]$, i.e.,

$$\mathrm{dist}_g(\gamma_\lambda(t), \overline{\gamma}(t)) < \iota, \quad \forall (\lambda,t) \in \hat{\Lambda} \times [0,\tau]. \tag{3.46}$$

Then the reduction in Section 3.1.1 is valid after we use $\hat{\Lambda}$ to replace Λ therein. Therefore for the functionals in (3.9), (3.45) implies that for some $\bar{\epsilon} > 0$ we have

$$\left.\begin{array}{l}x \in C^1_{V_0\times V_1}([0,\tau]; B^n_\iota(0)) \text{ satisfies } \|x\|_{C^2([0,\tau];\mathbb{R}^N)} \le \bar{\epsilon} \\ \text{and } d\tilde{\mathcal{E}}^*_\lambda(x) = 0 \text{ for some } \lambda \in \hat{\Lambda} \quad \Longrightarrow\ x = 0.\end{array}\right\} \tag{3.47}$$

It follows from this fact, (3.25) and Proposition 3.11 that there exists $\hat{\epsilon} > 0$ such that

$$\left.\begin{array}{l}x \in \mathcal{U} \text{ satisfies } \|x\|_{1,2} \le \hat{\epsilon} \text{ and} \\ d\check{\mathcal{E}}_\lambda(x) = 0 \text{ for some } \lambda \in \hat{\Lambda} \quad \Longrightarrow\ x = 0.\end{array}\right\} \tag{3.48}$$

Since we can shrink $\hat{\epsilon} > 0$ so that the ball $\bar{B}_{\hat{\epsilon}}(\mathbf{H}_{V_0\times V_1}) := \{\xi \in \mathbf{H}_{V_0\times V_1} \mid \|\xi\|_{1,2} \le \hat{\epsilon}\}$ is contained in $\mathcal{U}$, (3.48) means that $0 \in \mathcal{U}$ is a unique critical point of $\check{\mathcal{E}}_\lambda$ in $\bar{B}_{\hat{\epsilon}}(\mathbf{H}_{V_0\times V_1})$ for each $\lambda \in \hat{\Lambda}$.

Take $\bar{s} \in [0,1]$ such that $\alpha(\bar{s}) = \mu$. We have a connected compact neighborhood $N(\bar{s})$ of $\bar{s}$ in $[0,1]$ such that $\alpha(N(\bar{s})) \subset \hat{\Lambda}$. Because of (3.48), as in the proof of [37, Theorem C.5] we have a correspondent result of [37, (C.17)] and therefore obtain that for any Abel group $\mathbf{K}$ and any $s, s' \in N(\bar{s})$,

$$C_q(\check{\mathcal{E}}_{\alpha(s)}, 0; \mathbf{K}) \cong C_q(\check{\mathcal{E}}_{\alpha(s')}, 0; \mathbf{K}), \ \ \forall q \in \mathbb{N} \cup \{0\}. \tag{3.49}$$

As in the previous proofs we may use [21, Corollary 2.8] to deduce that

$$C_q(\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}, 0; \mathbf{K}) \cong C_q(\check{\mathcal{E}}_\lambda, 0; \mathbf{K}), \quad \forall \lambda \in \hat{\Lambda}.$$

This, (3.25) and (3.10) lead to

$$C_q(\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}, 0; \mathbf{K}) \cong C_q(\tilde{\mathcal{E}}^*_\lambda, 0; \mathbf{K}) \cong C_q(\mathcal{E}_\lambda, \gamma_\lambda; \mathbf{K}), \quad \forall \lambda \in \hat{\Lambda}.$$

and hence

$$C_q(\mathcal{E}_{\alpha(s)}, \gamma_{\alpha(s)}; \mathbf{K}) \cong C_q(\mathcal{E}_{\alpha(s')}, \gamma_{\alpha(s')}; \mathbf{K}), \ \ \forall s, s' \in N(\bar{s}), \ \forall q \in \mathbb{N} \cup \{0\}. \tag{3.50}$$

Because the point $\mu \in \alpha([0,1])$ is arbitrary, (3.50) implies

$$C_q(\mathcal{E}_\lambda, \gamma_\lambda; \mathbf{K}) \cong C_q(\mathcal{E}_{\lambda'}, \gamma_{\lambda'}; \mathbf{K}), \ \ \forall \lambda, \lambda' \in \alpha([0,1]), \ \forall q \in \mathbb{N} \cup \{0\}. \tag{3.51}$$

Almost repeating the arguments below (C.18) in the proof of [37, Theorem C.5] we may see that (3.51) contradicts to each of the conditions (i)-(iii) in Theorem 1.5. Hence the assumption above (3.45) is not true. Theorem 1.5 is proved. □

3.2 Proof of Theorem 1.9

We first admit the following version of the Morse index theorem due to Duistermaat [11], which can directly lead to the Morse index theorem in Finsler geometry (Corollary 9.7).

Theorem 3.14. *Let the functionals $\mathcal{L}_{S_0,s}$ be as in (1.8). Then*

$$m^-(\mathcal{L}_{S_0,\lambda},\gamma_\lambda)=\sum_{0<s<\lambda} m^0(\mathcal{L}_{S_0,s},\gamma_s),\quad \forall\lambda\in(0,\tau].$$

Theorem 3.14 implies that there only exist finitely many $\mu\in(0,\tau)$ where $m^0(\mathcal{L}_{S_0,\mu},\gamma_\mu)\neq 0$, and hence the conclusion (i) follows.

Proof of Theorem 1.9. In the arguments above (7.9) taking $\gamma_\mu=\gamma$ and $S_1=\{\gamma(\tau)\}$ we have a unique map

$$\mathbf{u}\in C^1_{V_0\times\{0\}}([0,\tau];B^n_\iota(0))\cap C^3([0,\tau];B^n_\iota(0))$$

such that $\mathbf{u}(0)=\mathbf{u}(\tau)=0$ and $\gamma(t)=\phi_{\overline{\gamma}}(t,\mathbf{u}(t))$ for all $t\in[0,\tau]$. For $\lambda\in(0,\tau]$ let

$$W^{1,2}_{V_0\times\{\mathbf{u}(\lambda)\}}([0,\lambda];B^n_{2\iota}(0))=\left\{u\in W^{1,2}_{V_0\times\{\mathbf{u}(\lambda)\}}([0,\lambda];\mathbb{R}^n)\,\middle|\,u([0,\lambda])\subset B^n_{2\iota}(0)\right\},$$

which contains $C^1_{V_0\times\{\mathbf{u}(\lambda)\}}([0,\lambda];B^n_{2\iota}(0))$ as a dense subset. As before using [48, Theorems 4.2, 4.3] we deduce that

$$W^{1,2}_{S_0\times\{\gamma(\lambda)\}}([0,\lambda];M):=\left\{\alpha\in W^{1,2}([0,\lambda];M)\,\middle|\,\alpha(0)\in S_0,\alpha(\lambda)=\gamma(\lambda)\right\}$$

is a C^4 Hilbert manifold and obtain a C^2 chart

$$\Phi_\lambda: W^{1,2}_{V_0\times\{0\}}([0,\lambda];B^n_\iota(0))\to W^{1,2}_{S_0\times\{\gamma(\lambda)\}}([0,\lambda];M)$$

given by $\Phi_\lambda(\xi)(t)=\phi_{\overline{\gamma}}(t,\mathbf{u}_\lambda(t)+\xi(t))$ $\forall t\in[0,\lambda]$ for each $\lambda\in(0,\tau]$, where $\mathbf{u}_\lambda:=\mathbf{u}|_{[0,\lambda]}$. Then $\Phi_\lambda(0)=\gamma_\lambda:=\gamma|_{[0,\lambda]}$. Note that $\Phi_\lambda: C^1_{V_0\times\{0\}}([0,\lambda];B^n_\iota(0))\to C^1_{S_0\times\{\gamma(\lambda)\}}([0,\lambda];M)$ is also a C^2 chart by the ω-Lemma. Clearly, the Banach space isomorphism

$$\Gamma_\lambda: W^{1,2}_{V_0\times\{0\}}([0,1];\mathbb{R}^n)\to W^{1,2}_{V_0\times\{0\}}([0,\lambda];\mathbb{R}^n),\ \xi\mapsto\xi(\lambda^{-1}\cdot)$$

maps $W^{1,2}_{V_0\times\{0\}}([0,1];B^n_\iota(0))$ onto $W^{1,2}_{V_0\times\{0\}}([0,\lambda];B^n_\iota(0))$. Put

$$\mathbf{L}_\lambda: W^{1,2}_{V_0\times\{0\}}([0,1];B^n_\iota(0))\to\mathbb{R},\ \xi\mapsto\mathcal{L}_{S_0,\lambda}\circ\Phi_\lambda\circ\Gamma_\lambda(\xi).$$

It is easy to check that

$$\begin{aligned}\mathbf{L}_\lambda(\xi)&=\int_0^\lambda L\left(t,\Phi_\lambda(\Gamma_\lambda(\xi))(t),\frac{d}{dt}\Phi_\lambda(\Gamma_\lambda(\xi))(t)\right)dt\\&=\int_0^1\lambda L\left(\lambda s,\Phi_\lambda(\Gamma_\lambda(\xi))(\lambda s),\frac{d}{dt}\Phi_\lambda(\Gamma_\lambda(\xi))(t)\Big|_{t=\lambda s}\right)ds.\end{aligned}\tag{3.52}$$

Note that $\Phi_\lambda(\Gamma_\lambda(\xi))(t)=\phi_{\overline{\gamma}}(t,\mathbf{u}_\lambda(t)+\Gamma_\lambda(\xi)(t))=\phi_{\overline{\gamma}}(t,\mathbf{u}_\lambda(t)+\xi(t/\lambda))$ for all $t\in[0,\lambda]$. We have

$$\frac{d}{dt}\Phi_\lambda(\Gamma_\lambda(\xi))(t)=D_1\phi_{\overline{\gamma}}(t,\mathbf{u}_\lambda(t)+\xi(t/\lambda))+D_2\phi_{\overline{\gamma}}(t,\mathbf{u}_\lambda(t)+\xi(t/\lambda))[\dot{\mathbf{u}}_\lambda(t)+\frac{1}{\lambda}\dot{\xi}(t/\lambda)].$$

It follows that for any $s \in [0,1]$,

$$\Phi_\lambda(\Gamma_\lambda(\xi))(\lambda s) = \phi_{\overline{\gamma}}(\lambda s, \mathbf{u}(\lambda s) + \xi(s)),$$
$$\frac{d}{dt}\Phi_\lambda(\Gamma_\lambda(\xi))(t)\Big|_{t=\lambda s} = D_1\phi_{\overline{\gamma}}(\lambda s, \mathbf{u}(\lambda s) + \xi(s)) + D_2\phi_{\overline{\gamma}}(\lambda s, \mathbf{u}(\lambda s) + \xi(s))[\dot{\mathbf{u}}(\lambda s) + \frac{1}{\lambda}\dot{\xi}(s)].$$

Define $\hat{L} : (0,\tau) \times [0,1] \times B^n_\iota(0) \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}&\hat{L}(\lambda, s, q, v) = \hat{L}_\lambda(s, q, v)\\ = \quad & \lambda L\left(\lambda s, \phi_{\overline{\gamma}}(\lambda s, \mathbf{u}(\lambda s) + q), D_1\phi_{\overline{\gamma}}(\lambda s, \mathbf{u}(\lambda s) + q) + \frac{1}{\lambda}D_2\phi_{\overline{\gamma}}(\lambda s, \mathbf{u}(\lambda s) + q)[v + \lambda\dot{\mathbf{u}}(\lambda s)]\right).\end{aligned} \tag{3.53}$$

Since both $\mathbf{u}$ and L are C^3, $\hat{L}$ is C^2 and fiberwise strictly convex. Clearly, (3.52) becomes

$$\mathbf{L}_\lambda(\xi) = \int_0^1 \hat{L}_\lambda\left(t, \xi(t), \dot{\xi}(t)\right) dt. \tag{3.54}$$

Let $\gamma_\lambda := \gamma|_{[0,\lambda]}$. Then $\Phi_\lambda \circ \Gamma_\lambda(0) = \gamma_\lambda$ and

$$d\mathbf{L}_\lambda(0) = d\mathcal{L}_{S_0,\lambda}(\gamma_\lambda) \circ d(\Phi_\lambda \circ \Gamma_\lambda)(0) = 0,$$
$$d^2\mathbf{L}_\lambda(0)[\xi, \eta] = d^2\mathcal{L}_{S_0,\lambda}(\gamma_\lambda)\big[d\Phi_\lambda(0)[\Gamma_\lambda(\xi)], d\Phi_\lambda(0)[\Gamma_\lambda(\eta)]\big], \quad \forall \xi, \eta \in W^{1,2}_{V_0\times\{0\}}([0,1];\mathbb{R}^n).$$

Let $m^-(\mathbf{L}_\lambda, 0)$ and $m^0(\mathbf{L}_\lambda, 0)$ be the Morse index and nullity of $\mathbf{L}_\lambda$ at 0, respectively. Then

$$m^-(\mathbf{L}_\lambda, 0) = m^-(\mathcal{L}_{S_0,\lambda}, \gamma_\lambda) \quad \text{and} \quad m^0(\mathbf{L}_\lambda, 0) = m^0(\mathcal{L}_{S_0,\lambda}, \gamma_\lambda) \tag{3.55}$$

because $d\Phi_\lambda(0) \circ \Gamma_\lambda\left(C^1_{V_0\times\{0\}}([0,1];\mathbb{R}^n)\right)$ is equal to

$$C^1_{S_0\times\{\gamma_\lambda(\lambda)\}}(\gamma_\lambda^*TM) := \left\{X \in C^1(\gamma_\lambda^*TM) \,\middle|\, X(0) \in T_{\gamma_\lambda(0)}S_0 = T_{\gamma(0)}S_0,\ X(\lambda) = 0\right\}.$$

Claim 3.15. *For $\mu \in (0,\tau]$, μ is a bifurcation instant for (S_0, γ) if and only if $(\mu, 0) \in (0,\tau] \times C^1_{V_0\times\{0\}}([0,1]; B^n_\iota(0))$ is a bifurcation point of the problem*

$$\frac{d}{dt}\left(\partial_v\hat{L}_\lambda(t, x(t), \dot{x}(t))\right) - \partial_x\hat{L}_\lambda(t, x(t), \dot{x}(t)) = 0, \tag{3.56}$$

$$\left.\begin{array}{l} x \in C^2([0,1]; B^n_\iota(0)),\ (x(0), x(1)) \in V_0 \times \{0\} \quad \textit{and} \\ \partial_v\hat{L}_\lambda(0, x(0), \dot{x}(0))[v_0] = 0\ \forall v_0 \in V_0. \end{array}\right\} \tag{3.57}$$

with respect to the trivial branch $\{(\lambda, 0) \,|\, \lambda \in (0,\tau]\}$ in $(0,\tau] \times C^1_{V_0\times\{0\}}([0,1]; B^n_\iota(0))$.

Proof. By Definition 1.8 a real $\mu \in (0,\tau]$ is a bifurcation instant for (S_0, γ) if and only if there exists a sequence $(\lambda_k) \subset (0,\tau]$ converging to μ and a sequence of Euler-Lagrange curves of L emanating perpendicularly from S_0, $\gamma^k : [0,\lambda_k] \to M$, such that (1.10) and (1.11) are satisfied, i.e.,

$$\gamma^k(t_k) = \gamma(\lambda_k) \text{ for all } k \in \mathbb{N}, \quad 0 < \|\gamma^k - \gamma|_{[0,\lambda_k]}\|_{C^1([0,t_k],\mathbb{R}^N)} \to 0 \ \text{ as } k \to \infty.$$

Since $\lambda_k \to \mu$, from the latter we deduce that for each k large enough each γ^k sits in the image of the map $\Phi_{\lambda_k} : C^1_{V_0\times\{0\}}([0,\lambda_k]; B^n_\iota(0)) \to C^1_{S_0\times\{\gamma(\lambda_k)\}}([0,\lambda_k]; M)$ and therefore there exists a unique $\mathbf{u}^k \in C^1_{V_0\times\{0\}}([0,\lambda_k]; B^n_\iota(0))$ such that $\Phi_{\lambda_k}(\mathbf{u}^k) = \gamma^k \in C^1_{S_0\times\{\gamma(\lambda_k)\}}([0,\lambda_k]; M)$.

Let $\mathbf{v}^k := (\Gamma_{\lambda_k})^{-1}(\mathbf{u}^k) \in C^1_{V_0\times\{0\}}([0,1]; B^n_\iota(0))$. It follows from $\Phi_{\lambda_k}(0) = \gamma|_{[0,\lambda_k]}$ that $0 < \|\mathbf{u}^k\|_{C^1([0,\lambda_k],\mathbb{R}^n)} \to 0$ and so $0 < \|\mathbf{v}^k\|_{C^1([0,1],\mathbb{R}^n)} \to 0$ as $k \to \infty$. Moreover, $\Phi_{\lambda_k} \circ \Gamma_{\lambda_k}(\mathbf{v}^k) = \gamma|_{[0,\lambda_k]}$ implies

$$d\mathbf{L}_{\lambda_k}(\mathbf{v}^k) = d\mathcal{L}_{S_0,\lambda_k}\left(\Phi_{\lambda_k} \circ \Gamma_{\lambda_k}(\mathbf{v}^k)\right) \circ d(\Phi_{\lambda_k} \circ \Gamma_{\lambda_k})(\mathbf{v}^k) = 0.$$

These affirm the necessary.

Carefully checking the above arguments it is easily seen that the sufficiency also holds. □

Suppose that $\mu \in (0,\tau]$ is a bifurcation instant for (S_0,γ). Then it follows from Claim 3.15 and Theorem 1.4 (or Theorem 3.5) that $m^0(\mathbf{L}_\mu, 0) > 0$ and therefore $m^0(\mathcal{L}_{S_0,\mu}, \gamma_\mu) > 0$ by (3.55). That is, μ is a S_0-focal point along γ. (ii) is proved.

Finally, let us prove (iii). Suppose that $\mu \in (0,\tau)$ is a S_0-focal point along γ. By (i) or Theorem 3.14 there only exist finitely many numbers in $(0,\tau)$, $0 < \mu_1 < \cdots < \mu_m < \tau$, such that $m^0(\mathcal{L}_{S_0,\mu_i}, \gamma_{\mu_i}) > 0$, $i = 1,\cdots,m$. Then $\mu \in \{\mu_1,\cdots,\mu_m\}$. Then by Theorem 3.14 and (3.55) we obtain that

$$\begin{aligned}
&m^0(\mathbf{L}_{\mu_i}, 0) \neq 0,\ i = 1,\cdots,m \quad \text{and} \quad m^0(\mathbf{L}_\lambda, 0) = 0 \quad \text{for} \quad \lambda \in (0,\tau)\setminus\{\mu_1,\cdots,\mu_m\},\\
&m^-(\mathbf{L}_\lambda, 0) = \sum_{0<s<\lambda} m^0(\mathbf{L}_s, 0), \quad \forall\lambda \in (0,\tau].
\end{aligned} \tag{3.58}$$

Let ρ be the distance from μ to the set $\{0,\mu_1,\cdots,\mu_m,\tau\}\setminus\{\mu\}$. Then for any $0 < \epsilon < \rho$ it holds that $m^-(\mathbf{L}_{\mu-\epsilon}, 0) \neq m^-(\mathbf{L}_{\mu+\epsilon}, 0)$ and that $m^0(\mathbf{L}_{\mu-\epsilon}, 0) = m^0(\mathbf{L}_{\mu+\epsilon}, 0) = 0$. By Theorem 3.5(II) and Claim 3.15 we deduce that μ is a bifurcation instant for (S_0,γ). This completes the proof of the first claim in (iii).

It remains to prove others in (iii). Note that (3.58) gives rise to

$$m^-(\mathbf{L}_\lambda, 0) = \begin{cases} m^-(\mathbf{L}_\mu, 0) & \text{for } \lambda < \mu \text{ near } \mu,\\ m^-(\mathbf{L}_\mu, 0) + m^0(\mathbf{L}_\mu, 0) & \text{for } \lambda > \mu \text{ near } \mu.\end{cases}$$

Then Theorem 3.7 may be applicable to $\tilde{\mathcal{E}}^*_\lambda = \mathbf{L}_\lambda$ and $V_1 = \{0\} \subset \mathbb{R}^n$, and therefore one of the following alternatives occurs:

(A) The problem (3.56)–(3.57) with $\lambda = \mu$ has a sequence of distinct solutions, $\mathbf{v}^k \neq 0$, $k = 1,2,\cdots$, which converges to 0 in $C^2_{V_0\times\{0\}}([0,1]; B^n_\iota(0))$.

(B) There exists a real $0 < \sigma < \min\{\mu, \tau-\mu\}$ such that for every $\lambda \in [\mu-\sigma, \mu+\sigma]\setminus\{\mu\}$ the problem (3.56)–(3.57) with parameter value λ has a solution $\mathbf{v}^\lambda \neq 0$ to satisfy $\|\mathbf{v}^\lambda\|_{C^2} \to 0$ as $\lambda \to \mu$.

(C) For a given neighborhood $\mathfrak{W}$ of $0 \in C^1_{V_0\times\{0\}}([0,1]; B^n_\iota(0))$, there exists a real $0 < \sigma < \min\{\mu, \tau-\mu\}$ such that for any $\lambda \in \Lambda^0\setminus\{\mu\}$, where $\Lambda^0 = [\mu-\sigma,\mu]$ or $[\mu,\mu+\sigma]$, the problem (3.56)–(3.57) with parameter value λ has at least two distinct solutions in $\mathfrak{W}$, $\mathbf{v}^\lambda_1 \neq 0$ and $\mathbf{v}^\lambda_2 \neq 0$, which can also be required to satisfy $\mathbf{L}_\lambda(\mathbf{v}^\lambda_1) \neq \mathbf{L}_\lambda(\mathbf{v}^\lambda_2)$ provided that $m^0(\mathbf{L}_\mu, 0) > 1$ and the problem (3.12)–(3.13) with parameter value λ has only finitely many solutions in $\mathfrak{W}$.

Let us prove that the cases (A) and (B) lead to (iii-1) and (iii-2) in Theorem 1.9, respectively.

In the case of (A) above, C^2 paths

$$\alpha_k : [0,\mu] \to M,\ t \mapsto (\Phi_\mu \circ \Gamma_\mu)(\mathbf{v}^k)(t) = \phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^k(t/\mu)\right), \quad k = 1,2,\cdots,$$

are a sequence distinct C^2 Euler-Lagrange curves of L emanating perpendicularly from S_0 and ending at $\gamma(\mu)$, and each of them is not equal to $\gamma|_{[0,\mu]}$. Moreover, since $\phi_{\overline{\gamma}} : [0,\tau] \times B^n_{2\iota}(0) \to M$ and $\mathbf{u} : [0,\tau] \to B^n_\iota(0)$ are C^5 and C^3, respectively, we have a continuous map

$$C^2([0,\mu]; [0,\tau] \times B^n_{2\iota}(0)) \to C^2([0,\mu]; M),\ w \mapsto \phi_{\overline{\gamma}} \circ w$$

by Exercise 10 on the page 64 of [16], and a C^∞ map

$$\Theta : C^2([0,1]; B^n_\iota(0)) \to C^2([0,\mu]; [0,\tau] \times B^n_{2\iota}(0))$$

given by $\Theta(v)(t) = (t, \mathbf{u}(t) + v(t/\mu))$. Hence the composition map

$$C^2([0,1]; B^n_\iota(0)) \to C^2([0,\mu]; M),\ v \mapsto \phi_{\overline{\gamma}} \circ \Theta(v) = (\Phi_\mu \circ \Gamma_\mu)(v)$$

is continuous, and therefore $\alpha_k = (\Phi_\mu \circ \Gamma_\mu)(\mathbf{v}^k) \to \Phi_\mu(0) = \gamma|_{[0,\mu]}$ in $C^2([0,\mu], \mathbb{R}^N)$ as $k \to \infty$.

In the case of (B) above, for each $\lambda \in [\mu - \sigma, \mu + \sigma] \setminus \{\mu\}$,

$$\alpha^\lambda : [0,\lambda] \to M,\ t \mapsto (\Phi_\lambda \circ \Gamma_\lambda)(\mathbf{v}^\lambda)(t) = \phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right)$$

is a C^2 Euler-Lagrange curve of L emanating perpendicularly from S_0 and ending at $\gamma(\lambda)$, and not equal to $\gamma|_{[0,\lambda]}$. We cannot prove the desired claim as above. But noting that we have assumed $M \subset \mathbb{R}^N$, $\phi_{\overline{\gamma}}$ can be viewed as a C^5 map from $[0,\tau] \times B^n_{2\iota}(0)$ to $\mathbb{R}^N$. A straight computation leads to

$$\begin{aligned}
(\alpha^\lambda)'(t) &= D_1\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right) + D_2\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right)[\mathbf{u}'(t) + \frac{1}{\lambda}(\mathbf{v}^\lambda)'(t/\lambda)],\\
(\alpha^\lambda)''(t) &= D_1D_1\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right) + D_2D_1\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right)[\mathbf{u}'(t) + \frac{1}{\lambda}(\mathbf{v}^\lambda)'(t/\lambda)]\\
&+ D_2D_1\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right)[\mathbf{u}'(t) + \frac{1}{\lambda}(\mathbf{v}^\lambda)'(t/\lambda)]\\
&+ D_2\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right)[\mathbf{u}''(t) + \frac{1}{\lambda^2}(\mathbf{v}^\lambda)''(t/\lambda)]\\
&+ D_2D_2\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda)\right)[\mathbf{u}'(t) + \frac{1}{\lambda}(\mathbf{v}^\lambda)'(t/\lambda), \mathbf{u}'(t) + \frac{1}{\lambda}(\mathbf{v}^\lambda)'(t/\lambda)],
\end{aligned}$$

where we denote by D_1 and D_2 the partial derivatives of $\phi_{\overline{\gamma}}(t,x)$ with respect to the arguments t and x, respectively, and in particular, the final term is equal to

$$\frac{\partial^2}{\partial s_1 \partial s_2}\phi_{\overline{\gamma}}\left(t, \mathbf{u}(t) + \mathbf{v}^\lambda(t/\lambda) + (s_1 + s_2)(\mathbf{u}'(t) + \frac{1}{\lambda}(\mathbf{v}^\lambda)'(t/\lambda))\right)\Bigg|_{s_1=s_2=0}.$$

Since $(\mu + \sigma)^i \le \lambda^i \le (\mu - \sigma)^i$, $i = 1, 2$, and $\|v^\lambda\|_{C^2} \to 0$ as $\lambda \to \mu$, it follows from the above expressions that $\|\alpha_\lambda - \gamma|_{[0,\lambda]}\|_{C^2([0,\lambda],\mathbb{R}^N)} \to 0$ as $\lambda \to \mu$.

(As pointed out below (1.7) the above Euler-Lagrange curves of L, α_k, α^λ and the following β^i_λ are actually C^3.)

Suppose that (iii-1) and (iii-2) in Theorem 1.9 do not hold. Then the above proofs show that the cases (A) and (B) do not occur. That is, the case (C) must hold. Let us prove that it implies (iii-3) in Theorem 1.9. By the proof of the case (B) above we have positive numbers δ and $\sigma' < \min\{\mu, \tau - \mu\}$ such that

$$\mathfrak{W}_0 := \left\{\mathbf{v} \in C^1_{V_0 \times \{0\}}([0,1]; B^n_\iota(0)) \,\middle|\, \|\mathbf{v}\|_{C^1} \le \delta\right\} \subset \mathfrak{W}$$

and that C^1-paths

$$\alpha_{\lambda,\mathbf{v}}:[0,\lambda]\to M,\ t\mapsto \phi_{\overline{\gamma}}\left(t,\mathbf{u}(t)+\mathbf{v}(t/\lambda)\right)$$

associated with $(\lambda,\mathbf{v})\in[\mu-\sigma',\mu+\sigma]\times\{\mathbf{v}\in C^1_{V_0\times\{0\}}([0,1];B^n_\iota(0))\,|\,\|\mathbf{v}\|_{C^1}\le\delta\}$ satisfy

$$\|\alpha_{\lambda,\mathbf{v}}-\gamma|_{[0,\lambda]}\|_{C^1}<\epsilon.$$

By (C) there exists a positive number $\sigma_0\le\min\{\sigma',\sigma\}$ such that the corresponding conclusions in (C) also hold true after $\mathfrak{W}$ and σ are replaced by $\mathfrak{W}_0$ and σ_0, respectively. For $\Lambda^*:=\Lambda^0\cap[\mu-\sigma_0,\mu+\sigma_0]$ and $\lambda\in\Lambda^*\setminus\{\mu\}$, let $\mathbf{v}_1^\lambda\in\mathfrak{W}_0\setminus\{0\}$ and $\mathbf{v}_2^\lambda\in\mathfrak{W}_0\setminus\{0\}$ be two distinct solutions of the problem (3.56)–(3.57) with parameter value λ. Then

$$\beta^i_\lambda:[0,\lambda]\to M,\ t\mapsto(\Phi_\lambda\circ\Gamma_\lambda)(\mathbf{v}_i^\lambda)(t)=\phi_{\overline{\gamma}}\left(t,\mathbf{u}(t)+\mathbf{v}_i^\lambda(t/\lambda)\right),\quad i=1,2,$$

are two distinct C^2 Euler-Lagrange curves of L emanating perpendicularly from S_0 and ending at $\gamma(\lambda)$, and not equal to $\gamma|_{[0,\lambda]}$. Moreover both satisfy $\|\beta^\lambda_i-\gamma|_{[0,\lambda]}\|_{C^1}<\epsilon$, $i=1,2$.

Suppose further that $m^0(\mathcal{L}_{S_0,\mu},\gamma_\mu)=m^0(\mathbf{L}_\mu,0)>1$. For $\lambda\in\Lambda^*\setminus\{\mu\}$, if there only exist finitely many distinct C^2 Euler-Lagrange curves of L emanating perpendicularly from S_0 and ending at $\gamma(\lambda)$, $\alpha_1,\cdots,\alpha_m$, such that $\|\alpha_i-\gamma|_{[0,\lambda]}\|_{C^1}<\epsilon$, $i=1,\cdots,m$. then the above arguments imply that the problem (3.56)–(3.57) with parameter value λ has only finitely many solutions in $\mathfrak{W}_0$. In this case the above $\mathbf{v}_1^\lambda$ and $\mathbf{v}_2^\lambda$ can be chosen to satisfy $\mathbf{L}_\lambda(\mathbf{v}_1^\lambda)\ne\mathbf{L}_\lambda(\mathbf{v}_2^\lambda)$, which implies (1.12). □

Proof of Theorem 3.14. Follow the first paragraph in the proof of Theorem 1.9. We have a C^2 chart

$$\Psi_\lambda:C^1_{V_0\times\{\mathbf{u}(\lambda)\}}([0,\lambda];B^n_{2\iota}(0))\to C^1_{S_0\times\{\gamma(\lambda)\}}([0,\lambda];M)$$

given by $\Psi_\lambda(\xi)(t)=\phi_{\overline{\gamma}}(t,\xi(t))$ $\forall t\in[0,\lambda]$ for each $\lambda\in(0,\tau]$. Then $\Psi_\lambda(\mathbf{u}_\lambda(t))=\gamma_\lambda$, where $\mathbf{u}_\lambda:=\mathbf{u}|_{[0,\lambda]}$ and $\gamma_\lambda=\gamma|_{[0,\lambda]}$. Define $\tilde{L}:[0,\tau]\times B^n_{2\iota}(0)\times\mathbb{R}^n\to\mathbb{R}$ by

$$\tilde{L}(t,q,v)=\tilde{L}(t,q,v)=L\left(t,\phi_{\overline{\gamma}}(t,q),D_t\phi_{\overline{\gamma}}(t,q)+D_q\phi_{\overline{\gamma}}(t,q)[v]\right).$$

By Assumption 1.7, $\tilde{L}$ is C^3 and $\tilde{L}(t,q,v)$ is strictly convex in v for each $(t,q)\in[0,\tau]\times B^n_{2\iota}(0)$. Therefore for each $\lambda\in(0,\tau]$ the functional

$$C^1_{V_0\times\{\mathbf{u}(\lambda)\}}([0,\lambda];B^n_{2\iota}(0))\ni x\mapsto\tilde{\mathcal{E}}_{V_0,\lambda}(x)=\int_0^\lambda\tilde{L}(t,x(t),\dot{x}(t))dt\in\mathbb{R}$$

is C^2, and satisfies $d\tilde{\mathcal{E}}_{V_0,\lambda}(\mathbf{u}_\lambda)=0$ and

$$\tilde{\mathcal{E}}_{V_0,\lambda}(\xi)=\mathcal{L}_{S_0,\lambda}\left(\Psi_{\overline{\gamma}}(\xi)\right)\quad\text{for all }\xi\in C^1_{V_0\times\{\mathbf{u}(\lambda)\}}([0,\lambda];B^n_{2\iota}(0)).$$

With the same reasoning as for (3.55) these yield

$$m^-(\tilde{\mathcal{E}}_{V_0,\lambda},\mathbf{u}_\lambda)=m^-(\mathcal{L}_{S_0,\lambda},\gamma_\lambda)\quad\text{and}\quad m^0(\tilde{\mathcal{E}}_{V_0,\lambda},\mathbf{u}_\lambda)=m^0(\mathcal{L}_{S_0,\lambda},\gamma_\lambda).\tag{3.59}$$

Therefore from now on we may assume that $M=\mathbb{R}^n$ and S_0 is a linear subspace in $\mathbb{R}^n$. Then for $0<\lambda\le\tau$ and $y,z\in C^1_{S_0\times\{0\}}([0,\lambda];\mathbb{R}^n)$,

$$D^2\mathcal{L}_{S_0,\lambda}(\gamma_\lambda)[y,z]=\int_0^\lambda\left[(\mathsf{P}\dot{y}+\mathsf{Q}y)\cdot\dot{z}+\ \mathsf{Q}^T\dot{y}\cdot z+\mathsf{R}y\cdot z\right]dt,\tag{3.60}$$

where $\mathsf{P}(t) = \partial_{vv} L\,(t, \gamma(t), \dot\gamma(t))$, $\mathsf{Q}(t) = \partial_{xv} L\,(t, \gamma(t), \dot\gamma(t))$ and $\mathsf{R}(t) = \partial_{xx} L\,(t, \gamma(t), \dot\gamma(t))$.

Let $v = v(t, x, \xi)$ be the solution of $\xi = \partial_v L(t, x, v)$. Define

$$H(t, x, \xi) = \langle v(t, x, \xi), \xi\rangle_{\mathbb{R}^n} - L(t, x, v(t, x, \xi)).$$

Writing $x(t) = \gamma(t)$ and $\xi(t) = \partial_v L(t, x(t), v(t))$, we get that (1.7) is equivalent to

$$\left.\begin{array}{l} \frac{d}{dt}x(t) = \partial_\xi H(t, x(t), \xi(t)), \\ \frac{d}{dt}\xi(t) = -\partial_x H(t, x(t), \xi(t)) \end{array}\right\} \tag{3.61}$$

with boundary condition

$$\left.\begin{array}{l} (x(0), x(\tau)) \in S_0 \times \{q\} \subset \mathbb{R}^n \times \mathbb{R}^n, \\ (\xi(0), -\xi(\tau)) \in S_0^\perp \times \mathbb{R}^n \end{array}\right\} \tag{3.62}$$

since $(T_{(x(0),x(\tau))}(S_0 \times \{q\})^\perp = (S_0 \times \{0\})^\perp = S_0^\perp \times \mathbb{R}^n$. Note that $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. The natural projection $\pi : T^*M \to M$ becomes the projection from $\mathbb{R}^n \times \mathbb{R}^n$ onto the first factor $\mathbb{R}^n$. The co-normal bundle of S_0, i.e., $N^*S_0 = \{(x, \xi) \in T^*M \,|\, x \in S_0, \langle \xi, v\rangle = 0\ \forall v \in T_x S_0\}$, becomes $S_0 \times S_0^\perp$ and therefore its tangent space $U = S_0 \times S_0^\perp$. Moreover the vertical space V is equal to $\{0\} \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$. By [11, Proposition 4.5] with $\rho = U \times V$ we have

$$m^-(D^2\mathcal{L}_{S_0,\lambda}(\gamma_\lambda)) = \sum_{0<s<\lambda} \dim(U \cap \Phi(0, s)^{-1}(V)), \tag{3.63}$$

where $\Phi(0, t)$, according to (1.19) and (1.20) in [11], is the fundamental matrix solution of

$$\begin{pmatrix} \dot x(t) \\ \dot y(t) \end{pmatrix} = A(0, t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{3.64}$$

with

$$A(0, t) = \begin{pmatrix} D^2_{x\xi} H(t, x(t), \xi(t)) & D^2_{\xi} H(t, x(t), \xi(t)) \\ -D^2_{xx} H(t, x(t), \xi(t)) & -D^2_{\xi x} H(t, x(t), \xi(t)) \end{pmatrix}$$

with $\xi(t) = \partial_v L(t, \gamma(t), \dot\gamma(t))$. Note that [11, (1.13)-(1.14)] with $\mu = 0$ corresponds to the Jacobi equation of the functional $\mathcal{L}_{S_0,\tau}$, namely, the following linearized problem of (1.7)

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\mathsf{Q}(t) \cdot x(t) + \mathsf{P}(t) \cdot \dot x(t)\Big) = \mathsf{R}(t) \cdot x(t) + \mathsf{Q}^T(t) \cdot \dot x(t), \\ x(0) \in S_0,\ x(\tau) = 0 \quad \text{and} \quad \mathsf{Q}(0) \cdot x(0) + \mathsf{P}(0) \cdot \dot x(0) \in S_0^\perp \end{array}\right\} \tag{3.65}$$

whose solution space is equal to the kernel of $D^2\mathcal{L}_{S_0,\tau}(\gamma)$. It was claimed in [11, page 179] that (3.65) is equivalent to (3.64) plus with boundary condition

$$\left.\begin{array}{l} (x(0), x(\tau)) \in S_0 \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n, \\ y(0) \in S_0^\perp. \end{array}\right\} \tag{3.66}$$

According to the deduction from [11, (1.13)-(1.14)] with $\mu = 0$ to [11, (1.19)-(1.21)] with $\mu = 0$, (3.64) was obtained by putting $y(t) := \mathsf{Q}(t) \cdot x(t) + \mathsf{P}(t) \cdot \dot x(t)$ in (3.66). Hence (3.64) is exactly

$$\begin{pmatrix} \dot x(t) \\ \dot y(t) \end{pmatrix} = \begin{pmatrix} -[\mathsf{P}(t)]^{-1}\mathsf{Q}(t) & [\mathsf{P}(t)]^{-1} \\ \mathsf{R}(t) - [\mathsf{Q}(t)]^T[\mathsf{P}(t)]^{-1}\mathsf{Q}(t) & [\mathsf{Q}(t)]^T[\mathsf{P}(t)]^{-1} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{3.67}$$

that is,

$$\left(\begin{array}{cc} -[\mathsf{P}(t)]^{-1}\mathsf{Q}(t) & [\mathsf{P}(t)]^{-1} \\ \mathsf{R}(t)-[\mathsf{Q}(t)]^T[\mathsf{P}(t)]^{-1}\mathsf{Q}(t) & [\mathsf{Q}(t)]^T[\mathsf{P}(t)]^{-1} \end{array}\right) = A(0,t)$$

with $\xi(t)=\partial_v L(t,\gamma(t),\dot\gamma(t))$. (Hence $\Phi(0,t)=d\phi^t(\gamma(0),\partial_v L(t,\gamma(0),\dot\gamma(0))):\mathbb{R}^{2n}\to\mathbb{R}^{2n}$, where ϕ^t is the flow of the Hamiltonian system (3.61); see [11, page 192].)

Note that the map sending $(\bar x,\bar y)\in U\cap\Phi(0,\tau)^{-1}(V)$ to x, where $(x(t),y(t))=\Phi(0,t)(\bar x,\bar y)$ with $0\le t\le\tau$, is a linear isomorphism between $U\cap\Phi(0,\tau)^{-1}(V)$ and the space of solutions of (3.65). We obtain

$$\dim(U\cap\Phi(0,\tau)^{-1}(V))=m^0(D^2\mathcal{L}_{S_0,\lambda}(\gamma_\tau))$$

and so $\dim(U\cap\Phi(0,s)^{-1}(V))=m^0(D^2\mathcal{L}_{S_0,s}(\gamma_s))$ since τ may be replaced by any $0<\lambda\le\tau$. The desired conclusion follows from these and (3.63). □

4 Proofs of Theorems 1.13, 1.14, 1.15

The proofs are completely similar to those of Section 3. We only outline main procedures.

4.1 Reduction to Euclidean spaces

As in Section 3.1.1, we have a positive number ι such that 3ι is less than the injectivity radius of g at each point on $\gamma_\mu([0,\tau])$, and a path $\overline{\gamma}\in C^1_{\mathbb{I}_g}([0,\tau];M)\cap C^7([0,\tau];M)$ such that (3.2) is satisfied. Then the injectivity radius of g at each point on $\overline{\gamma}([0,\tau])$ is at least 2ι. Then we assume that (3.3) is satisfied. (For cases of Theorems 1.13, 1.15, it is naturally satisfied after shrinking Λ toward μ.)

As in [32, §3], starting with a unit orthogonal frame at $T_{\gamma_\mu(0)}M$ and using the parallel transport along $\overline{\gamma}$ with respect to the Levi-Civita connection of the Riemannian metric g we get a unit orthogonal parallel C^5 frame field $[0,\tau]\to\overline{\gamma}^*TM,\ t\mapsto(e_1(t),\cdots,e_n(t))$. Note that there exists a unique orthogonal matrix $E_{\overline{\gamma}}$ such that $(e_1(\tau),\cdots,e_n(\tau))=(\mathbb{I}_{g*}e_1(0),\cdots,\mathbb{I}_{g*}e_n(0))E_{\overline{\gamma}}$. Let $B^n_{2\iota}(0):=\{x\in\mathbb{R}^n\,|\,|x|<2\iota\}$ and exp denote the exponential map of g. Then

$$\phi_{\overline{\gamma}}:[0,\tau]\times B^n_{2\iota}(0)\to M,\ (t,x)\mapsto\exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n x_ie_i(t)\right)\tag{4.1}$$

is a C^5 map and satisfies

$$\phi_{\overline{\gamma}}(\tau,x)=\mathbb{I}_g\left(\phi_{\overline{\gamma}}(0,(E_{\overline{\gamma}}x^T)^T)\right)\quad\text{and}\quad d\phi_{\overline{\gamma}}(\tau,x)[(1,v)]=d\phi_{\overline{\gamma}}(0,(E_{\overline{\gamma}}x^T)^T)[(\tau,(E_{\overline{\gamma}}v^T)^T)]$$

for any $(t,x,v)\in[0,\tau]\times B^n_{2\iota}(0)\times\mathbb{R}^n$. (Note that the tangent map $d\phi_{\overline{\gamma}}:T([0,\tau]\times B^n_{2\iota}(0))\to TM$ is C^4.) Consider the Hilbert subspace

$$W^{1,2}_{E_{\overline{\gamma}}}([0,\tau];\mathbb{R}^n):=\{u\in W^{1,2}([0,\tau];\mathbb{R}^n)\,|\,u(\tau)=E_{\overline{\gamma}}u(0)\}\tag{4.2}$$

of $W^{1,2}([0,\tau];\mathbb{R}^{2n})$ equipped with $W^{1,2}$-inner product (2.2), and Banach spaces

$$C^i_{E_{\overline{\gamma}}}([0,\tau];\mathbb{R}^n)=\{u\in C^i([0,\tau];\mathbb{R}^n)\,|\,u(\tau)=E_{\overline{\gamma}}u(0)\}\tag{4.3}$$

with the induced norm $\|\cdot\|_{C^i}$ from $C^i([0,\tau],\mathbb{R}^n)$ for $i\in\mathbb{N}\cup\{0\}$. By [48, Theorem 4.3], we get a C^2 coordinate chart around $\overline{\gamma}$ on the C^4 Banach manifold $C^1_{\mathbb{I}_g}([0,\tau];M)$,

$$\Phi_{\overline{\gamma}}:C^1_{E_{\overline{\gamma}}}([0,\tau],B^n_{2\iota}(0))=\{\xi\in C^1_{E_{\overline{\gamma}}}([0,\tau],\mathbb{R}^n)\,|\,\|\xi\|_{C^0}<2\iota\}\to C^1_{\mathbb{I}_g}([0,\tau];M)\tag{4.4}$$

given by $\Phi_{\overline{\gamma}}(\xi)(t) = \phi_{\overline{\gamma}}(t, \xi(t))$, and

$$d\Phi_{\overline{\gamma}}(0) : C^1_{E_{\overline{\gamma}}}([0,\tau], \mathbb{R}^n) \to C^1_{\mathbb{I}_g}(\overline{\gamma}^*TM),\ \xi \mapsto \sum_{j=1}^n \xi_j e_j.$$

By (3.2) and (3.3), for each $\lambda \in \Lambda$ there exists a unique map $\mathbf{u}_\lambda : [0,\tau] \to B^n_\iota(0)$ such that

$$\gamma_\lambda(t) = \phi_{\overline{\gamma}}(t, \mathbf{u}_\lambda(t)) = \exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n \mathbf{u}^i_\lambda e_i(t)\right), \quad t \in [0,\tau].$$

As in the proofs of Lemma 3.1, 3.2 we have:

Lemma 4.1. $\mathbf{u}_\lambda \in C^2([0,\tau]; B^n_\iota(0))$, $\mathbf{u}_\mu(0) = 0 = \mathbf{u}_\mu(\tau)$ *(and so* $\mathbf{u}_\lambda \in C^1_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0)))$ *and*

$$(\lambda, t) \mapsto \mathbf{u}_\lambda(t), \quad (\lambda, t) \mapsto \dot{\mathbf{u}}_\lambda(t) \quad \textit{and} \quad (\lambda, t) \mapsto \ddot{\mathbf{u}}_\lambda(t)$$

are continuous as maps from $\Lambda \times [0,\tau]$ *to* $\mathbb{R}^n$.

Let $\tilde{L}^* : \Lambda \times [0,\tau] \times B^n_\iota(0) \times \mathbb{R}^n \to \mathbb{R}$ by (7.9). It satisfies Proposition 3.3. Each functional

$$\tilde{\mathcal{E}}^*_\lambda : C^1_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0)) \to \mathbb{R},\ x \mapsto \int_0^1 \tilde{L}^*_\lambda(t, x(t), \dot{x}(t))dt \tag{4.5}$$

is C^2, and satisfies

$$\tilde{\mathcal{E}}^*_\lambda(x) = \mathcal{E}_\lambda\left(\Phi_{\overline{\gamma}}(x + \mathbf{u}_\lambda)\right)\ \forall x \in C^1_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0)) \quad \text{and} \quad d\tilde{\mathcal{E}}^*_\lambda(0) = 0. \tag{4.6}$$

Hence for each $\lambda \in \Lambda$, $x \in C^1_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0))$ satisfies $d\tilde{\mathcal{E}}^*_\lambda(x) = 0$ if and only if $\gamma := \Phi_{\overline{\gamma}}(x + \mathbf{u}_\lambda)$ satisfies $d\mathcal{E}_\lambda(\gamma) = 0$; and in this case γ and x have the same Morse indexes and nullities. In particular, for each $\lambda \in \Lambda$, it holds that

$$m^-(\tilde{\mathcal{E}}^*_\lambda, 0) = m^-(\mathcal{E}_\lambda, \gamma_\lambda) \quad \text{and} \quad m^0(\tilde{\mathcal{E}}^*_\lambda, 0) = m^0(\mathcal{E}_\lambda, \gamma_\lambda). \tag{4.7}$$

The critical points of $\tilde{\mathcal{E}}^*_\lambda$ correspond to the solutions of the following boundary problem:

$$\frac{d}{dt}\Big(\partial_v \tilde{L}^*_\lambda(t, x(t), \dot{x}(t))\Big) - \partial_q \tilde{L}^*_\lambda(t, x(t), \dot{x}(t)) = 0, \tag{4.8}$$

$$\left.\begin{array}{l} x \in C^2_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0)) \quad \text{and} \\ (E^T_{\overline{\gamma}})^{-1}\partial_v \tilde{L}^*_\lambda(0, x(0), \dot{x}(0)) = \partial_v \tilde{L}^*_\lambda(\tau, x(\tau), \dot{x}(\tau)) \end{array}\right\} \tag{4.9}$$

([6, Proposition 4.2]). Corresponding to Theorems 3.5, 3.6, 3.7, we have the following three theorems, which also hold true provided that $\tilde{L}^*$ satisfies (a) in Proposition 3.3 and the weaker (b') in Remark 3.4 as noted in Remark 3.4.

Theorem 4.2. (I) (Necessary condition): *Suppose that* $(\mu, 0) \in \Lambda \times C^1_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0))$ *is a bifurcation point along sequences of the problem (4.8)–(4.9) with respect to the trivial branch* $\{(\lambda, 0) \mid \lambda \in \Lambda\}$ *in* $\Lambda \times C^1_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0))$. *Then* $m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 0$.

(II) (Sufficient condition): *Suppose that* Λ *is first countable and that there exist two sequences in* Λ *converging to* μ, (λ^-_k) *and* (λ^+_k), *such that one of the following conditions is satisfied:*

(II.1) *For each* $k \in \mathbb{N}$, *either* 0 *is not an isolated critical point of* $\tilde{\mathcal{E}}^*_{\lambda_k^+}$, *or* 0 *is not an isolated critical point of* $\tilde{\mathcal{E}}^*_{\lambda_k^-}$, *or* 0 *is an isolated critical point of* $\tilde{\mathcal{E}}^*_{\lambda_k^+}$ *and* $\tilde{\mathcal{E}}^*_{\lambda_k^-}$ *and* $C_m(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0; \mathbf{K})$ *and* $C_m(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0; \mathbf{K})$ *are not isomorphic for some Abel group* $\mathbf{K}$ *and some* $m \in \mathbb{Z}$.

(II.2) *For each* $k \in \mathbb{N}$, *there exists* $\lambda \in \{\lambda_k^+, \lambda_k^-\}$ *such that* 0 *is an either nonisolated or homological visible critical point of* $\tilde{\mathcal{E}}^*_{\lambda}$, *and*

$$[m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0)] = \emptyset.$$

(II.3) $[m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0)] = \emptyset$, *and either* $m^0(\tilde{\mathcal{E}}^*_{\lambda_k^-}, 0) = 0$ *or* $m^0(\tilde{\mathcal{E}}^*_{\lambda_k^+}, 0) = 0$ *for each* $k \in \mathbb{N}$.

Then $(\mu, 0)$ *is a bifurcation point of the problem (4.8)–(4.9) in* $\hat{\Lambda} \times C^2_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0))$ *with respect to the branch* $\{(\lambda, 0) \,|\, \lambda \in \hat{\Lambda}\}$ *(and so* $\{(\lambda, 0) \,|\, \lambda \in \Lambda\}$*), where* $\hat{\Lambda} = \{\mu, \lambda_k^+, \lambda_k^- \,|\, k \in \mathbb{N}\}$.

Theorem 4.3 (Existence for bifurcations). *Let* Λ *be connected. For* $\lambda^-, \lambda^+ \in \Lambda$ *suppose that one of the following conditions is satisfied:*

(i) *Either* 0 *is not an isolated critical point of* $\tilde{\mathcal{E}}^*_{\lambda^+}$, *or* 0 *is not an isolated critical point of* $\tilde{\mathcal{E}}^*_{\lambda^-}$, *or* 0 *is an isolated critical point of* $\tilde{\mathcal{E}}^*_{\lambda^+}$ *and* $\tilde{\mathcal{E}}^*_{\lambda^-}$ *and* $C_m(\tilde{\mathcal{E}}^*_{\lambda^+}, 0; \mathbf{K})$ *and* $C_m(\tilde{\mathcal{E}}^*_{\lambda^-}, 0; \mathbf{K})$ *are not isomorphic for some Abel group* $\mathbf{K}$ *and some* $m \in \mathbb{Z}$.

(ii) $[m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^+}, 0)] = \emptyset$, *and there exists* $\lambda \in \{\lambda^+, \lambda^-\}$ *such that* 0 *is an either nonisolated or homological visible critical point of* $\mathcal{E}^*_{\lambda}$.

(iii) $[m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^-}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^-}, 0)] \cap [m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0), m^-(\tilde{\mathcal{E}}^*_{\lambda^+}, 0) + m^0(\tilde{\mathcal{E}}^*_{\lambda^+}, 0)] = \emptyset$, *and either* $m^0(\mathcal{E}^*_{\lambda^+}, 0) = 0$ *or* $m^0(\mathcal{E}^*_{\lambda^-}, 0) = 0$.

Then for any path $\alpha : [0,1] \to \Lambda$ *connecting* λ^+ *to* λ^- *there exists a sequence* $(t_k) \subset [0,1]$ *converging to some* $\bar{t} \in [0,1]$, *and a nonzero solution* x_k *of the problem (4.8)–(4.9) with* $\lambda = \alpha(t_k)$ *for each* $k \in \mathbb{N}$ *such that* $\|x_k\|_{C^2([0,\tau];\mathbb{R}^n)} \to 0$ *as* $k \to \infty$. *Moreover,* $\alpha(\bar{t})$ *is not equal to* λ^+ *(resp.* λ^-*) if* $m^0(\tilde{\mathcal{E}}^*_{\lambda^+}, 0) = 0$ *(resp.* $m^0(\tilde{\mathcal{E}}^*_{\lambda^-}, 0) = 0$*).*

Theorem 4.4 (Alternative bifurcations of Rabinowitz's type). *Let* Λ *be a real interval and* $\mu \in \mathrm{Int}(\Lambda)$. *Suppose that* $m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 0$, *and that* $m^0(\tilde{\mathcal{E}}^*_\lambda, 0) = 0$ *for each* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ, *and* $m^-(\tilde{\mathcal{E}}^*_\lambda, 0)$ *take, respectively, values* $m^-(\tilde{\mathcal{E}}^*_\mu, 0)$ *and* $m^-(\tilde{\mathcal{E}}^*_\mu, 0) + m^0(\tilde{\mathcal{E}}^*_\mu, 0)$ *as* $\lambda \in \Lambda$ *varies in two deleted half neighborhoods of* μ. *Then one of the following alternatives occurs:*

(i) *The problem (4.8)–(4.9) with* $\lambda = \mu$ *has a sequence of solutions,* $x_k \neq 0$, $k = 1, 2, \cdots$, *which converges to* 0 *in* $C^2([0,\tau], \mathbb{R}^n)$.

(ii) *For every* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ *there is a solution* $y_\lambda \neq 0$ *of (4.8)–(4.9) with parameter value* λ, *such that* y_λ *converges to zero in* $C^2([0,\tau], \mathbb{R}^n)$ *as* $\lambda \to \mu$.

(iii) *For a given neighborhood* $\mathfrak{W}$ *of* $0 \in C^1_{E_{\overline{\gamma}}}([0,\tau], B^n_\iota(0))$, *there is an one-sided neighborhood* Λ^0 *of* μ *such that for any* $\lambda \in \Lambda^0 \setminus \{\mu\}$, *the problem (4.8)–(4.9) with parameter value* λ *has at least two distinct solutions in* $\mathfrak{W}$, $y^1_\lambda \neq 0$ *and* $y^2_\lambda \neq 0$, *which can also be required to satisfy* $\tilde{\mathcal{E}}^*_\lambda(y^1_\lambda) \neq \tilde{\mathcal{E}}^*_\lambda(y^2_\lambda)$ *provided that* $m^0(\tilde{\mathcal{E}}^*_\mu, 0) > 1$ *and the problem (4.8)–(4.9) with parameter value* λ *has only finitely many solutions in* $\mathfrak{W}$.

As in the proofs of Theorems 1.4, 1.6, Theorems 1.13, 1.15 are derived from Theorems 4.2, 4.4, respectively.

4.2 Proofs of Theorems 1.13, 1.14, 1.15

Let us write

$$\begin{aligned}
&\mathbf{H}_{E_{\overline{\gamma}}} := W^{1,2}_{E_{\overline{\gamma}}}([0,\tau];\mathbb{R}^n) \quad \text{and} \quad \mathbf{X}_{E_{\overline{\gamma}}} := C^1_{E_{\overline{\gamma}}}([0,\tau];\mathbb{R}^n),\\
&\mathcal{U} := \left\{u \in W^{1,2}\left([0,\tau]; B^n_{\iota/2}(0)\right) \,|\, u(\tau) = E_{\overline{\gamma}}u(0)\right\},\\
&\mathcal{U}^X := \mathcal{U} \cap \mathbf{X}_{E_{\overline{\gamma}}} = \left\{u \in C^1\left([0,\tau]; B^n_{\iota/2}(0)\right) \,|\, u(\tau) = E_{\overline{\gamma}}u(0)\right\}
\end{aligned}$$

The latter two sets are open subsets in the spaces $\mathbf{H}_{E_{\overline{\gamma}}}$ and $\mathbf{X}_{E_{\overline{\gamma}}}$, respectively. Let the continuous function $\check{L} : \hat{\Lambda} \times [0,\tau] \times B^n_{3\iota/4}(0) \times \mathbb{R}^n \to \mathbb{R}$ be given by Lemma 3.8. Define a family of functionals $\check{\mathcal{E}}_\lambda : \mathcal{U} \to \mathbb{R}$ given by

$$\check{\mathcal{E}}_\lambda(x) = \int_0^\tau \check{L}_\lambda(t, x(t), \dot{x}(t))dt, \quad \lambda \in \hat{\Lambda}. \tag{4.10}$$

Then (L1) in Lemma 3.8 implies

$$\tilde{\mathcal{E}}^*_\lambda = \check{\mathcal{E}}_\lambda|_{\mathcal{U}^X} \quad \text{in} \quad \{x \in \mathcal{U}^X \,|\, \|x\|_{C^1} < \rho_0\} \subset \mathcal{U}^X, \tag{4.11}$$

and hence

$$m^\star(\tilde{\mathcal{E}}^*_\lambda, 0) = m^\star(\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}, 0) = m^\star(\check{\mathcal{E}}_\lambda, 0), \quad \star = -, 0. \tag{4.12}$$

Since $\mathbf{H}_{E_{\overline{\gamma}}}$ contains the subspace $\left\{u \in W^{1,2}([0,\tau];\mathbb{R}^n) \,\middle|\, u(\tau) = 0 = u(0)\right\}$, carefully checking the computation of [31, (4.14)] it is easily seen that replacing H_V by $\mathbf{H}_{E_{\overline{\gamma}}}$ we also obtain that the gradient $\nabla\check{\mathcal{E}}_\lambda(x)$ of $\check{\mathcal{E}}_\lambda$ at $x \in \mathcal{U}$ is still given by

$$\begin{aligned}
\nabla\check{\mathcal{E}}_\lambda(x)(t) \;=\; & e^t \int_0^t \left[e^{-2s} \int_0^s e^r f_{\lambda,x}(r)dr\right] ds + c_1(\lambda,x)e^t + c_2(\lambda,x)e^{-t}\\
& + \int_0^t \partial_v \check{L}_\lambda(s, x(s), \dot{x}(s))ds,
\end{aligned} \tag{4.13}$$

where $c_1(\lambda,x), c_2(\lambda,x) \in \mathbb{R}^n$ are suitable constant vectors and $f_\lambda(t)$ is given by (3.29).

Proposition 4.5. *Proposition 3.9 is still effective after making the following substitutions:*

- *The functionals* $\tilde{\mathcal{E}}^*_\lambda$ *and* $\check{\mathcal{E}}_\lambda$ *are changed into* $\tilde{\mathcal{E}}^*_\lambda$ *and* $\check{\mathcal{E}}_\lambda$*, respectively.*
- *The spaces* $\mathbf{H}_{V_0\times V_1}$ *and* $\mathbf{X}_{V_0\times V_1}$ *are changed into* $\mathbf{H}_{E_{\overline{\gamma}}}$ *and* $\mathbf{X}_{E_{\overline{\gamma}}}$*, respectively.*
- *The boundary problem (3.27) is changed into*

$$\left.\begin{array}{l}
\frac{d}{dt}\Big(\partial_v \check{L}_\lambda(t, x(t), \dot{x}(t))\Big) - \partial_q \check{L}_\lambda(t, x(t), \dot{x}(t)) = 0,\\
x(\tau) = E_{\overline{\gamma}}x(0).
\end{array}\right\} \tag{4.14}$$

Then repeating proofs of Propositions 3.10, 3.11 we can obtain:

Proposition 4.6. *Both maps* $\hat{\Lambda}\times\mathcal{U}^X \ni (\lambda,x) \mapsto \check{\mathcal{E}}_\lambda(x) \in \mathbb{R}$ *and* $\hat{\Lambda}\times\mathcal{U}^X \ni (\lambda,x) \mapsto A_\lambda(x) \in \mathbf{X}_{E_{\overline{\gamma}}}$ *are continuous.*

Proposition 4.7. *For any given $\epsilon > 0$ there exists $\varepsilon > 0$ such that if a critical point x of $\check{\mathcal{E}}_\lambda$ satisfies $\|x\|_{1,2} < \varepsilon$ then $\|x\|_{C^2} < \epsilon$. (**Note**: ε is independent of $\lambda \in \hat{\Lambda}$.) Consequently, if $0 \in \mathcal{U}^X$ is an isolated critical point of $\check{\mathcal{E}}_\lambda|_{\mathcal{U}^X}$ then $0 \in \mathcal{U}$ is also an isolated critical point of $\check{\mathcal{E}}_\lambda$.*

Having these we can prove

- Theorem 4.2(I) with [34, Theorem 3.1] ([37, Theorem C.6]) as in the proof of Theorem 3.5(I),
- Theorem 4.4 with [37, Theorem C.7] ([36, Theorem 3.6]) as in the proof of Theorem 3.7,
- a corresponding result of Theorem 3.13, from which Theorem 4.2(II) and Theorem 4.3 may be derived,
- Theorem 1.14 as in the proof of Theorem 1.5 in Section 3.1.4.

5 Proof of Theorem 1.20

Step 1 (*Reduction of the problem (1.23) to one on open subsets of* $\mathbb{R}^n$). We can assume $\Lambda = [\mu - \varepsilon, \mu + \varepsilon]$ for some $\varepsilon > 0$. (1.24) implies that each γ_λ is a constant solution of (1.23) for any $\tau > 0$. Since $\mathbb{I}_g^l = id_M$, each solution of (1.23) is $l\tau$-periodic. All solutions of (1.23) near γ_μ sit in a compact neighborhood of $\gamma_\mu \in M$. Let $e_1, \cdots, e_n$ be a unit orthogonal frame at $T_{\gamma_\mu}M$. Then $(\mathbb{I}_{g*}e_1, \cdots, \mathbb{I}_{g*}e_n) = (e_1, \cdots, e_n)E_{\gamma_\mu}$ for a unique orthogonal matrix E_{γ_μ}. Clearly, $E_{\gamma_\mu}^l = I_n$ Let $B^n_{2\iota}(0) := \{x \in \mathbb{R}^n \,|\, |x| < 2\iota\}$ and exp denote the exponential map of g. Then

$$\phi : B^n_{2\iota}(0) \to M,\ x \mapsto \exp_{\gamma_\mu}\left(\sum_{i=1}^n x_i e_i\right) \tag{5.1}$$

is a C^5 embedding of codimension zero and satisfies $\phi(E_{\gamma_\mu}x) = \mathbb{I}_g\phi(x)$ and $d\phi(0)[y] = \sum_{i=1}^n y_i e_i$ for any $y \in \mathbb{R}^n$. Shrinking Λ toward μ (if necessary) we may assume that $\gamma_\lambda \in \phi(B^n_{2\iota}(0))$ for all $\lambda \in \Lambda$. (This is possible because $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto \gamma_\lambda(t) \in M$ is continuous.) Therefore there exists a unique $x_\lambda \in B^n_{2\iota}(0)$ such that $\phi(x_\lambda) = \gamma_\lambda$ for all $\lambda \in \Lambda$. Clearly, $x_\mu = 0$, $E_{\gamma_\mu}x_\lambda = x_\lambda\ \forall\lambda$ and $\Lambda \ni \lambda \to x_\lambda \in B^n_{2\iota}(0)$ is continuous. Define

$$L^* : \Lambda \times B^n_{2\iota}(0) \times \mathbb{R}^n \to \mathbb{R},\ (\lambda, x, y) \mapsto L(\lambda, \phi(x), d\phi(x)[y]).$$

It is C^2 with respect to (x, y) and strictly convex with respect to y, and all its partial derivatives also depend continuously on (λ, x, y). Moreover, $L^*_\lambda = L^*(\lambda, \cdot)$ is also E_{γ_μ}-invariant. From (1.24) we derive that $L^*_\lambda(x, 0) = L^*(\lambda, x, 0)$ has the differential at $x_\lambda \in \mathbb{R}^n$,

$$\partial_x L^*_\lambda(x_\lambda, 0)[y] = \partial_q L(\lambda, \gamma_\lambda, 0)[d\phi(0)[y]] = 0 \quad \forall\lambda \in \Lambda.$$

These show that L^* satisfies

$$\left.\begin{array}{l}\Lambda \ni \lambda \to x_\lambda \in U \cap \mathrm{Ker}(E - I_n) \text{ be continuous and} \\ \partial_q L_\lambda(x_\lambda, 0) = 0\ \forall\lambda \in \Lambda.\end{array}\right\} \tag{5.2}$$

Recall that the Banach spaces

$$\mathcal{X}^i_\tau(\mathbb{R}^n, E_{\gamma_\mu}) := \{\gamma \in C^i(\mathbb{R}, \mathbb{R}^n) \,|\, E_{\gamma_\mu}(\gamma(t)) = \gamma(t + \tau)\ \forall t\}$$

with the induced norm $\|\xi\|_{C^i}$ from $C^1([0, \tau], \mathbb{R}^n)$, $i = 0, 1, 2, \cdots$. Consider the functional on the open subset $\mathcal{X}^1_\tau(B^n_{2\iota}(0)), E_{\gamma_\mu})$ of $\mathcal{X}^1_\tau(\mathbb{R}^n, E_{\gamma_\mu})$,

$$x \mapsto \mathfrak{E}^*_\lambda(x) = \int_0^\tau L^*_\lambda(x(t), \dot{x}(t))dt. \tag{5.3}$$

Note that the map ϕ induces an C^2 embedding of codimension zero

$$\Phi_{\gamma_\mu}:\mathcal{X}^1_\tau(B^n_{2\iota}(0)), E_{\gamma_\mu})\to\mathcal{X}^1_\tau(M,\mathbb{I}_g),\ x\mapsto\phi\circ x.$$

It is not hard to see that $\Phi_{\gamma_\mu}(x_\lambda)=\gamma_\lambda$ and

$$D^2\mathfrak{E}^*_\lambda(x_\lambda)[\xi,\eta]=D^2\mathfrak{E}_\lambda(\gamma_\lambda)[d\Phi_{\gamma_\mu}(x_\lambda)\xi,d\Phi_{\gamma_\mu}(x_\lambda)\eta],\quad\forall\xi,\eta\in\mathcal{X}^1_\tau(\mathbb{R}^n,E_{\gamma_\mu}).\tag{5.4}$$

We conclude that the conditions (a),(b) and (c) in Theorem 1.20 are, respectively, equivalent to the following three conditions:

(a') $\partial_{xx}L^*_\mu(x_\mu,0)$ is positive definite;

(b') $\partial_{xx}L^*_\mu(x_\mu,0)(a_1,\cdots,a_n)^T=0$ and $E_{\gamma_\mu}(a_1,\cdots,a_n)^T=(a_1,\cdots,a_n)^T$ has only the zero solution in $\mathbb{R}^n$.

(c') $m^0_\tau(\mathfrak{E}^*_\mu,0)\ne0$, $m^0_\tau(\mathfrak{E}^*_\lambda,x_\lambda)=0$ for each $\lambda\in\Lambda\setminus\{\mu\}$ near μ, and $m^-_\tau(\mathfrak{E}^*_\lambda,x_\lambda)$ takes, respectively, values $m^-_\tau(\mathfrak{E}^*_\mu,0)$ and $m^-_\tau(\mathfrak{E}^*_\mu,0)+m^0_\tau(\mathfrak{E}^*_\mu,0)$ as $\lambda\in\Lambda$ varies in two deleted half neighborhoods of μ.

In fact, let $\mathcal{H}_{\lambda,\gamma_\lambda}$ be the Hessian bilinear form of the function $M\ni q\mapsto L(\lambda,q,0)$ at $\gamma_\lambda\in M$. Then

$$\mathcal{H}_{\lambda,\gamma_\lambda}(u,v)=g_{\gamma_\lambda}(\partial_{qq}L_\mu\,(\gamma_\lambda,0)\,u,v)\quad\forall u,v\in T_{\gamma_\lambda}M.$$

View $(x_1,\cdots,x_n)\in B^n_{2\iota}(0)$ as local coordinates at $\phi(x)\in M$ and write $u=\sum_{i=1}^n a_i\frac{\partial}{\partial x_i}|_{\gamma_\mu}$ and $v=\sum_{i=1}^n b_i\frac{\partial}{\partial x_i}|_{\gamma_\mu}$. We have

$$\begin{aligned}&\mathbb{I}_{g*}u=u\quad\iff\quad E_{\gamma_\mu}(a_1,\cdots,a_n)^T=(a_1,\cdots,a_n)^T,\\ &g_{\gamma_\mu}\left(\partial_{qq}L_\mu(\gamma_\mu,0)u,v\right)=\mathcal{H}_{\mu,\gamma_\mu}(u,v)=\textstyle\sum_{i,j=1}^n a_ib_j\frac{\partial^2L^*_\mu}{\partial x_i\partial x_j}(0,0)\\ &\qquad=\left(\partial_{xx}L^*_\mu(x_\mu,0)(a_1,\cdots,a_n)^T,(b_1,\cdots,b_n)^T\right)_{\mathbb{R}^n}.\end{aligned}$$

Hence the conditions (a) and (b) in Theorem 1.20 are equivalent to (a') and (b'), respectively. Clearly, (5.4) implies the equivalence between (c) and (c').

Since $\lambda\mapsto x_\lambda$ is continuous and $x_\mu=0$ we can shrink $\varepsilon>0$ in $\Lambda=[\mu-\varepsilon,\mu+\varepsilon]$ so that $x_\lambda\in B^n_\iota(0)$ for all $\lambda\in\Lambda$. Define $\tilde{L}^*:\Lambda\times B^n_\iota(0)\times\mathbb{R}^n\to\mathbb{R}$ by

$$\tilde{L}^*(\lambda,x,y)=\tilde{L}^*_\lambda(x,y)=L^*(\lambda,x+x_\lambda,y).$$

For a given positive $\rho_0>0$, by Lemma 2.4 we have a continuous function $\tilde{L}^*:\Lambda\times B^n_\iota(0)\times\mathbb{R}^n\to\mathbb{R}$ and a constant $\kappa>0$ satisfying the following properties:

(i) $\tilde{L}^*$ is equal to L^* on $\Lambda\times B^n_\iota(0)\times B^n_{\rho_0}(0)$.

(ii) $\tilde{L}^*$ is C^2 with respect to (x,y) and strictly convex with respect to y, and all its partial derivatives also depend continuously on (λ,x,y). Moreover, each $\tilde{L}^*_\lambda=\tilde{L}^*(\lambda,\cdot)$ is also E_{γ_μ}-invariant.

(iii) There exists a constant $C>0$ such that

$$\tilde{L}^*_\lambda(x,y)\ge\kappa|y|^2-C,\quad\forall(\lambda,x,y)\in\Lambda\times B^n_{3\iota/4}(0)\times\mathbb{R}^n.$$

(iv) $\partial_{xx}\tilde{L}^*_\mu(0,0)$ is positive definite;

(v) $\partial_{xx}\tilde{L}^*_\mu(0,0)(a_1,\cdots,a_n)^T = 0$ and $E_{\gamma_\mu}(a_1,\cdots,a_n)^T = (a_1,\cdots,a_n)^T$ has only the zero solution in $\mathbb{R}^n$.

(**Note**: Applying Theorem 8.12 to $\tilde{L}^*$ we can also complete the required proof.)

Since $\Lambda = [\mu-\varepsilon, \mu+\varepsilon]$ is compact, by shrinking $\varepsilon > 0$ (if necessary) as in Lemma 3.8 we can modify $\tilde{L}^*$ to get a continuous function $\check{L} : \Lambda \times B^n_{3\iota/4}(0) \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties for some constants $\check{\kappa} > 0$ and $0 < \check{c} < \check{C}$:

(L0) $\tilde{L}^*$ in the above (ii) and (iv)-(v) is changed into $\check{L}$.

(L1) $\check{L}$ and $\tilde{L}^*$ are equal in $\Lambda \times B^n_{3\iota/4}(0) \times B^n_{\rho_0}(0)$.

(L2) $\partial_{yy}\check{L}_\lambda(x,y) \ge \check{c}I_n, \quad \forall(\lambda,x,y) \in \Lambda \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

(L3) $\left|\frac{\partial^2}{\partial x_i\partial x_j}\check{L}_\lambda(x,y)\right| \le \check{C}(1+|y|^2), \quad \left|\frac{\partial^2}{\partial x_i\partial y_j}\check{L}_\lambda(x,y)\right| \le \check{C}(1+|y|),$ and $\left|\frac{\partial^2}{\partial y_i\partial y_j}\check{L}_\lambda(x,y)\right| \le \check{C}, \quad \forall(\lambda,x,y) \in \Lambda \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

(L4) $\check{L}(\lambda,x,y) \ge \check{\kappa}|y|^2 - \check{C}, \quad \forall(\lambda,x,y) \in \Lambda \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

(L5) $|\partial_q\check{L}(\lambda,x,y)| \le \check{C}(1+|y|^2)$ and $|\partial_y\check{L}(\lambda,x,y)| \le \check{C}(1+|y|)$ for all $(\lambda,x,y) \in \Lambda\times B^n_{3\iota/4}(0)\times\mathbb{R}^n$.

(L6) $|\check{L}_\lambda(x,y)| \le \check{C}(1+|y|^2), \quad \forall(\lambda,x,y) \in \Lambda \times B^n_{3\iota/4}(0) \times \mathbb{R}^n$.

Consider the Hilbert space

$$\mathbf{H}_{E_{\gamma_\mu}} := \{\xi \in W^{1,2}_{\rm loc}(\mathbb{R};\mathbb{R}^n) \,|\, E^T_{\gamma_\mu}(\xi(t)) = \xi(t+\tau)\ \forall t \in \mathbb{R}\}$$

equipped with $W^{1,2}$-inner product as in (2.2). Since $E^l_{\gamma_\mu} = I_n$, the spaces $\mathbf{X}_{E_{\gamma_\mu}} := \mathcal{X}^1_\tau(\mathbb{R}^n, E_{\gamma_\mu})$ and $\mathbf{H}_{E_{\gamma_\mu}}$ carry a natural S^1-action with $S^1 = \mathbb{R}/(lT\mathbb{Z})$ given by

$$(\theta\cdot x)(t) = x(t+\theta), \quad \theta \in \mathbb{R}, \tag{5.5}$$

and have the following S^1-invariant open subsets

$$\begin{aligned}\mathcal{U}_{E_{\gamma_\mu}} &:= \{\xi \in W^{1,2}_{\rm loc}(\mathbb{R}; B^n_{3\iota/4}(0)) \,|\, E^T_{\bar\gamma}(\xi(t)) = \xi(t+\tau)\ \forall t \in \mathbb{R}\},\\ \mathcal{U}^X_{E_{\gamma_\mu}} &:= \mathcal{U}_{E_{\gamma_\mu}} \cap \mathbf{X}_{E_{\gamma_\mu}} = \mathcal{X}^1_\tau(B^n_{3\iota/4}(0)), E_{\gamma_\mu})\end{aligned}$$

respectively. For each $\lambda \in \Lambda$, define functionals $\mathfrak{E}^*_\lambda : \mathcal{X}^1_\tau(B^n_\iota(0)), E_{\gamma_\mu}) \to \mathbb{R}$ and $\check{\mathcal{L}}_\lambda : \mathcal{U}_{E_{\gamma_\mu}} \to \mathbb{R}$ by

$$\tilde{\mathfrak{E}}^*_\lambda(x) = \int_0^\tau \tilde{L}^*_\lambda(x(t),\dot{x}(t))dt, \tag{5.6}$$

$$\check{\mathcal{L}}_\lambda(x) = \int_0^\tau \check{L}_\lambda(x(t),\dot{x}(t))dt. \tag{5.7}$$

They are invariant for the above S^1-action.

Corresponding to Proposition 4.5, 4.6, 4.7 we have

Proposition 5.1. *Proposition 3.9 is still effective after making the following substitutions:*

- *The functionals $\tilde{\mathcal{E}}^*_\lambda$ and $\check{\mathcal{E}}_\lambda$ are changed into $\mathfrak{E}^*_\lambda$ and $\check{\mathcal{L}}_\lambda$, respectively.*

- *The spaces $\mathbf{H}_{V_0\times V_1}$ and $\mathbf{X}_{V_0\times V_1}$ are changed into $\mathbf{H}_{E_{\gamma_\mu}}$ and $\mathbf{X}_{E_{\gamma_\mu}}$, respectively.*
- *The boundary problem (3.27) is changed into*

$$\left.\begin{array}{l}\frac{d}{dt}\Big(\partial_v\check{L}_\lambda(x(t),\dot{x}(t))\Big)-\partial_q\check{L}_\lambda(x(t),\dot{x}(t))=0\ \forall t\in\mathbb{R}\\ E_{\gamma_\mu}x(t)=x(t+\tau)\quad\forall t\in\mathbb{R}.\end{array}\right\}\tag{5.8}$$

Proposition 5.2. *Both maps $\Lambda\times\mathcal{U}^X\ni(\lambda,x)\mapsto\check{\mathcal{L}}_\lambda(x)\in\mathbb{R}$ and $\Lambda\times\mathcal{U}^X\ni(\lambda,x)\mapsto A_\lambda(x)\in\mathbf{X}_{E_{\gamma_\mu}}$ are continuous.*

Proposition 5.3. *For any given $\bar{\epsilon}>0$ there exists $\bar{\varepsilon}>0$ such that if a critical point x of $\check{\mathcal{L}}_\lambda$ satisfies $\|x\|_{1,2}<\bar{\varepsilon}$ then $\|x\|_{C^2}<\bar{\epsilon}$. (**Note**: ε is independent of $\lambda\in\hat{\Lambda}$.) Consequently, if $0\in\mathcal{U}^X_{E_{\gamma_\mu}}$ is an isolated critical point of $\check{\mathfrak{E}}_\lambda|_{\mathcal{U}^X}$ then $0\in\mathcal{U}_{E_{\gamma_\mu}}$ is also an isolated critical point of $\check{\mathcal{L}}_\lambda$.*

Step 2 (*Prove that [37, Theorem C.6] ([37, Theorem 3.7] or [34, Theorem 5.12]) can be used for $\check{\mathcal{L}}_\lambda$*). As before, we have $m^\star_\tau(\mathfrak{E}_\lambda,\gamma_\lambda)=m^\star_\tau(\mathfrak{E}^*_\lambda,x_\lambda)=m^\star_\tau(\check{\mathfrak{E}}^*_\lambda,0)=m^\star_\tau(\check{\mathcal{L}}_\lambda,0)$ for $\star=-,0$. Because of the assumption (c), we obtain

(c") $m^0_\tau(\check{\mathcal{L}}_\mu,0)\neq0$, $m^0_\tau(\check{\mathcal{L}}_\lambda,x_\lambda)=0$ for each $\lambda\in\Lambda\setminus\{\mu\}$ near μ, and $m^-_\tau(\check{\mathcal{L}}_\lambda,x_\lambda)$ takes, respectively, values $m^-_\tau(\check{\mathcal{L}}_\mu,0)$ and $m^-_\tau(\check{\mathcal{L}}_\mu,0)+m^0_\tau(\check{\mathcal{L}}_\mu,0)$ as $\lambda\in\Lambda$ varies in two deleted half neighborhoods of μ.

Next, let us prove that

$$\text{the fixed point set of the induced } S^1\text{-action on } (\mathbf{H}_{E_{\gamma_\mu}})^0_\mu:=\mathrm{Ker}(\check{\mathcal{L}}''_\mu(0)) \text{ is } \{0\},\tag{5.9}$$

Note that $\xi\in\mathbf{H}_{E_{\gamma_\mu}}$ belongs to $(\mathbf{H}_{E_{\gamma_\mu}})^0_\mu$ if and only if it is C^2 and satisfies

$$\partial_{xy}\check{L}_\lambda(0,0)\dot{\xi}+\partial_{yy}\check{L}_\lambda(0,0)\ddot{\xi}-\partial_{xx}\check{L}_\lambda(0,0)\xi-\partial_{yx}\check{L}_\lambda(0,0)\dot{\xi}=0.\tag{5.10}$$

Suppose that ξ is also a fixed point for the action in (5.5). Then it is equal to a constant vector in $\mathbb{R}^n$ and $E_{\gamma_\mu}\xi=\xi$. By (5.10) we obtain $\partial_{xx}\check{L}_\lambda(0,0)\xi=0$ and hence $\xi=0$ because $\partial_{xx}\check{L}_\mu(0,0)$ is positive definite by (L0). (5.9) is proved.

By [37, Theorem C.6] ([37, Theorem 3.7] or [34, Theorem 5.12]) one of the following alternatives occurs:

(I) $(\mu,0)$ is not an isolated solution in $\{\mu\}\times\mathcal{U}_{E_{\gamma_\mu}}$ of $\nabla\check{\mathcal{L}}_\mu=0$.

(II) There exist left and right neighborhoods Λ^- and Λ^+ of μ in Λ and integers $n^+,n^-\geq0$, such that $n^++n^-\geq\frac{1}{2}\dim(\mathbf{H}_{E_{\gamma_\mu}})^0_\mu$, and that for $\lambda\in\Lambda^-\setminus\{\mu\}$ (resp. $\lambda\in\Lambda^+\setminus\{\mu\}$) the functional $\check{\mathcal{L}}_\lambda$ has at least n^- (resp. n^+) distinct critical S^1-orbits disjoint with 0, which converge to 0 in $\mathcal{U}^X_{E_{\gamma_\mu}}$ as $\lambda\to\mu$.

Moreover, if $\dim(\mathbf{H}_{E_{\gamma_\mu}})^0_\mu\geq3$, then (ii) may be replaced by the following alternatives:

(III) For every $\lambda\in\Lambda\setminus\{0\}$ near $0\in\Lambda$ there is a S^1-orbit $S^1\cdot\bar{w}_\lambda\neq\{0\}$ near $0\in\mathcal{U}^X_{E_{\gamma_\mu}}$ such that $\nabla\check{\mathcal{L}}_\lambda(\bar{w}_\lambda)=0$ and that $S^1\cdot\bar{w}_\lambda\to0$ in $\mathcal{U}^X_{E_{\gamma_\mu}}$ as $\lambda\to\mu$.

(IV) For any small S^1-invariant neighborhood $\mathcal{N}$ of 0 in $\mathcal{U}^X_{E_{\gamma_\mu}}$ there is an one-sided deleted neighborhood Λ^0 of $\mu \in \Lambda$ such that for any $\lambda \in \Lambda^0$, $\nabla\check{\mathcal{L}}_\lambda = 0$ has either infinitely many S^1-orbits of solutions in $\mathcal{N}$, $S^1 \cdot \bar{w}^j_\lambda$, $j = 1, 2, \cdots$, or at least two S^1-orbits of solutions in $\mathcal{N}$, $S^1 \cdot \hat{w}^1_\lambda \neq \{0\}$ and $S^1 \cdot \hat{w}^2_\lambda \neq \{0\}$, such that $\check{\mathcal{L}}_\lambda(\hat{w}^1_\lambda) \neq \check{\mathcal{L}}_\lambda(\hat{w}^2_\lambda)$. Moreover, these orbits converge to 0 in $\mathcal{U}^X_{E_{\gamma_\mu}}$ as $\lambda \to \mu$.

Step 3 (*Complete the proof of Theorem 1.20*).

In the case of (I), we have a sequence $(w_j) \subset \mathbf{H}_{E_{\gamma_\mu}} \setminus \{0\}$ such that $\|w_j\|_{1,2} \to 0$ as $j \to \infty$ and that $\nabla\check{\mathcal{L}}_\mu(w_j) = 0$ for each $j \in \mathbb{N}$. By Proposition 5.3 these w_j are C^2 and $\|w_j\|_{C^2} \to 0$. Because 0 is a fixed point for the S^1-action, the S^1-orbits are compact and different S^1-orbits are not intersecting, by passing to a subsequence we can assume that any two of w_j, $j = 0, 1, \cdots$, do not belong the same S^1-orbit. Using the chart ϕ in (5.1) we define $\mathbb{R} \ni t \mapsto \gamma^k(t) := \phi(w_k(t))$, $k = 1, \cdots$. They satisfy (i) of Theorem 1.20.

In the case of (II), note firstly that $\dim(\mathbf{H}_{E_{\gamma_\mu}})^0_\mu$ is equal to $m^0_\tau(\check{\mathcal{L}}_\mu, 0) = m^0_\tau(\mathfrak{E}_\mu, \gamma_\mu)$. For $\lambda \in \Lambda^- \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^+ \setminus \{\mu\}$) let $S^1 \cdot w^i_\lambda$, $i = 1, \cdots, n^-$ (resp. n^+) be distinct critical S^1-orbits of $\check{\mathcal{L}}_\lambda$ disjoint with 0, which converge to 0 in $\mathcal{U}^X_{E_{\gamma_\mu}}$ as $\lambda \to \mu$. Proposition 5.3 implies that $\|w^i_\lambda\|_{C^2} \to 0$ as $\lambda \to \mu$. Then $\mathbb{R} \ni t \mapsto \gamma^i_\lambda(t) = \phi(x_\lambda(t) + w^i_\lambda(t))$, $i = 1, \cdots, n^-$ (resp. n^+) are the required solutions.

In the case of (III), let $\alpha_\lambda(t) = \phi(x_\lambda(t) + \bar{w}_\lambda(t))$ for $t \in \mathbb{R}$. It satisfies (1.23) with parameter value λ and Proposition 5.3 implies that $\alpha_\lambda - \gamma_\lambda$ converges to zero on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $\lambda \to \mu$. Since $S^1 \cdot \bar{w}_\lambda \neq \{0\}$, for any $t \in \mathbb{R}$ we have $\bar{w}_\lambda(t) \neq 0$ and so $\alpha_\lambda(t) \neq \gamma_\lambda(t)$. Note that all γ_λ are constant solutions. Hence $\alpha_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$.

In the case of (IV), for the first case let $\alpha^j_\lambda(t) = \phi(x_\lambda(t) + \bar{w}^j_\lambda(t))$ for $t \in \mathbb{R}$ and $j = 1, 2, \cdots$, and for the second case let $\beta^i_\lambda(t) = \phi(x_\lambda(t) + \hat{w}^i_\lambda(t))$ for $t \in \mathbb{R}$ and $i = 1, 2$. They satisfy (1.23) with parameter value λ. For a given small $\epsilon > 0$, by Proposition 5.3 there is an one-sided neighborhood Λ^0 of μ in Λ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$,

- for the first case $\|\bar{\alpha}^k_\lambda|_{[0,\tau]} - \gamma_\lambda|_{[0,\tau]}\|_{C^2([0,\tau];\mathbb{R}^N)} < \epsilon$, $k = 1, 2, \cdots$,
- for the second case $\|\beta^i_\lambda|_{[0,\tau]} - \gamma_\lambda|_{[0,\tau]}\|_{C^2([0,\tau];\mathbb{R}^N)} < \epsilon$, $i = 1, 2$.

Moreover, each γ_λ is constant, orbits $\mathbb{R} \cdot \bar{\alpha}^k_\lambda = S^1 \cdot \bar{\alpha}^k_\lambda$ are distinct, and $\bar{\alpha}^k_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$ as above. Similarly, β^1_λ and β^2_λ are $\mathbb{R}$-distinct, and $\beta^1_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$ and $\beta^2_\lambda \notin \mathbb{R} \cdot \gamma_\lambda$. Finally, $\mathfrak{E}_\lambda(\beta^1_\lambda) \neq \mathfrak{E}_\lambda(\beta^2_\lambda)$ because $\check{\mathcal{L}}_\lambda(\hat{w}^1_\lambda) \neq \check{\mathcal{L}}_\lambda(\hat{w}^2_\lambda)$. The desired assertions are proved. $\square$

6 Proofs of Theorems 1.23, 1.24, 1.25, 1.26

6.1 Reduction of the problem (1.23) to one on open subsets of $\mathbb{R}^n$

Since $\mathbb{I}_g(\bar{\gamma}(t)) = \bar{\gamma}(t + \tau)$ $\forall t \in \mathbb{R}$ and $\mathbb{I}_g$ is an isometry, the closure $Cl(\bar{\gamma}(\mathbb{R}))$ of $\bar{\gamma}(\mathbb{R})$ is compact. As in Section 3.1.1 we may choose a number $\iota > 0$ such that the following holds:

(♣6) the closure $\bar{\mathbf{U}}_{3\iota}(Cl(\bar{\gamma}(\mathbb{R})))$ of $\mathbf{U}_{3\iota}(Cl(\bar{\gamma}(\mathbb{R}))) := \{p \in M \,|\, d_g(p, Cl(\bar{\gamma}(\mathbb{R}))) < 3\iota\}$ is a compact neighborhood of $\gamma_\mu([0,\tau])$ in M, and $\bar{\mathbf{U}}_{3\iota}(Cl(\bar{\gamma}(\mathbb{R}))) \times \bar{\mathbf{U}}_{3\iota}(Cl(\bar{\gamma}(\mathbb{R})))$ is contained in the image of $\mathbb{F}|_{\mathcal{W}(0_{TM})}$, where $\mathbb{F}$ is as in (6.1).

(♠6) $\{(q, v) \in TM \,|\, q \in \bar{\mathbf{U}}_{3\iota}(Cl(\bar{\gamma}(\mathbb{R}))),\ |v|_q \leq 3\iota\} \subset \mathcal{W}(0_{TM})$.

Then 3ι is less than the injectivity radius of g at each point on $\bar{\mathbf{U}}_{3\iota}(Cl(\bar{\gamma}(\mathbb{R})))$.

Let us choose the C^6 Riemannian metric g on M so that S_0 (resp. S_1) is totally geodesic near $\gamma_\mu(0)$ (resp. $\gamma_\mu(\tau)$). There exists a fibrewise convex open neighborhood $\mathcal{U}(0_{TM})$ of the zero section of TM such that the exponential map of g gives rise to C^5 immersion

$$\mathbb{F} : \mathcal{U}(0_{TM}) \to M \times M,\ (q, v) \mapsto (q, \exp_q(v)), \tag{6.1}$$

(cf. Appendix A). By (A.3), $d\mathbb{F}(q, 0_q) : T_{(q,0_q)} \to T_{(q,q)}(M \times M) = T_qM \times T_qM$ is an isomorphism for each $q \in M$. Since $\mathbb{E}$ is injective on the closed subset $0_{TM} \subset TM$, it follows from Exercise 7 in [16, page 41] that $\mathbb{F}|_{\mathcal{W}(0_{TM})}$ is a C^5 embedding of some smaller open neighborhood $\mathcal{W}(0_{TM}) \subset \mathcal{U}(0_{TM})$ of 0_{TM}. Note that $\mathbb{F}(0_{TM})$ is equal to the diagonal Δ_M in $M \times M$, and that $\gamma_\mu([0, \tau])$ is compact.

Since $\bar{\gamma}$ is C^6, as in [32, §3], starting with a unit orthogonal frame at $T_{\bar{\gamma}(0)}M$ and using the parallel transport along $\bar{\gamma}$ with respect to the Levi-Civita connection of the Riemannian metric g we get a unit orthogonal parallel C^5 frame field $\mathbb{R} \to \bar{\gamma}^*TM,\ t \mapsto (e_1(t), \cdots, e_n(t))$, such that

$$(e_1(t + \tau), \cdots, e_n(t + \tau)) = (\mathbb{I}_{g*}(e_1(t)), \cdots, \mathbb{I}_{g*}(e_n(t)))E_{\bar{\gamma}} \quad \forall t \in \mathbb{R},$$

where $E_{\bar{\gamma}}$ is an orthogonal matrix of order n.

By [17, Corollary 2.5.11] there exists an orthogonal matrix Ξ such that

$$\Xi^{-1}E_{\bar{\gamma}}\Xi = \mathrm{diag}(S_1, \cdots, S_\sigma) \in \mathbb{R}^{n\times n}, \tag{6.2}$$

where each S_j is either 1, or -1, or $\begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}$, $0 < \theta_j < \pi$, and their orders satisfy: $\mathrm{ord}(S_1) \ge \cdots \ge \mathrm{ord}(S_\sigma)$. Replacing $(e_1, \cdots, e_n)$ by $(e_1, \cdots, e_n)\Xi$ we may assume

$$E_{\bar{\gamma}} = \mathrm{diag}(S_1, \cdots, S_\sigma) \in \mathbb{R}^{n\times n}. \tag{6.3}$$

Let $B^n_r(0) := \{x \in \mathbb{R}^n \,|\, |x| < r\}$ and $\bar{B}^n_r(0) := \{x \in \mathbb{R}^n \,|\, |x| \le r\}$ for $r > 0$. Then

$$\phi_{\bar{\gamma}} : \mathbb{R} \times B^n_{3\iota}(0) \to M,\ (t, x) \mapsto \exp_{\bar{\gamma}(t)}\left(\sum_{i=1}^n x_i e_i(t)\right) \tag{6.4}$$

is a C^5 map and satisfies

$$\begin{aligned}&\phi_{\bar{\gamma}}(t + \tau, x) = \mathbb{I}_g\left(\phi_{\bar{\gamma}}(t, E_{\bar{\gamma}}x)\right) \quad \text{and}\\ &d\phi_{\bar{\gamma}}(t + \tau, x)[(1, v)] = d\mathbb{I}_g\left(\phi_{\bar{\gamma}}(t, E_{\bar{\gamma}}x)\right) \circ d\phi_{\bar{\gamma}}(t, E_{\bar{\gamma}}x)[(1, E_{\bar{\gamma}}v)]\end{aligned}$$

for any $(t, x, v) \in \mathbb{R} \times B^n_{3\iota}(0) \times \mathbb{R}^n$. By [48, Theorem 4.3], from $\phi_{\bar{\gamma}}$ we get a C^2 coordinate chart around $\bar{\gamma}$ on the C^4 Banach manifold $\mathcal{X}_\tau(M, \mathbb{I}_g)$ modeled on the Banach space

$$\mathcal{X}^1_\tau(\mathbb{R}^n, E_{\bar{\gamma}}) = \{\xi \in C^1(\mathbb{R}; \mathbb{R}^n) \,|\, E^T_{\bar{\gamma}}\xi(t) = \xi(t + \tau)\ \forall t \in \mathbb{R}\}$$

with the induced norm $\|\xi\|_{C^1}$ from $C^1([0, \tau], \mathbb{R}^n)$,

$$\Phi_{\bar{\gamma}} : \mathcal{X}^1_\tau(B^n_{2\iota}(0), E_{\bar{\gamma}}) = \{\xi \in \mathcal{X}^1_\tau(\mathbb{R}^n, E_{\bar{\gamma}}) \,|\, \|\xi\|_{C^0} < 2\iota\} \to \mathcal{X}^1_\tau(M, \mathbb{I}_g) \tag{6.5}$$

given by $\Phi_{\bar{\gamma}}(\xi)(t) = \phi_{\bar{\gamma}}(t, \xi(t))$. Moreover

$$d\Phi_{\bar{\gamma}}(0) : \mathcal{X}^1_\tau(\mathbb{R}^n, E_{\bar{\gamma}}) \to T_{\bar{\gamma}}\mathcal{X}_\tau(M, \mathbb{I}_g),\ \xi \mapsto \sum_{j=1}^n \xi_j e_j$$

is a Banach space isomorphism. (Actually, we have

$$|\xi(t)|^2_{\mathbb{R}^n} = \sum_{j=1}^{n}(\xi_j(t))^2 = g\left(\sum_{j=1}^{n}\xi_j(t)e_j(t), \sum_{j=1}^{n}\xi_j(t)e_j(t)\right) = |d\Phi_{\bar\gamma}(0)[\xi](t)|^2_g \tag{6.6}$$

for any $\xi:\mathbb{R}\to\mathbb{R}^n$ and $t\in\mathbb{R}$.) Therefore there exists a unique $\zeta_0\in\mathcal{X}^1_\tau(\mathbb{R}^n, E_{\bar\gamma})$ satisfying $d\Phi_{\bar\gamma}(0)[\zeta_0]=\dot{\bar\gamma}$, that is, $\dot{\bar\gamma}(t)=\sum_{j=1}^n\zeta_{0j}(t)e_j(t)$, $\forall t\in\mathbb{R}$, where

$$\zeta_{0j}(t)=g(\dot{\bar\gamma}(t), e_j(t))\ \forall t\in\mathbb{R},\quad j=1,\cdots,n.$$

Clearly, $\zeta_0\in\mathcal{X}^5_\tau(\mathbb{R}^n,E_{\bar\gamma})$ and $\zeta_0\neq 0$ because $\bar\gamma$ is nonconstant. If $\gamma\in\mathcal{X}^1_\tau(M,\mathbb{I}_g)$ is nonconstant and C^l $(2\le l\le 5)$, by the arguments above [37, Proposition 4.1] the orbit $\mathcal{O}:=\mathbb{R}\cdot\gamma$ is either an one-to-one C^{l-1} immersion submanifold of dimension one or a C^{l-1}-embedded circle; moreover $T_\gamma\mathcal{O}=\dot\gamma\mathbb{R}\subset T_\gamma\mathcal{X}^1_\tau(M,\mathbb{I}_g)$. Then for any reals $a<b$, $[a,b]\cdot\bar\gamma$ is a C^4 embedded submanifold of dimension one. Take $a>0$ such that $[-a,a]\cdot\bar\gamma\subset\mathrm{Im}(\Phi_{\bar\gamma})$. Then

$$S_0:=\Phi^{-1}_{\bar\gamma}\big([-a,a]\cdot\bar\gamma\cap\mathrm{Im}(\Phi_{\bar\gamma})\big) \tag{6.7}$$

is an one-dimensional compact C^2 submanifold of $\mathcal{X}^1_\tau(B^n_{2\iota}(0),E_{\bar\gamma})$ containing 0 as an interior point, and

$$T_0S_0=(d\Phi_{\bar\gamma}(0))^{-1}(T_{\bar\gamma}S_0)=(d\Phi_{\bar\gamma}(0))^{-1}(\mathbb{R}\dot{\bar\gamma})=\mathbb{R}\zeta_0.$$

Define the function $L^\star:\Lambda\times\mathbb{R}\times B^n_{3\iota}(0)\times\mathbb{R}^n\to\mathbb{R}$ by

$$L^\star(\lambda,t,x,v)=L\big(\lambda,\phi_{\bar\gamma}(t,x),d\phi_{\bar\gamma}(t,x)[(1,v)]\big).$$

It is continuous and satisfies (2.7) with $U=B^n_{3\iota}(0)$ and $E=E_{\bar\gamma}$, i.e.,

$$\begin{aligned}
&L^\star(\lambda,t+\tau,x,v)=L\big(\lambda,\phi_{\bar\gamma}(t+\tau,x),d\phi_{\bar\gamma}(t+\tau,x)[(1,v)]\big)\\
=\ &L\left(\lambda,\mathbb{I}_g\left(\phi_{\bar\gamma}(t,E_{\bar\gamma}x)\right),d\mathbb{I}_g\left(\phi_{\bar\gamma}(t,E_{\bar\gamma}x)\right)\circ d\phi_{\bar\gamma}(t,E_{\bar\gamma}x)[(1,E_{\bar\gamma}v)]\right)\\
=\ &L\left(\lambda,\phi_{\bar\gamma}(t,E_{\bar\gamma}x),d\phi_{\bar\gamma}(t,E_{\bar\gamma}x)[(1,E_{\bar\gamma}v)]\right)\\
=\ &L^\star(\lambda,t,E_{\bar\gamma}x,E_{\bar\gamma}v)
\end{aligned}\tag{6.8}$$

for all $(\lambda,t,x,v)\in\Lambda\times\mathbb{R}\times B^n_{3\iota}(0)\times\mathbb{R}^n$, and thus

$$\begin{aligned}
\partial_xL^\star(\lambda,t+\tau,x,v)&=E_{\bar\gamma}\partial_xL^\star\big(\lambda,t,E_{\bar\gamma}x,E_{\bar\gamma}v\big),\\
\partial_vL^\star(\lambda,t+\tau,x,v)&=E_{\bar\gamma}\partial_vL^\star\big(\lambda,t,E_{\bar\gamma}x,E_{\bar\gamma}v\big).
\end{aligned}$$

(Here $\partial_xL^\star$ and $\partial_vL^\star$ denote the gradients of $L^\star$ with respect to x and v, respectively. Recall that all vectors in $\mathbb{R}^n$ in this paper are understood as column vectors.) Each $L^\star(\lambda,\cdot)$ is C^4 and all its partial derivatives of order no more than two depend continuously on $(\lambda,t,x,v)\in\Lambda\times\mathbb{R}\times B^n_{3\iota}(0)\times\mathbb{R}^n$. Moreover, $d\phi_{\bar\gamma}(t,x)[(1,v)]=\partial_x\phi_{\bar\gamma}(t,x)[v]+\partial_t\phi_{\bar\gamma}(t,x)$ implies that

$$\mathbb{R}^n\ni v\mapsto L^\star(\lambda,t,x,v)=L\big(\lambda,\phi_{\bar\gamma}(t,x),d\phi_{\bar\gamma}(t,x)[(1,v)]\big)\in\mathbb{R}$$

is strictly convex.

Consider the functional $\mathfrak{E}^\star_\lambda:\mathcal{X}^1_\tau(B^n_{2\iota}(0),E_{\bar\gamma})\to\mathbb{R}$,

$$\mathfrak{E}^\star_\lambda(\xi)=\int_0^\tau L^\star(\lambda,t,\xi(t),\dot\xi(t))dt=\mathfrak{E}_\lambda\circ\Phi_{\bar\gamma}(\xi), \tag{6.9}$$

where $\mathfrak{E}_\lambda$ is given by (1.19). It is C^2 by Proposition **??**. Since $\mathfrak{E}_\lambda$ is invariant under the continuous $\mathbb{R}$-action on $\mathcal{X}^1_\tau(M, \mathbb{I}_g)$ given by

$$\chi : \mathcal{X}^1_\tau(M, \mathbb{I}_g) \times \mathbb{R} \to \mathcal{X}^1_\tau(M, \mathbb{I}_g),\ (s, \gamma) \mapsto s \cdot \gamma, \tag{6.10}$$

where $(s \cdot \gamma)(t) = \gamma(s + t)\ \forall s, t \in \mathbb{R}$, S_0 is a critical submanifold of each $\mathfrak{E}^\star_\lambda$, and there holds

$$D^2\mathfrak{E}^\star_\lambda(0)[\xi, \eta] = D^2\mathfrak{E}_\lambda(\bar\gamma)\big[d\Phi_{\bar\gamma}(0)[\xi], d\Phi_{\bar\gamma}(0)[\eta]\big] \quad \forall \xi, \eta \in \mathcal{X}^1_\tau(B^n_{2\iota}(0), E_{\bar\gamma}),$$

which imply

$$m^-_\tau(\mathfrak{E}^\star_\lambda, 0) = m^-_\tau(\mathfrak{E}_\lambda, \bar\gamma) \quad \text{and} \quad m^0_\tau(\mathfrak{L}^\star_\lambda, 0) = m^0_\tau(\mathfrak{E}_\lambda, \bar\gamma). \tag{6.11}$$

Since S_0 is compact, there exists $\rho_0 > 3\iota$ such that $\sup_t |\dot x(t)| < \rho_0$ for all $x \in S_0$. For the Lagrangian $L^\star$, as in Lemma 3.8 we can modify it to obtain:

Lemma 6.1. *For a given subset $\hat\Lambda \subset \Lambda$ which is either compact or sequential compact, There exists a continuous function $\check L : \hat\Lambda \times \mathbb{R} \times \bar B^n_{2\iota}(0) \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions for some constants $\kappa > 0$ and $0 < c < C$:*

(L0) $\check L(\lambda, t+\tau, x, v) = \check L(\lambda, t, E_{\bar\gamma}x, E_{\bar\gamma}v)$ *for all* (λ, t, x, v)*, and each the function* $\check L_\lambda(\cdot) = \check L(\lambda, \cdot)$ ($\lambda \in \hat\Lambda$) *is* C^4 *and partial derivatives*

$$\begin{aligned}&\partial_t \check L_\lambda(\cdot), \quad \partial_q \check L_\lambda(\cdot), \quad \partial_v \check L_\lambda(\cdot), \quad \partial_{qv} \check L_\lambda(\cdot), \quad \partial_{qq} \check L_\lambda(\cdot), \quad \partial_{vv} \check L_\lambda(\cdot)\\ &\textit{and} \quad \partial_{tt} \check L_\lambda(\cdot), \quad \partial_{tq} \check L_\lambda(\cdot), \quad \partial_{tv} \check L_\lambda(\cdot)\end{aligned}$$

depend continuously on $(\lambda, t, q, v) \in \hat\Lambda \times \mathbb{R} \times \bar B^n_{2\iota}(0) \times \mathbb{R}^n$.

(L1) $\check L$ *and* $L^\star$ *are same on* $\hat\Lambda \times \mathbb{R} \times B^n_{3\iota/2}(0) \times B^n_{\rho_0}(0)$;

(L2) $\partial_{vv} \check L_\lambda(t, q, v) \ge c I_n, \quad \forall (\lambda, t, q, v) \in \hat\Lambda \times [0, \tau] \times B^n_{3\iota/2}(0) \times \mathbb{R}^n$.

(L3) $\left|\frac{\partial^2}{\partial q_i \partial q_j} \check L_\lambda(t, q, v)\right| \le C(1 + |v|^2), \quad \left|\frac{\partial^2}{\partial q_i \partial v_j} \check L_\lambda(t, q, v)\right| \le C(1 + |v|),$ *and*
$\left|\frac{\partial^2}{\partial v_i \partial v_j} \check L_\lambda(t, q, v)\right| \le C, \quad \forall (\lambda, t, q, v) \in \hat\Lambda \times [0, \tau] \times B^n_{3\iota/2}(0) \times \mathbb{R}^n$.

(L4) $\check L(\lambda, t, q, v) \ge \kappa |v|^2 - C, \quad \forall (\lambda, t, q, v) \in \hat\Lambda \times \mathbb{R} \times B^n_{3\iota/2}(0) \times \mathbb{R}^n$.

(L5) $|\partial_q \check L(\lambda, t, q, v)| \le C(1 + |v|^2)$ *and* $|\partial_q \check L(\lambda, t, q, v)| \le C(1 + |v|)$ *for all* $(\lambda, t, q, v) \in \hat\Lambda \times \mathbb{R} \times B^n_{3\iota/2}(0) \times \mathbb{R}^n$.

(L6) $|\check L_\lambda(t, q, v)| \le C(1 + |v|^2), \quad \forall (\lambda, t, q, v) \in \hat\Lambda \times \mathbb{R} \times \bar B^n_{3\iota/2}(0) \times \mathbb{R}^n$.

Consider the Banach space

$$\mathbf{X} := \mathcal{X}^1_\tau(\mathbb{R}^n, E_{\bar\gamma})$$

with the induced norm $\|\cdot\|_{C^1}$ from $C^1([0, \tau], \mathbb{R}^n)$, and the Hilbert space

$$\mathbf{H} := \{\xi \in W^{1,2}_{\rm loc}(\mathbb{R}; \mathbb{R}^n) \,|\, E^T_{\bar\gamma}(\xi(t)) = \xi(t + \tau)\ \forall t \in \mathbb{R}\}$$

equipped with $W^{1,2}$-inner product as in (2.2). Both carry a natural $\mathbb{R}$-action given by $(\theta \cdot x)(t) = x(t + \theta)$ for $\theta \in \mathbb{R}$. The spaces $\mathbf{H}$ and $\mathbf{X}$ have the following $\mathbb{R}$-invariant open subsets

$$\mathcal{U} : \ = \ \{\xi \in W^{1,2}_{\rm loc}(\mathbb{R}; B^n_{3\iota/2}(0)) \,|\, E^T_{\bar\gamma}(\xi(t)) = \xi(t + \tau)\ \forall t \in \mathbb{R}\},$$

$$\mathcal{U}^X : \quad = \quad \mathcal{U} \cap \mathbf{X} = \mathcal{X}^1_\tau(B^n_{3\iota/2}(0), E_{\bar\gamma})$$

respectively. Define a family of functionals $\check{\mathcal{L}}_\lambda : \mathcal{U} \to \mathbb{R}$ by

$$\check{\mathcal{L}}_\lambda(x) = \int_0^\tau \check{L}_\lambda(t, x(t), \dot{x}(t))dt, \quad \lambda \in \hat\Lambda, \tag{6.12}$$

and put $\mathbf{H}^\perp := \{x \in \mathbf{H} \,|\, (\zeta_0, x)_{1,2} = 0\}$ and

$$\mathbf{X}^\perp := \{x \in \mathbf{X} \,|\, (\zeta_0, x)_{1,2} = 0\} = \mathbf{X} \cap \mathbf{H}^\perp, \tag{6.13}$$
$$\check{\mathcal{L}}^\perp_\lambda : \mathcal{U} \cap \mathbf{H}^\perp \to \mathbb{R}, \ x \mapsto \check{\mathcal{L}}_\lambda(x). \tag{6.14}$$

Remark 6.2. Because of Lemma 6.1, by [6, Theorem 4.5] we deduce that every critical point of $\check{\mathcal{L}}_\lambda$ is C^4.

6.2 Properties of functionals $\check{\mathcal{L}}$ and $\check{\mathcal{L}}^\perp_\lambda$

In order to use the abstract theorems developed in [34, 35, 37] we need to study some properties of the functionals $\check{\mathcal{L}}$ and $\check{\mathcal{L}}^\perp_\lambda$ near 0. By [32, §3] and [30], we have (i)-(iii) of the following corresponding result of Proposition 3.9.

Proposition 6.3. (i) *$\check{\mathcal{L}}_\lambda$ is C^{2-0} and the gradient map $\nabla\check{\mathcal{L}}_\lambda : \mathcal{U} \to \mathbf{H}$ has the Gâteaux derivative $B_\lambda(x) \in \mathcal{L}_s(\mathbf{H})$ at $x \in \mathcal{U}$.*

(ii) *$\nabla\check{\mathcal{L}}_\lambda$ restricts to a C^1 map $A_\lambda : \mathcal{U}^X \to \mathbf{X}$.*

(iii) *(D1) of [34, Hypothesis 1.1] and (C) of [34, Hypothesis 1.3] hold near the origin $0 \in \mathbf{H}$, i.e.,*

$$\{u \in \mathbf{H} \,|\, B_\lambda(0)u = su, \ s \le 0\} \subset \mathbf{X} \quad \text{and} \quad \{u \in \mathbf{H} \,|\, B_\lambda(0)u \in X\} \subset \mathbf{X}. \tag{6.15}$$

(iv) *Since $\check{L} = \tilde{L} = L^\star$ on $\Lambda \times \mathbb{R} \times B^n_{3\iota/2}(0) \times B^n_{\rho_0}(0)$, it holds that*

$$\check{\mathcal{L}}_\lambda(x) = \int_0^\tau \check{L}_\lambda(t, x(t), \dot{x}(t))dt = \mathcal{L}^\star_\lambda(x) \tag{6.16}$$

for each x in an open subset $\{x \in \mathcal{X}^1_\tau(B^n_{3\iota/2}(0), E_{\bar\gamma}) \,|\, \sup_t |\dot{x}(t)| < \rho_0\}$ of $\mathcal{U}^X$. Clearly, $0 \in S_0$ has an open neighborhood S_{00} in S_0 contained in the open subset. It follows that $d\check{L}_\lambda(x) = 0 \ \forall x \in S_{00}$ and that

$$m^-_\tau(\check{\mathcal{L}}_\lambda, 0) = m^-_\tau(\mathfrak{E}^\star_\lambda, 0) \quad \text{and} \quad m^0_\tau(\check{\mathcal{L}}_\lambda, 0) = m^0_\tau(\mathfrak{E}^\star_\lambda, 0). \tag{6.17}$$

By (iv) and (6.9), a point $x \in \mathcal{X}^1_\tau(B^n_{3\iota/2}(0), E_{\bar\gamma})$ near 0 is a critical point of $\mathfrak{E}^\star_\lambda$ if and only if $\gamma = \Phi_{\bar\gamma}(x) \in \mathcal{X}_\tau(M, \mathbb{I}_g)$ is a critical point of $\mathfrak{E}_\lambda$ near $\bar\gamma$. However, if $\gamma \in \mathrm{Im}(\Phi_{\bar\gamma})$ is a critical point of $\mathfrak{E}_\lambda$, so is each point in $\mathbb{R}\cdot\gamma$. Therefore for $|s|$ small enough $(\Phi_{\bar\gamma})^{-1}(s\cdot\gamma)$ is also a critical point of $\mathfrak{E}^\star_\lambda$. Such a critical point is said to be the $\mathbb{R}$-same as γ. We need to study behavior of $\mathbb{R}$-distinct critical points of $\mathfrak{E}^\star_\lambda$ near 0. Clearly, $d\check{\mathcal{L}}_\lambda(0) = 0$ and $d\check{\mathcal{L}}^\perp_\lambda(0) = 0 \ \forall\lambda$.

Denote by $\Pi : \mathbf{H} \to \mathbf{H}^\perp$ the orthogonal projection. Then $\Pi(x) = x - \frac{(x,\zeta_0)_{1,2}}{\|\zeta_0\|_{1,2}}\zeta_0$ for $x \in \mathbf{H}$, and

$$\nabla\check{\mathcal{L}}^\perp_\lambda(x) = \nabla\check{\mathcal{L}}_\lambda(x) - \frac{(\nabla\check{\mathcal{L}}_\lambda(x), \zeta_0)_{1,2}}{\|\zeta_0\|_{1,2}}\zeta_0 \quad \forall x \in \mathcal{U} \cap \mathbf{H}^\perp, \tag{6.18}$$

where by [32, (3.10)-(3.11)] (all vectors in $\mathbb{R}^n$ are understood as column vectors now)

$$
\begin{aligned}
\nabla\check{\mathcal{L}}_\lambda(\xi)(t) &= \frac{1}{2}\int_t^\infty e^{t-s}\left[\partial_q\check{L}_\lambda\big(s,\xi(s),\dot{\xi}(s)\big)-\mathfrak{R}^\xi_\lambda(s)\right]ds\\
&+\frac{1}{2}\int_{-\infty}^t e^{s-t}\left[\partial_q\check{L}_\lambda\big(s,\xi(s),\dot{\xi}(s)\big)-\mathfrak{R}^\xi(s)\right]ds+\mathfrak{R}^\xi_\lambda(t)
\end{aligned}\tag{6.19}
$$

where $\mathfrak{R}^\xi_\lambda$, (provided $2=\mathrm{ord}(S_p)>\mathrm{ord}(S_{p+1})$ for some $p\in\{0,\cdots,\sigma\}$ in (6.3)), is given by

$$
\begin{gathered}
\mathfrak{R}^\xi_\lambda(t)=\textstyle\int_0^t\partial_v\check{L}_\lambda\big(s,\xi(s),\dot{\xi}(s)\big)ds+\\
\left[\left(\oplus_{l\le p}\frac{\sin\theta_l}{2-2\cos\theta_l}\begin{pmatrix}0&-1\\1&0\end{pmatrix}-\frac{1}{2}I_{2p}\right)\oplus\mathrm{diag}(a_{p+1}(t),\cdots,a_\sigma(t))\right]\textstyle\int_0^\tau\partial_v\check{L}_\lambda\big(s,\xi(s),\dot{\xi}(s)\big)ds
\end{gathered}\tag{6.20}
$$

with $a_j(t)=\frac{2tS_j+2t+1-S_j}{4}$, $j=p+1,\cdots,\sigma=n-p$. (As usual $p=0$ (resp. $p=\sigma$) means that $\mathrm{ord}(S_1)=\cdots=\mathrm{ord}(S_\sigma)=1$ (resp. $\mathrm{ord}(S_1)=\cdots=\mathrm{ord}(S_\sigma)=2$) and hence there is no the first (resp. second) term in the square brackets in (6.20).) Note that

$$
\frac{d}{dt}\mathfrak{R}^\xi_\lambda(t)=\partial_v\check{L}_\lambda\big(t,\xi(t),\dot{\xi}(t)\big)+\mathfrak{M}\int_0^\tau\partial_v\check{L}_\lambda\big(s,\xi(s),\dot{\xi}(s)\big)ds,\tag{6.21}
$$

where $\mathfrak{M}\in\mathbb{R}^{n\times n}$ is a matrix only depending on E.

Since ζ_0 is C^1, we derive from (6.18) that $\nabla\check{\mathcal{L}}^\perp_\lambda(x)\in\mathbf{X}^\perp$ for any $x\in\mathbf{X}^\perp\cap\mathcal{U}$, and

$$
\mathbb{A}_\lambda:\mathcal{U}^X\cap\mathbf{X}^\perp\to\mathbf{X}^\perp,\ x\mapsto\nabla\check{\mathcal{L}}^\perp_\lambda(x)\tag{6.22}
$$

is C^1. By [32, §3] $\nabla\check{\mathcal{L}}^\perp_\lambda$ has the Gâteaux derivative $\mathbb{B}_\lambda(x)\in\mathcal{L}_s(\mathbf{H}^\perp)$ at $x\in\mathcal{U}\cap\mathbf{H}^\perp$ given by

$$
\mathbb{B}_\lambda(x)v=B_\lambda(x)v-\frac{(B_\lambda(x)v,\zeta_0)_2}{\|\zeta_0\|_2}\zeta_0\quad\forall v\in\mathbf{H}^\perp,\tag{6.23}
$$

where $B_\lambda(\zeta)\in\mathcal{L}_s(\mathbf{H})$ for $\zeta\in\mathcal{U}$ is a self-adjoint Fredholm operator defined by

$$
\begin{aligned}
(B_\lambda(\zeta)\xi,\eta)=\int_0^\tau\Big(&\partial_{vv}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\dot{\xi}(t),\dot{\eta}(t)\big]+\partial_{qv}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\xi(t),\dot{\eta}(t)\big]\\
&+\partial_{vq}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\dot{\xi}(t),\eta(t)\big]\\
&+\partial_{qq}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)\big[\xi(t),\eta(t)\big]\Big)\,dt
\end{aligned}\tag{6.24}
$$

for any $\xi,\eta\in\mathbf{H}$. The map B_λ has a decomposition $B_\lambda=P_\lambda+Q_\lambda$, where $P_\lambda(\zeta)\in\mathcal{L}_s(H)$ is a positive definitive linear operator defined by

$$
(P_\lambda(\zeta)\xi,\eta)_{1,2}=\int_0^\tau\Big(\partial_{vv}\check{L}_\lambda\big(t,\zeta(t),\dot{\zeta}(t)\big)[\dot{\xi}(t),\dot{\eta}(t)]+\big(\xi(t),\eta(t)\big)_{\mathbb{R}^n}\Big)\,dt,\tag{6.25}
$$

and $Q_\lambda(\zeta)\in\check{\mathcal{L}}_s(H)$ is a compact linear operator. Then

$$
\mathbb{P}_\lambda(x):=\Pi\circ P_\lambda(x)|_{\mathbf{H}^\perp}\quad\text{and}\quad\mathbb{Q}_\lambda(x):=\Pi\circ Q_\lambda(x)|_{\mathbf{H}^\perp}\tag{6.26}
$$

are positive definitive and compact, respectively, and $\mathbb{B}_\lambda(x)=\mathbb{P}_\lambda(x)+\mathbb{Q}_\lambda(x)$ by (6.23). Since $\mathbb{R}\zeta_0\subset\mathrm{Ker}(B_\lambda(0))$ we get

$$
m^-_\tau(\check{\mathcal{L}}^\perp_\lambda,0)=m^-_\tau(\check{\mathcal{L}}_\lambda,0)\quad\text{and}\quad m^0_\tau(\check{\mathcal{L}}^\perp_\lambda,0)=m^0_\tau(\check{\mathcal{L}}_\lambda,0)-1.\tag{6.27}
$$

Clearly, (L1) and (6.25) yield

$$(P_\lambda(\zeta)\xi,\xi)_{1,2} \ge \min\{c,1\}\|\xi\|^2_{1,2}, \quad \forall x\in\mathcal{U},\ \forall\xi\in\mathbf{H}, \tag{6.28}$$

and hence

$$(\mathbb{P}_\lambda(x)\xi,\xi)_{1,2} \ge \min\{c,1\}\|\xi\|^2_{1,2}, \quad \forall\xi\in\mathbf{H}^\perp, \quad \forall x\in\mathcal{U}\cap\mathbf{H}^\perp. \tag{6.29}$$

Proposition 6.4. *Let $(\lambda_k)\subset\Lambda$ and $(\zeta_k)\subset\mathcal{U}$ converge to $\mu\in\Lambda$ and $0\in\mathcal{U}$, respectively. Then $\|P_{\lambda_k}(\zeta_k)\xi - P_\mu(0)\xi\|_{1,2}\to 0$ for each $\xi\in H$. In particular, if $(\zeta_k)\subset\mathcal{U}\cap\mathbf{H}^\perp$ converges to zero, then $\|\mathbb{P}_{\lambda_k}(\zeta_k)\xi - \mathbb{P}_\mu(0)\xi\|_{1,2}\to 0$ for any $\xi\in\mathbf{H}^\perp$.*

Proof. By (6.25) we have

$$\|[P_{\lambda_k}(\zeta_k) - P_\mu(0)]\xi\|^2_{1,2} \le \int_0^\tau \left|[\partial_{vv}\check{L}_{\lambda_k}\big(t,\zeta_k(t),\dot\zeta_k(t)\big) - \partial_{vv}\check{L}_\mu\big(t,0,0\big)]\dot\xi(t)\right|^2_{\mathbb{R}^n} dt.$$

Note that $\|\zeta_k\|_{1,2}\to 0$ implies $\|\zeta_k\|_{C^0}\to 0$. Since $(\lambda,t,x,v)\mapsto\partial_{vv}\check{L}_\lambda(t,x,v)$ is continuous, by the third inequality in (L2) in Lemma 6.1 we may apply [37, Prop. B.9] ([35, Prop. C.1]) to

$$f(t,\eta;\lambda) = \partial_{vv}\check{L}(\lambda,t,\zeta_k(t),\dot\zeta_k(t))\eta$$

to get that

$$\int_0^\tau \left|[\partial_{vv}\check{L}_{\lambda_k}\big(t,\zeta_k(t),\dot\zeta_k(t)\big) - \partial_{vv}\check{L}_\mu\big(t,0,0\big)]\dot\xi(t)\right|^2_{\mathbb{R}^n} dt \to 0.$$

Moreover, the Lebesgue dominated convergence theorem also leads to

$$\int_0^\tau \left|[\partial_{vv}\check{L}_{\lambda_k}\big(t,0,0\big) - \partial_{vv}\check{L}_\mu\big(t,0,0\big)]\dot\xi(t)\right|^2_{\mathbb{R}^n} dt \to 0.$$

Hence $\|[P_{\lambda_k}(\zeta_k) - P_\mu(0)]\xi\|_{1,2}\to 0$. The final claim follows from this and (6.26). □

Proposition 6.5. *$\mathcal{U}\ni\zeta\mapsto Q_\lambda(\zeta)\in\mathcal{L}_s(\mathbf{H})$ is uniformly continuous at 0 with respect to $\lambda\in\Lambda$. Moreover, if $(\lambda_k)\subset\Lambda$ converges to $0\in\Lambda$ then $\|Q_{\lambda_k}(0) - Q_\mu(0)\|\to 0$.*

Proof. Write $Q_\lambda(\zeta) := Q_{\lambda,1}(\zeta) + Q_{\lambda,2}(\zeta) + Q_{\lambda,3}(\zeta)$, where

$$\begin{aligned}
(Q_{\lambda,1}(\zeta)\xi,\eta)_{1,2} &= \int_0^\tau \partial_{vq}\check{L}_\lambda\big(t,\zeta(t),\dot\zeta(t)\big)[\dot\xi(t),\eta(t)]dt,\\
(Q_{\lambda,2}(\zeta)\xi,\eta)_{1,2} &= \int_0^\tau \partial_{qv}\check{L}_\lambda\big(t,\zeta(t),\dot\zeta(t)\big)[\xi(t),\dot\eta(t)]dt,\\
(Q_{\lambda,3}(\zeta)\xi,\eta)_{1,2} &= \int_0^\tau \Big(\partial_{qq}\check{L}_\lambda\big(t,\zeta(t),\dot\zeta(t)\big)[\xi(t),\eta(t)] - \big(\xi(t),\eta(t)\big)_{\mathbb{R}^n}\Big)\,dt.
\end{aligned}$$

As above the first claim follows from (L2) in Lemma 6.1 and [37, Prop. B.9] ([35, Prop. C.1]) directly.

In order to prove the second claim, as in the proof of [30, page 571] we have

$$\begin{aligned}
&\|Q_{\lambda_k,1}(0) - Q_{\mu,1}(0)\|_{\mathcal{L}(\mathbf{H})}\\
\le\ & 2(e^\tau+1)\left(\int_0^\tau \left|\partial_{vq}\check{L}_{\lambda_k}(s,0,0) - \partial_{vq}\check{L}_\mu(s,0,0)\right|^2 ds\right)^{1/2}.
\end{aligned}$$

Because of the second inequality in (**L2**), it follows from the Lebesgue dominated convergence theorem that $\|Q_{\lambda_k,1}(0)-Q_{\mu,1}(0)\|_{\mathcal{L}(\mathbf{H})}\to 0$. Observe that $(Q_{\lambda,2}(\zeta)\xi,\eta)_{1,2}=\big(\xi,(Q_{\lambda,1}(\zeta))^*\eta\big)_{1,2}$. Hence $\|Q_{\lambda_k,2}(0)-Q_{\mu,2}(0)\|_{\mathcal{L}(\mathbf{H})}\to 0$. Finally, it is easy to deduce that

$$\|Q_{\lambda_k,3}(0)-Q_{\mu,3}(0)\|^2_{\mathcal{L}(\mathbf{H})}\le\int_0^\tau\Big|\partial_{qq}\check{L}_{\lambda_k}(t,0,0)-\partial_{qq}\check{L}_{\mu}(t,0,0)\Big|^2\,dt.$$

By the Lebesgue dominated convergence theorem the right side converges to zero. Then $\|Q_{\lambda_k,3}(0)-Q_{\mu,3}(0)\|_{\mathcal{L}(\mathbf{H})}\to 0$ and therefore $\|Q_{\lambda_k}(0)-Q_{\mu}(0)\|\to 0$. □

By (6.26) and Proposition 6.5, maps $\mathcal{U}\cap\mathbf{H}^\perp\ni\zeta\mapsto\mathbb{Q}_\lambda(\zeta)\in\mathcal{L}_s(\mathbf{H}^\perp)$ is uniformly continuous at 0 with respect to $\lambda\in\Lambda$. Moreover, if $(\lambda_k)\subset\Lambda$ converges to $0\in\Lambda$ then $\|\mathbb{Q}_{\lambda_k}(0)-\mathbb{Q}_{\mu}(0)\|\to 0$.

Proposition 6.6. $(\mathbf{H}^\perp,\mathbf{X}^\perp,\check{\mathcal{L}}^\perp_\lambda,\mathbb{A}_\lambda=\nabla\check{\mathcal{L}}^\perp_\lambda,\mathbb{B}_\lambda)$ *satisfies (C) of [37, Hypothesis B.2] ([34, Hypothesis 1.3]) and (D1) of [37, Hypothesis B.1] ([34, Hypothesis 1.1]) at the origin* $0\in\mathbf{H}^\perp$*, namely*

(C) $\{u\in\mathbf{H}^\perp\,|\,\mathbb{B}_\lambda(0)u\in\mathbf{X}^\perp\}\subset\mathbf{X}^\perp$,

(D1) $\{u\in\mathbf{H}^\perp\,|\,\mathbb{B}_\lambda(0)u=su,\ s\le 0\}\subset\mathbf{X}^\perp$.

Proof. In order to prove (C) let $u\in\mathbf{H}^\perp$ be such that $v:=\mathbb{B}_\lambda(0)u\in\mathbf{X}^\perp$. By (6.23)

$$B_\lambda(0)u-\frac{(B_\lambda(0)u,\zeta_0)_2}{\|\zeta_0\|_2}\zeta_0=\mathbb{B}_\lambda(0)u=v\in\mathbf{X}^\perp.$$

Since $\zeta_0\in\mathbf{X}$, it follows from this and (6.15) that $u\in\mathbf{X}$ and hence $u\in\mathbf{X}^\perp$.

Next, let $u\in\mathbf{H}^\perp$ satisfy $\mathbb{B}_\lambda(0)u=su$ for some $s\le 0$. Then (6.23) leads to

$$B_\lambda(0)u=su+\frac{(B_\lambda(0)u,\zeta_0)_2}{\|\zeta_0\|_2}\zeta_0.$$

Since $T_0S_0=\mathbb{R}\zeta_0$ and $\nabla\check{\mathcal{L}}_\lambda(x)=0\,\forall x\in S_0$, we deduce $B_\lambda(0)\zeta_0=0$ and therefore $B_\lambda(0)u=su$. By (6.15) this implies $u\in\mathbf{X}$ and so $u\in\mathbf{X}^\perp$.

□

We also need the following two corresponding results of Propositions 3.10, 3.11.

Proposition 6.7. *Both* $\hat{\Lambda}\times\mathcal{U}\ni(\lambda,x)\mapsto\check{\mathcal{L}}_\lambda(x)\in\mathbb{R}$ *and* $\hat{\Lambda}\times\mathcal{U}^X\ni(\lambda,x)\mapsto A_\lambda(x)\in\mathbf{X}$ *are continuous.*

Proofs of this proposition and the following key result are similar to those of Propositions 3.10, 3.11. For completeness their proof are put off until Section 6.4 because they are rather long.

Proposition 6.8. *Let* $\bar{\varepsilon}>0$ *be such that* $B_{\mathbf{H}}(0,\bar{\varepsilon})\subset\mathcal{U}$*. For any given* $\epsilon>0$ *there exists* $0<\varepsilon\le\bar{\varepsilon}$ *such that if* $x\in B_{\mathbf{H}^\perp}(0,\varepsilon):=\{x\,|\,x\in\mathbf{H}^\perp,\ \|x\|_{1,2}<\varepsilon\}$ *is a critical point of* $\check{\mathcal{L}}^\perp_\lambda$ *with some* $\lambda\in\hat{\Lambda}$ *then* x *is a critical point of* $\check{\mathcal{L}}_\lambda$*, belongs to* $C^4(\mathbb{R};\mathbb{R}^n)$ *and also satisfies* $\|x\|_{C^2}<\epsilon$*. In particular, for* $x\in B_{\mathbf{H}^\perp}(0,\varepsilon)$*,* $d\check{\mathcal{L}}^\perp_\lambda(x)=0$ *if and only if* $d\check{\mathcal{L}}_\lambda(x)=0$*.*

6.3 Completing the proofs of Theorems 1.23, 1.24, 1.25, 1.26

The ideas are the same as the proofs of [37, Theorems 1.18, 1.19, 1.21]. But the corresponding checks and computations are much more complex and difficult.

Proposition 6.9 ([5, Proposition 3.5])**.** *Let P be a finite-dimensional manifold, N a (possibly infinite dimensional) Banach manifold, $Q \subset N$ a Banach submanifold, and A a topological space. Assume that $\chi : A \times P \to N$ is a continuous function such that there exist $a_0 \in A$ and $m_0 \in P$ with:*

(a) $\chi(a_0, m_0) \in Q$*;*

(b) $\chi(a_0, \cdot) : P \to N$ *is of class* C^1*;*

(c) $\partial_2\chi(a_0, m_0)(T_{m_0}P) + T_{\chi(a_0,m_0)}Q = T_{\chi(a_0,m_0)}N$.

Then, for $a \in A$ near a_0, $\chi(a, P) \cap Q \neq \emptyset$.

For $0 < \delta \le 3\iota/2$ put

$$B_{\mathbf{X}^\perp}(0, \delta) := \{\xi \in C^1_{E_{\bar\gamma},\tau}(\mathbb{R}, B^n_{3\iota/2}(0)) \,|\, \|\xi\|_{C^1} < \delta\} \quad \text{and} \quad \Omega_\delta := \Phi_{\bar\gamma}\left(B_{\mathbf{X}^\perp}(0, \delta)\right).$$

Clearly, Ω_δ is a C^2 Banach submanifold of $\mathcal{X}_\tau(M, \mathbb{I}_g)$. For the action χ in (6.10), since $\bar\gamma$ is nonconstant and C^6,

$$\chi(\bar\gamma, \cdot) : \mathbb{R} \to \mathcal{X}_\tau(M, \mathbb{I}_g),\ s \mapsto \chi(\bar\gamma, s)$$

is a C^4 one-to-one immersion, and

$$\partial_2\chi(\bar\gamma, 0)(T_0\mathbb{R}) = \mathbb{R}\dot{\bar\gamma} = d\Phi_{\bar\gamma}(0)(\mathbb{R}\zeta_0) \quad \text{and} \quad T_{\chi(\bar\gamma,0)}\Omega_\delta = T_{\bar\gamma}\Omega_\delta = d\Phi_{\bar\gamma}(0)(\mathbf{X}^\perp),$$

we have $\partial_2\chi(\bar\gamma, 0)(T_0\mathbb{R}) + T_{\chi(\bar\gamma,0)}\Omega_\delta = T_{\chi(\bar\gamma,)}\mathcal{X}_\tau(M, \mathbb{I}_g)$. Applying Proposition 6.9 to $A = N = \mathcal{X}_\tau(M, \mathbb{I}_g)$, $P = \mathbb{R}$, $Q = \Omega_\delta$, $a_0 = \bar\gamma$, $m_0 = 0$ we get:

Proposition 6.10. *For any given $0 < \delta \le 2\iota$, if $\gamma \in \mathcal{X}_\tau(M, \mathbb{I}_g)$ is close to $\bar\gamma$, then $(\mathbb{R}\cdot\gamma) \cap \Omega_\delta \neq \emptyset$, that is, $\mathbb{R} \cdot \Omega_\delta$ is a neighborhood of the orbit $\mathcal{O} = \mathbb{R} \cdot \bar\gamma$ in $\mathcal{X}_\tau(M, \mathbb{I}_g)$.*

Proof of Theorem 1.23. By the assumptions there exists a sequence $(\lambda_k) \subset \Lambda$ converging to $\mu \in \Lambda$ such that the problem (1.23) with $\lambda = \lambda_k$ has solutions γ_k, $k = 1, 2, \cdots$, which are $\mathbb{R}$-distinct each other and satisfy $\gamma_k|_{[0,\tau]} \to \bar\gamma|_{[0,\tau]}$ in $C^1([0, \tau]; M)$. Then $\hat\Lambda = \{\mu, \lambda_k \,|\, k \in \mathbb{N}\}$ is compact and sequential compact. Take a decreasing sequence of positive numbers $\delta_m \le 2\iota$ such that $\delta_m \to 0$. For each δ_m, by Proposition 6.10 we have $\gamma_{k_m} \in \mathbb{R} \cdot \Omega_{\delta_m}$ and thus $\beta_m := s_m \cdot \gamma_{k_m} \in \Omega_{\delta_m}$ for some $s_m \in \mathbb{R}$. Note that each β_m is a critical point of $\mathfrak{E}_{\lambda_{k_m}}$ on $\Phi_{\bar\gamma}(\mathcal{U}^X)$. Since any two of (γ_k) are $\mathbb{R}$-distinct, so are any two of (β_m).

Note that $\Omega_\delta \subset \Phi_{\bar\gamma}\left(\mathcal{U}^X\right) \subset \Phi_{\bar\gamma}\left(C^1_{E_{\bar\gamma},\tau}(\mathbb{R}, B^n_{3\iota/2}(0))\right)$, and by (6.9) and (6.16) we have

$$\check{\mathcal{L}}_\lambda(x) = \mathfrak{E}_\lambda\left(\Phi_{\bar\gamma}(x)\right) \quad \forall x \in B_{\mathbf{X}^\perp}(0, \delta) \tag{6.30}$$

because $\delta \le 2\iota < \rho_0$ and $x \in B_{\mathbf{X}^\perp}(0, \delta)$ imply that for all $t \in \mathbb{R}$,

$$(t, x(t), \dot{x}(t)) \in \mathbb{R} \times B^n_{2\iota}(0) \times B^n_{\rho_0}(0)$$

and so $\check{L}(\lambda, t, x(t), \dot{x}(t)) = L^\star(\lambda, t, x(t), \dot{x}(t))$.

It follows that each

$$x_m := (\Phi_{\bar{\gamma}})^{-1}(\beta_m) \in \{\xi \in \mathbf{X}^\perp \,|\, \|\xi\|_{C^1} < \delta_m\}$$

is a critical point of $\check{\mathcal{L}}_{\lambda_{k_m}}$ in $\mathbf{H}$ (and hence that of $\check{\mathcal{L}}^\perp_{\lambda_{k_m}}$ in $\mathbf{H}^\perp$) and they are distinct each other. Moreover $\|x_m\|_{1,2} \le \sqrt{\tau}\|x_m\|_{C^1} \to 0$. Hence $(\mu, 0) \in \Lambda \times (\mathcal{U} \cap \mathbf{H}^\perp)$ is a bifurcation point of $\nabla\check{\mathcal{L}}^\perp_\lambda = 0$ in $\Lambda \times (\mathcal{U} \cap \mathbf{H}^\perp)$.

Since $\check{\mathcal{L}}_\lambda$ is C^{2-0} and $\nabla\check{\mathcal{L}}_\lambda : \mathcal{U} \to \mathbf{H}$ has the Gâteaux derivative $B_\lambda(x) \in \mathcal{L}_s(\mathbf{H})$ at $x \in \mathcal{U}$, $\check{\mathcal{L}}^\perp_\lambda$ is C^{2-0} and $\nabla\check{\mathcal{L}}^\perp_\lambda$ has a Gâteaux derivative $\mathbb{B}_\lambda(x) \in \mathcal{L}_s(\mathbf{H}^\perp)$ at $x \in \mathcal{U} \cap \mathbf{H}^\perp$ given by (6.23). From (6.26), (6.29) and Propositions 6.4, 6.5 it easily follows that the conditions (i)-(iv) of [34, Theorem 3.1] ([37, Theorem C.6]) are satisfied with $\mathcal{F}_\lambda = \check{\mathcal{L}}^\perp_\lambda$ and $H = X = \mathbf{H}^\perp$ and $U = \mathcal{U} \cap \mathbf{H}^\perp$. Therefore $m^0_\tau(\check{\mathcal{L}}^\perp_\mu, 0) \ge 1$. This and (6.11), (6.17) and (6.27) lead to $m^0_\tau(\mathfrak{E}_\mu, \bar{\gamma}) = m^0_\tau(\check{\mathcal{L}}_\mu, 0) \ge 2$. □

Note that Propositions 6.8, 6.7 are not used in the proof of Theorem 1.23. However, they are necessary for proofs of Theorems 1.24,1.26.

Proof of Theorem 1.24. The original Λ can be replaced by compact and sequential compact $\hat{\Lambda} = \{\mu, \lambda^+_k, \lambda^-_k \,|\, k \in \mathbb{N}\}$. Follow the notations above. By Propositions 6.7, 6.4, 6.5, and (6.15) and (6.28), the conditions of [36, Theorem 3.3] (or [34, Theorem A.3]) are satisfied with $\mathcal{L}_\lambda = \check{\mathcal{L}}_\lambda$, $H = \mathbf{H}$, $X = \mathbf{X}$, $U = \mathcal{U}$ and $\lambda^* = \mu$. From these, (6.22)-(6.23) and (6.26), (6.29) and Proposition 6.6 it follows that $\mathcal{L}_\lambda = \check{\mathcal{L}}^\perp_\lambda$, $H = \mathbf{H}^\perp$, $X = \mathbf{X}^\perp$, $U = \mathcal{U} \cap \mathbf{H}^\perp$ and $\lambda^* = \mu$ satisfy the conditions of [36, Theorem 3.3] (or [34, Theorem A.3]).

By the assumptions (a)–(b) of Theorem 1.24 we may use (6.11), (6.17) and (6.27) to deduce that $m^0_\tau(\check{\mathcal{L}}^\perp_\mu, 0) \ge 1$ and that for each $k \in \mathbb{N}$,

$$[m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^-_k}, 0), m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^-_k}, 0) + m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^-_k}, 0)] \cap [m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^+_k}, 0), m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^+_k}, 0) + m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^+_k}, 0)] = \emptyset$$

and either $m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^-_k}, 0) = 0$ or $m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^+_k}, 0) = 0$.

Thus [37, Theorem C.4] concludes that there exists an infinite sequence $(\lambda_k, x_k) \subset \hat{\Lambda} \times \mathbf{H}^\perp \setminus \{(\mu, 0)\}$ converging to $(\mu, 0)$ such that each $x_k \ne 0$ and satisfies $\check{\mathcal{L}}^\perp_{\lambda_k}(x_k) = 0$ for all $k \in \mathbb{N}$.

Fix $0 < \delta \le 2\iota$. Let Ω_δ be as in Proposition 6.10. By Proposition 6.8, passing to a subsequence (if necessary) we may assume: each x_k is C^4 and a critical point of $\check{\mathcal{L}}_{\lambda_k}$, $\|x_k\|_{C^2} < \delta$ $\forall k$ and $\|x_k\|_{C^2} \to 0$. Then each $\gamma_k := \Phi_{\bar{\gamma}}(x_k) \in \Omega_\delta$ is a C^6 solution of the corresponding problem (1.18) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, and (γ_k) converges to $\bar{\gamma}$ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $k \to \infty$.

Since $d\Phi_{\bar{\gamma}}(0)[\zeta_0] = \dot{\bar{\gamma}}$ and $T_{\bar{\gamma}}\Omega_\delta = d\Phi_{\bar{\gamma}}(0)(\mathbf{X}^\perp)$, $\mathbb{R}\zeta_0 + \mathbf{X}^\perp = \mathbf{X}$ implies $\mathbb{R}\dot{\bar{\gamma}} + T_{\bar{\gamma}}\Omega_\delta = T_{\bar{\gamma}}\mathcal{X}^1_\tau(M, \mathbb{I}_g)$, that is, the C^4 embedded circle $\mathcal{O} = \mathbb{R} \cdot \bar{\gamma}$ (because of periodicity of $\bar{\gamma}$) and Ω_δ are transversely intersecting at $\bar{\gamma}$. It follows that there exists a neighborhood $\mathcal{V}$ of $\bar{\gamma}$ in $\mathcal{X}^1_\tau(M, \mathbb{I}_g)$ such that $\mathcal{V} \cap \mathcal{O} \cap \Omega_\delta = \{\bar{\gamma}\}$. Because $\|x_k\|_{C^2} \to 0$, there exists $k_0 > 0$ such that for each $k > k_0$, $\gamma_k = \Phi_{\bar{\gamma}}(x_k) \in \mathcal{V} \cap \Omega_\delta \setminus \{\bar{\gamma}\}$ and the C^4 immersed submanifold $\mathbb{R} \cdot \gamma_k$ transversely intersect with Ω_δ at γ_k. Hence

$$\mathbb{R} \cdot \gamma_k \ne \mathcal{O} \quad \text{for any } k > k_0. \tag{6.31}$$

(Otherwise, $\mathcal{O}$ and Ω_δ have at least two distinct intersecting points γ_k and $\bar{\gamma}$ in $\mathcal{V}$.)

We conclude that $\{\mathbb{R} \cdot \gamma_k \,|\, k \in \mathbb{N}\}$ is an infinite set. (Thus (γ_k) has a subsequence which only consists of $\mathbb{R}$-distinct elements. The proof is completed.) Otherwise, passing to a subsequence we may assume that all γ_k are $\mathbb{R}$-same, i.e., $\gamma_k = s_k \cdot \gamma^*$ for some $s_k \in \mathbb{R}$, where $\gamma^* : \mathbb{R} \to M$ satisfies (1.23) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$. Since all its partial derivatives of $L(\lambda, \cdot)$ of order no more than two depend continuously on $(\lambda, x, v) \in \Lambda \times TM$, it easily follows that γ^* satisfies (1.23) with $\lambda = \mu$. Clearly, $\bar{\gamma}$ sits in the intersection of $\mathcal{O}$ and the closure of $\mathbb{R}\cdot\gamma^*$. By the assumption (c) of Theorem 1.24 $\mathbb{R}\cdot\gamma^*$ is closed and so equal to $\mathcal{O}$, namely $\mathbb{R}\cdot\gamma_k = \mathcal{O}\ \forall k$, which contradicts (6.31).

□

In order to prove Theorem 1.26 we also need some preparations. Consider the C^4 Hilbert–Riemannian manifold

$$\Lambda_\tau(M, \mathbb{I}_g) = \{\gamma \in W^{1,2}_{loc}(\mathbb{R}, M)\,|\,\gamma(t+\tau) = \mathbb{I}_g(\gamma(t))\ \forall t\}$$

with the natural Riemannian metric given by (1.3); see [48, Theorem 4.2] (or [46, Theorem(8)]). Let $\|\cdot\|_1 = \sqrt{\langle\cdot,\cdot\rangle_1}$ be the induced norm.

Since $\mathbb{R}_{\bar{\gamma}}$ is an infinite cyclic subgroup of $\mathbb{R}$ with generator $p > 0$, i.e., $\bar{\gamma}$ has the least period p, the orbit $\mathcal{O} := \mathbb{R}\cdot\bar{\gamma}$ is an $\mathbb{R}$-invariant compact connected C^3 submanifold of $\Lambda_\tau(M, \mathbb{I}_g)$, precisely an C^3 embedded circle $S^1(p) := \mathbb{R}/p\mathbb{Z}$. Let $\pi : N\mathcal{O} \to \mathcal{O}$ be the normal bundle of $\mathcal{O}$ in $\Lambda_\tau(M, \mathbb{I}_g)$. It is a C^2 Hilbert vector bundle over $\mathcal{O}$ (because $T\mathcal{O}$ is a C^2 subbundle of $T_{\mathcal{O}}\Lambda_\tau(M, \mathbb{I}_g)$), and

$$XN\mathcal{O} := T_{\mathcal{O}}\mathcal{X}_\tau(M, \mathbb{I}_g) \cap N\mathcal{O}$$

is a C^2 Banach vector subbundle of $T_{\mathcal{O}}\mathcal{X}_\tau(M, \mathbb{I}_g)$ by [32, Proposition 5.1]. Recall that 3ι is less than the injectivity radius of g at each point on $\bar{\gamma}(\mathbb{R})$. For $0 < \nu \le 3\iota$ we define

$$N\mathcal{O}(\nu) := \{(\gamma, v) \in N\mathcal{O}\,|\,\|v\|_{1,2} < \nu\} \quad \text{and} \quad XN\mathcal{O}(\nu) := \{(\gamma, v) \in XN\mathcal{O}\,|\,\|v\|_{C^1} < \nu\}.$$

Clearly, $XN\mathcal{O}(\nu) \subset N\mathcal{O}(\sqrt{\tau}\nu)$ and there exist natural induced $\mathbb{R}$-actions on these bundles given by

$$(\gamma, v) \mapsto (s\cdot\gamma, s\cdot v) \quad \forall s \in \mathbb{R}.$$

Using the exponential map exp of g we define the map

$$\mathrm{EXP} : T\Lambda_\tau(M, \mathbb{I}_g)(\sqrt{\tau}\nu) = \{(\gamma, v) \in T\Lambda_\tau(M, \mathbb{I}_g)\,|\,\|v\|_{1,2} < \sqrt{\tau}\nu\} \to \Lambda_\tau(M, \mathbb{I}_g) \tag{6.32}$$

by $\mathrm{EXP}(\gamma, v)(t) = \exp_{\gamma(t)} v(t)\ \forall t \in \mathbb{R}$. Clearly, EXP is equivariant, i.e.,

$$s\cdot(\mathrm{EXP}(\gamma, v)) = \mathrm{EXP}(s\cdot\gamma, s\cdot v) \quad \forall s \in \mathbb{R}.$$

It follows from [32, Lemma 5.2] that EXP is C^2. For sufficiently small $\nu > 0$, EXP gives rise to a C^2 diffeomorphism $\digamma : N\mathcal{O}(\sqrt{\tau}\nu) \to \mathcal{N}(\mathcal{O}, \sqrt{\tau}\nu)$, where $\mathcal{N}(\mathcal{O}, \sqrt{\tau}\nu)$ is an open neighborhood of $\mathcal{O}$ in $\Lambda_\tau(M, \mathbb{I}_g)$. Let $\mathcal{X}(\mathcal{O}, \nu) := \digamma(XN\mathcal{O}(\nu))$, which is contained in $\mathcal{N}(\mathcal{O}, \sqrt{\tau}\nu)$.

Lemma 6.11. *Suppose that $(\mathbb{I}_g)^l = id_M$ for some $l \in \mathbb{N}$, and that $0 < \delta \le 2\iota$ is so small that*

$$\Omega_\delta = \Phi_{\bar{\gamma}}\left(B_{\mathbf{X}^\perp}(0, \delta)\right) \subset \mathcal{X}(\mathcal{O}, \nu).$$

Then for any two different points $\gamma_i \in \Omega_\delta$, $i = 1, 2$, either they are $\mathbb{R}$-distinct, or there exists an integer $0 < m \le l$ such that $\gamma_2 = (m\tau)\cdot\gamma_2$ or $\gamma_1 = (m\tau)\cdot\gamma_1$. In particular, if $l = 1$ and τ is equal to the minimal period p of $\bar{\gamma}$, then any two different points in Ω_δ are $\mathbb{R}$-distinct.

Proof. Let different points $\xi_1, \xi_2 \in B_{\mathbf{X}^\perp}(0, \delta)$ be such that $\gamma_1 = \Phi_{\bar{\gamma}}(\xi_1)$ and $\gamma_2 = \Phi_{\bar{\gamma}}(\xi_2)$ are $\mathbb{R}$-same. Then we have $s \ge 0$ such that $s\cdot\gamma_1 = \gamma_2$. Since $\mathbb{I}_g^l = id_M$ implies $(kl\tau)\cdot\gamma_1 = \gamma_1$ and

$(s-kl\tau)\cdot\gamma_1 = s\cdot\gamma_1$ for any $k\in\mathbb{Z}$ we can assume $0\le s<l\tau$. By (6.6), $(\bar\gamma,\sum_{i=1}^n\xi_2^ie_i)$ and $(s\cdot\bar\gamma,\sum_{i=1}^n(s\cdot\xi_1^i)(s\cdot e_i))$ belong to $N\mathcal{O}(\sqrt{\tau}\nu)$. Note that

$$\begin{aligned}(s\cdot\gamma_1)(t) &= \gamma_1(s+t)=\exp_{\bar\gamma(s+t)}\left(\sum_{i=1}^n\xi_1^i(s+t)e_i(s+t)\right)\\ &= \mathrm{EXP}\left(s\cdot\bar\gamma,\sum_{i=1}^n(s\cdot\xi_1^i)(s\cdot e_i)\right)(t)\end{aligned}$$

and so

$$s\cdot\gamma_1=\digamma\left(s\cdot\bar\gamma,\sum_{i=1}^n(s\cdot\xi_1^i)(s\cdot e_i)\right).$$

Similarity, we have

$$\gamma_2=\digamma\left(\bar\gamma,\sum_{i=1}^n\xi_2^ie_i\right).$$

Then $s\cdot\bar\gamma=\bar\gamma$ and

$$\sum_{i=1}^n(s\cdot\xi_1^i)(s\cdot e_i)=\sum_{i=1}^n\xi_2^ie_i.$$

The former implies $s\in\mathbb{R}_{\bar\gamma}\subset\{[0],\cdots,[(l-1)\tau]\}$, where $[q\tau]=q\tau+l\mathbb{Z}$. Hence $s\in\{0,\cdots,l-1\}$. Combing with the latter we obtain $\xi_1=\xi_2$, and therefore a contradiction.

When $l=1$ and τ is equal to the minimal period p of $\bar\gamma$, $\mathbb{R}_{\bar\gamma}=\{0\}$ and so $\xi_1=\xi_2$. A contradiction is obtained.

□

Proof of Theorem 1.26. The original Λ may be replaced by $\hat\Lambda=[\mu-\varepsilon,\mu+\varepsilon]$. By the first paragraph in the proof of Theorem 1.24 we have checked that $\mathcal{L}_\lambda=\check{\mathcal{L}}_\lambda^\perp$, $H=\mathbf{H}^\perp$, $X=\mathbf{X}^\perp$, $U=\mathcal{U}\cap\mathbf{H}^\perp$ and $\lambda^*=\mu$ satisfy the conditions of [36, Theorem 3.6] except for the condition (f).

By the assumptions of Theorem 1.26 and (6.11), (6.17) and (6.27) we get that $m_\tau^0(\check{\mathcal{L}}_\mu^\perp,0)\ge1$ and $m_\tau^0(\check{\mathcal{L}}_\lambda^\perp,0)=0$ for each $\lambda\in\hat\Lambda\setminus\{\mu\}$ near μ, and that $m_\tau^-(\mathcal{L}_\lambda^\perp,0)$ takes, respectively, values $m_\tau^0(\check{\mathcal{L}}_\mu^\perp,0)$ and $m_\tau^-(\check{\mathcal{L}}_\mu^\perp,0)+m_\tau^0(\check{\mathcal{L}}_\mu^\perp,0)-1$ as $\lambda\in\hat\Lambda$ varies in two deleted half neighborhoods of μ. These mean that the condition (f) of [37, Theorem C.7] ([36, Theorem 3.6]) is satisfied. Therefore one of the following alternatives occurs:

(i) There exists a sequence $(x_k)\subset\mathbf{H}^\perp\setminus\{0\}$ converging to 0 in $\mathbf{H}^\perp$ such that $\nabla\check{\mathcal{L}}_\mu^\perp(x_k)=0$ for all k.

(ii) For each $\lambda\in\Lambda\setminus\{\mu\}$ near μ, $\nabla\check{\mathcal{L}}_\lambda^\perp(w)=0$ has a solution $x_\lambda\in\mathbf{X}^\perp$ different from 0, which converges to 0 in $\mathbf{X}^\perp$ as $\lambda\to\mu$.

(iii) Given a neighborhood $\mathfrak{W}$ of 0 in $\mathbf{X}^\perp$, there is an one-sided neighborhood Λ^0 of μ such that for any $\lambda\in\Lambda^0\setminus\{\mu\}$, $\nabla\check{\mathcal{L}}_\lambda^\perp(w)=0$ has at least two nonzero solutions in $\mathfrak{W}$, x_λ^1 and x_λ^2, which can also be required to satisfy $\check{\mathcal{L}}_\lambda^\perp(x_\lambda^1)\ne\check{\mathcal{L}}_\lambda^\perp(x_\lambda^2)$ provided that $m_\tau^0(\check{\mathcal{L}}_\mu^\perp,0)\ge2$ and $\nabla\check{\mathcal{L}}_\lambda^\perp(w)=0$ has only finitely many nonzero solutions in $\mathfrak{W}$.

Let $\delta>0$ satisfy Proposition 6.10 and Lemma 6.11. By Proposition 3.11, we obtain:

- In case (i), passing to a subsequence (if necessary) all x_k are C^4 and satisfy: $\nabla\check{\mathcal{L}}_\mu(x_k)=0$, $0<\|x_k\|_{C^2}<\delta$ and $\|x_k\|_{C^2}\to0$. Therefore each $\gamma_k:=\Phi_{\bar\gamma}(x_k)\in\Omega_\delta$ is a C^6 solution of the corresponding problem (1.23) with $\lambda=\mu$, and (γ_k) converges to $\bar\gamma$ on any compact interval $I\subset\mathbb{R}$ in C^2-topology as $k\to\infty$.

- In case (ii), when $\lambda \in \Lambda \setminus \{\mu\}$ is close to μ, all x_λ are C^4 and satisfy: $\nabla \check{\mathcal{L}}_\lambda(x_\lambda) = 0$, $0 < \|x_\lambda\|_{C^2} < \delta$ and $\|x_\lambda\|_{C^2} \to 0$ as $\lambda \to \mu$. Hence each $\gamma_\lambda := \Phi_{\bar{\gamma}}(x_\lambda) \in \Omega_\delta \setminus \{\bar{\gamma}\}$ is a C^6 solution of the corresponding problem (1.23), γ_λ converges to $\bar{\gamma}$ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $\lambda \to \mu$, and $\mathbb{R} \cdot \gamma_\lambda \neq \mathcal{O}$ by Lemma 6.11.

- In case (iii), we can require that the neighborhood $\mathfrak{W}$ so small that $\Phi_{\bar{\gamma}}(\mathfrak{W}) \subset \mathcal{W}$ and $\mathfrak{W} \subset B_{\mathbf{X}^\perp}(0, \delta)$. The latter implies $\Phi_{\bar{\gamma}}(\mathfrak{W}) \subset \Omega_\delta$. Then all x^1_λ and x^2_λ are C^4 and critical points of $\check{\mathcal{L}}_\lambda$, and satisfy: $0 < \|x^i_\lambda\|_{C^2} < \delta$ and $\|x^i_\lambda\|_{C^2} \to 0$, $i = 1, 2$. Consequently, $\gamma^1_\lambda := \Phi_{\bar{\gamma}}(x^1_\lambda)$ and $\gamma^2_\lambda := \Phi_{\bar{\gamma}}(x^2_\lambda)$ belong to $\Omega_\delta \cap \mathcal{W} \setminus \mathbb{R} \cdot \gamma_0$, are C^6 solutions of the corresponding problem (1.23). When $m^0_\tau(\mathfrak{E}_\mu, \bar{\gamma}) = m^0_\tau(\check{\mathcal{L}}^\perp_\mu, 0) + 1 \geq 3$, and (1.23) with parameter value λ has only finitely many $\mathbb{R}$-distinct solutions in $\mathcal{W}$ which are $\mathbb{R}$-distinct from $\bar{\gamma}$, it is clear that $\nabla \check{\mathcal{L}}^\perp_\lambda(w) = 0$ has only finitely many nonzero solutions in $\mathfrak{W}$, and therefore we can require that x^1_λ and x^2_λ satisfy $\check{\mathcal{L}}^\perp_\lambda(x^1_\lambda) \neq \check{\mathcal{L}}^\perp_\lambda(x^2_\lambda)$, which implies $\mathfrak{E}_\lambda(\gamma^1_\lambda) \neq \mathfrak{E}_\lambda(\gamma^2_\lambda)$,

□

Proof of Theorem 1.25. The original Λ may be replaced by $\hat{\Lambda} = \alpha([0,1])$. By the first paragraph in the proof of Theorem 1.24 we have checked that $\mathcal{L}_\lambda = \check{\mathcal{L}}^\perp_\lambda$, $H = \mathbf{H}^\perp$, $X = \mathbf{X}^\perp$, $U = \mathcal{U} \cap \mathbf{H}^\perp$ satisfy the assumptions a)-c) and (i)-(v) of [37, Theorem C.5] for any $\lambda^* \in \hat{\Lambda}$. The condition (d) of Theorem 1.25 can be translated into:

$$[m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^-}, 0), m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^-}, 0) + m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^-}, 0)] \cap [m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^+}, 0), m^-_\tau(\check{\mathcal{L}}^\perp_{\lambda^+}, 0) + m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^+}, 0)] = \emptyset$$

and either $m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^-}, 0) = 0$ or $m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^+}, 0) = 0$.

Hence the condition (e.3) of [37, Theorem C.5] is satisfied. Thus [37, Theorem C.4] concludes that there exists $\mu \in \alpha([0,1])$ and an infinite sequence $(\lambda_k, x_k) \subset \hat{\Lambda} \times \mathbf{H}^\perp \setminus \{(\mu, 0)\}$ converging to $(\mu, 0)$ such that each $x_k \neq 0$ and satisfies $\check{\mathcal{L}}^\perp_{\lambda_k}(x_k) = 0$ for all $k \in \mathbb{N}$. Moreover, μ is not equal to λ^+ (resp. λ^-) if $m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^+}, 0) = 0$ (resp. $m^0_\tau(\check{\mathcal{L}}^\perp_{\lambda^-}, 0) = 0$). We can assume $\lambda_k = \alpha(t_k)$ for some $(t_k) \subset [0,1]$ converging to $\bar{t} \in [0,1]$.

Fix $0 < \delta \leq 2\iota$. Let Ω_δ satisfy Proposition 6.10 and Lemma 6.11 By Proposition 6.8, passing to a subsequence (if necessary) we may assume: each x_k is C^4 and a critical point of $\check{\mathcal{L}}_{\lambda_k}$, $\|x_k\|_{C^2} < \delta$ $\forall k$ and $\|x_k\|_{C^2} \to 0$. Then each $\gamma_k := \Phi_{\bar{\gamma}}(x_k) \in \Omega_\delta$ is a C^6 solution of the corresponding problem (1.18) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, and (γ_k) converges to $\bar{\gamma}$ on any compact interval $I \subset \mathbb{R}$ in C^2-topology as $k \to \infty$. We can assume that all x_k are distinct each other. By Lemma 6.11 each γ_k has at most l $\mathbb{R}$-same points in $\{\gamma_k \,|\, k \in \mathbb{N}\}$. Hence (γ_k) has a subsequence (γ_{k_i}) consisting of completely $\mathbb{R}$-distinct points. The required assertions are proved. □

6.4 Proofs of Propositions 6.7,6.8

Proof of Proposition 6.7. Step 1(*Prove that* $\hat{\Lambda} \times \mathcal{U} \ni (\lambda, x) \mapsto \check{\mathcal{L}}_\lambda(x) \in \mathbb{R}$ *is continuous*). Indeed, for any two points (λ, x) and (λ_0, x_0) in $\hat{\Lambda} \times \mathcal{U}$ we can write

$$\begin{aligned} \check{\mathcal{L}}_\lambda(x) - \check{\mathcal{L}}_{\lambda_0}(x_0) &= \left[\int_0^\tau \check{L}_\lambda(t, x(t), \dot{x}(t))dt - \int_0^\tau \check{L}_\lambda(t, x_0(t), \dot{x}_0(t))dt\right] \\ &\quad + \left[\int_0^\tau \check{L}_\lambda(t, x_0(t), \dot{x}_0(t))dt - \int_0^\tau \check{L}_{\lambda_0}(t, x_0(t), \dot{x}_0(t))dt\right]. \end{aligned}$$

As $(\lambda, x) \to (\lambda_0, x_0)$, we derive from (L6) in Lemma 6.1 and [37, Prop.B.9] or [35, Proposition C.1] (resp. (L6) in Lemma 6.1 and the Lebesgue dominated convergence theorem) that the first (resp. second) bracket on the right side converges to the zero.

(**Actually**, we only need that $\hat\Lambda\times\mathcal{U}^X\ni(\lambda,x)\mapsto\check{\mathcal{L}}_\lambda(x)\in\mathbb{R}$ is continuous. This can easily be proved as follows. For any fixed point $x_0\in\mathcal{U}^X$, we can take a positive $\rho>0$ such that $\rho>\sup_t|\dot{x}_0(t)|$. Since $E_{\bar\gamma}$ is an orthogonal matrix, and $\check{L}:\hat\Lambda\times\mathbb{R}\times\bar{B}^n_{2\iota}(0)\times\mathbb{R}^n\to\mathbb{R}$ is continuous, we deduce that $\check{L}$ is uniformly continuous in $\hat\Lambda\times\mathbb{R}\times\bar{B}^n_{2\iota}(0)\times\bar{B}^n_\rho(0)$. This and (6.12) lead to the desired claim.)

Step 2(*Prove that* $\hat\Lambda\times\mathcal{U}^X\ni(\lambda,x)\mapsto A_\lambda(x)\in\mathbf{X}$ *is continuous*). For $(\lambda_1,x),(\lambda_2,y)\in\hat\Lambda\times\mathcal{U}^X$, and $\xi\in\mathbf{H}$, since

$$\begin{aligned}d\check{\mathcal{L}}_{\lambda_1}(x)[\xi]-d\check{\mathcal{L}}_{\lambda_2}(y)[\xi] &= \int_0^\tau\big(\partial_q\check{L}(\lambda_1,t,x(t),\dot{x}(t))-\partial_q\check{L}(\lambda_2,t,y(t),\dot{y}(t))\big)\cdot\xi(t)dt\\ &+\int_0^\tau\Big(\partial_v\check{L}\left(\lambda_1,t,x(t),\dot{x}(t)\right)-\partial_v\check{L}(\lambda_2,t,y(t),\dot{y}(t))\big)\cdot\dot\xi(t)\Big)\,dt,\end{aligned}$$

we have

$$\begin{aligned}\|\nabla\check{\mathcal{L}}_{\lambda_1}(x)-\nabla\check{\mathcal{L}}_{\lambda_2}(y)\|_{1,2}\le&\left(\int_0^\tau\left|\partial_q\check{L}(\lambda_1,t,x(t),\dot{x}(t))-\partial_q\check{L}(\lambda_2,t,y(t),\dot{y}(t))\right|^2dt\right)^{1/2}\\ &+\left(\int_0^\tau\left|\partial_v\check{L}(\lambda_1,t,x(t),\dot{x}(t))-\partial_v\check{L}(\lambda_2,t,y(t),\dot{y}(t))\right|^2dt\right)^{1/2}.\end{aligned}$$

Fix a point $(\lambda_1,x)\in\hat\Lambda\times\mathcal{U}^X$. Then $\{(\lambda_1,t,x(t),\dot{x}(t))\,|\,t\in[0,\tau]\}$ is a compact subset of $\hat\Lambda\times[0,\tau]\times B^n_{2\iota}(0)\times\mathbb{R}^n$. Since $\partial_q\check{L}$ and $\partial_v\check{L}$ are uniformly continuous in any compact neighborhood of this compact subset we deduce that

$$\|\nabla\check{\mathcal{L}}_{\lambda_1}(x)-\nabla\check{\mathcal{L}}_{\lambda_2}(y)\|_{C^0}\le C_\tau\|\nabla\check{\mathcal{L}}_{\lambda_1}(x)-\nabla\check{\mathcal{L}}_{\lambda_2}(y)\|_{1,2}\to 0 \tag{6.33}$$

provided $(\lambda_2,y)\in\hat\Lambda\times\mathcal{U}^X$ converges to (λ_1,x) in $\hat\Lambda\times\mathcal{U}^X$.

By (6.19) and (6.21), we have

$$\begin{aligned}\frac{d}{dt}\nabla\check{\mathcal{L}}_\lambda(x)(t) &= \frac{e^t}{2}\int_t^\infty e^{-s}\left(\partial_q\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\mathfrak{R}^x_\lambda(s)\right)\,ds\\ &-\frac{e^{-t}}{2}\int_{-\infty}^t e^{s}\left(\partial_q\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\mathfrak{R}^x_\lambda)(s)\right)\,ds\\ &+\partial_v\check{L}_\lambda\big(t,x(t),\dot{x}(t)\big)+\mathfrak{M}\int_0^\tau\partial_v\check{L}_\lambda(s,x(s),\dot{x}(s))ds,\end{aligned}\tag{6.34}$$

where $\mathfrak{R}^x_\lambda$ is given by (6.20). Let $\mathfrak{T}^x_\lambda(t)$ denote a column vector

$$\left[\left(\oplus_{l\le p}\frac{\sin\theta_l}{2-2\cos\theta_l}\begin{pmatrix}0&-1\\1&0\end{pmatrix}-\frac{1}{2}I_{2p}\right)\oplus\operatorname{diag}(a_{p+1}(t),\cdots,a_\sigma(t))\right]\int_0^\tau\partial_v\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)ds.$$

Then we have a constant $C(E_{\bar\gamma})>0$ only depending on $E_{\bar\gamma}$ such that for all t,

$$\left|\mathfrak{T}^x_{\lambda_1}(t)-\mathfrak{T}^y_{\lambda_2}(t)\right|\le C(E_{\gamma_0})(1+|t|)\int_0^\tau\left|\partial_v\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_v\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\right|ds. \tag{6.35}$$

By (6.20) and (6.21) we observe

$$\mathfrak{R}^x_\lambda(t)=\int_0^t\partial_v\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)ds+\mathfrak{T}^x_\lambda(t), \tag{6.36}$$

$$\frac{d}{dt}\mathfrak{T}^x_\lambda(t) = \mathfrak{M}\int_0^\tau \partial_v \check{L}_\lambda\big(s, x(s), \dot{x}(s)\big)ds. \tag{6.37}$$

For $0 \le t \le \tau$, let

$$\Gamma^+_\lambda(x)(t) : = \frac{e^t}{2}\int_t^\infty e^{-s}\left(\partial_q \check{L}_\lambda\big(s, x(s), \dot{x}(s)\big) - \mathfrak{R}^x_\lambda(s)\right)ds, \tag{6.38}$$

$$\Gamma^-_\lambda(x)(t) : = \frac{e^{-t}}{2}\int_{-\infty}^t e^{s}\left(\partial_q \check{L}_\lambda\big(s, x(s), \dot{x}(s)\big) - \mathfrak{R}^x_\lambda(s)\right)ds. \tag{6.39}$$

It follows from (6.34) that

$$\begin{aligned}
&\left|\frac{d}{dt}\nabla\check{\mathcal{L}}_{\lambda_1}(x)(t) - \frac{d}{dt}\nabla\check{\mathcal{L}}_{\lambda_2}(y)(t)\right| \\
\le\ & |\Gamma^+_{\lambda_1}(x)(t) - \Gamma^+_{\lambda_2}(y)(t)| + |\Gamma^-_{\lambda_1}(x)(t) - \Gamma^-_{\lambda_2}(y)(t)| \\
& + |\partial_v \check{L}_{\lambda_1}\big(t, x(t), \dot{x}(t)\big) - \partial_v \check{L}_{\lambda_2}\big(t, y(t), \dot{y}(t)\big)| \\
& + |\mathfrak{M}|\int_0^\tau |\partial_v \check{L}_{\lambda_1}(s, x(s), \dot{x}(s)) - \partial_v \check{L}_{\lambda_2}(s, y(s), \dot{y}(s))|ds.
\end{aligned} \tag{6.40}$$

Let us estimate terms in the right side.

Suppose $k\tau \le t \le (k+1)\tau$ for some integer $k \ge 0$. By $\check{L}(\lambda, t+\tau, x, v) = \check{L}(\lambda, t, E_{\bar\gamma}x, E_{\bar\gamma}v)$ and (6.35) we deduce

$$\begin{aligned}
& |\mathfrak{R}^x_{\lambda_1}(t) - \mathfrak{R}^y_{\lambda_2}(t)| \\
\le\ & \int_0^{(k+1)\tau}\left|\partial_v \check{L}_{\lambda_1}\big(s, x(s), \dot{x}(s)\big) - \partial_v \check{L}_{\lambda_2}\big(s, y(s), \dot{y}(s)\big)\right|ds + \left|\mathfrak{T}^x_{\lambda_1}(t) - \mathfrak{T}^y_{\lambda_2}(t)\right| \\
\le\ & (k+1)\int_0^\tau \left|\partial_v \check{L}_{\lambda_1}\big(s, x(s), \dot{x}(s)\big) - \partial_v \check{L}_{\lambda_2}\big(s, y(s), \dot{y}(s)\big)\right|ds \\
+\ & C(E_{\bar\gamma})(1+(k+1)\tau)\int_0^\tau \left|\partial_v \check{L}_{\lambda_1}\big(s, x(s), \dot{x}(s)\big) - \partial_v \check{L}_{\lambda_2}\big(s, y(s), \dot{y}(s)\big)\right|ds.
\end{aligned} \tag{6.41}$$

Similarly, for $(-k-1)\tau \le t \le -k\tau$ for some integer $k \ge 0$. We have also

$$\begin{aligned}
& |\mathfrak{R}^x_{\lambda_1}(t) - \mathfrak{R}^y_{\lambda_2}(t)| \\
\le\ & \int_{(-k-1)\tau}^{0}\left|\partial_v \check{L}_{\lambda_1}\big(s, x(s), \dot{x}(s)\big) - \partial_v \check{L}_{\lambda_2}\big(s, y(s), \dot{y}(s)\big)\right|ds + \left|\mathfrak{T}^x_{\lambda_1}(t) - \mathfrak{T}^y_{\lambda_2}(t)\right| \\
\le\ & (k+1)\int_0^\tau \left|\partial_v \check{L}_{\lambda_1}\big(s, x(s), \dot{x}(s)\big) - \partial_v \check{L}_{\lambda_2}\big(s, y(s), \dot{y}(s)\big)\right|ds \\
+\ & C(E_{\bar\gamma})(1+(k+1)\tau)\int_0^\tau \left|\partial_v \check{L}_{\lambda_1}\big(s, x(s), \dot{x}(s)\big) - \partial_v \check{L}_{\lambda_2}\big(s, y(s), \dot{y}(s)\big)\right|ds.
\end{aligned} \tag{6.42}$$

For $0 \le t \le \tau$, it follows from (6.38) and (6.41) that

$$\begin{aligned}
& |\Gamma^+_{\lambda_1}(x)(t) - \Gamma^+_{\lambda_2}(y)(t)| \\
\le\ & \frac{e^\tau}{2}\sum_{k=0}^\infty \int_{k\tau}^{(k+1)\tau} e^{-s}\left|\partial_q \check{L}_{\lambda_1}\big(s, x(s), \dot{x}(s)\big) - \partial_q \check{L}_{\lambda_2}\big(s, y(s), \dot{y}(s)\big)\right|ds \\
& + \frac{e^\tau}{2}\sum_{k=0}^\infty \int_{k\tau}^{(k+1)\tau} e^{-s}\left|\mathfrak{R}^x_{\lambda_1}(s) - \mathfrak{R}^y_{\lambda_2}(s)\right|ds
\end{aligned} \tag{6.43}$$

$$
\begin{aligned}
\le\ & \frac{e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}\int_0^\tau \left|\partial_q\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_q\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\right|ds\\
&+\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(k+1)\int_0^\tau \left|\partial_v\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_v\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\right|ds\\
&+C(E_{\bar{\gamma}})\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(1+(k+1)\tau)\int_0^\tau \Big|\partial_v\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_v\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\Big|ds.
\end{aligned}
$$

Similarly, for $0\le t\le\tau$, (6.39) and (6.42) lead to

$$
\begin{aligned}
&|\Gamma^-_{\lambda_1}(x)(t)-\Gamma^-_{\lambda_2}(y)(t)| \qquad (6.44)\\
\le\ & \frac{1}{2}\sum_{k=0}^\infty\int_{-k\tau}^{(-k+1)\tau} e^s\left|\partial_q\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_q\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\right|ds\\
&+\frac{1}{2}\sum_{k=0}^\infty\int_{-k\tau}^{(-k+1)\tau} e^s\left|\mathfrak{R}^x_{\lambda_1}(s)-\mathfrak{R}^y_{\lambda_2}(s)\right|ds\\
\le\ & \frac{e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}\int_0^\tau \left|\partial_q\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_q\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\right|ds\\
&+\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(k+1)\int_0^\tau \left|\partial_v\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_v\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\right|ds\\
&+C(E_{\bar{\gamma}})\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(1+(k+1)\tau)\int_0^\tau \Big|\partial_v\check{L}_{\lambda_1}\big(s,x(s),\dot{x}(s)\big)-\partial_v\check{L}_{\lambda_2}\big(s,y(s),\dot{y}(s)\big)\Big|ds.
\end{aligned}
$$

From these and (6.40) we get

$$
\begin{aligned}
&\left|\frac{d}{dt}\nabla\check{\mathcal{L}}_{\lambda_1}(x)(t)-\frac{d}{dt}\nabla\check{\mathcal{L}}_{\lambda_2}(y)(t)\right|\\
\le\ & |\partial_v\check{L}_{\lambda_1}\big(t,x(t),\dot{x}(t)\big)-\partial_v\check{L}_{\lambda_2}\big(t,y(t),\dot{y}(t)\big)|\\
&+(C(E_{\bar{\gamma}})C_\tau+|\mathfrak{M}|)\int_0^\tau |\partial_v\check{L}_{\lambda_1}(s,x(s),\dot{x}(s))-\partial_v\check{L}_{\lambda_2}(s,y(s),\dot{y}(s))|ds\\
&+C^*_\tau\int_0^\tau |\partial_q\check{L}_{\lambda_1}(s,x(s),\dot{x}(s))-\partial_q\check{L}_{\lambda_2}(s,y(s),\dot{y}(s))|ds \qquad (6.45)
\end{aligned}
$$

for some constant $C^*_\tau>0$ and for all $0\le t\le\tau$.

As above, for a fixed $(\lambda_1,x)\in\hat{\Lambda}\times\mathcal{U}^X$, by uniform continuity of $\partial_q\check{L}$ and $\partial_v\check{L}$ on a compact neighborhood of a compact subset $\{(\lambda_1,t,x(t),\dot{x}(t))\,|\,t\in[0,\tau]\}$ of $\hat{\Lambda}\times[0,\tau]\times B^n_{2\iota}(0)\times\mathbb{R}^n$, we can derive from these and (6.45) that

$$
\left\|\frac{d}{dt}\nabla\check{\mathcal{L}}_{\lambda_1}(x)-\frac{d}{dt}\nabla\check{\mathcal{L}}_{\lambda_2}(y)\right\|_{C^0}\to 0
$$

provided $(\lambda_2,y)\in\hat{\Lambda}\times\mathcal{U}^X$ converges to (λ_1,x) in $\hat{\Lambda}\times\mathcal{U}^X$. This and (6.33) lead to the second claim. □

Proof of Proposition 3.11. It is enough to prove sufficiency. Since $x\in B_{\mathbf{H}^\perp}(0,\varepsilon)$ is a critical point of the restriction of $\check{\mathcal{L}}_\lambda$ to $B_{\mathbf{H}^\perp}(0,\varepsilon)$,

$$
d\check{\mathcal{L}}_\lambda(x)[\xi]=0\ \ \forall\xi\in T_xB_{\mathbf{H}^\perp}(0,\varepsilon)=\mathbf{H}^\perp. \qquad (6.46)
$$

We shall prove $d\check{\mathcal{L}}_\lambda(x) = 0$ in four steps.

Step 1. Prove that x is C^4. By (6.46) we have $\mu(\lambda, x) \in \mathbb{R}$ such that

$$\nabla\check{\mathcal{L}}_\lambda(x) = \mu(\lambda, x)\zeta_0. \tag{6.47}$$

That is, $d\check{\mathcal{L}}_\lambda(x)[\xi] - \mu(\lambda, x)(\zeta_0, \xi)_{1,2} = 0$ for all $\xi \in \mathbf{H}$. It follows that

$$\begin{aligned}&\int_0^\tau [(\partial_q \check{L}_\lambda(x(t), \dot{x}(t)), \xi(t))_{\mathbb{R}^n} + (\partial_v \check{L}_\lambda(x(t), \dot{x}(t), \dot{\xi}(t))_{\mathbb{R}^n}]dt\\ &-\int_0^\tau [\mu(\lambda, x)(\zeta_0(t), \xi(t))_{\mathbb{R}^n} + \mu(\lambda, x)(\dot{\zeta}_0(t), \dot{\xi}(t))_{\mathbb{R}^n}]dt = 0 \quad \forall \xi \in \mathbf{H}.\end{aligned}$$

Define $\mathbf{L}_\lambda(t, q, v) = \check{L}_\lambda(t, q, v) - \mu(\lambda, x)(\zeta_0(t), q)_{\mathbb{R}^n} + \mu(\lambda, x)(\dot{\zeta}_0(t), v)_{\mathbb{R}^n}$. Then $x(t)$ satisfies

$$\int_0^\tau [(\partial_q \mathbf{L}_\lambda(t, x(t), \dot{x}(t)), \xi(t))_{\mathbb{R}^n} + (\partial_v \mathbf{L}_\lambda(t, x(t), \dot{x}(t)), \dot{\xi}(t))_{\mathbb{R}^n}]dt = 0 \quad \forall \xi \in \mathbf{H}. \tag{6.48}$$

Since ζ_0 is C^5, $\mathbf{L}_\lambda$ is C^4 and satisfies the conditions in Lemma 6.1, by Remark 6.2 we obtain that x is C^4.

Step 2. For any $\epsilon > 0$ there exists $\delta > 0$ such that $\|x\|_{1,2} < \delta$ implies $|\mu(\lambda, x)| < \epsilon$. Since $\hat{\Lambda} \subset \mathbb{R}$ is compact and sequential compact, $\check{L}(\lambda, t + \tau, q, v) = \check{L}(\lambda, t, E_{\bar{\gamma}}q, E_{\bar{\gamma}}v)$, and partial derivatives

$$\partial_q \check{L}_\lambda(\cdot), \quad \partial_v \check{L}_\lambda(\cdot), \quad \partial_{qv} \check{L}_\lambda(\cdot), \quad \partial_{qq} \check{L}_\lambda(\cdot), \quad \partial_{vv} \check{L}_\lambda(\cdot), \quad \partial_{vt} \check{L}_\lambda(\cdot)$$

depend continuously on $(\lambda, t, q, v) \in \hat{\Lambda} \times \mathbb{R} \times \bar{B}^n_{2\iota}(0) \times \mathbb{R}^n$, by shrinking $\iota > 0$ (if necessary) it follows from (L3) in Lemma 6.1 that

$$\begin{aligned}&|\partial_q \check{L}_\lambda(t, q, v) - \partial_q \check{L}_\lambda(t, 0, 0)|\\ \le\ & |\partial_q \check{L}_\lambda(t, q, v) - \partial_q \check{L}_\lambda(t, q, 0)| + |\partial_q \check{L}_\lambda(t, q, 0) - \partial_q \check{L}_\lambda(t, 0, 0)|\\ \le\ & \sup_{0\le s\le 1} |\partial_{qv} \check{L}_\lambda(t, q, sv)| \cdot |v| + \sup_{0\le s\le 1} |\partial_{qq} \check{L}_\lambda(t, sq, 0)| \cdot |q|\\ \le\ & C(|v| + |v|^2) + C|q|.\end{aligned}$$

Hence we have a constant $C' > 0$ such that

$$|\partial_q \check{L}_\lambda(t, q, v)| \le C'(1 + |v|^2), \quad \forall (\lambda, t, q, v) \in \hat{\Lambda} \times \mathbb{R} \times \bar{B}^n_{2\iota}(0) \times \mathbb{R}^n. \tag{6.49}$$

Similarly, we can increase the constant $C' > 0$ so that

$$|\partial_v \check{L}_\lambda(t, q, v)| \le C'(1 + |v|), \quad \forall (\lambda, t, q, v) \in \hat{\Lambda} \times \mathbb{R} \times \bar{B}^n_{2\iota}(0) \times \mathbb{R}^n. \tag{6.50}$$

Since $\nabla\check{\mathcal{L}}_\lambda(0) = 0$, by (6.47) we have

$$\begin{aligned}\mu(\lambda, x)(\zeta_0, \xi)_{1,2} &= d\check{\mathcal{L}}_\lambda(x)[\xi] - d\check{\mathcal{L}}_\lambda(0)[\xi]\\ &= \int_0^\tau [(\partial_q \check{L}_\lambda(t, x(t), \dot{x}(t)), \xi(t))_{\mathbb{R}^n} - (\partial_q \check{L}_\lambda(t, 0, 0), \xi(t))_{\mathbb{R}^n}]dt\\ &\quad + \int_0^\tau [(\partial_v \check{L}_\lambda(t, x(t), \dot{x}(t), \dot{\xi}(t))_{\mathbb{R}^n} - (\partial_v \check{L}_\lambda(t, 0, 0), \dot{\xi}(t))_{\mathbb{R}^n}]dt.\end{aligned}$$

For the first integral, by the mean value theorem, (L3) in Lemma 6.1 and (2.3) we derive

$$\Big|\int_0^\tau [(\partial_q \check{L}_\lambda(t, x(t), \dot{x}(t)) - \partial_q \check{L}_\lambda(t, 0, 0), \xi(t))_{\mathbb{R}^n}]dt\Big|$$

$$\begin{aligned}
&\leq \Big|\int_0^\tau\int_0^1[(\partial_{qq}\check{L}_\lambda(t,sx(t),s\dot{x}(t))x(t),\xi(t))_{\mathbb{R}^n}]dtds\Big|\\
&\qquad +\Big|\int_0^\tau\int_0^1[(\partial_{qv}\check{L}_\lambda(t,sx(t),s\dot{x}(t))\dot{x}(t),\xi(t))_{\mathbb{R}^n}]dtds\Big|\\
&\leq C\int_0^\tau\int_0^1(1+|s\dot{x}(t))|^2)|x(t)||\xi(t)|dtds + C\int_0^\tau\int_0^1(1+|s\dot{x}(t))|)|\dot{x}(t)||\xi(t)|dtds\\
&\leq C(C_\tau)^2\|x\|_{1,2}\|\xi\|_{1,2}\big(\tau+2\|x\|_{1,2}^2\big)+CC_\tau\|\xi\|_{1,2}(\sqrt{\tau}\|x\|_{1,2}+\|x\|_{1,2}^2). \qquad (6.51)
\end{aligned}$$

Similarly, we may estimate the second integral as follows:

$$\begin{aligned}
&\Big|\int_0^\tau[(\partial_v\check{L}_\lambda(t,x(t),\dot{x}(t),\dot{\xi}(t))_{\mathbb{R}^n}-(\partial_v\check{L}_\lambda(t,0,0),\dot{\xi}(t))_{\mathbb{R}^n}]dt\Big|\\
&\leq \Big|\int_0^\tau\int_0^1[(\partial_{vv}\check{L}_\lambda(t,sx(t),s\dot{x}(t))\dot{x}(t),\dot{\xi}(t))_{\mathbb{R}^n}]dtds\Big|\\
&\qquad +\Big|\int_0^\tau\int_0^1[(\partial_{qv}\check{L}_\lambda(t,sx(t),s\dot{x}(t))x(t),\dot{\xi}(t))_{\mathbb{R}^n}]dtds\Big|\\
&\leq C\|x\|_{1,2}\|\xi\|_{1,2}+CC_\tau\|x\|_{1,2}(\sqrt{\tau}\|\xi\|_{1,2}+\|x\|_{1,2}\|\xi\|_{1,2}). \qquad (6.52)
\end{aligned}$$

Hence we get

$$\begin{aligned}
|\mu(\lambda,x)|\|\zeta_0\|_{1,2} &\leq C(C_\tau)^2\|x\|_{1,2}\big(\tau+2\|x\|_{1,2}^2\big)+CC_\tau(\sqrt{\tau}\|x\|_{1,2}+\|x\|_{1,2}^2)\\
&\quad +C\|x\|_{1,2}+CC_\tau\|x\|_{1,2}(\sqrt{\tau}+\|x\|_{1,2}).
\end{aligned}$$

The desired claim immediately follows because $\zeta_0\neq 0$.

Step 3. For any $\epsilon>0$ there exists $\varepsilon>0$ such that $\|x\|_{1,2}\leq\varepsilon$ implies $\|x\|_{C^2}<\epsilon$. By (L2) in Lemma 6.1 and the mean value theorem of integrals we deduce

$$\begin{aligned}
c|v|^2 &\leq \int_0^1\big(\partial_{vv}\check{L}_\lambda(t,q,sv)[v],v\big)_{\mathbb{R}^n}ds\\
&= \big(\partial_v\check{L}_\lambda(t,q,v)-\partial_v\check{L}_\lambda(t,q,0),v\big)_{\mathbb{R}^n}
\end{aligned}$$

and so

$$c|v|\leq\big|\partial_v\check{L}_\lambda(t,q,v)-\partial_v\check{L}_\lambda(t,q,0)\big| \qquad (6.53)$$

for any $(\lambda,t,q,v)\in\hat{\Lambda}\times\mathbb{R}\times\bar{B}^n_{2\iota}(0)\times\mathbb{R}^n$. Since we have proved that x is C^4 in Step 1, (6.47) also holds in the sense of pointwise, i.e.,

$$\nabla\check{\mathcal{L}}_\lambda(x)(t)=\mu(\lambda,x)\zeta_0(t)\ \forall t. \qquad (6.54)$$

From this and (6.34) it follows that

$$\begin{aligned}
\mu(\lambda,x)\dot{\zeta}_0(t) &= \frac{e^t}{2}\int_t^\infty e^{-s}\left(\partial_q\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\mathfrak{R}^x_\lambda(s)\right)ds\\
&\quad -\frac{e^{-t}}{2}\int_{-\infty}^t e^s\left(\partial_q\check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\mathfrak{R}^x_\lambda)(s)\right)ds\\
&\quad +\partial_v\check{L}_\lambda\big(t,x(t),\dot{x}(t)\big)+\mathfrak{M}\int_0^\tau\partial_v\check{L}_\lambda(s,x(s),\dot{x}(s))ds, \qquad (6.55)
\end{aligned}$$

where $\mathfrak{R}^x_\lambda$ is given by (6.20). In particular, taking $x=0$ we get

$$\mu(\lambda,0)\dot{\zeta}_0(t) = \frac{e^t}{2}\int_t^\infty e^{-s}\left(\partial_q\check{L}_\lambda\big(s,0,0\big)-\mathfrak{R}^0_\lambda(s)\right)ds$$

$$-\frac{e^{-t}}{2}\int_{-\infty}^{t} e^{s}\left(\partial_q \check{L}_\lambda\big(s,0,0\big)-\mathfrak{R}_\lambda^0)(s)\right)\,ds$$
$$+\partial_v \check{L}_\lambda\big(t,0,0\big)+\mathfrak{M}\int_0^\tau \partial_v \check{L}_\lambda(s,0,0)ds. \tag{6.56}$$

(6.55) minus (6.56) gives rise to

$$\begin{aligned}
&\partial_v \check{L}_\lambda\big(t,0,0\big)-\partial_v \check{L}_\lambda\big(t,x(t),\dot{x}(t)\big)\\
=\;& \frac{e^t}{2}\int_t^\infty e^{-s}\left(\partial_q \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_q \check{L}_\lambda\big(s,0,0\big)\right)\,ds+\frac{e^t}{2}\int_t^\infty e^{-s}\left(\mathfrak{R}_\lambda^0(s)-\mathfrak{R}_\lambda^x(s)\right)\,ds\\
&-\frac{e^{-t}}{2}\int_{-\infty}^t e^{s}\left(\partial_q \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_q \check{L}_\lambda\big(s,0,0\big)\right)\,ds+\frac{e^{-t}}{2}\int_{-\infty}^t e^{s}\left(\mathfrak{R}_\lambda^x(s)-\mathfrak{R}_\lambda^0(s)\right)\,ds\\
&+\mathfrak{M}\int_0^\tau[\partial_v \check{L}_\lambda(s,x(s),\dot{x}(s))-\partial_v \check{L}_\lambda(s,0,0)]ds\\
&+\mu(\lambda,0)\dot{\zeta}_0(t)-\mu(\lambda,x)\dot{\zeta}_0(t)
\end{aligned} \tag{6.57}$$

For $0\le t\le\tau$, it follows from (6.43) and (6.44) that

$$\begin{aligned}
&\left|\frac{e^t}{2}\int_t^\infty e^{-s}\left(\partial_q \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_q \check{L}_\lambda\big(s,0,0\big)\right)\,ds+\frac{e^t}{2}\int_t^\infty e^{-s}\left(\mathfrak{R}_\lambda^0(s)-\mathfrak{R}_\lambda^x(s)\right)\,ds\right|\\
\le\;&\left[\frac{e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}+\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(k+1)+C(E_{\bar{\gamma}})\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(1+(k+1)\tau)\right]\times\\
&\times\left|\int_0^\tau\big[\partial_v \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_v \check{L}_\lambda\big(s,0,0\big)\big]ds\right|
\end{aligned} \tag{6.58}$$

and

$$\begin{aligned}
&\left|-\frac{e^{-t}}{2}\int_{-\infty}^t e^{s}\left(\partial_q \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_q \check{L}_\lambda\big(s,0,0\big)\right)\,ds+\frac{e^{-t}}{2}\int_{-\infty}^t e^{s}\left(\mathfrak{R}_\lambda^x(s)-\mathfrak{R}_\lambda^0(s)\right)\,ds\right|\\
\le\;&\left[\frac{e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}+\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(k+1)+C(E_{\bar{\gamma}})\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(1+(k+1)\tau)\right]\times\\
&\times\left|\int_0^\tau\big[\partial_v \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_v \check{L}_\lambda\big(s,0,0\big)\big]ds\right|.
\end{aligned} \tag{6.59}$$

An elementary calculation yields

$$\begin{aligned}
&\left[\frac{e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}+\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(k+1)+C(E_{\bar{\gamma}})\frac{\tau e^\tau}{2}\sum_{k=0}^\infty e^{-k\tau}(1+(k+1)\tau)\right]\\
<\;& C(E_{\bar{\gamma}},\tau):=(2+C(E_{\bar{\gamma}})\tau+C(E_{\bar{\gamma}})\tau^2)\frac{e^{2\tau}}{e^\tau-1}+(1+C(E_{\bar{\gamma}})\tau^2)\frac{e^{4\tau}}{(e^\tau-1)^2}.
\end{aligned}$$

Hence (6.57), (6.58) and (6.59) lead to

$$\begin{aligned}
&\big|\partial_v \check{L}_\lambda\big(t,x(t),\dot{x}(t)\big)-\partial_v \check{L}_\lambda\big(t,0,0\big)\big|\\
\le\;& 2C(E_{\bar{\gamma}},\tau)\left|\int_0^\tau\big[\partial_v \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_v \check{L}_\lambda\big(s,0,0\big)\big]ds\right|\\
&+|\mathfrak{M}|\left|\int_0^\tau\big[\partial_v \check{L}_\lambda\big(s,x(s),\dot{x}(s)\big)-\partial_v \check{L}_\lambda\big(s,0,0\big)\big]ds\right|+|\mu(\lambda,0)-\mu(\lambda,x)|\cdot|\dot{\zeta}_0(t)|.
\end{aligned}$$

From this, (6.53) and (6.57), (6.58) and (6.59) we deduce

$$\begin{aligned}
c|\dot x(t)| &\le \big|\partial_v\check L_\lambda\big(t,x(t),\dot x(t)\big)-D_v\check L_\lambda(t,x(t),0)\big|\\
&\le \big|\partial_v\check L_\lambda\big(t,x(t),\dot x(t)\big)-\partial_v\check L_\lambda\big(t,0,0\big)\big|+\big|D_v\check L_\lambda(t,x(t),0)-\partial_v\check L_\lambda\big(t,0,0\big)\big|\\
&\le (2C(E_{\bar\gamma},\tau)+|\mathfrak{M}|)\Big|\int_0^\tau\big[\partial_v\check L_\lambda\big(s,x(s),\dot x(s)\big)-\partial_v\check L_\lambda\big(s,0,0\big)\big]ds\Big|\\
&\quad +|\mu(\lambda,0)-\mu(\lambda,x)|\cdot|\dot\zeta_0(t)|+\big|D_v\check L_\lambda(t,x(t),0)-\partial_v\check L_\lambda\big(t,0,0\big)\big| \qquad (6.60)
\end{aligned}$$

for all $0\le t\le\tau$.

Since $|\dot\zeta_0(t)|$ are bounded on $[0,\tau]$, by Step 2 we have $|\mu(\lambda,0)-\mu(\lambda,x)|\cdot\sup_t|\dot\zeta_0(t)|\to 0$ as $\|x\|_{1,2}\to 0$. It follows from (6.50) and [37, Prop.B.9] ([35, Proposition C.1]) that

$$\Big|\int_0^\tau\big[\partial_v\check L_\lambda\big(s,x(s),\dot x(s)\big)-\partial_v\check L_\lambda\big(s,0,0\big)\big]ds\Big|\to 0$$

uniformly in λ as $\|x\|_{1,2}\to 0$. These and (6.60) show:

For any $\epsilon>0$ there exists $\varepsilon>0$ such that $\|x\|_{1,2}\le\varepsilon$ implies $\|x\|_{C^1}<\epsilon$. (6.61)

By (6.48) we have

$$\begin{aligned}
0 &= \frac{d}{dt}\Big(\partial_v\mathbf{L}_\lambda(t,x(t),\dot x(t))\Big)-\partial_q\mathbf{L}_\lambda(t,x(t),\dot x(t))\\
&= \frac{d}{dt}\Big(\partial_v\check L_\lambda(t,x(t),\dot x(t))+\mu(\lambda,x)\dot\zeta_0(t)\Big)-\partial_q\check L_\lambda(t,x(t),\dot x(t))+\mu(\lambda,x)\zeta_0(t)\\
&= \partial_{vv}\check L_\lambda(t,x(t),\dot x(t))\ddot x(t)+\partial_{vq}\check L_\lambda(t,x(t),\dot x(t))\dot x(t)+\partial_{vt}\check L_\lambda(t,x(t),\dot x(t))\\
&\quad -\partial_q\check L_\lambda(t,x(t),\dot x(t))+\mu(\lambda,x)\ddot\zeta_0(t)+\mu(\lambda,x)\zeta_0(t). \qquad (6.62)
\end{aligned}$$

In particular, taking $x=0$ we get

$$0=\partial_{vt}\check L_\lambda(t,0,0)-\partial_q\check L_\lambda(t,0,0)+\mu(\lambda,0)\ddot\zeta_0(t)+\mu(\lambda,0)\zeta_0(t). \qquad (6.63)$$

(6.62) minus (6.63) gives rise to

$$\begin{aligned}
0 &= \partial_{vv}\check L_\lambda(t,x(t),\dot x(t))\ddot x(t)+\partial_{vq}\check L_\lambda(t,x(t),\dot x(t))\dot x(t)\\
&\quad +\partial_{vt}\check L_\lambda(t,x(t),\dot x(t))-\partial_{vt}\check L_\lambda(t,0,0)\\
&\quad -\partial_q\check L_\lambda(t,x(t),\dot x(t))+\partial_q\check L_\lambda(t,0,0)\\
&\quad +\mu(\lambda,x)\ddot\zeta_0(t)-\mu(\lambda,0)\ddot\zeta_0(t)+\mu(\lambda,x)\zeta_0(t)-\mu(\lambda,0)\zeta_0(t). \qquad (6.64)
\end{aligned}$$

Note that (L2) of Lemma 6.1 implies $|[\partial_{vv}\check L_\lambda(t,x(t),\dot x(t))]^{-1}\xi|\le\frac{1}{c}|\xi|\;\forall\xi\in\mathbb{R}^n$. (6.64) leads to

$$\begin{aligned}
|\ddot x(t)| &\le \frac{1}{c}|\partial_{vq}\check L_\lambda(t,x(t),\dot x(t))|\cdot|\dot x(t)|+\frac{1}{c}|\partial_q\check L_\lambda(t,x(t),\dot x(t))-\partial_q\check L_\lambda(t,0,0)|\\
&\quad +\frac{1}{c}|\partial_{vt}\check L_\lambda(t,x(t),\dot x(t))-\partial_{vt}\check L_\lambda(t,0,0)|+\frac{1}{c}|\partial_q\check L_\lambda(t,x(t),\dot x(t))-\partial_q\check L_\lambda(t,0,0)|\\
&\quad +|\mu(\lambda,x)-\mu(\lambda,0)||\ddot\zeta_0(t)|+|\mu(\lambda,x)-\mu(\lambda,0)||\zeta_0(t)|. \qquad (6.65)
\end{aligned}$$

Since $|\zeta_0(t)|$ and $|\ddot\zeta_0(t)|$ are bounded on $[0,\tau]$, the desired claim may follow from this, Step 2 and (6.61).

Step 4. Prove $d\check{\mathcal{L}}_\lambda(x)=0$. Since $\|\zeta_0\|_{C^0}>0$, by Step 3 we get $\varepsilon>0$ such that $\|x\|_{1,2}\le\varepsilon$ implies $\|x\|_{C^2}<\rho_0$. In particular, $x\in\mathcal{U}^X$. Since $\check{L}=L^\star$ on $\hat{\Lambda}\times\mathbb{R}\times B^n_{3\iota/2}(0)\times B^n_{\rho_0}(0)$, it holds that

$$\check{\mathcal{L}}_\lambda(x)=\int_0^\tau \check{L}_\lambda(t,x(t),\dot{x}(t))dt=\mathcal{L}^\star_\lambda(x)=\mathfrak{E}_\lambda(\Phi_{\bar\gamma}(x)).$$

Note that $\Phi_{\bar\gamma}(x)\in\mathcal{X}^6_\tau(M,\mathbb{I}_g)$. $[-a,a]\cdot\Phi_{\bar\gamma}(x)$ is a C^4 submanifold of dimension one. Therefore by these and the arguments below (6.4) we get that

$$S_x:=\Phi_{\bar\gamma}^{-1}\big([-a,a]\cdot\Phi_{\bar\gamma}(x)\cap\mathrm{Im}(\Phi_{\bar\gamma})\cap\mathcal{X}^1_\tau(M,\mathbb{I}_g)\big)$$

is a C^2 submanifold of $\mathcal{X}^1_\tau(B^n_{2\iota}(0),E_{\bar\gamma})$ containing 0 as an interior point. Observe that

$$T_{\Phi_{\bar\gamma}(x)}\big([-a,a]\cdot\Phi_{\bar\gamma}(x)\cap\mathrm{Im}(\Phi_{\bar\gamma})\cap\mathcal{X}^1_\tau(M,\mathbb{I}_g)\big)=\mathbb{R}\frac{d}{ds}(s\cdot\Phi_{\bar\gamma}(x))\Big|_{s=0}=\mathbb{R}(\Phi_{\bar\gamma}(x))^{\cdot}$$

and that $\mathfrak{E}_\lambda$ is constant on $[-a,a]\cdot\Phi_{\bar\gamma}(x)$. Hence

$$d\mathfrak{E}_\lambda(\Phi_{\bar\gamma}(x))[(\Phi_{\bar\gamma}(x))^{\cdot}]=0.$$

Let $\zeta_x:=(d\Phi_{\bar\gamma}(x)))^{-1}((\Phi_{\bar\gamma}(x))^{\cdot})\in T_xS_x$. Then

$$d\check{\mathcal{L}}_\lambda(x)[\zeta_x]=d\mathfrak{E}_\lambda(\Phi_{\bar\gamma}(x))[(\Phi_{\bar\gamma}(x))^{\cdot}]=0. \tag{6.66}$$

Since $\Phi_{\bar\gamma}(\xi)(t)=\phi_{\bar\gamma}(t,\xi(t))$ and $d\phi_{\bar\gamma}(t,p)[(1,v)]=\partial_2\phi_{\bar\gamma}(t,p)[v]+\partial_1\phi_{\bar\gamma}(t,p)$ we get

$$\begin{aligned}&\partial_2\phi_{\bar\gamma}(t,x(t))[\dot{x}(t)]+\partial_1\phi_{\bar\gamma}(t,x(t))=(\Phi_{\bar\gamma}(x))^{\cdot}(t)=(d\Phi_{\bar\gamma}(x)[\zeta_x])(t)\\ =\;&\frac{d}{ds}\Big|_{s=0}\Phi_{\bar\gamma}(x+s\zeta_x)(t)=\frac{d}{ds}\Big|_{s=0}\phi_{\bar\gamma}(t,x(t)+s\zeta_x(t))\\ =\;&\partial_2\phi_{\bar\gamma}(t,x(t))[\zeta_x(t)]\end{aligned}$$

and thus

$$\partial_1\phi_{\bar\gamma}(t,x(t))=\partial_2\phi_{\bar\gamma}(t,x(t))[\zeta_x(t)-\dot{x}(t)]. \tag{6.67}$$

As $\|x\|_{C^1}\to0$ we deduce that $\|\zeta_x-\zeta_0\|_{C^0}\to0$, that is,

$$\zeta_x(t)=\dot{x}(t)+(\partial_2\phi_{\bar\gamma}(t,x(t)))^{-1}\big(\partial_1\phi_{\bar\gamma}(t,x(t))\big)\to(\partial_2\phi_{\bar\gamma}(t,0))^{-1}\big(\dot{\bar\gamma}(t)\big)=\zeta_0(t) \tag{6.68}$$

uniformly on $[0,\tau]$.

We hope to prove that $\|\zeta_x-\zeta_0\|_{C^1}\to0$ as $\|x\|_{1,2}\to0$.

Fix $\bar{t}\in[0,\tau]$ and a C^7 coordinate chart (Θ,W) around $\bar\gamma(\bar{t})$ on M, where W is an open neighborhood of $\bar\gamma(\bar{t})$ and Θ is a C^7 diffeomorphism from W to an open subset in $\mathbb{R}^n$. Then there exists a closed neighborhood J of $\bar{t}$ in $[0,\tau]$ such that $\bar\gamma(t)\in W$ for any $t\in J$.

Shrinking J we have a positive number $\nu<2\iota$ such that $x([0,\tau])\subset B^n_\nu(0)$ and that

$$\phi_{\bar\gamma,t}:=\phi_{\bar\gamma}(t,\cdot)\text{ maps }B^n_\nu(0)\text{ into }W\text{ for each }t\in J. \tag{6.69}$$

Define $\Upsilon:J\times B^n_\nu(0)\to\mathbb{R}^n,\ (t,p)\mapsto\Theta(\phi_{\bar\gamma}(t,p))$. It is C^5, and for each $t\in J$, $\Upsilon(t,\cdot)$ is a C^5 diffeomorphism from $B^n_\nu(0)$ onto an open subset in $\mathbb{R}^n$. Let $\partial_1\Upsilon$ and $\partial_2\Upsilon$ denote differentials of Υ with respect to t and p, respectively. For $t\in J$, since

$$\begin{aligned}&d\Theta(\phi_{\bar\gamma}(t,x(t)))\circ(\partial_2\phi_{\bar\gamma}(t,x(t)))=d(\Theta\circ\phi_{\bar\gamma,t})(x(t))=\partial_2\Upsilon(t,x(t)),\\ &d\Theta(\phi_{\bar\gamma}(t,x(t)))[\partial_1\phi_{\bar\gamma}(t,x(t))]=\frac{d}{ds}\Theta(\phi_{\bar\gamma}(s,p))\Big|_{(s,p)=(t,x(t))}=\partial_1\Upsilon(t,x(t)),\end{aligned}$$

composing $d\Theta(\phi_{\bar\gamma}(t,x(t)))$ in two sides of (6.67) we obtain

$$\partial_1\Upsilon(t,x(t))=\partial_2\Upsilon(t,x(t))[\zeta_x(t)-\dot x(t)],\quad \forall t\in J. \tag{6.70}$$

Differentiating this equality with respect to t gives rise to

$$\begin{aligned}&\partial_1\partial_1\Upsilon(t,x(t))+\partial_2\partial_1\Upsilon(t,x(t))[\dot x(t)]\\ =\;&\partial_1\partial_2\Upsilon(t,x(t))[\zeta_x(t)-\dot x(t)]+\partial_2\partial_2\Upsilon(t,x(t))[\dot x(t),\zeta_x(t)-\dot x(t)]\\ &+\partial_2\Upsilon(t,x(t))[\dot\zeta_x(t)-\ddot x(t)]\end{aligned}$$

and so

$$\begin{aligned}\dot\zeta_x(t)\;=\;&\ddot x(t)-\big(\partial_2\Upsilon(t,x(t))\big)^{-1}\partial_1\partial_2\Upsilon(t,x(t))[\zeta_x(t)-\dot x(t)]\\ -\;&\big(\partial_2\Upsilon(t,x(t))\big)^{-1}\partial_2\partial_2\Upsilon(t,x(t))[\dot x(t),\zeta_x(t)-\dot x(t)]\\ +\;&\big(\partial_2\Upsilon(t,x(t))\big)^{-1}\partial_1\partial_1\Upsilon(t,x(t))+\big(\partial_2\Upsilon(t,x(t))\big)^{-1}\partial_2\partial_1\Upsilon(t,x(t))[\dot x(t)],\end{aligned}$$

where $\partial_2\Upsilon(t,x(t))$ may be understand as matrixes. As $\|x\|_{C^2}\to 0$ it follows that

$$\dot\zeta_x(t)\;\to\;\big(\partial_2\Upsilon(t,0)\big)^{-1}\partial_1\partial_1\Upsilon(t,0)=\dot\zeta_0(t)$$

uniformly on J, and hence that $\|\zeta_x-\zeta_0\|_{C^1(J)}\to 0$ by (6.68).

Note that $[0,\tau]$ can be covered by finitely many neighborhoods of form J. We arrive at $\|\zeta_x-\zeta_0\|_{C^1}\to 0$, in particular, $\|\zeta_x-\zeta_0\|_{1,2}\to 0$. Then for $\varepsilon>0$ small enough, the orthogonal decomposition $\mathbf{H}=(\mathbb{R}\zeta_0)\oplus\mathbf{H}^\perp$ of Hilbert spaces implies a direct sum decomposition of Banach spaces $\mathbf{H}=(\mathbb{R}\zeta_x)\dot{+}\mathbf{H}^\perp$. From this, (6.66) and (6.46) we may deduce that $d\check{\mathcal{L}}_\lambda(x)=0$ because $T_xB_{\mathbf{H}^\perp}(0,\varepsilon)=T_0B_{\mathbf{H}^\perp}(0,\varepsilon)=\mathbf{H}^\perp$.

□

6.5 Strengthening of Theorems 1.23, 1.24, 1.25, 1.26 for Lagrangian systems on $\mathbb{R}^n$

When M is an open subset U of $\mathbb{R}^n$ and $\mathbb{I}_g$ is an orthogonal matrix E of order n which maintains U invariant, Assumption 1.21 in Theorems 1.23, 1.24, 1.25, 1.26 can be replaced by the following weaker:

Assumption 6.12. For an orthogonal matrix E of order n, and an E-invariant open subset $U\subset\mathbb{R}^n$ let $L:\Lambda\times U\times\mathbb{R}^n\to\mathbb{R}$ be a continuous function such that for each $\lambda\in\Lambda$ the function $L_\lambda(\cdot)=L(\lambda,\cdot)$ is C^4 and partial derivatives

$$\partial_qL_\lambda(\cdot),\quad \partial_vL_\lambda(\cdot),\quad \partial_{qv}L_\lambda(\cdot),\quad \partial_{qq}L_\lambda(\cdot),\quad \partial_{vv}L_\lambda(\cdot)$$

depend continuously on $(\lambda,q,v)\in\Lambda\times U\times\mathbb{R}^n$. Moreover, for each $(\lambda,q)\in\Lambda\times U$, $L(\lambda,q,v)$ is convex in v, and satisfies

$$L(\lambda,Eq,Ev)=L(\lambda,q,v)\quad\forall(\lambda,q,v)\in\Lambda\times U\times\mathbb{R}^n. \tag{6.71}$$

A nonconstant C^2 map $\bar\gamma:\mathbb{R}\to U$ satisfies (5.8) for each $\lambda\in\Lambda$ (so $\bar\gamma$ is C^4), and the closure of $\bar\gamma(\mathbb{R})$ has an E-invariant compact neighborhood U_0 contained in U (thus there exists $\nu_0>0$ such that $Cl(\bar\gamma(\mathbb{R}))+\bar B^n_{\nu_0}(0)\subset U_0$). For some real $\rho>\sup_t|\dot{\bar\gamma}(t)|$ and each $(\lambda,q)\in\Lambda\times U$, $L(\lambda,q,v)$ is strictly convex in v in $B^n_\rho(0)$.

In fact, taking an orthogonal matrix Ξ such that $\Xi^{-1}E\Xi$ is equal to the right side of (6.2) and replacing L_λ and $\bar\gamma$ by

$$(\Xi^{-1}U)\times\mathbb{R}^n\to\mathbb{R},\ (x,v)\mapsto L_\lambda(\Xi x,\Xi v)$$

and $\Xi^{-1}\bar\gamma$, respectively, we may assume $E=\mathrm{diag}(S_1,\cdots,S_\sigma)\in\mathbb{R}^{n\times n}$ as in (6.3).

Take $\iota=\nu_0/3$ and define $\phi_{\bar\gamma}:\mathbb{R}\times B^n_{3\iota}(0)\to U$ by $\phi_{\bar\gamma}(t,x)=\bar\gamma(t)+x$, which is clearly C^4. Then $\phi_{\bar\gamma}$ gives rise to a C^∞ coordinate chart around $\bar\gamma$ on the C^∞ Banach manifold $\mathcal{X}_\tau(U,E)$,

$$\Phi_{\bar\gamma}:\mathcal{X}^1(B^n_{2\iota}(0),E^T)\to\mathcal{X}^1_\tau(U,E)$$

given by $\Phi_{\bar\gamma}(\xi)(t)=\bar\gamma(t)+\xi(t)$. For any $\xi\in\mathcal{X}^1(B^n_{2\iota}(0),E^T)$ and $\gamma\in\mathcal{X}^1_\tau(U,E)$ there holds

$$T_\xi\mathcal{X}^1(B^n_{2\iota}(0),E^T)=T_\gamma\mathcal{X}^1_\tau(U,E)=\mathbf{X}:=\mathcal{X}^1(\mathbb{R}^n,E^T).$$

Clearly, $d\Phi_{\bar\gamma}(0)=id_{\mathbf{X}}$. Since $d\phi_{\bar\gamma}(t,x)[(1,v)]=\dot{\bar\gamma}(t)+v$, we define

$$L^\star:\Lambda\times\mathbb{R}\times B^n_{3\iota}(0)\times\mathbb{R}^n\to\mathbb{R},\ (\lambda,t,x,v)\mapsto L\big(\lambda,\bar\gamma(t)+x,\dot{\bar\gamma}(t)+v\big).$$

It is continuous and satisfies (6.8) with $E_{\bar\gamma}=E$. Moreover, each $L^\star_\lambda$ is C^4. Take $\rho_0>3\iota$ such that $\rho>\rho_0>\sup_t|\dot{\bar\gamma}(t)|$. Then for a given subset $\hat\Lambda\subset\Lambda$ which is either compact or sequential compact, Lemma 6.1 yields a continuous function $\check L:\hat\Lambda\times\mathbb{R}\times\bar B^n_{2\iota}(0)\times\mathbb{R}^n\to\mathbb{R}$. Put

$$\begin{aligned}&\mathbf{H}:=\{\xi\in W^{1,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)\,|\,E(\xi(t))=\xi(t+\tau)\ \forall t\in\mathbb{R}\}\quad\text{and}\\ &\mathcal{U}:=\{\xi\in W^{1,2}_{\mathrm{loc}}(\mathbb{R};B^n_{2\iota}(0))\,|\,E(\xi(t))=\xi(t+\tau)\ \forall t\in\mathbb{R}\},\quad \mathcal{U}^X:=\mathcal{U}\cap\mathbf{X}.\end{aligned}$$

Define $\check{\mathcal{L}}_\lambda:\mathcal{U}\to\mathbb{R}$ as in (6.12), and $\check{\mathcal{L}}^\perp_\lambda:\mathcal{U}\cap\mathbf{H}^\perp\to\mathbb{R}$ as in (6.14), where

$$\mathbf{H}^\perp:=\{x\in\mathbf{H}\,|\,(\dot{\bar\gamma},x)_{1,2}=0\}\quad\text{and}\quad\mathbf{X}^\perp:=\mathbf{X}\cap\mathbf{H}^\perp.$$

The other arguments are same, except that "C^6" is replaced by "C^4", $\mathbb{I}_g$ by E, etc. Actually, the proof of Proposition 3.11 are much simpler in this situation.

7 Proofs of Theorems 1.29, 1.30, 1.31

Following the paragraph above (3.2) we take a path $\overline{\gamma}\in EC^7(S_\tau;M)$ such that

$$\mathrm{dist}_g(\gamma_\mu(t),\overline{\gamma}(t))<\iota\ \forall t\in\mathbb{R}. \tag{7.1}$$

We first assume:

$$d_g(\gamma_\lambda(t),\overline{\gamma}(t))<\iota,\quad\forall(\lambda,t)\in\Lambda\times\mathbb{R}. \tag{7.2}$$

(For cases of Theorems 1.29, 1.31, by contradiction we may use nets to prove that (7.2) is satisfied after shrinking Λ toward μ.) Then (7.1) and (7.2) imply

$$d_g(\gamma_\lambda(t),\gamma_\mu([0,\tau]))\le d_g(\gamma_\lambda(t),\gamma_\mu(t))<2\iota,\quad\forall(\lambda,t)\in\Lambda\times\mathbb{R}. \tag{7.3}$$

By Claim A.2 we have a unit orthogonal parallel C^5 frame field $\mathbb{R}\to\overline{\gamma}^*TM,\ t\mapsto(e_1(t),\cdots,e_n(t))$ to satisfy

$$(e_1(\tau\pm t),\cdots,e_n(\tau\pm t))=(e_1(t),\cdots,e_n(t))\quad\forall t\in\mathbb{R}. \tag{7.4}$$

Let $B^n_{2\iota}(0) := \{x \in \mathbb{R}^n \,|\, |x| < 2\iota\}$ and exp denote the exponential map of g. Then

$$\phi_{\overline{\gamma}} : \mathbb{R} \times B^n_{2\iota}(0) \to M,\ (t,x) \mapsto \exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n x_i e_i(t)\right)$$

is a C^5 map and satisfies

$$\phi_{\overline{\gamma}}(\tau \pm t, x) = \phi_{\overline{\gamma}}(t,x) \quad \text{and} \quad \frac{d}{dt}\phi_{\overline{\gamma}}(\tau \pm t, x) = \pm(\phi_{\overline{\gamma}})'_1(\tau \pm t, x) \tag{7.5}$$

for any $(t,x) \in \mathbb{R} \times B^n_{2\iota}(0)$. By [48, Theorem 4.3], we have a C^2 coordinate chart around $\overline{\gamma}$ on the C^4 Banach manifold $EC^1(S_\tau; M)$,

$$\Phi_{\overline{\gamma}} : EC^1(S_\tau; B^n_{2\iota}(0)) = \{\xi \in C^1(S_\tau; \mathbb{R}^n) \,|\, \|\xi\|_{C^0} < 2\iota\} \to EC^1(S_\tau; M) \tag{7.6}$$

given by $\Phi_{\overline{\gamma}}(\xi)(t) = \phi_{\overline{\gamma}}(t, \xi(t))$, and

$$d\Phi_{\overline{\gamma}}(0) : EC^1(S_\tau; \mathbb{R}^n) \to T_{\overline{\gamma}}EC^1(S_\tau; M),\ \xi \mapsto \sum_{j=1}^n \xi_j e_j.$$

It is easy to proved that

$$\Phi_{\overline{\gamma}}(EC^1(S_\tau; B^n_{2\iota}(0))) = \mathcal{U}(\overline{\gamma}, 2\iota) := \{\gamma \in EC^1(S_\tau, M) \,|\, d_\infty(\gamma, \overline{\gamma}) < 2\iota\},$$

where $d_\infty(\gamma_1, \gamma_2) = \max_{t \in S_\tau} \mathrm{dist}_g(\gamma_1(t), \gamma_2(t))$ for $\gamma_i \in EC^1(S_\tau, M)$, $i = 1, 2$.

Since the injectivity radius of g at each point on $\overline{\gamma}(S_\tau)$ is at least 2ι, by (7.2) there exists a unique map $\mathbf{u}_\lambda : \mathbb{R} \to B^n_\iota(0)$ such that

$$\gamma_\lambda(t) = \phi_{\overline{\gamma}}(t, \mathbf{u}_\lambda(t)) = \exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n \mathbf{u}^i_\lambda(t) e_i(t)\right) \quad \forall t \in \mathbb{R},$$

Note that $\overline{\gamma}$ is even and τ-periodic. From (7.4) we derive

$$\gamma_\lambda(\tau \pm t) = \exp_{\overline{\gamma}(\tau \pm t)}\left(\sum_{i=1}^n \mathbf{u}^i_\lambda(\tau \pm t) e_i(\tau \pm t)\right) = \exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n \mathbf{u}^i_\lambda(\tau \pm t) e_i(t)\right).$$

Hence $\mathbf{u}_\lambda \in EC^1(S_\tau; B^n_\iota(0)) \cap C^2(S_\tau; B^n_\iota(0))$. By Lemma 3.1, 3.2 we also see that

$$(\lambda, t) \mapsto \mathbf{u}_\lambda(t), \quad (\lambda, t) \mapsto \dot{\mathbf{u}}_\lambda(t) \quad \text{and} \quad (\lambda, t) \mapsto \ddot{\mathbf{u}}_\lambda(t)$$

are continuous maps from $\Lambda \times \mathbb{R}$ to $\mathbb{R}^n$.

Define $L^*_\lambda : \mathbb{R} \times B^n_{2\iota}(0) \times \mathbb{R}^n \to \mathbb{R}$ by

$$L^*_\lambda(t, x, v) = L_\lambda\left(t, \phi_{\overline{\gamma}}(t,x), (\phi_{\overline{\gamma}})'_1(t,x) + (\phi_{\overline{\gamma}})'_2(t,x)[v]\right). \tag{7.7}$$

Then it follows from (7.5) that $L^*_\lambda(\tau + t, x, v) = L^*_\lambda(t, x, v)$ and

$$\begin{aligned}
L^*_\lambda(\tau - t, x, -v) &= L_\lambda\left(\tau - t, \phi_{\overline{\gamma}}(\tau - t, x), (\phi_{\overline{\gamma}})'_1(\tau - t, x) + (\phi_{\overline{\gamma}})'_2(\tau - t, x)[-v]\right) \\
&= L_\lambda\left(\tau - t, \phi_{\overline{\gamma}}(t, x), -(\phi_{\overline{\gamma}})'_1(t, x) - (\phi_{\overline{\gamma}})'_2(t, x)[v]\right) \\
&= L_\lambda\left(t, \phi_{\overline{\gamma}}(t, x), (\phi_{\overline{\gamma}})'_1(t, x) + (\phi_{\overline{\gamma}})'_2(t, x)[v]\right) \\
&= L^*_\lambda(t, x, v) \quad \forall (t, x, v) \in \mathbb{R} \times B^n_{2\iota}(0) \times \mathbb{R}^n.
\end{aligned} \tag{7.8}$$

As before we also define $\tilde{L}^*:\Lambda\times\mathbb{R}\times B^n_\iota(0)\times\mathbb{R}^n\to\mathbb{R}$ by

$$\tilde{L}^*(\lambda,t,q,v)=\tilde{L}^*_\lambda(t,q,v)=L^*(\lambda,t,q+\mathbf{u}_\lambda(t),v+\dot{\mathbf{u}}_\lambda(t)). \tag{7.9}$$

It satisfies Proposition 3.3 after the interval $[0,\tau]$ is replaced by $\mathbb{R}$. Moreover, (7.8) leads to

$$\begin{aligned}\tilde{L}^*(\lambda,\tau-t,q,-v) &= L^*(\lambda,\tau-t,q+\mathbf{u}_\lambda(\tau-t),-v+\dot{\mathbf{u}}_\lambda(\tau-t))\\ &= L^*(\lambda,\tau-t,q+\mathbf{u}_\lambda(t),-v-\dot{\mathbf{u}}_\lambda(t))\\ &= L^*(\lambda,t,q+\mathbf{u}_\lambda(t),v+\dot{\mathbf{u}}_\lambda(t))\\ &= \tilde{L}^*(\lambda,t,q,v)\end{aligned}$$

because $\mathbf{u}_\lambda(\tau-t)=\mathbf{u}_\lambda(t)$ implies $-\dot{\mathbf{u}}_\lambda(\tau-t)=\dot{\mathbf{u}}_\lambda(t)$.

Now we have a family of C^2 functionals

$$\tilde{\mathcal{L}}^E_\lambda: EC^1(S_\tau;B^n_\iota(0))\to\mathbb{R},\ x\mapsto\int_0^\tau\tilde{L}^*_\lambda(t,x(t),\dot{x}(t))dt,\quad \lambda\in\Lambda$$

(see the proof of the first claim in [35, Proposition 4.2]). For $\xi\in EC^1(S_\tau;B^n_\iota(0))$, since

$$\frac{d}{dt}\Phi_{\overline{\gamma}}(\xi+\mathbf{u}_\lambda)(t)=(\phi_{\overline{\gamma}})'_1(t,\xi(t)+\mathbf{u}_\lambda(t))+(\phi_{\overline{\gamma}})'_2(t,\xi(t)+\mathbf{u}_\lambda(t))[\dot{\xi}(t)+\dot{\mathbf{u}}_\lambda(t)]\quad\forall t\in\mathbb{R}, \tag{7.10}$$

we deduce

$$\tilde{\mathcal{L}}^E_\lambda(\xi)=\mathcal{L}^E_\lambda\left(\Phi_{\overline{\gamma}}(\xi+\mathbf{u}_\lambda)\right)\quad\forall\xi\in EC^1(S_\tau;B^n_\iota(0)) \tag{7.11}$$

and therefore that

$$m^-_\tau(\tilde{\mathcal{L}}^E_\lambda,0)=m^-_\tau(\mathcal{L}^E_\lambda,\gamma_\lambda)\quad\text{and}\quad m^0_\tau(\tilde{\mathcal{L}}^E_\lambda,0)=m^0_\tau(\mathcal{L}^E_\lambda,\gamma_\lambda).$$

For the function $\tilde{L}^*$ in (7.9), a positive number $\rho_0>0$ and a subset $\hat{\Lambda}\subset\Lambda$ which is either compact or sequential compact, as in Lemma 3.8 using Lemma 2.4, 2.7 we can construct a continuous function $\check{L}:\hat{\Lambda}\times\mathbb{R}\times B^n_{3\iota/4}(0)\times\mathbb{R}^n\to\mathbb{R}$ satisfying the properties (L0)-(L6) in Lemma 3.8 on $\hat{\Lambda}\times\mathbb{R}\times B^n_{3\iota/4}(0)\times\mathbb{R}^n$ and the following equality

$$\check{L}(\lambda,-t,q,-v)=\check{L}(\lambda,t,q,v)=\check{L}(\lambda,\tau+t,q,v)$$

for all $(\lambda,t,q,v)\in\hat{\Lambda}\times\mathbb{R}\times B^n_{3\iota/4}(0)\times\mathbb{R}^n$. Let us write

$$\begin{aligned}&\mathbf{H}:=W^{1,2}(S_\tau;\mathbb{R}^n),\quad \mathbf{X}:=C^1(S_\tau;\mathbb{R}^n),\\ &\mathcal{U}:=W^{1,2}(S_\tau;B^n_{\iota/2}(0),\quad \mathcal{U}^X:=\mathcal{U}\cap\mathbf{X}=EC^1\left(S_\tau;B^n_{\iota/2}(0)\right),\\ &\mathbf{H}_e:=EW^{1,2}(S_\tau;\mathbb{R}^n)=\{x\in W^{1,2}(S_\tau;\mathbb{R}^n)\,|\,x(-t)=x(t)\ \forall t\},\quad \mathbf{X}_e:=EC^1(S_\tau;\mathbb{R}^n),\\ &\mathcal{U}_e:=\left\{u\in W^{1,2}\left(S_\tau;B^n_{\iota/2}(0)\right)\,|\,u(-t)=u(t)\ \forall t\right\},\\ &\mathcal{U}^X_e:=\mathcal{U}\cap\mathbf{X}=\left\{u\in C^1\left(S_\tau;B^n_{\iota/2}(0)\right)\,|\,u(-t)=u(t)\ \forall t\right\}.\end{aligned}$$

For each $\lambda\in\hat{\Lambda}$ define functionals

$$\check{\mathcal{L}}_\lambda:\mathcal{U}\to\mathbb{R},\ x\mapsto\int_0^\tau\check{L}_\lambda(t,x(t),\dot{x}(t))dt\quad\text{and}\quad\check{\mathcal{L}}^E_\lambda:\mathcal{U}_e\to\mathbb{R},\ x\mapsto\int_0^\tau\check{L}_\lambda(t,x(t),\dot{x}(t))dt.$$

Clearly, $\check{\mathcal{L}}_\lambda|_{\mathcal{U}_e}=\check{\mathcal{L}}^E_\lambda$. As a special case, $\{(\check{\mathcal{L}}_\lambda,\mathcal{U},\mathcal{U}^X)\,|\,\lambda\in\hat{\Lambda}\}$ has the same properties as $\{(\check{\mathcal{E}}_\lambda,\mathcal{U},\mathcal{U}^X)\,|\,\lambda\in\hat{\Lambda}\}$ in Section 4.2. By the arguments in [29], $\{(\check{\mathcal{L}}^E_\lambda,\mathcal{U}_e,\mathcal{U}^X_e)\,|\,\lambda\in\hat{\Lambda}\}$ has also the same properties. (In fact, from the expression of $\nabla\check{\mathcal{L}}_\lambda$ it is not hard to prove that $\nabla\check{\mathcal{L}}_\lambda(x)$ is even for each $x\in\mathcal{U}_e$. Therefore $\nabla\check{\mathcal{L}}^E_\lambda(x)=\nabla\check{\mathcal{L}}_\lambda(x)$ for each $x\in\mathcal{U}_e$.) This implies that for each $x\in\mathcal{U}_e$ the operator $B_\lambda(x)\in\mathcal{L}_s(\mathbf{H})$ defined by (6.24) maps $\mathbf{H}_e$ into $\mathbf{H}_e$. Almost repeating the proofs in Section 3 we can obtain the required results. Of course, the next section also provides a way.

8 An alternate method for bifurcations of Lagrangian systems on open subsets in $\mathbb{R}^n$

In this section, we will show that for the main results in Section 1, except for Theorems 1.23, 1.24, 1.25, 1.26, all others, when restricting to Lagrangian systems on open subsets in $\mathbb{R}^n$, may be almost derived from bifurcation results for Hamiltonian systems in [37].

Assumption 8.1. Let Assumption 2.2, 2.5 be satisfied and let E be a real orthogonal matrix of order n such that $(EU)\cap U\neq\emptyset$. Consider the Lagrangian boundary value problem (2.5)-(2.6) on U. Suppose that each x_λ in Assumption 2.5 also satisfies (2.6).

Then $C^1_E([0,\tau];U)$ is a nonempty open subset in $C^1_E([0,\tau];\mathbb{R}^n)$ [cf. (4.3)],

$$\mathcal{L}_\lambda: C^1_E([0,\tau];U)\to\mathbb{R},\ x\mapsto\int_0^\tau L_\lambda(t,x(t),\dot{x}(t))dt \tag{8.1}$$

is a C^2 functional, each $x_\lambda\in C^2_E([0,\tau];U)$ satisfies $d\mathfrak{L}_\lambda(x_\lambda)=0$, and the second variation at x_λ is given by

$$\mathcal{L}''_\lambda(x_\lambda)[y,z]=\int_0^\tau\left[(\mathsf{P}_\lambda\dot{y}+\mathsf{Q}_\lambda y)\cdot\dot{z}+\ \mathsf{Q}_\lambda^T\dot{y}\cdot z+\mathsf{R}_\lambda y\cdot z\right]dt \tag{8.2}$$

for all $y,z\in C^1_E([0,\tau];\mathbb{R}^n)$, where $\mathsf{P}_\lambda(t)=\partial_{vv}L_\lambda\left(t,x_\lambda(t),\dot{x}_\lambda(t)\right)$ and

$$\mathsf{Q}_\lambda(t)=\partial_{xv}L_\lambda\left(t,x_\lambda(t),\dot{x}_\lambda(t)\right),\qquad \mathsf{R}_\lambda(t)=\partial_{xx}L_\lambda\left(t,x_\lambda(t),\dot{x}_\lambda(t)\right). \tag{8.3}$$

The form $\mathcal{L}''_\lambda(x_\lambda)$ can be extended into a continuous symmetric bilinear form on $W^{1,2}_E([0,\tau];\mathbb{R}^n)$, whose Morse index and nullity are called the **Morse index** and **nullity** of $\mathcal{L}_\lambda$ at x_λ, denoted by

$$m^-_\tau(\mathcal{L}_\lambda,x_\lambda)\quad\text{and}\quad m^0_\tau(\mathcal{L}_\lambda,x_\lambda) \tag{8.4}$$

respectively. Both are finite because $\mathsf{P}_\lambda(t)$ is positive definite.

For a given compact or sequential compact subset $\hat{\Lambda}\subset\Lambda$ let $\hat{L}:\hat{\Lambda}\times[0,\tau]\times B^n_\delta(0)\times\mathbb{R}^n\to\mathbb{R}$ be given by (2.14). Define

$$\hat{\mathcal{L}}_\lambda: C^1_E([0,\tau];B^n_\delta(0))\to\mathbb{R},\ x\mapsto\int_0^\tau \hat{L}_\lambda(t,x(t),\dot{x}(t))dt. \tag{8.5}$$

It is C^2 because $\hat{\mathcal{L}}_\lambda(x)=\mathcal{L}_\lambda(x+x_\lambda)$ for all $x\in C^1_E([0,\tau];B^n_\delta(0))$. It is clear that $x^0:[0,\tau]\to\mathbb{R}^n,\ t\mapsto 0$, satisfies the following boundary value problem:

$$\tfrac{d}{dt}\left(\partial_v\hat{L}_\lambda(t,x(t),\dot{x}(t))\right)-\partial_q\hat{L}_\lambda(t,x(t),\dot{x}(t))=0, \tag{8.6}$$

$$E(x(0))=x(\tau)\quad\text{and}\quad (E^T)^{-1}\left[\partial_v\hat{L}_\lambda(0,x(0),\dot{x}(0))\right]=\partial_v\hat{L}_\lambda(\tau,x(\tau),\dot{x}(\tau)). \tag{8.7}$$

Therefore $d\hat{\mathcal{L}}_\lambda(x^0)=0$, $\hat{\mathcal{L}}''_\lambda(x^0)=\mathcal{L}''_\lambda(x_\lambda)$ and

$$m^-_\tau(\hat{\mathcal{L}}_\lambda,x^0)=m^-_\tau(\mathcal{L}_\lambda,x_\lambda)\quad\text{and}\quad m^0_\tau(\hat{\mathcal{L}}_\lambda,x^0)=m^0_\tau(\mathcal{L}_\lambda,x_\lambda). \tag{8.8}$$

The following is easily proved.

Claim 8.2. *For $\lambda\in\hat{\Lambda}$, a curve $x\in C^2([0,\tau],U)$ satisfies (2.5)-(2.6) and $\|x-x_\lambda\|_{C^1}<\rho_0-\rho_{00}$ if and only if $x-x_\lambda\in C^2([0,\tau],\mathbb{R}^n)$ is a solution of (8.6)–(8.7) satisfying $\|x-x_\lambda-x^0\|_{C^1}<\rho_0-\rho_{00}$. Specially, the bifurcation problem (2.5)–(2.6) around (μ,x_μ) with respect to the trivial branch $\{(\lambda,x_\lambda)\,|\,\lambda\in\Lambda\}$ is equivalent to that of the corresponding problem (8.6)–(8.7) around (μ,x^0) with respect to the trivial branch $\{(\lambda,x^0)\,|\,\lambda\in\hat{\Lambda}\}$.*

For each $(\lambda, t, q) \in \hat{\Lambda} \times [0,\tau] \times B^n_\delta(0)$, by [12, Exercise 1.3.4]), (2.15) implies that $v \mapsto \hat{L}_\lambda(t, q, v)$ is superlinear, and therefore the associated Legendre transform

$$\mathfrak{L}_{\lambda,t,q} : \mathbb{R}^n \to (\mathbb{R}^n)^* \equiv \mathbb{R}^n,\ v \mapsto \partial_v \hat{L}_\lambda(t, q, v)$$

is a C^1 diffeomorphism ([12, Corollary 1.4.7]). (Here the dual space $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$ is naturally identified with $\mathbb{R}^n$.) Since

$$\hat{\Lambda} \times [0,\tau] \times B^n_\delta(0) \times \mathbb{R}^n \to \mathbb{R}^n,\ (\lambda, t, q, v) \mapsto \partial_v \hat{L}_\lambda(t, q, v)$$

is continuous, we derive from [12, Lemma 2.7.2] that

$$\mathfrak{L} : \hat{\Lambda} \times [0,\tau] \times B^n_\delta(0) \times \mathbb{R}^n \to \hat{\Lambda} \times [0,\tau] \times B^n_\delta(0) \times \mathbb{R}^n,\ (\lambda, t, q, v) \mapsto (\lambda, t, q, \partial_v \hat{L}_\lambda(t, q, v))$$

is a homeomorphism. Similarly, for each $\lambda \in \hat{\Lambda}$, the Legendre transform

$$\mathfrak{L}_\lambda : [0,\tau] \times B^n_\delta(0) \times \mathbb{R}^n \to [0,\tau] \times B^n_\delta(0) \times \mathbb{R}^n,\ (t, q, v) \mapsto (t, q, \partial_v \hat{L}_\lambda(t, q, v)),$$

is a C^1 diffeomorphism, and $(\lambda, t, q, v) \mapsto (t, q, \partial_v \hat{L}_\lambda(t, q, v))$ is continuous by the implicit function theorem. Hence for $(\lambda, t, q, p) \in \hat{\Lambda} \times [0,\tau] \times B^n_\delta(0) \times \mathbb{R}^n$ we have a unique $v \in \mathbb{R}^n$ such that $\partial_v \hat{L}_\lambda(t, q, v) = p$. Define $\hat{H} : \hat{\Lambda} \times [0,\tau] \times \mathbb{R}^n \times B^n_\delta(0) \to \mathbb{R}$ by

$$\hat{H}(\lambda, t, p, q) = (p, v)_{\mathbb{R}^n} - \hat{L}_\lambda(t, q, v). \tag{8.9}$$

Claim 8.3. *$\hat{H}$ is continuous, each $\hat{H}(\lambda, t, \cdot)$ is C^2 and all possible partial derivatives of it depend continuously on $(\lambda, t, p, q) \in \hat{\Lambda} \times [0,\tau] \times \mathbb{R}^n \times B^n_\delta(0)$. For $\lambda \in \hat{\Lambda}$, if a curve $x \in C^2([0,\tau], U)$ satisfies (2.5)-(2.6) and $\|x - x_\lambda\|_{C^1} < \rho_0 - \rho_{00}$, then*

$$[0,\tau] \ni t \mapsto z_\lambda(t) := \begin{pmatrix} y_\lambda(t) \\ x(t) - x_\lambda(t) \end{pmatrix} \in \mathbb{R}^{2n}$$

where $y_\lambda(t) := \partial_v \hat{L}_\lambda(t, x(t) - x_\lambda(t), \dot{x}(t) - \dot{x}_\lambda(t)) = \partial_v L_\lambda(t, x(t), \dot{x}(t))$, satisfies

$$\dot{z}(t) = J\nabla \hat{H}_{\lambda,t}(z(t)) \quad \text{and} \quad \mathbb{E}z(0) = z(\tau) \tag{8.10}$$

write $\mathbb{E} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$ (a symplectic orthogonal matrix of order $2n$), and

$$\int_0^\tau \left[\frac{1}{2}(J\dot{z}_\lambda(t), z_\lambda(t))_{\mathbb{R}^{2n}} + \hat{H}(\lambda, t, z_\lambda(t))\right] dt = -\mathcal{L}_\lambda(x) = -\int_0^\tau L_\lambda(t, x(t), \dot{x}(t))dt; \tag{8.11}$$

in particular, for $y_{0,\lambda}(t) := \partial_v \hat{L}_\lambda(t, 0, 0) = \partial_v \hat{L}_\lambda(t, x^0, \dot{x}^0) = \partial_v L_\lambda(t, x_\lambda(t), \dot{x}_\lambda(t))$, the curve

$$[0,\tau] \ni t \mapsto u_{0,\lambda}(t) := (y_{0,\lambda}(t)^T, 0)^T \in \mathbb{R}^{2n} \tag{8.12}$$

satisfies (8.10). Conversely, if $z(t) = (p(t)^T, x(t)^T)^T$ is a solution of (8.10) near $u_{0,\lambda}$ (which can be required to be close in C^1-topology by [37, Proposition 1.3]), then $x + x_\lambda$ satisfies (2.5)-(2.6) and

$$\mathcal{L}_\lambda(x + x_\lambda) = \hat{\mathcal{L}}_\lambda(x) = -\int_0^\tau \left[\frac{1}{2}(J\dot{z}_\lambda(t), z_\lambda(t))_{\mathbb{R}^{2n}} + \hat{H}(\lambda, t, z_\lambda(t))\right] dt. \tag{8.13}$$

Proof. *Step 1*[Prove the assertions for $\hat{H}$]. Because of dependence on λ we cannot directly use [12, Proposition 2.6.3]. But the proof is almost same as that of [12, Proposition 2.6.3]. By (8.9), it holds that for any $(\lambda, t, q, v) \in \hat{\Lambda} \times [0, \tau] \times B^n_\delta(0) \times \mathbb{R}^n$,

$$\hat{H}(\lambda, t, \partial_v \hat{L}_\lambda(t, q, v), q) = \sum_{j=1}^{n} \partial_{v_j} \hat{L}_\lambda(t, q, v) v_j - \hat{L}_\lambda(t, q, v). \tag{8.14}$$

Differentiating both sides with respect to the variable v_i, we get

$$\sum_{j=1}^{n} \partial_{p_j} \hat{H}(\lambda, t, \partial_v \hat{L}_\lambda(t, q, v), q) \partial_{v_i v_j} \hat{L}_\lambda(t, q, v) = \sum_{j=1}^{n} v_j \partial_{v_i v_j} \hat{L}_\lambda(t, q, v), \quad i = 1, \cdots, n.$$

(**Note**: hereafter we understand $\partial_{v_i v_j} L = \partial_{v_i}(\partial_{v_j} L)$. Otherwise, the desired result cannot be derived.) But $\partial_{vv} \hat{L}_\lambda(t, q, v)$ is invertible. It follows that

$$\partial_{p_j} \hat{H}(\lambda, t, \partial_v \hat{L}_\lambda(t, q, v), q) = v_j, \quad j = 1, \cdots, n. \tag{8.15}$$

Differentiating both sides of (8.14) with respect to the variable q_i, and using (8.15) we obtain

$$\partial_{q_i} \hat{H}(\lambda, t, \partial_v \hat{L}_\lambda(t, q, v), q) + \sum_{j=1}^{n} v_j \partial_{q_i v_j} \hat{L}_\lambda(t, q, v) = \sum_{j=1}^{n} v_j \partial_{q_i v_j} \hat{L}_\lambda(t, q, v) - \partial_{q_i} \hat{L}_\lambda(t, q, v)$$

and hence

$$\partial_{q_i} \hat{H}(\lambda, t, \partial_v \hat{L}_\lambda(t, q, v), q) = -\partial_{q_i} \hat{L}_\lambda(t, q, v), \quad i = 1, \cdots, n. \tag{8.16}$$

Differentiating both sides of (8.14) with respect to the variable t and using (8.15) lead to

$$\partial_t \hat{H}(\lambda, t, \partial_v \hat{L}_\lambda(t, q, v), q) + \sum_{j=1}^{n} v_j \partial_{t v_j} \hat{L}_\lambda(t, q, v) = \sum_{j=1}^{n} v_j \partial_{t v_j} \hat{L}_\lambda(t, q, v) - \partial_t \hat{L}_\lambda(t, q, v)$$

and so

$$\partial_t \hat{H}(\lambda, t, p, q) = -\partial_t \hat{L}_\lambda(t, q, v). \tag{8.17}$$

By the implicit function theorem we have a continuous map

$$v : \hat{\Lambda} \times [0, \tau] \times B^n_\delta(0) \times \mathbb{R}^n \to \mathbb{R}^n, \ (\lambda, t, q, p) \mapsto v(\lambda, t, q, p),$$

such that the following holds:

(i) $\partial_v \hat{L}_\lambda(t, q, v(\lambda, t, q, p)) = p$ for all $(\lambda, t, q, p) \in \hat{\Lambda} \times [0, \tau] \times B^n_\delta(0) \times \mathbb{R}^n$;

(ii) $v(\lambda, t, q, p)$ is C^1 in (t, q, p), and $\partial_t v(\lambda, t, q, p)$, $\partial_{q_i} v(\lambda, t, q, p)$ and $\partial_{p_i} v(\lambda, t, q, p)$ are continuous (λ, t, q, p).

(iii) $\partial_q v(\lambda, t, q, p) = -[\partial_{vv} \hat{L}_\lambda(t, q, v(\lambda, t, q, p))]^{-1} \partial_{qv} \hat{L}_\lambda(t, q, v(\lambda, t, q, p))$ and $\partial_p v(\lambda, t, q, p) = [\partial_{vv} \hat{L}_\lambda(t, q, v(\lambda, t, q, p))]^{-1}$.

It follows from (8.15) and (8.16) and (8.17) that

$$\partial_{p_j}\hat{H}(\lambda,t,p,q) = v_j(\lambda,t,q,p), \tag{8.18}$$
$$\partial_{q_j}\hat{H}(\lambda,t,p,q) = -\partial_{q_j}\hat{L}_\lambda(t,q,v(\lambda,t,q,p)), \tag{8.19}$$
$$\partial_t\hat{H}(\lambda,t,p,q) = -\partial_t\hat{L}_\lambda(t,q,v(\lambda,t,q,p)) \tag{8.20}$$

for $(\lambda,t,p,q)\in\hat{\Lambda}\times[0,\tau]\times\mathbb{R}^n\times B^n_\delta(0)$ and $j=1,\cdots,n$. These imply that

$$\partial_t\hat{H}(\lambda,t,p,q),\quad \partial_{p_j}\hat{H}(\lambda,t,p,q)\quad\text{and}\quad \partial_{q_j}\hat{H}(\lambda,t,p,q)$$

are continuous in $(\lambda,t,p,q)\in\hat{\Lambda}\times[0,\tau]\times\mathbb{R}^n\times B^n_\delta(0)$, and C^1 in (p,q), and hence that

$\hat{H}$ is continuous, and C^2 in (p,q) and and all its partial derivatives depend continuously on $(\lambda,t,p,q)\in\hat{\Lambda}\times[0,\tau]\times\mathbb{R}^n\times B^n_\delta(0)$.

Step 2[Prove that z_λ satisfies (8.10)-(8.11)]. Suppose a curve $x\in C^2([0,\tau],U)$ satisfies (2.5)-(2.6) and $\|x-x_\lambda\|_{C^1}<\rho_0-\rho_{00}$ By Claim 8.2, $x-x_\lambda\in C^2_E([0,\tau],\mathbb{R}^n)$ is a solution of (8.6)–(8.7) satisfying $\|x-x_\lambda\|_{C^1}<\rho_0-\rho_{00}$. Put

$$y_{\lambda,i}(t)=\partial_{v_i}\hat{L}_\lambda(t,x(t)-x_\lambda(t),\dot{x}(t)-\dot{x}_\lambda(t)),\quad i=1,\cdots,n. \tag{8.21}$$

Then $v_i(\lambda,t,x(t)-x_\lambda(t),y_\lambda(t))=\dot{x}_{\lambda,i}(t)-\dot{x}_{\lambda,i}(t)$, $i=1,\cdots,n$. From (8.18), (8.19) and (8.6) we derive

$$\dot{x}_i(t)-\dot{x}_{\lambda,i}(t)=\partial_{p_i}\hat{H}(\lambda,t,y_\lambda(t),x(t)-x_\lambda(t)), \tag{8.22}$$
$$\dot{y}_{\lambda,i}(t)=-\partial_{q_i}\hat{H}(\lambda,t,y_\lambda(t),x(t)-x_\lambda(t),) \tag{8.23}$$

for $i=1,\cdots,n$, that is, $z_\lambda(t):=(y_\lambda(t)^T,x(t)^T-(x_\lambda(t))^T)^T$ satisfies the first equation in (8.10). Moreover, that $x-x_\lambda$ satisfies (8.6)–(8.7) implies $Ey_\lambda(0)=(E^T)^{-1}y_\lambda(0)=y_\lambda(\tau)$ by (8.21). This and $E(x(0)-x_\lambda(0))=x(\tau)-x_\lambda(0)$ show that $z_\lambda(t)$ satisfies the second equation in (8.10).

In order to prove (8.11), note that (8.14) and (8.21) yield

$$\hat{L}_\lambda(t,x(t)-x_\lambda(t),\dot{x}(t)-\dot{x}_\lambda(t))=(x(t)-x_\lambda(t),y_\lambda(t))_{\mathbb{R}^n}-\hat{H}_\lambda(t,\dot{x}(t)-\dot{x}_\lambda(t),x(t)-x_\lambda(t)) \tag{8.24}$$

Since E is an orthogonal matrix, $E(x(0)-x_\lambda(0))=x(\tau)-x_\lambda(0)$ and $Ey_\lambda(0)=y_\lambda(\tau)$, a direct computation leads to

$$\int_0^\tau(\dot{x}(t)-\dot{x}_\lambda(t),y_\lambda(t))_{\mathbb{R}^n}dt=-\frac{1}{2}\int_0^\tau(J\dot{z}_\lambda(t),z_\lambda(t))_{\mathbb{R}^{2n}}dt$$

as in [43, pages 36-37]. Then (8.11) follows from this and (8.24).

Step 3[Prove the converse part]. Suppose that $z(t)=(p(t)^T,x(t)^T)^T$ near $u_{0,\lambda}$ is a solution of (8.10), i.e., $\mathbb{E}z(0)=z(\tau)$, and for $i=1,\cdots,n$ it holds that

$$\dot{x}_i(t)=\partial_{p_i}\hat{H}(\lambda,t,p(t),x(t))\quad\text{and}\quad \dot{p}_i(t)=-\partial_{q_i}\hat{H}(\lambda,t,p(t),x(t)). \tag{8.25}$$

Then the former and (8.18) lead to $\dot{x}(t)=v(\lambda,t,x(t),p(t))$, and so

$$\partial_v\hat{L}_\lambda(t,x(t),\dot{x}(t))=\partial_v\hat{L}_\lambda(t,x(t),v(\lambda,t,x(t),p(t)))=p(t) \tag{8.26}$$

by (i) below (8.17). From this, (8.25) and (8.19) we derive

$$\frac{d}{dt}\Big(\partial_v\hat{L}_\lambda(t,x(t),\dot{x}(t))\Big)=\dot{p}(t)=\partial_q\hat{L}_\lambda(t,x(t),v(\lambda,t,x(t),p(t)))=\partial_q\hat{L}_\lambda(t,x(t),\dot{x}(t)).$$

Moreover, E is an orthogonal matrix, and $\mathbb{E}z(0) = z(\tau)$ if and only if

$$Ex(0) = x(\tau) \quad \text{and} \quad E\partial_v \hat{L}_\lambda(0, x(0), \dot{x}(0)) = \partial_v \hat{L}_\lambda(\tau, x(\tau), \dot{x}(\tau))$$

by (8.26). Therefore x satisfies (8.7). By Claim 8.2 $x + x_\lambda$ satisfies (2.5)-(2.6). With the same proof as (8.11) we may obtain (8.13). □

Let $\mathsf{P}(\lambda, t, p, q) = \partial_{vv}\hat{L}_\lambda(t, q, v)$, $\mathsf{Q}(\lambda, t, p, q) = \partial_{qv}\hat{L}_\lambda(t, q, v)$ and $\mathsf{R}(\lambda, t, p, q) = \partial_{qq}\hat{L}_\lambda(t, q, v)$. with $v = v(\lambda, t, q, p)$. It follows from (iii) in Step 1 and (8.18)-(8.19) that

$$\begin{aligned}
\partial_{qp}\hat{H}(\lambda, t, p, q) &= \partial_q(\partial_p \hat{H}(\lambda, t, p, q)) = \partial_q v(\lambda, t, q, p) = -[\mathsf{P}(\lambda, t, p, q)]^{-1}\mathsf{Q}(\lambda, t, p, q),\\
\partial_{pp}\hat{H}(\lambda, t, p, q) &= \partial_p(\partial_p \hat{H}(\lambda, t, p, q)) = \partial_p v(\lambda, t, q, p) = [\mathsf{P}(\lambda, t, p, q)]^{-1},\\
\partial_{pq}\hat{H}(\lambda, t, p, q) &= \partial_p(\partial_q \hat{H}(\lambda, t, p, q)) = -\mathsf{Q}(\lambda, t, p, q)^T[\mathsf{P}(\lambda, t, p, q)]^{-1},\\
\partial_{qq}\hat{H}(\lambda, t, p, q) &= [\mathsf{Q}(\lambda, t, p, q)]^T[\mathsf{P}(\lambda, t, p, q)]^{-1}\mathsf{Q}(\lambda, t, p, q) - \mathsf{R}(\lambda, t, p, q).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\hat{H}''_{(p,q)}(\lambda, t, p, q) = \begin{pmatrix} \partial_{pp}\hat{H}(\lambda, t, p, q) & \partial_{qp}\hat{H}(\lambda, t, p, q) \\ \partial_{pq}\hat{H}(\lambda, t, p, q) & \partial_{qq}\hat{H}(\lambda, t, p, q) \end{pmatrix}\\
&= \begin{pmatrix} [\mathsf{P}(\lambda, t, p, q)]^{-1} & -[\mathsf{P}(\lambda, t, p, q)]^{-1}\mathsf{Q}(\lambda, t, p, q) \\ -[\mathsf{Q}(\lambda, t, p, q)]^T[\mathsf{P}(\lambda, t, p, q)]^{-1} & [\mathsf{Q}(\lambda, t, p, q)]^T[\mathsf{P}(\lambda, t, p, q)]^{-1}\mathsf{Q}(\lambda, t, p, q) - \mathsf{R}(\lambda, t, p, q) \end{pmatrix}.
\end{aligned} \tag{8.27}$$

By Claim 8.3, $u_{0,\lambda}(t) = (y_{0,\lambda}(t)^T, 0)^T$ satisfies (8.10). It follows that $u_{0,\lambda}$ is C^2 and that $u_{0,\lambda}(t)$, $\dot{u}_{0,\lambda}(t)$ and $\ddot{u}_{0,\lambda}(t)$ are continuous in (λ, t). These imply that the Hamiltonian H : $\hat{\Lambda} \times [0, \tau] \times \mathbb{R}^n \times B^n_\delta(0) \to \mathbb{R}$ defined by

$$H(\lambda, t, z) = H_{\lambda,t}(z) := (J\dot{u}_{0,\lambda}(t), z)_{\mathbb{R}^{2n}} + \hat{H}_\lambda(t, z + u_{0,\lambda}(t)) \tag{8.28}$$

is continuous and each $H_{\lambda,t}(\cdot)$ is C^2 and all its partial derivatives depend continuously on (λ, t, z). Clearly, (8.28) implies

$$\nabla H_{\lambda,t}(z) = J\dot{u}_{0,\lambda}(t) + \nabla \hat{H}_{\lambda,t}(z + u_{0,\lambda}(t)), \tag{8.29}$$

$$H''_{\lambda,t}(z) = \hat{H}''_{\lambda,t}(z + u_{0,\lambda}(t)). \tag{8.30}$$

Recall that $y_{0,\lambda}(t) := \partial_v \hat{L}_\lambda(t, 0, 0) = \partial_v \hat{L}_\lambda(t, x^0, \dot{x}^0) = \partial_v L_\lambda(t, x_\lambda(t), \dot{x}_\lambda(t))$. We have $q = 0$, $p = y_{0,\lambda}(t)$ and $v(\lambda, t, 0, y_{0,\lambda}(t)) = 0$, and hence

$$\begin{aligned}
\mathsf{P}(\lambda, t, y_{0,\lambda}(t), 0) &= \partial_{vv}\hat{L}_\lambda(t, 0, 0) = \partial_{vv}L_\lambda(t, x_\lambda(t), \dot{x}_\lambda(t)) = \mathsf{P}_\lambda(t),\\
\mathsf{Q}(\lambda, t, y_{0,\lambda}(t), 0) &= \partial_{qv}\hat{L}_\lambda(t, 0, 0) = \partial_{qv}L_\lambda(t, x_\lambda(t), \dot{x}_\lambda(t)) = \mathsf{Q}_\lambda(t),\\
\mathsf{R}(\lambda, t, y_{0,\lambda}(t), 0) &= \partial_{qq}\hat{L}_\lambda(t, 0, 0) = \partial_{qq}L_\lambda(t, x_\lambda(t), \dot{x}_\lambda(t)) = \mathsf{R}_\lambda(t).
\end{aligned}$$

by (8.3). These, (8.27) and (8.30) lead to

$$\begin{aligned}
H''_{\lambda,t}(0) &= \hat{H}''_{\lambda,t}(u_{0,\lambda}(t)) = \hat{H}''_{(p,q)}(\lambda, t, y_{0,\lambda}(t), 0)\\
&= \begin{pmatrix} \mathsf{P}_\lambda(t)^{-1} & -\mathsf{P}_\lambda(t)^{-1}\mathsf{Q}_\lambda(t) \\ -\mathsf{Q}_\lambda(t)^T\mathsf{P}_\lambda(t)^{-1} & \mathsf{Q}_\lambda(t)^T\mathsf{P}_\lambda(t)^{-1}\mathsf{Q}_\lambda(t) - \mathsf{R}_\lambda(t) \end{pmatrix}.
\end{aligned} \tag{8.31}$$

It follows from Claim 8.3 and (8.29) that

$$\dot{z}_\lambda(t) - \dot{u}_{0,\lambda}(t) \;=\; J\nabla \hat{H}_{\lambda,t}(z_\lambda(t)) - \dot{u}_{0,\lambda}(t)$$

$$
\begin{aligned}
&= J\nabla\hat{H}_{\lambda,t}\big(z_\lambda(t)-u_{0,\lambda}(t)+u_{0,\lambda}(t)\big)-\dot{u}_{0,\lambda}(t)\\
&= J\big(\nabla H_{\lambda,t}(z_\lambda(t)-u_{0,\lambda}(t))-J\dot{u}_{0,\lambda}(t)\big)-\dot{u}_{0,\lambda}(t)\\
&= J\nabla H_{\lambda,t}(z_\lambda(t)-u_{0,\lambda}(t)).
\end{aligned}
$$

Hence we obtain:

Claim 8.4. *For any $\lambda\in\hat{\Lambda}$, by (8.29) the constant path $z^0:[0,\tau]\to\mathbb{R}^{2n}$, $t\mapsto 0$, satisfies*

$$
\dot{z}(t)=J\nabla H_{\lambda,t}(z(t))\quad\textit{and}\quad \mathbb{E}z(0)=z(\tau); \tag{8.32}
$$

and (μ,x_μ) is a bifurcation point of the problem (2.5)–(2.6) in $\hat{\Lambda}\times C^1_E([0,\tau];\mathbb{R}^n)$ with respect to the trivial branch $\{(\lambda,x_\lambda)\,|\,\lambda\in\hat{\Lambda}\}$ if and only if (μ,z^0) is that of (8.32) in $\hat{\Lambda}\times C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$ (or equivalently $\hat{\Lambda}\times W^{1,2}_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$) with respect to the trivial branch $\{(\lambda,z^0)\,|\,\lambda\in\hat{\Lambda}\}$. Precisely, if a sequence (λ_k,x^k) in $\hat{\Lambda}\times C^2([0,\tau],U)$ converges to (μ,x_μ) in $\hat{\Lambda}\times C^1([0,\tau],U)$ and each x^k satisfies (2.5)-(2.6) with $\lambda=\lambda_k$, $k=1,2,\cdots$, then

$$
z^k(t):=\begin{pmatrix} x^k(t)-x_{\lambda^k}(t)\\ \partial_vL_\lambda(t,x^k(t),\dot{x}^k(t))\end{pmatrix}-u_{0,\lambda_k}(t)=\begin{pmatrix} x^k(t)-x_{\lambda_k}(t)\\ \partial_vL_\lambda(t,x^k(t),\dot{x}^k(t))-y_{0,\lambda_k}(t)\end{pmatrix}
$$

satisfies (8.32) with $\lambda=\lambda_k$ and

$$
-\int_0^\tau\left[\frac{1}{2}(J\dot{z}^k(t),z^k(t))_{\mathbb{R}^{2n}}+H(\lambda_k,t,z^k(t))\right]dt=\mathfrak{L}_\lambda(x)=\int_0^\tau L_{\lambda_k}(t,x^k(t),\dot{x}^k(t))dt
$$

for each $k\in\mathbb{N}$, and $z^k\neq z^0$ (if $x^k\neq x_{\lambda_k}$), $z^k\to z^0$ in $C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$ (by [37, Proposition 1.3]); conversely, if a sequence $z^k(t)=(p^k(t)^T,x^k(t)^T)^T$ in $C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$ converges to z^0 in $C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$ and each z^k satisfies (8.32) with $\lambda=\lambda_k$, then $x^k+x_{\lambda_k}$ satisfies (2.5)-(2.6) with $\lambda=\lambda_k$ and

$$
\mathfrak{L}_{\lambda_k}(x^k+x_{\lambda_k})=-\int_0^\tau\left[\frac{1}{2}(J\dot{z}^k(t),z^k(t))_{\mathbb{R}^{2n}}+H(\lambda_k,t,z^k(t))\right]dt, \tag{8.33}
$$

$$
p^k(t)+y_{0,\lambda_k}(t)=\partial_v\hat{L}_{\lambda_k}(t,x^k(t),\dot{x}^k(t))=\partial_vL_{\lambda_k}(t,x^k(t)+x_{\lambda_k}(t),\dot{x}^k(t)+\dot{x}_{\lambda_k}(t)). \tag{8.34}
$$

(The latter implies that $p^k=0$ if and only if $x^k=0$, and hence that $z^k=0$ if and only if $x^k=0$.)

Let $\delta'\in(0,\delta)$ be close to δ. We can choose a C^∞ function $\phi:\mathbb{R}^n\to[0,1]$ such that it is equal to 1 in $B^n_{\delta'}(0)$, has support $\mathrm{supp}(\phi)\subset B^n_\delta(0)$ and satisfies $\phi(q)=\phi(|q|)$ for all $q\in\mathbb{R}^n$. Clearly, Claim 8.4 is still true if H is replaced by

$$
\Lambda\times[0,\tau]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\ (\lambda,t,p,q)\mapsto\begin{cases}\phi(q)H(\lambda,t,p,q), & q\in B^n_\delta(0),\\ 0, & q\in\mathbb{R}^n\setminus B^n_\delta(0).\end{cases} \tag{8.35}
$$

Following the notations below (8.1) let $S_\lambda(t)$ denote the matrix in (8.31). We have obtained $H''_{\lambda,t}(z^0(t))=\hat{H}''_{\lambda,t}(u_{0,\lambda}(t))=S_\lambda(t)$. Let $\Upsilon_\lambda:[0,\tau]\to\mathrm{Sp}(2n,\mathbb{R})$ be the fundamental solution of the problem $\dot{\mathbf{u}}(t)=JS_\lambda(t)\mathbf{u}$. Then

$$
m^0_\tau(\mathfrak{L}_\lambda,x_\lambda)=m^0_\tau(\hat{\mathfrak{L}}_\lambda,x^0)=\dim\mathrm{Ker}(\Upsilon_\lambda(\tau)-\mathbb{E}) \tag{8.36}
$$

by [18, Lemma 3.1], and Υ_λ gives rise to a path of Lagrangian subspaces

$$
[0,\tau]\ni t\mapsto\mathrm{Gr}(\Upsilon_\lambda)(t):=\{(v^T,(\Upsilon_\lambda(t)v)^T)^T\,|\,v\in\mathbb{R}^{2n}\}
$$

in the symplectic vector space $(F,\Omega) := (\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}, (-\omega_0)\oplus\omega_0)$. (Recall that all vectors in $\mathbb{R}^m$ are understand as column vectors.) Let $\mathcal{L}(F,\Omega)$ denote the manifold of Lagrangian subspaces of (F,Ω). Then $\mathrm{Gr}(\mathbb{E})\in\mathcal{L}(F,\Omega)$. Recall that the Cappell-Lee-Miller index μ^{CLM} characterized by properties I-VI of [7, pp. 127-128] assigns an integer $\mu^{\mathrm{CLM}}_F(\Lambda,\Lambda')$ to every pair of Lagrangian paths $\Lambda,\Lambda' : [a,b]\to\mathcal{L}(F,\Omega)$.

Let $\mathcal{P}_\tau(2n) = \{\Upsilon\in C([0,\tau],\mathrm{Sp}(2n,\mathbb{R}))\,|\,\Upsilon(0) = I_{2n}\}$. As extensions of the Maslov-type index $(i_\tau(\Upsilon),\nu_\tau(\Upsilon))$ of $\Upsilon\in\mathcal{P}_\tau(2n)$, Dong [10] and Liu [24] defined the **Maslov type index of** $\Upsilon\in\mathcal{P}_\tau(2n)$ **relative to** $P\in\mathrm{Sp}(2n,\mathbb{R})$ via different methods, respectively denoted by

$$(i_{\tau,P}(\Upsilon),\nu_{\tau,P}(\Upsilon))\quad\text{and}\quad(i^P_\tau(\Upsilon),\nu^P_\tau(\Upsilon))\tag{8.37}$$

for the sake of clearness (though both were written as $(i_P(\Upsilon),\nu_P(\Upsilon))$ in [10] and [24]), where

$$\nu_{\tau,P}(\Upsilon) = \dim\mathrm{Ker}(\Upsilon(\tau)-P) = \nu^P_\tau(\Upsilon).\tag{8.38}$$

Lemma 8.5. (i) $m^-_\tau(\mathcal{L}_\lambda,x_\lambda)+\dim\mathrm{Ker}(E-I_n) = \mu^{\mathrm{CLM}}_F(\mathrm{Gr}(\mathbb{E}),\mathrm{Gr}(\Upsilon_\lambda))$.

(ii) $i^P_\tau(\Upsilon) = \mu^{\mathrm{CLM}}_F(\mathrm{Gr}(P),\mathrm{Gr}(\Upsilon))\ \ \forall(P,\Upsilon)\in\mathrm{Sp}(2n,\mathbb{R})\times\mathcal{P}_\tau(2n)$.

(iii) $i_{\tau,P}(\Upsilon) = i^P_\tau(\Upsilon)\ \ \forall(P,\Upsilon)\in\mathrm{Sp}(2n,\mathbb{R})\times\mathcal{P}_\tau(2n)$.

These three equalities come from [18, Theorem 1.2], [26, Theorem 5.18(2)] and [27], respectively. Therefore from this lemma, (8.36) and (8.38) we derive

$$\left.\begin{array}{l}m^0_\tau(\mathcal{L}_\lambda,x_\lambda) = m^0_\tau(\hat{\mathcal{L}}_\lambda,x^0) = \nu_{\tau,\mathbb{E}}(\Upsilon_\lambda),\\ m^-_\tau(\mathcal{L}_\lambda,x_\lambda)+\dim\mathrm{Ker}(E-I_n) = i_{\tau,\mathbb{E}}(\Upsilon_\lambda).\end{array}\right\}\tag{8.39}$$

Remark 8.6. (i) Under Assumptions 2.2, 2.5, suppose also that U is a symmetric open neighborhood of the origin in $\mathbb{R}^n$, and that for each (λ,t) the function $L(\lambda,t,q,v)$ is even in (q,v). If $x_\lambda = 0\ \forall\lambda$, then H defined by (8.28) can be required to be even in z. In fact, by Lemma 2.4(vi) $\tilde{L}(\lambda,t,q,v)$ can be chosen to be even in (q,v). Since $x_\lambda = 0\ \forall\lambda$, the function $\hat{L}$ defined by (2.14) is even in (q,v), and so $\partial_v\hat{L}_\lambda(t,-q,-v) = -\partial_v\hat{L}_\lambda(t,q,v)$. (This implies $y_{0,\lambda}(t) = \partial_v\hat{L}_\lambda(t,0,0) = 0$ and hence $u_{0,\lambda}(t) = (y_{0,\lambda}(t)^T,0)^T = 0$.) Suppose that $\partial_v\hat{L}_\lambda(t,q,v) = p$. Then $-p = \partial_v\hat{L}_\lambda(t,-q,-v)$. By (8.9) we deduce that

$$\begin{array}{rcl}\hat{H}(\lambda,t,-p,-q) &=& (-p,-v)_{\mathbb{R}^n}-\hat{L}_\lambda(t,-q,-v)\\ &=& (p,v)_{\mathbb{R}^n}-\hat{L}_\lambda(t,q,v) = \hat{H}(\lambda,t,p,q)\end{array}$$

and therefore that $H(\lambda,t,z)$ is even in z by (8.28).

(ii) We make the following assumption:

Assumption 8.7. Under Assumption 2.3 with $E = I_n$, suppose that L also satisfies

$$L(\lambda,-t,q,-v) = L(\lambda,t,q,v)\quad\forall(t,q,v)\in\Lambda\times\mathbb{R}\times U\times\mathbb{R}^n.\tag{8.40}$$

For each $\lambda\in\Lambda$, let $x_\lambda:\mathbb{R}\to U$ be a C^2 map satisfying

$$\left.\begin{array}{l}\frac{d}{dt}\Big(\frac{\partial L_\lambda}{\partial v}(t,x(t),\dot{x}(t))\Big)-\frac{\partial L_\lambda}{\partial q}(t,x(t),\dot{x}(t)) = 0\ \forall t\in\mathbb{R}\\ x(-t) = x(t) = x(t+\tau)\quad\forall t\in\mathbb{R}\end{array}\right\}\tag{8.41}$$

and such that maps $\Lambda\times\mathbb{R}\ni(\lambda,t)\to x_\lambda(t)\in U$ and $\Lambda\times\mathbb{R}\ni(\lambda,t)\mapsto\dot{x}_\lambda(t)\in\mathbb{R}^n$ are also continuous. Moreover, for any compact or sequential compact subset $\hat{\Lambda}\subset\Lambda$ there exists $\rho>0$ such that $\sup\{|\dot{x}_\lambda(t)|\,|\,(\lambda,t)\in\hat{\Lambda}\times[0,\tau]\}<\rho$ and that $\hat{\Lambda}\times[0,\tau]\times U\times B^n_\rho(0)\ni(\lambda,t,q,v)\mapsto L_\lambda(t,q,v)$ is strictly convex with respect to v.

Then we have a corresponding result to Lemma 2.4, in which $[0,\tau]$ is replaced by $\mathbb{R}$ and $\tilde{L}:\Lambda\times\mathbb{R}\times U\times\mathbb{R}^n\to\mathbb{R}$ is defined by

$$\tilde{L}(\lambda,t,q,v)=L(\lambda,t,q,v)+\psi_{\rho_0,\rho_1}(|v|^2) \tag{8.42}$$

as in (2.8). Clearly, it is τ-periodic in t and (8.40) implies

$$\tilde{L}(\lambda,-t,q,-v)=\tilde{L}(\lambda,t,x,v)\quad\forall(\lambda,t,q,v)\in\Lambda\times\mathbb{R}\times U\times\mathbb{R}^n. \tag{8.43}$$

As in (2.14) we define $\hat{L}:\Lambda\times\mathbb{R}\times B^n_\delta(0)\times\mathbb{R}^n\to\mathbb{R}$ by

$$\hat{L}(\lambda,t,q,v)\mapsto\tilde{L}(\lambda,t,q+x_\lambda(t),v+\dot{x}_\lambda(t)). \tag{8.44}$$

By Assumption 8.7 and (8.43) it is τ-periodic and

$$\begin{aligned}\hat{L}(\lambda,-t,q,-v)&=L(\lambda,-t,q+x_\lambda(-t),-v-\dot{x}_\lambda(t))\\&=L(\lambda,t,q+x_\lambda(t),v+\dot{x}_\lambda(t))=\hat{L}(\lambda,t,q,v)\end{aligned} \tag{8.45}$$

for any $(\lambda,t,q,v)\in\Lambda\times\mathbb{R}\times B^n_\delta(0)\times\mathbb{R}^n$. Thus $\partial_v\hat{L}(\lambda,-t,q,-v)=-\partial_v\hat{L}(\lambda,t,q,v)$. Let

$$v:\Lambda\times\mathbb{R}\times B^n_\delta(0)\times\mathbb{R}^n\to\mathbb{R}^n,\ (\lambda,t,q,p)\mapsto v(\lambda,t,q,p)$$

be the unique solution of

$$\partial_v\hat{L}_\lambda(t,q,v(\lambda,t,q,p))=p\quad\forall(\lambda,t,q,p)\in\Lambda\times\mathbb{R}\times B^n_\delta(0)\times\mathbb{R}^n \tag{8.46}$$

obtained by the implicit function theorem as in (8.46). Then we have

$$v(\lambda,-t,q,-p)=-v(\lambda,t,q,p)\quad\forall(\lambda,t,q,p)\in\Lambda\times\mathbb{R}\times B^n_\delta(0)\times\mathbb{R}^n. \tag{8.47}$$

As in (8.9) we define $\hat{H}:\Lambda\times\mathbb{R}\times\mathbb{R}^n\times B^n_\delta(0)\to\mathbb{R}$ by

$$\hat{H}(\lambda,t,p,q)=(p,v)_{\mathbb{R}^n}-\hat{L}_\lambda(t,q,v),$$

where $(\lambda,t,q,v)\in\Lambda\times\mathbb{R}\times B^n_\delta(0)\times\mathbb{R}^n$ satisfies $\partial_v\hat{L}_\lambda(t,q,v)=p$, that is,

$$\hat{H}(\lambda,t,p,q)=(p,v(\lambda,t,q,p))_{\mathbb{R}^n}-\hat{L}_\lambda(t,q,v(\lambda,t,q,p)). \tag{8.48}$$

Then from (8.45) and (8.47) we derive

$$\begin{aligned}\hat{H}(\lambda,-t,-p,q,)&=(-p,v(\lambda,-t,q,-p))_{\mathbb{R}^n}-\hat{L}_\lambda(-t,q,v(\lambda,-t,q,-p))\\&=(p,v(\lambda,t,q,p))_{\mathbb{R}^n}-\hat{L}_\lambda(-t,q,-v(\lambda,t,q,p))\\&=(p,v(\lambda,t,q,p))_{\mathbb{R}^n}-\hat{L}_\lambda(t,q,v(\lambda,t,q,p))\\&=\hat{H}(\lambda,t,p,q)\end{aligned} \tag{8.49}$$

for any $(\lambda,t,p,q)\in\Lambda\times\mathbb{R}\times\mathbb{R}^n\times B^n_\delta(0)$. Let us define $H:\Lambda\times\mathbb{R}\times\mathbb{R}^n\times B^n_\delta(0)\to\mathbb{R}$ by

$$H(\lambda,t,z):=(J\dot{u}_{0,\lambda}(t),z)_{\mathbb{R}^{2n}}+\hat{H}_\lambda(t,z+u_{0,\lambda}(t)) \tag{8.50}$$

as in (8.28), where $z=(q^T,p^T)^T$ and $u_{0,\lambda}(t)=(0,y_{0,\lambda}(t)^T)^T$ with $y_{0,\lambda}(t)=\partial_v\hat{L}_\lambda(t,0,0)$. It is clear that $H(\lambda,t+\tau,z)=H(\lambda,t,z)$. Because $y_{0,\lambda}(-t)=\partial_v\hat{L}_\lambda(-t,0,0)=-\partial_v\hat{L}_\lambda(t,0,0)=-y_{0,\lambda}(t)$ we have $u_{0,\lambda}(-t)=-u_{0,\lambda}(t)$ and hence

$$H(\lambda,-t,Nz)\quad=\quad(J\dot{u}_{0,\lambda}(-t),Nz)_{\mathbb{R}^{2n}}+\hat{H}_\lambda(-t,Nz+u_{0,\lambda}(-t))$$

$$= \quad H(\lambda, t, z) \tag{8.51}$$

by (8.49), where $N = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$.According to (8.35) we modify this H to get a new function on $\Lambda \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, also denoted by H, which satisfies

$$H(\lambda, -t, Nz) = H(\lambda, t, z) = H(\lambda, t + \tau, z), \quad \forall(\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n}. \tag{8.52}$$

By (8.31) we have $H''_{\lambda,t}(0) = S_\lambda(t)$. Let $\Upsilon_\lambda : [0, \tau] \to \mathrm{Sp}(2n, \mathbb{R})$ be as above (8.36), and let $\Upsilon_\lambda(\tau/2) = \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{pmatrix}$, where $A_\lambda, B_\lambda, C_\lambda, D_\lambda \in \mathbb{R}^{n\times n}$. For $U_1 = \{0\} \times \mathbb{R}^n$, Long, Zhang and Zhu [28] used the Cappell-Lee-Miller index μ^{CLM} to define

$$\mu_{1,\tau}(\Upsilon_\lambda) = \mu^{\mathrm{CLM}}_{\mathbb{R}^{2n}}(U_1, \Upsilon_\lambda U_1, [0, \tau/2]) \quad \text{and} \quad \nu_{1,\tau}(\Upsilon_\lambda) = \dim \mathrm{Ker}(B_\lambda). \tag{8.53}$$

The author and Wang proved in [41, Theorem 3.4]:

$$m^-_\tau(\hat{\mathcal{L}}^E_\lambda, 0) = m^-_\tau(\mathcal{L}^E_\lambda, x_\lambda) = \mu_{1,\tau}(\Upsilon_\lambda) \quad \text{and} \quad m^0_\tau(\hat{\mathcal{L}}^E_\lambda, 0) = m^0_\tau(\mathcal{L}^E_\lambda, x_\lambda) = \nu_{1,\tau}(\Upsilon_\lambda),$$

where for $x \in EC^1(S_\tau; U)$ and $y \in EC^1(S_\tau; B^n_\iota(0))$,

$$\mathcal{L}^E_\lambda(x) = \int_0^\tau L_\lambda(t, x(t), \dot{x}(t))dt \quad \text{and} \quad \hat{\mathcal{L}}^E_\lambda(y) = \int_0^\tau \hat{L}_\lambda(t, y(t), \dot{y}(t))dt.$$

Corresponding to Claim 8.4, we have: For any $\lambda \in \hat{\Lambda}$, the constant path $z^0 : [0, \tau] \to \mathbb{R}^{2n}$, $t \mapsto 0$, satisfies

$$\dot{z}(t) = J\nabla H_{\lambda,t}(z(t)), \quad z(t + \tau) = z(t) \quad \text{and} \quad z(-t) = Nz(t), t \in \mathbb{R}; \tag{8.54}$$

and (μ, x_μ) is a bifurcation point of the problem (8.41) in $\hat{\Lambda} \times EC^1(S_\tau; \mathbb{R}^n)$ with respect to the trivial branch $\{(\lambda, x_\lambda) \,|\, \lambda \in \hat{\Lambda}\}$ if and only if (μ, z^0) is that of (8.54) in $\hat{\Lambda} \times C^1(S_\tau; \mathbb{R}^{2n})$ (or equivalently $\hat{\Lambda} \times W^{1,2}(S_\tau; \mathbb{R}^{2n})$) with respect to the trivial branch $\{(\lambda, z^0) \,|\, \lambda \in \hat{\Lambda}\}$.

For the bifurcation problem of (2.5)–(2.6), using the above arguments the following results can directly be derived from Theorems 1.4, 1.7 in [37] about bifurcations for Hamiltonian systems respectively.

Theorem 8.8 (Necessary condition). *Under Assumption 8.1, suppose for some $\mu \in \Lambda$ that (μ, x_μ) is a bifurcation point along sequences of the problem (2.5)–(2.6) with respect to the trivial branch $\{(\lambda, x_\lambda) \,|\, \lambda \in \Lambda\}$. Then $m^0_\tau(\mathcal{L}_\mu, x_\mu) > 0$.*

Theorem 8.9 (Sufficient condition). *Under Assumption 8.1, let Λ be first countable. Suppose for some $\mu \in \Lambda$ that there exist two sequences in Λ converging to μ, (λ^-_k) and (λ^+_k), such that for each $k \in \mathbb{N}$,*

$$[m^-_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k}), m^-_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k}) + m^0_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k})] \cap [m^-_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k}), m^-_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k}) + m^0_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k})] = \emptyset$$

and either $m^0_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k}) = 0$ or $m^0_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k}) = 0$. Let $\hat{\Lambda} := \{\mu, \lambda^+_k, \lambda^-_k \,|\, k \in \mathbb{N}\}$. Then (μ, x_μ) is a bifurcation point of the problem (2.5)–(2.6) in $\hat{\Lambda} \times C^1_E([0, \tau]; U)$ with respect to the trivial branch $\{(\lambda, x_\lambda) \,|\, \lambda \in \Lambda\}$ (and thus a bifurcation point along sequences of the problem (2.5)–(2.6) in $\Lambda \times C^1_E([0, \tau]; U)$).

Theorem 8.10 (Existence for bifurcations). *Under Assumptions 2.2, 2.5, let Λ be path-connected. Suppose that there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that*

$$[m^-_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-}), m^-_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-}) + m^0_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-})] \cap [m^-_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+}), m^-_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+}) + m^0_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+})]$$

is empty, and either $m^0_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+}) = 0$ or $m^0_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-}) = 0$. Then there exists $\mu \in \Lambda$ such that (μ, x_μ) is a bifurcation point along sequences of the problem (2.5)–(2.6) in $\Lambda \times C^1_E([0,\tau]; U)$ with respect to the trivial branch $\{(\lambda, x_\lambda) \,|\, \lambda \in \Lambda\}$, and μ is not equal to λ^+ (resp. λ^-) if $m^0_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+}) = 0$ (resp. $m^0_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-}) = 0$).

Theorem 8.11 (Alternative bifurcations of Rabinowitz's type and of Fadell-Rabinowitz's type). *Under Assumptions 2.2, 2.5 with Λ being a real interval, let $m^0_\tau(\mathcal{L}_\mu, x_\mu) > 0$ for some $\mu \in \mathrm{Int}(\Lambda)$. Suppose that $m^0_\tau(\mathcal{L}_\lambda, x_\lambda) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near μ, and that $m^-_\tau(\mathcal{L}_\lambda, x_\lambda)$ take, respectively, values $m^-_\tau(\mathcal{L}_\mu, x_\mu)$ and $m^-_\tau(\mathcal{L}_\mu, x_\mu) + m^0_\tau(\mathcal{L}_\mu, x_\mu)$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of μ. Then one of the following alternatives occurs:*

(i) *The problem (2.5)–(2.6) with $\lambda = \mu$ has a sequence of solutions, $x_k \neq x_\mu$, $k = 1, 2, \cdots$, which converges to x_μ in $C^2([0,\tau], U)$.*

(ii) *For every $\lambda \in \Lambda \setminus \{\mu\}$ near μ there is a solution $y_\lambda \neq x_\lambda$ of (2.5)–(2.6) with parameter value λ, such that $\|y_\lambda - x_\lambda\|_{C^2} \to 0$ as $\lambda \to \mu$.*

(iii) *For a given neighborhood $\mathcal{W}$ of x_μ in $C^2([0,\tau], U)$, there is an one-sided neighborhood Λ^0 of μ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (2.5)–(2.6) with parameter value λ has at least two distinct solutions in $\mathcal{W}$, $y^1_\lambda \neq x_\lambda$ and $y^2_\lambda \neq x_\lambda$, which can also be chosen to satisfy $\mathcal{L}_\lambda(y^1_\lambda) \neq \mathcal{L}_\lambda(y^2_\lambda)$ provided that $m^0_\tau(\mathcal{L}_\mu, x_\mu) > 1$ and (2.5)–(2.6) with parameter value λ has only finitely many distinct solutions in $\mathcal{W}$.*

In addition, if we also assume that U is a symmetric open neighborhood of the origin in $\mathbb{R}^n$, $x_\lambda \equiv 0\ \forall \lambda$ and each $L(\lambda, t, \cdot)$ is even, then either (i) holds or the following occurs:

(iv) *There exist left and right neighborhoods Λ^- and Λ^+ of μ in Λ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq m^0_\tau(\mathcal{L}_\mu, 0)$, and for $\lambda \in \Lambda^- \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^+ \setminus \{\mu\}$), (2.5)–(2.6) with parameter value λ has at least n^- (resp. n^+) distinct pairs of nontrivial solutions, $\{y^i_\lambda, -y^i_\lambda\}$, $i = 1, \cdots, n^-$ (resp. n^+), which converge to zero in $C^2([0,\tau], U)$ as $\lambda \to \mu$.*

Proof of Theorem 8.8. By the assumption (cf. Definition 1.3) there exists a sequence $(\lambda_k) \subset \Lambda$ converging to μ and solutions $x^k \neq x_{\lambda_k}$ of (2.5)–(2.6) with $\lambda = \lambda_k$ such that $x^k \to x_\mu$ in $C^1([0,\tau], \mathbb{R}^n)$. By Lemma 2.6(ii) we have also $\|x_k - x_{\lambda_k}\|_{C^2} \to 0$ as $k \to \infty$. Let $\hat{\Lambda} = \{\mu, \lambda_k \,|\, k \in \mathbb{N}\}$. It is a compact and sequential compact subset of Λ. For this $\hat{\Lambda}$ let $\hat{L}$ be as above (8.5). By Claim 8.4 (μ, z^0) is a bifurcation point of (8.32) in $\hat{\Lambda} \times C^1_{\mathbb{E}}([0,\tau]; \mathbb{R}^{2n})$ with respect to the trivial branch $\{(\lambda, z^0) \,|\, \lambda \in \hat{\Lambda}\}$. We conclude $\nu_{\tau,\mathbb{E}}(\Upsilon_\lambda) > 0$ by [37, Theorem 1.4(I)], and hence $m^0_\tau(\mathcal{L}_\mu, x_\mu) > 0$ by (8.39). $\square$

Proof of Theorem 8.9. Note that $\hat{\Lambda} = \{\mu, \lambda^+_k, \lambda^+_k \,|\, k \in \mathbb{N}\}$ is first countable, and not only compact but also sequential compact in Λ. For this $\hat{\Lambda}$ let $\hat{L}$ be as above (8.5). Under the assumptions of Theorem 8.9, from (8.39) we deduce that for each $k \in \mathbb{N}$,

$$\begin{aligned}
& [i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-_k}), i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-_k}) + \nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-_k})] \cap [i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^+_k}), i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^+_k}) + \nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda^+_k})] \\
= \; & [m^-_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k}) + \dim\mathrm{Ker}(E - I_n), m^-_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k}) + \dim\mathrm{Ker}(E - I_n) + m^0_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k})] \\
& \cap [m^-_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k}) + \dim\mathrm{Ker}(E - I_n), m^-_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k}) + \dim\mathrm{Ker}(E - I_n) + m^0_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k})]
\end{aligned}$$

$$= \emptyset$$

and either $\nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda_k +}) = m^0_\tau(\mathcal{L}_{\lambda^+_k}, x_{\lambda^+_k}) = 0$ or $\nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-_k}) = m^0_\tau(\mathcal{L}_{\lambda^-_k}, x_{\lambda^-_k}) = 0$. By [37, Theorem 1.4(II)] we get that (μ, z^0) is a bifurcation point of (8.32) in $\hat{\Lambda} \times C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$ (or equivalently $\hat{\Lambda} \times W^{1,2}_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$) with respect to the trivial branch $\{(\lambda, z^0)\,|\,\lambda \in \hat{\Lambda}\}$. Then the required conclusion follows from Claim 8.4. $\Box$

Proof of Theorem 8.10. Since Λ is path-connected, there exists a path $\alpha : [0,1] \to \Lambda$ from $\lambda^+ = \alpha(0)$ to $\lambda^- = \alpha(1)$. Let $\hat{\Lambda} = \alpha([0,1])$, which is a compact and sequential compact subset of Λ. For this $\hat{\Lambda}$ let $\hat{L}$ be as above (8.5). From the assumptions of Theorem 8.10 and (8.39) we derive that

$$\begin{aligned}
& [i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-}), i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-}) + \nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-})] \cap [i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^+}), i_{\tau,\mathbb{E}}(\Upsilon_{\lambda^+}) + \nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda^+})] \\
= \; & [m^-_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-}) + \dim\operatorname{Ker}(E - I_n), m^-_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-}) + \dim\operatorname{Ker}(E - I_n) + m^0_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-})] \\
& \cap [m^-_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+}) + \dim\operatorname{Ker}(E - I_n), m^-_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+}) + \dim\operatorname{Ker}(E - I_n) + m^0_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+})] \\
= \; & \emptyset
\end{aligned}$$

and either $\nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda^+}) = m^0_\tau(\mathcal{L}_{\lambda^+}, x_{\lambda^+}) = 0$ or $\nu_{\tau,\mathbb{E}}(\Upsilon_{\lambda^-}) = m^0_\tau(\mathcal{L}_{\lambda^-}, x_{\lambda^-}) = 0$. As above the required conclusions follows from Claim 8.4 and [37, Theorem 1.4(III)]. $\Box$

Proof of Theorem 8.11. Since Λ is a real interval and $\mu \in \operatorname{Int}(\Lambda)$ we can assume $\Lambda = \hat{\Lambda} = [\mu - \varepsilon, \mu + \varepsilon]$ for some $\varepsilon > 0$, which is compact and sequential compact. For this $\hat{\Lambda}$ let $\hat{L}$ be as above (8.5). It follows from the assumptions of Theorem 8.11 and (8.39) that

- $\nu_{\tau,\mathbb{E}}(\Upsilon_\mu) = m^0_\tau(\mathcal{L}_\mu, x_\mu) \neq 0$, $\nu_{\tau,\mathbb{E}}(\Upsilon_\lambda) = m^0_\tau(\mathcal{L}_\lambda, x_\lambda) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near μ,
- as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of μ, $i_{\tau,\mathbb{E}}(\Upsilon_\lambda) = m^-_\tau(\mathcal{L}_\lambda, x_\lambda) + \dim\operatorname{Ker}(E - I_n)$ take, respectively, values $m^-_\tau(\mathcal{L}_\mu, x_\mu) + \dim\operatorname{Ker}(E - I_n) = i_{\tau,\mathbb{E}}(\Upsilon_\mu)$ and

  $$m^-_\tau(\mathcal{L}_\mu, x_\mu) + m^0_\tau(\mathcal{L}_\mu, x_\mu) + \dim\operatorname{Ker}(E - I_n) = i_{\tau,\mathbb{E}}(\Upsilon_\mu) + m^0_\tau(\mathcal{L}_\mu, x_\mu) = i_{\tau,\mathbb{E}}(\Upsilon_\mu) + \nu_{\tau,\mathbb{E}}(\Upsilon_\mu).$$

Hence by [37, Theorem 1.7] one of the following assertions holds:

(A) The problem (8.32) with $\lambda = \mu$ has a sequence of distinct nontrivial solutions converging to z^0 in $C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$, $z^k(t) = (p^k(t)^T, x^k(t)^T)^T$, $k = 1, 2, \cdots$.

(B) For every $\lambda \in \Lambda \setminus \{\mu\}$ near μ there is a solution $z^\lambda \neq z^0$ of (8.32) with parameter value λ, such that z^λ converges to zero in $C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$ as $\lambda \to \mu$.

(C) For a given neighborhood $\mathcal{V}$ of z^0 in $C^1_{\mathbb{E}}([0,\tau];\mathbb{R}^{2n})$ there is an one-sided neighborhood Λ^0 of μ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (8.32) with parameter value λ has at least two distinct solutions $z^{1\lambda} \neq z^0$ and $z^{2\lambda} \neq z^0$ in $\mathcal{V}$, which can also be required to satisfy

$$\begin{aligned}
& \int_0^\tau \left[\frac{1}{2}(J\dot{z}^{1\lambda}(t), z^{1\lambda}(t))_{\mathbb{R}^{2n}} + H(\lambda, t, z^{1\lambda}(t))\right] dt \\
\neq & \int_0^\tau \left[\frac{1}{2}(J\dot{z}^{2\lambda}(t), z^{2\lambda}(t))_{\mathbb{R}^{2n}} + H(\lambda, t, z^{2\lambda}(t))\right] dt
\end{aligned}$$

provided that $\nu_{\tau,\mathbb{E}}(\Upsilon_\mu) > 1$ and (8.32) with parameter value λ has only finitely many solutions in $\mathcal{V}$.

Moreover, if all $H(\lambda, t, \cdot)$ are even, then either (A) holds or the following occurs:

(D) There exist left and right neighborhoods Λ^- and Λ^+ of μ in Λ and integers $n^+, n^- \ge 0$, such that $n^+ + n^- \ge \nu_{\tau,\mathbb{E}}(\Upsilon_\mu)$, and for $\lambda \in \Lambda^- \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^+ \setminus \{\mu\}$), (8.32) with parameter value λ has at least n^- (resp. n^+) distinct pairs of nontrivial solutions, $\{z^{i\lambda}, -z^{i\lambda}\}$, $i = 1, \cdots, n^-$ (resp. n^+), which converge to zero in $C^1_{\mathbb{E}}([0,\tau]; \mathbb{R}^{2n})$ as $\lambda \to \mu$.

In the case of (A), Claim 8.4 shows that $x^k + x_\mu \ne x_\mu$ satisfies (2.5)-(2.6) with $\lambda = \mu$ and

$$\mathfrak{L}_\mu(x^k + x_\mu) = -\int_0^\tau \left[\frac{1}{2}(J\dot{z}^k(t), z^k(t))_{\mathbb{R}^{2n}} + H(\mu, t, z^k(t))\right] dt, \quad k \in \mathbb{N}.$$

Hence (i) of Theorem 8.11 occurs.

In the case of (B), let $z^\lambda(t) = (p^\lambda(t)^T, x^\lambda(t)^T)^T$. By Claim 8.4, $x^\lambda \ne 0$, $x^\lambda + x_\lambda$ satisfies (2.5)-(2.6), and $\|x^\lambda - x_\lambda\|_{C^2} \to 0$ as $\lambda \to \mu$ because of Lemma 2.6. Namely, (ii) of Theorem 8.11 occurs.

In the case of (C), for a given neighborhood $\mathcal{W}$ of x_μ in $C^2([0,\tau], U)$, by Lemma 2.6 we can choose a neighborhood Λ^* of μ and a positive number $\epsilon > 0$ such that if $x \in C^1_E([0,\tau]; \mathbb{R}^n)$ satisfies $\|x\|_{C^1} < \epsilon$ and $x + x_\lambda$ solves (2.5)–(2.6) with parameter value $\lambda \in \Lambda^*$ then $x + x_\lambda \in \mathcal{W}$. Let us take the above neighborhood $\mathcal{V}$ of z^0 in $C^1_{\mathbb{E}}([0,\tau]; \mathbb{R}^{2n})$ as $\mathcal{V} = \{z \in C^1_{\mathbb{E}}([0,\tau]; \mathbb{R}^{2n}) \,|\, \|z\|_{C^1} < \epsilon\}$. We can require that the corresponding Λ^0 is contained in Λ^*. Let $z^{i\lambda}(t) = (p^{i\lambda}(t)^T, x^{i\lambda}(t)^T)^T$, $i = 1, 2$. Since (8.34) implies that $x^{1\lambda} = x^{2\lambda}$ if and only if $p^{1\lambda} = p^{2\lambda}$, we obtain $x^{1\lambda} \ne x^{2\lambda}$. Clearly, for $i = 1, 2$, $\|x^{i\lambda}\|_{C^1} < \epsilon$ and $x^{i\lambda} + x_\lambda$ solves (2.5)–(2.6) with parameter value $\lambda \in \Lambda^*$, and therefore $x^{i\lambda} + x_\lambda \in \mathcal{W}$,

$$\mathfrak{L}_\lambda(x^{i\lambda} + x_\lambda) = -\int_0^\tau \left[\frac{1}{2}(J\dot{z}^{i\lambda}(t), z^{i\lambda}(t))_{\mathbb{R}^{2n}} + H(\lambda, t, z^{i\lambda}(t))\right] dt.$$

Suppose $m^0_\tau(\mathfrak{L}_\mu, x_\mu) > 1$, which implies $\nu_{\tau,\mathbb{E}}(\Upsilon_\mu) > 1$, and that (2.5)–(2.6) with some parameter value $\lambda \in \Lambda^0$ has only finitely many distinct solutions in $\mathcal{W}$. Then by Lemma 2.6, (8.32) with this parameter value λ has only finitely many distinct solutions in $\mathcal{V}$. Hence $\mathfrak{L}_\lambda(x^{1\lambda} + x_\lambda) \ne \mathfrak{L}_\lambda(x^{2\lambda} + x_\lambda)$. These show that (iii) of Theorem 8.11 occurs.

When U is a symmetric open neighborhood of the origin in $\mathbb{R}^n$, $x_\lambda \equiv 0$ $\forall \lambda$ and each $L(\lambda, t, \cdot)$ is even, it has been proved in Remark 8.6(i) that all $H(\lambda, t, \cdot)$ are even. Let $z^{i\lambda}(t) = (p^{i\lambda}(t)^T, x^{i\lambda}(t)^T)^T$, $i = 1, \cdots, n^-$ (resp. n^+), be as in (D). As above, using Claim 8.4 we deduce that $\{x^{i\lambda}, -x^{i\lambda}\}$, $i = 1, \cdots, n^-$ (resp. n^+), satisfy (iv) of Theorem 8.11. $\square$

Clearly, Theorems 8.8, 8.9, 8.11, 8.11 imply Theorems 3.5, 3.5(II.3), 3.6(iii), 3.7, respectively.

Similarly, Under Assumption 8.7, by Remark 8.6(ii) some corresponding bifurcation results of the problem (8.41) may follow from Theorems 1.23, 1.24, 1.26 in [37] directly.

For Lagrangian systems on $\mathbb{R}^n$ we may use [37, Theorem 1.14] to derive the following strengthening version of Theorem 1.20.

Theorem 8.12 (Alternative bifurcations of Fadell-Rabinowitz's type and of Rabinowitz's type)**.** *Under Assumption 2.3 with Λ being a real interval, suppose also that L is independent of t, the orthogonal matrix E satisfies $E^l = I_n$ for some $l \in \mathbb{N}$. Let*

$$\left.\begin{array}{l}\Lambda \ni \lambda \to x_\lambda \in U \cap \mathrm{Ker}(E - I_n) \textit{ be continuous and} \\ \partial_q L_\lambda(x_\lambda, 0) = 0\ \forall \lambda \in \Lambda.\end{array}\right\} \tag{8.55}$$

Suppose that for some $\mu \in \mathrm{Int}(\Lambda)$ and $\tau > 0$,

(a) $\partial_{vv}L_\mu(x_\mu, 0)$ *is positive definite;*

(b) $\partial_{qq}L_\mu(x_\mu, 0)\, y = 0$ *and* $Ey = y$ *have only the zero solution in* $\mathbb{R}^n$*;*

(c) $m^0_\tau(\mathcal{L}_\mu, x_\mu) \neq 0$, $m^0_\tau(\mathcal{L}_\lambda, x_\lambda) = 0$ *for each* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ, *and* $m^-_\tau(\mathcal{L}_\lambda, x_\lambda)$ *take, respectively, values* $m^-_\tau(\mathcal{L}_\mu, x_\mu)$ *and* $m^-_\tau(\mathcal{L}_\mu, x_\mu) + m^0_\tau(\mathcal{L}_\mu, x_\mu)$ *as* $\lambda \in \Lambda$ *varies in two deleted half neighborhoods of* μ.

Then one of the following alternatives occurs for the problem:

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\partial_v L_\lambda(x(t), \dot{x}(t))\Big) - \partial_q L_\lambda(x(t), \dot{x}(t)) = 0 \ \forall t \in \mathbb{R} \\ Ex(t) = x(t+\tau) \quad \forall t \in \mathbb{R}. \end{array}\right\} \tag{8.56}$$

(i) *Equation (8.56) with* $\lambda = \mu$ *has a sequence of* $\mathbb{R}$*-distinct solutions,* x_k, $k = 1, 2, \cdots$, *which are* $\mathbb{R}$*-distinct with* x_μ *and converges to* x_μ *in* $\mathcal{X}^2_\tau(\mathbb{R}^n, E)$.

(ii) *There exist left and right neighborhoods* Λ^- *and* Λ^+ *of* μ *in* Λ *and integers* $n^+, n^- \geq 0$, *such that* $n^+ + n^- \geq m^0_\tau(\mathcal{L}_\mu, x_\mu)/2$, *and for* $\lambda \in \Lambda^- \setminus \{\mu\}$ *(resp.* $\lambda \in \Lambda^+ \setminus \{\mu\}$*), (8.56) with parameter value* λ *has at least* n^- *(resp.* n^+*)* $\mathbb{R}$*-distinct solutions solutions,* $x^i_\lambda \notin \mathbb{R} \cdot x_\lambda$, $i = 1, \cdots, n^-$ *(resp.* n^+*) such that all* $x^i_\lambda - x_\lambda$ *converge to zero in* $\mathcal{X}^2_\tau(\mathbb{R}^n, E)$ *as* $\lambda \to \mu$.

Moreover, if $m^0_\tau(\mathcal{L}_\mu, x_\mu) \geq 3$, *then (ii) may be replaced by the following alternatives:*

(iii) *For every* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ *there is a solution* $y_\lambda \notin \mathbb{R} \cdot x_\lambda$ *of (8.56) with parameter value* λ, *such that* $y_\lambda - x_\lambda$ *converges to zero in* $\mathcal{X}^2_\tau(\mathbb{R}^n, E)$ *as* $\lambda \to \mu$.

(iv) *For a given* $\varepsilon > 0$ *there is an one-sided neighborhood* Λ^0 *of* μ *in* Λ *such that for any* $\lambda \in \Lambda^0 \setminus \{\mu\}$, *(8.56) with parameter value* λ *has either infinitely many* $\mathbb{R}$*-distinct solutions* $\bar{y}^k_\lambda \notin \mathbb{R} \cdot x_\lambda$ *such that* $\|\bar{y}^k_\lambda|_{[0,\tau]} - x_\lambda|_{[0,\tau]}\|_{C^2} < \varepsilon$, $k = 1, 2, \cdots$, *or at least two* $\mathbb{R}$*-distinct solutions* $y^1_\lambda \notin \mathbb{R} \cdot x_\lambda$ *and* $y^2_\lambda \notin \mathbb{R} \cdot x_\lambda$ *such that* $\|y^i_\lambda|_{[0,\tau]} - x_\lambda|_{[0,\tau]}\|_{C^2} < \varepsilon$, $i = 1, 2$, *and that* $\mathcal{L}_\lambda(y^1_\lambda) \neq \mathcal{L}_\lambda(y^2_\lambda)$.

Proof of Theorem 8.12. Take a small $\varepsilon > 0$ so that $\hat{\Lambda} = [\mu - \varepsilon, \mu + \varepsilon] \subset \Lambda$. By Assumption 2.3, $\partial_{vv}L_\lambda(x, v)$ continuously depends on $(\lambda, x, v) \in \Lambda \times U \times \mathbb{R}^n$. Because of this, (8.55) and the condition (a), by shrinking $\varepsilon > 0$ and U toward x_μ, we can assume that there exists some small real $\rho > 0$ such that $L(\lambda, q, v)$ is strictly convex in v in $B^n_\rho(0)$ for each $(\lambda, q) \in \Lambda \times U$.

Since $E^l = I_n$ implies $\mathbb{E}^l = I_{2n}$, where $\mathbb{E}$ is as in (8.10), each solution of (8.56) is $l\tau$-periodic. Note that all solutions of (8.56) near x_μ sit in a compact neighborhood of $x_\mu \in \mathbb{R}^n$. We have the corresponding Lagrangian

$$\hat{L} : \Lambda \times B^n_\delta(0) \times \mathbb{R}^n \to \mathbb{R},\ (\lambda, q, v) \mapsto \tilde{L}(\lambda, q + x_\lambda, v)$$

as in (2.14). By Lemma 2.4 and (8.55), for any $(\lambda, q, v) \in \Lambda \times B^n_\delta(0) \times \mathbb{R}^n$ it holds that

$$\hat{L}(\lambda, Eq, Ev) = \tilde{L}(\lambda, Eq + x_\lambda, Ev) = \tilde{L}(\lambda, E(q + x_\lambda), Ev) = \tilde{L}(\lambda, q + x_\lambda, v) = \hat{L}(\lambda, q, v).$$

This shows that $\partial_v \hat{L}_\lambda(Eq, Ev) = E\partial_v \hat{L}_\lambda(q, v)$ because E is an orthogonal matrix. Obverse that for $(\lambda, p, q) \in \Lambda \times \mathbb{R}^n \times B^n_\delta(0)$ we have a unique $v = v(\lambda, q, p) \in \mathbb{R}^n$ such that $\partial_v \hat{L}_\lambda(q, v(\lambda, q, p)) = p$. It follows that $v(\lambda, Eq, Ep) = Ev(\lambda, q, p)$. Let us define $\hat{H} : \Lambda \times \mathbb{R}^n \times B^n_\delta(0) \to \mathbb{R}$ by

$$\hat{H}(\lambda, p, q) = (p, v(\lambda, q, p))_{\mathbb{R}^n} - \hat{L}_\lambda(q, v(\lambda, q, p))$$

as in (8.9). Clearly, $\hat{H}(\lambda, Ep, Eq) = \hat{H}(\lambda, p, q)$ since E is an orthogonal.

Note that (8.55) implies the constant path x_λ to satisfy (8.56) for each $\lambda \in \Lambda$. By Claim 8.3 $u_{0,\lambda} := (y_{0,\lambda}^T, 0)^T$ with $y_{0,\lambda} := \partial_v \hat{L}_\lambda(0,0) = \partial_v L_\lambda(x_\lambda, 0)$ satisfies (8.10). As in (8.28) we get a corresponding Hamiltonian $H : \Lambda \times \mathbb{R}^n \times B^n_\delta(0) \to \mathbb{R}$,

$$H(\lambda, z) = H_\lambda(z) := \hat{H}_\lambda(z + u_{0,\lambda}). \tag{8.57}$$

From $Ey_{0,\lambda} = E\partial_v L_\lambda(x_\lambda, 0) = \partial_v L_\lambda(Ex_\lambda, 0) = \partial_v L_\lambda(x_\lambda, 0) = y_{0,\lambda}$, it follows that $\mathbb{E}u_{0,\lambda} = u_{0,\lambda}$ and $H(\lambda, \mathbb{E}z) = H(\lambda, z)$ for all $(\lambda, z) \in \Lambda \times \mathbb{R}^n \times B^n_\delta(0)$.

By (a corresponding result of) Claim 8.4, for any $\lambda \in \Lambda$, $z^0 : \mathbb{R} \to \mathbb{R}^{2n}$, $t \mapsto 0$, satisfies

$$\dot{z}(t) = J\nabla H_\lambda(z(t)) \quad \text{and} \quad \mathbb{E}z(t) = z(t+\tau)\ \forall t \in \mathbb{R}; \tag{8.58}$$

and $x \in \mathcal{X}^2_\tau(\mathbb{R}^n, E)$ near x_μ satisfies (8.56) if and only if

$$z(t) := ((\partial_v L_\lambda(t, x(t), \dot{x}(t)))^T, x(t)^T - x_\lambda^T)^T$$

satisfies (8.58). As in (8.31) we have also

$$(H_\mu)''(z^0) = (\hat{H}_\mu)''(u_{0,\lambda}) = S_\mu = \begin{pmatrix} P_\mu^{-1} & -P_\mu^{-1}Q_\mu \\ -Q_\mu^T P_\mu^{-1} & Q_\mu^T P_\mu^{-1} Q_\mu - R_\mu \end{pmatrix},$$

where $P_\mu = \partial_{vv} L_\mu(x_\mu, 0)$ and $Q_\mu = \partial_{qv} L_\mu(x_\mu, 0)$ and $R_\mu = \partial_{qq} L_\mu(x_\mu, 0)$. We claim

$$\mathrm{Ker}(\mathbb{E} - I_{2n}) \cap \mathrm{Ker}((H_\mu)''(z^0)) = \{0\}.$$

Indeed, suppose that $(u^T, v^T)^T$ belongs to the left side. Then

$$Eu = u, \quad Ev = v, \quad P_\mu^{-1}u - P_\mu^{-1}Q_\mu v = 0, \quad -Q_\mu^T P_\mu^{-1}u + Q_\mu^T P_\mu^{-1} Q_\mu v - R_\mu v = 0.$$

The latter two equations imply $R_\mu v = 0$. By the assumption (b) we get $v = 0$ and so $u = 0$.

Let $\Upsilon_\lambda(t) = \exp(tJH''_\lambda(z^0))$ for $t \in \mathbb{R}$. By (8.36), (8.38) and the assumption (c), we get

$$\nu_{\tau,\mathbb{E}}(\Upsilon_\mu|_{[0,\tau]}) = m^0_\tau(\mathcal{L}_\mu, x_\mu) \neq 0 \quad \text{and} \quad \nu_{\tau,\mathbb{E}}(\Upsilon_\lambda|_{[0,\tau]}) = m^0_\tau(\mathcal{L}_\lambda, x_\lambda) = 0$$

for each $\lambda \in \Lambda \setminus \{\mu\}$ near μ. As in the proof of Theorem 8.11 we may also derive from (8.39) that $i_{\tau,\mathbb{E}}(\Upsilon_\lambda|_{[0,\tau]})$ takes, respectively, values $i_{\tau,\mathbb{E}}(\Upsilon_\mu|_{[0,\tau]})$ and $i_{\tau,\mathbb{E}}(\Upsilon_\mu|_{[0,\tau]}) + \nu_{\tau,\mathbb{E}}(\Upsilon_\mu|_{[0,\tau]})$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of μ. By Claim 8.4, a solution x^λ near $x_\mu \in \mathcal{X}^2_\tau(\mathbb{R}^n, E)$ of (8.56) gives rise to a solution

$$z^\lambda(t) := \begin{pmatrix} x^\lambda(t) - x_\lambda \\ \partial_v L_\lambda(t, x^\lambda(t), \dot{x}^\lambda(t)) - y_{0,\lambda}(t) \end{pmatrix} = \begin{pmatrix} x^\lambda(t) - x_\lambda \\ \partial_v L_\lambda(t, x^\lambda(t), \dot{x}^\lambda(t)) - \partial_v L_\lambda(x_\lambda, 0) \end{pmatrix}$$

of (8.58) near $z^0 \in \mathcal{X}^1_\tau(\mathbb{R}^{2n}, \mathbb{E})$. Clearly, $\mathbb{R}$-different x^λ corresponds to $\mathbb{R}$-different z^λ and vice versa. Using these [37, Theorem 1.14] may lead to the required conclusions. □

Assumption 8.13. Let Assumption 2.2, 2.5 be satisfied and let V_0, V_1 be two linear subspaces in $\mathbb{R}^n$. Consider the boundary value problem on U:

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\partial_v L_\lambda(t, x(t), \dot{x}(t))\Big) - \partial_q L_\lambda(t, x(t), \dot{x}(t)) = 0, \\ \partial_v L_\lambda(0, x(0), \dot{x}(0))[v_0] = 0 \quad \forall v_0 \in V_0, \\ \partial_v L_\lambda(\tau, x(\tau), \dot{x}(\tau))[v_1] = 0 \quad \forall v_1 \in V_1. \end{array}\right\} \tag{8.59}$$

Suppose that each x_λ in Assumption 2.5 also satisfies (8.59).

The cotangent space $T^*\mathbb{R}^n$ of the vector space $\mathbb{R}^n$ is naturally identified with the symplectic space $\mathbb{R}^{2n} = (\mathbb{R}^{2n}(p,q), \omega_0)$, where the standard symplectic form $\omega_0 = dp \wedge dq$ is given by

$$\omega_0\left[(p_1,q_1),(p_2,q_2)\right] := (p_1,q_2)_{\mathbb{R}^n} - (p_2,q_1)_{\mathbb{R}^n}.$$

If V is a linear subspace of $\mathbb{R}^n$, and $V^\perp$ is the orthogonal complementary of V in $\mathbb{R}^n$ with respect to the standard inner product, then the conormal space N^*V of V is the linear subspace of $\mathbb{R}^{2n}$ defined by $N^*V := V^\perp \times V$ via the above $T^*\mathbb{R}^n \equiv \mathbb{R}^{2n}$. N^*V is a Lagrangian subspace of $\mathbb{R}^{2n}$.

Let the Hamiltonian $H : \hat{\Lambda} \times [0,\tau] \times \mathbb{R}^n \times B^n_\delta(0) \to \mathbb{R}$ be as in (8.28). We have the following analogue of Claim 8.4.

Claim 8.14. *Under Assumption 8.13, for any $\lambda \in \hat{\Lambda}$, the constant path $z^0 : [0,\tau] \to \mathbb{R}^{2n},\ t \mapsto 0$, satisfies*

$$\dot{z}(t) = J\nabla H_{\lambda,t}(z(t)), \quad z(0) \in N^*V_0 \quad \text{and} \quad z(\tau) \in N^*V_1; \tag{8.60}$$

and (μ, x_μ) is a bifurcation point of the problem (8.59) in $\hat{\Lambda}\times C^1_{V_0\times V_1}([0,\tau];\mathbb{R}^n)$ with respect to the trivial branch $\{(\lambda, x_\lambda)\,|\,\lambda\in\hat{\Lambda}\}$ if and only if (μ, z^0) is that of (8.60) in $\hat{\Lambda}\times C^1_{N^*V_0\times N^*V_1}([0,\tau];\mathbb{R}^{2n})$ (or equivalently $\hat{\Lambda}\times W^{1,2}_{N^*V_0\times N^*V_1}([0,\tau];\mathbb{R}^{2n})$) with respect to the trivial branch $\{(\lambda, z^0)\,|\,\lambda\in\hat{\Lambda}\}$.

Under Assumption 8.13, each x_λ is a critical point of the functional

$$\mathcal{E}_\lambda : C^1_{V_0\times V_1}([0,\tau];\mathbb{R}^n) \to \mathbb{R},\ \gamma \mapsto \int_0^\tau L_\lambda(t,\gamma(t),\dot{\gamma}(t))dt. \tag{8.61}$$

Let $m^-(\mathcal{E}_\lambda, x_\lambda)$ and $m^0(\mathcal{E}_\lambda, x_\lambda)$ be the Morse index and nullity of $\mathcal{E}_\lambda$ at x_λ. Let $\Upsilon_\lambda : [0,\tau] \to \mathrm{Sp}(2n,\mathbb{R})$ be as above (8.36). We have the (N^*V_0, N^*V_1)-index of it

$$\left(i^{N^*V_1}_{N^*V_0}(\Upsilon_\lambda), \nu^{N^*V_1}_{N^*V_0}(\Upsilon_\lambda)\right) \in \mathbb{Z}\times\{0,1,\cdots,2n\} \tag{8.62}$$

introduced by Liu-Wang-Lin [25], where $\nu^{N^*V_1}_{N^*V_0}(\Upsilon_\lambda) = \dim(\Upsilon_\lambda(\tau)N^*V_0 \cap N^*V_1)$. According to [25], [26, Theorem 5.18] and [7, 11] it holds that

$$m^0(\mathcal{E}_\lambda, x_\lambda) = \nu^{N^*V_1}_{N^*V_0}(\Upsilon_\lambda) \quad m^-(\mathcal{E}_\lambda, x_\lambda) = i^{N^*V_1}_{N^*V_0}(\Upsilon_\lambda) + \ell(V_0,V_1,n),$$

where $\ell(V_0,V_1,n)$ is an integer only depending on (V_0,V_1,n). Using these, under Assumption 8.13, the corresponding results with Theorems 8.8, 8.9, 8.10, 8.11 may follow from Theorems 1.33, 1.34 in [37].

Part II
Bifurcations of geodesics

As pointed out in Remark 1.32, bifurcations of geodesics on Riemannian manifolds may be obtained as examples for those of solutions of Lagrangian systems in Part I; for instance, some of them are listed in Section 14 for clearness. The focus in this part is to study bifurcations of geodesics on Finsler manifolds. After reviewing some necessary definitions and preliminary results on Finsler geometry in Section 9, by refining techniques in [31] we can directly derive some bifurcation results of geodesics on Finsler manifolds in Sections 10, 11, 12, 13 from theorems in Part I.

9 Preliminaries for Finsler geometry

Without special statements, let (M, g) be as in "Basic assumptions and conventions" in Introduction and let $\mathbb{I}_g$ be a C^7 isometry on (M, g). Let P and Q be two connected C^7 submanifolds in M of dimension less than $n = \dim M$ and without boundary. For an integer $2 \le \ell \le 6$, a C^ℓ Finsler metric on M is a continuous function $F : TM \to \mathbb{R}$ satisfying the following properties:

(i) F is C^ℓ and positive in $TM \setminus 0_{TM}$, where 0_{TM} is the zero section of TM.

(ii) $F(x, tv) = tF(x, v)$ for every $t > 0$ and any $(x, v) \in TM$.

(iii) $L := F^2$ is fiberwise strongly convex, that is, for any $(x, v) \in TM \setminus 0_{TM}$ the symmetric bilinear form (the fiberwise Hessian operator)

$$g^F_v : T_xM \times T_xM \to \mathbb{R},\ (u, w) \mapsto \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[L(x, v + su + tw)\right]\Big|_{s=t=0} \tag{9.1}$$

is positive definite. (g^F_v is called the fundamental tensor of F at v.)

A Finsler metric F is said to be reversible (or absolute homogeneous) if $F(x, -v) = F(x, v)$ for all $(x, v) \in TM$. We say a differentiable curve $\gamma : [a, b] \to M$ to be admissible (or regular) if $\dot\gamma(t) \in TM \setminus 0_{TM}$ for all t. Such an admissible curve $\gamma = \gamma(t)$ in (M, F) is said to have constant speed if $F(\gamma(t), \dot\gamma(t))$ is constant along γ. The length of an admissible piecewise C^1 curve $\gamma : [a, b] \to M$ with respect to F is defined by $l_F(\gamma) = \int_a^b F(\gamma(t), \dot\gamma(t))dt$. According to [4, Proposition 5.1.1(a)], an admissible piecewise C^1 curve γ is called a F-geodesic in (M, F) if it minimizes the length between two sufficiently close points on the curve (hence C^1). The distance between any pair of points $p, q \in M$ is defined by

$$d_F(p, q) = \inf\{l_F(\gamma) \,|\, \gamma : [a, b] \to M \text{ is a piecewise } C^1 \text{ curve from } p \text{ to } q\}.$$

Let $W^{1,2}([0, \tau], M)$ denote the space of absolutely continuous curves γ from $[0, \tau]$ to M such that $\int_0^\tau \langle\dot\gamma(t), \dot\gamma(t)\rangle dt < \infty$, where $\langle u, v\rangle = g_x(u, v)$ for $u, v \in T_xM$. By [48, Theorem 4.3], $W^{1,2}([0, \tau], M)$ is a C^4 Riemannian–Hilbert manifold. A C^7 submanifold $\mathbf{N} \subset M\times M$ determines a Riemannian–Hilbert submanifold of $W^{1,2}([0, \tau], M)$,

$$\Lambda_{\mathbf{N}}(M) := \{\gamma \in W^{1,2}([0, \tau], M) \,|\, (\gamma(0), \gamma(\tau)) \in \mathbf{N}\}$$

with tangent space $T_\gamma\Lambda_{\mathbf{N}}(M) = W^{1,2}_{\mathbf{N}}(\gamma^*TM) = \{\xi \in W^{1,2}(\gamma^*TM) \,|\, (\xi(0), \xi(\tau)) \in T_{(\gamma(0),\gamma(\tau))}\mathbf{N}\}$. We also consider the C^4 Banach manifold

$$\mathcal{C}_{\tau,\mathbf{N}}(M) = \{\gamma \in C^1([0, \tau], M) \,|\, (\gamma(0), \gamma(\tau)) \in \mathbf{N}\},$$

which is equal to $C^1_{P\times Q}([0, \tau]; M)$ [resp. $C^1_{\mathbb{I}_g}([0, \tau]; M)$] if $\mathbf{N}$ is the product $P \times Q$ (resp. the graph of the isometry $\mathbb{I}_g$). It has the following open subset

$$\mathcal{C}_{\tau,\mathbf{N}}(M)_{\rm reg} = \{\gamma \in \mathcal{C}_{\tau,\mathbf{N}}(M) \,|\, \gamma \text{ is admissible}, i.e., \dot\gamma(t) \ne 0\ \forall t \in [0, \tau]\}. \tag{9.2}$$

Claim 9.1 ([8, 23, 44])**.** *For a C^ℓ Finsler metric F on M with $3 \le \ell \le 6$, a curve $\gamma \in \Lambda_{\mathbf{N}}(M)$ is a constant (non-zero) speed F-geodesic satisfying the boundary condition*

$$g^F_{\dot\gamma(0)}(u, \dot\gamma(0)) = g^F_{\dot\gamma(\tau)}(v, \dot\gamma(\tau)) \quad \forall (u, v) \in T_{(\gamma(0),\gamma(\tau))}\mathbf{N} \tag{9.3}$$

if and only if it is a (nontrivial) critical point of the C^{2-0} energy functional of F given by

$$\mathcal{L} : \Lambda_{\mathbf{N}}(M) \to \mathbb{R},\ \gamma \mapsto \int_0^\tau F^2(\gamma(t), \dot\gamma(t))dt. \tag{9.4}$$

(In this case γ must be C^ℓ.)

Therefore under Claim 9.1 a curve $\gamma \in \Lambda_{\mathbf{N}}(M)$ is a constant (non-zero) speed F-geodesic satisfying the boundary condition (9.3) if and only if it belongs to $\mathcal{C}_{\tau,\mathbf{N}}(M)_{\rm reg}$ and is a critical point of the following C^2 functional

$$\mathcal{E}_{\mathbf{N}}: \mathcal{C}_{\tau,\mathbf{N}}(M)_{\rm reg} \to \mathbb{R},\ \gamma \mapsto \int_0^\tau F^2(\gamma(t), \dot\gamma(t))dt.$$

We may denote the Morse index and nullity of $\mathcal{E}$ at a critical point $\gamma \in \mathcal{C}_{\tau,\mathbf{N}}(M)_{\rm reg}$ by

$$m^-(\mathcal{E}_{\mathbf{N}}, \gamma) \quad \text{and} \quad m^0(\mathcal{E}_{\mathbf{N}}, \gamma), \tag{9.5}$$

respectively. (See the explanations above Assumption 1.2 [resp. (1.16)] for $\mathbf{N} = P \times Q$ [resp. $\mathbf{N} = \mathrm{Graph}(\mathbb{I}_g)$].) In particular, for $\mathbf{N} = P \times Q$ we write $\mathcal{E}_{\mathbf{N}}$ as

$$\mathcal{E}_{P,Q}: C^1_{P\times Q}([0,\tau]; M)_{\rm reg} \to \mathbb{R},\ \gamma \mapsto \int_0^\tau [F(\gamma(t), \dot\gamma(t))]^2 dt, \tag{9.6}$$

whose critical point γ corresponds to a C^ℓ constant (non-zero) speed F-geodesic with boundary condition

$$\left\{\begin{array}{ll} g^F_{\dot\gamma(0)}(u, \dot\gamma(0)) = 0 & \forall u \in T_{\gamma(0)}P, \\ g^F_{\dot\gamma(\tau)}(v, \dot\gamma(\tau)) = 0 & \forall v \in T_{\gamma(\tau)}Q \end{array}\right. \tag{9.7}$$

(cf. [6, Chap.1, §1], [8, Proposition 2.1] and [20, Prop. 3.1, Cor.3.7]). (Such geodesics are said to be $g_{\dot\gamma}$-orthogonal (or perpendicular) to P and Q.) When $\ell = 6$, the geodesic γ, $m^-(\mathcal{E}_{P,Q}, \gamma)$ and $m^0(\mathcal{E}_{P,Q}, \gamma)$ have direct geometric explanations. See the second half of this section.

Assumption 9.2. $\{F_\lambda \,|\, \lambda \in \Lambda\}$ is a family of C^ℓ Finsler metrics on M with $3 \le \ell \le 6$ parameterized by a topological space Λ, such that $\Lambda \times TM \ni (\lambda, x, v) \to F_\lambda(x, v) \in \mathbb{R}$ is a continuous, and that all partial derivatives of each F_λ of order less than three depend continuously on $(\lambda, x, v) \in \Lambda \times (TM \setminus 0_{TM})$.

Assumption 9.3. Under Assumption 9.2 with an integer $4 \le \ell \le 6$, for each $\lambda \in \Lambda$ let $\gamma_\lambda : [0, \tau] \to M$ be a constant (non-zero) speed F_λ-geodesic satisfying the boundary condition

$$g^{F_\lambda}_{\dot\gamma_\lambda(0)}(u, \dot\gamma_\lambda(0)) = g^{F_\lambda}_{\dot\gamma_\lambda(\tau)}(v, \dot\gamma_\lambda(\tau)) \quad \forall (u, v) \in T_{(\gamma_\lambda(0), \gamma_\lambda(\tau))}\mathbf{N}, \tag{9.8}$$

where $\mathbf{N} \subset M \times M$ is a C^7 submanifold. (Therefore γ_λ is C^ℓ by Claim 9.1.) It is also required that the maps $\Lambda \times [0, \tau] \ni (\lambda, t) \to \gamma_\lambda(t) \in M$ and $\Lambda \times [0, \tau] \ni (\lambda, t) \mapsto \dot\gamma_\lambda(t) \in TM$ are continuous.

Let $m^-(\mathcal{E}_{\lambda,\mathbf{N}}, \gamma_\lambda)$ and $m^0(\mathcal{E}_{\lambda,\mathbf{N}}, \gamma_\lambda)$ denote the Morse index and nullity at γ_λ of the C^2 functional

$$\mathcal{E}_{\lambda,\mathbf{N}}: \mathcal{C}_{\tau,\mathbf{N}}(M)_{\rm reg} \to \mathbb{R},\ \gamma \mapsto \int_0^\tau [F_\lambda(\gamma(t), \dot\gamma(t))]^2 dt. \tag{9.9}$$

For conveniences, a constant (non-zero) speed F_λ-geodesic satisfying the boundary condition (9.8) is called a constant (non-zero) speed $(F_\lambda, \mathbf{N})$-geodesic.

Definition 9.4. Under Assumptions 9.2, 9.3, constant (non-zero) speed $(F_\lambda, \mathbf{N})$-geodesics with a parameter $\lambda \in \Lambda$ is said bifurcating at $\mu \in \Lambda$ along sequences with respect to the branch $\{\gamma_\lambda \,|\, \lambda \in \Lambda\}$ if there exists an infinite sequence $\{(\lambda_k, \gamma^k)\}_{k=1}^\infty$ in $\Lambda \times C^1([0, \tau], M) \setminus \{(\mu, \gamma_\mu)\}$ converging to (μ, γ_μ), such that each $\gamma^k \ne \gamma_{\lambda_k}$ is a constant (non-zero) speed $(F_{\lambda_k}, \mathbf{N})$-geodesic, $k = 1, 2, \cdots$. (Actually it is not hard to prove that $\gamma^k \to \gamma_\mu$ in $C^2([0, \tau], M)$.)

Here are the problems we study and answers:

- For a constant (nonzero) speed F-geodesic γ which is perpendicular to P at $\gamma(0)$, we shall provide where γ bifurcates and depict a rough bifurcation diagram near it.
- Under Assumptions 9.2, 9.3, using the Morse index $m^-(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda)$, the nullity $m^0(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda)$ and critical groups $C_*(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda;\mathbf{K})$, we give the conditions under which constant (non-zero) speed $(F_\lambda,\mathbf{N})$-geodesics with a parameter $\lambda\in\Lambda$ bifurcate at some $\mu\in\Lambda$ along sequences with respect to the branch $\{\gamma_\lambda\,|\,\lambda\in\Lambda\}$, characterize the location of such a parameter μ, and depict the bifurcation diagram near μ.

Our ideas are suitably modifying F_λ and converting the above questions into those studied in Part 1. Firstly, suitably modifying the proof of [31, Proposition 2.2] we have:

Proposition 9.5. *Under Assumption 9.2 let $L_\lambda := (F_\lambda)^2$. Suppose that*

$$\alpha_g := \inf_{\lambda\in\Lambda}\inf_{(x,v)\in TM,\,|v|_x=1}\inf_{u\ne0}\frac{g^{F_\lambda}_v(u,u)}{g_x(u,u)}\quad\textit{and}\quad \beta_g := \sup_{\lambda\in\Lambda}\sup_{(x,v)\in TM,\,|v|_x=1}\sup_{u\ne0}\frac{g^{F_\lambda}_v(u,u)}{g_x(u,u)}$$

are positive numbers, and that for some constant $C_1>0$,

$$|v|^2_x\le L_\lambda(x,v)\le C_1|v|^2_x\quad\forall(\lambda,x,v)\in\Lambda\times TM.\tag{9.10}$$

Hereafter $|v|_x=\sqrt{g_x(v,v)}$. For each $\lambda\in\Lambda$ define C^ℓ functions $L^\star_\lambda: TM\to\mathbb{R}$ by

$$L^\star_\lambda(x,v)=\psi_{\varepsilon,\delta}(L_\lambda(x,v))+\phi_{\mu,b}(|v|^2_x)-b\tag{9.11}$$

and by $L^*_\lambda(x,v)=(L^\star_\lambda(x,v)-\varrho_0)/\kappa$, where $\psi_{\varepsilon,\delta}$, $\phi_{\mu,b}$ and $\kappa,\varrho,\varrho_0,b$ are as in Lemma 2.1. Then for a given $c>0$ we can choose $\kappa>0$ so large that these L^*_λ satisfy the following:

(i) $L^*_\lambda(x,v)=L_\lambda(x,v)$ *if* $L_\lambda(x,v)\ge\frac{2c}{3C_1}$,

(ii) L^*_λ *attains the minimum, and* $L^*_\lambda(x,v)=\min L^*_\lambda\Longleftrightarrow v=0$,

(iii) $L^*_\lambda(x,v)\le L_\lambda(x,v)$ *for all* $(x,v)\in TM$,

(iv) $\partial_{vv}L^*_\lambda(x,v)[u,u]\ge\min\{\frac{2\mu}{\kappa},\frac12\alpha_g\}|u|^2_x$.

(v) *If F_λ is reversible, i.e. $F_\lambda(x,-v)=F_\lambda(x,v)\ \forall(x,v)\in TM$, so is L^*_λ.*

(vi) *If F_λ is $\mathbb{I}_g$-invariant for a g-isometry $\mathbb{I}_g: M\to M$, (i.e., it satisfies $F_\lambda(\mathbb{I}_g(x),\mathbb{I}_{g*}(u))=F_\lambda(x,u)$ for all $(x,u)\in TM$), so is L^*_λ.*

Moreover, $\Lambda\times TM\ni(\lambda,x,v)\to L^*_\lambda(x,v)\in\mathbb{R}$ is continuous and all partial derivatives of each L^*_λ of order less than three depend continuously on $(\lambda,x,v)\in\Lambda\times TM$.

Proof. By the assumptions, for any $(x,v)\in TM\setminus\{0\}$ and $(x,u)\in TM$ we have

$$\alpha_g|u|^2_x\le g^{F_\lambda}_v(u,u)\le\beta_g|u|^2_x,\quad\forall\lambda\in\Lambda.\tag{9.12}$$

Suppose that (2.4) is satisfied and that $\kappa\ge\mu$. Since $\phi''_{\mu,b}\le0$, $\phi''_{\mu,b}(|v|^2_x)=0$ for $|v|^2_x\ge\frac{2c}{3C_1}$, and $\phi''_{\mu,b}(|v|^2_x)$ is bounded for $|v|^2_x\in[\frac{\delta}{3C_1},\frac{2c}{3C_1}]$, we may choose $\kappa>0$ so large that

$$2\kappa\alpha_g+\frac{8c}{3C_1}\phi''_{\mu,b}(|v|^2_x)\ge\frac12\kappa\alpha_g.$$

By the proof of [31, Proposition 2.2], $L^\star_\lambda$ satisfies [31, Proposition 2.2] and therefore L^*_λ meets conditions (i)-(iv) in Proposition 9.5. Clearly, (9.11) implies (v)-(vi).

Since $\Lambda\times TM\ni(\lambda,x,v)\to F_\lambda(x,v)\in\mathbb{R}$ is a continuous, by (9.11) we see that $\Lambda\times TM\ni(\lambda,x,v)\to L^*_\lambda(x,v)\in\mathbb{R}$ is continuous. Note that $\{(\lambda,x,v)\in\Lambda\times TM\,|\,L_\lambda(x,v)<\varepsilon\}$ is an open neighborhood of $\Lambda\times 0_{TM}$ in $\Lambda\times TM$ and that $\psi_{\varepsilon,\delta}(L_\lambda(x,v))=0$ for all (λ,x,v) in this neighborhood. It follows from this and (9.11) that all partial derivatives of each L^*_λ of order less than three depend continuously on $(\lambda,x,v)\in\Lambda\times TM$ because all partial derivatives of each F_λ of order less than three depend continuously on $(\lambda,x,v)\in\Lambda\times(TM\setminus 0_{TM})$. (Actually, $\Lambda\times TM\ni(\lambda,x,v)\to L^*_\lambda(x,v)\in\mathbb{R}$ is C^ℓ in $\{(\lambda,x,v)\in\Lambda\times TM\,|\,L_\lambda(x,v)<\varepsilon\}$.) □

(*Note*: If M and Λ are compact, for any Riemannian metric g on M both α_g and β_g are positive numbers, and (9.10) always holds if g is replaced by a small scalar multiple of g.)

Under Assumption 9.3, let $\hat\Lambda\subset\Lambda$ be either compact or sequential compact. Since the map $\Lambda\times[0,\tau]\ni(\lambda,t)\to\gamma_\lambda(t)\in M$ is continuous, the image of $\hat\Lambda\times[0,\tau]$ under it is a compact subset of M and therefore there exists an open subset $\hat M$ of M with compact closure such that $\gamma_\lambda([0,\tau])\subset\hat M$ for all $\lambda\in\hat\Lambda$. Then the conditions in Proposition 9.5 can be satisfied. By Assumption 9.3 $\hat\Lambda\times[0,\tau]\ni(\lambda,t)\mapsto F_\lambda(\gamma_\lambda(t),\dot\gamma_\lambda(t))$ is continuous and positive. Therefore we have $c>0$ such that

$$[F_\lambda(\gamma_\lambda(t),\dot\gamma_\lambda(t))]^2>\frac{2c}{C_1},\quad \forall(\lambda,t)\in\hat\Lambda\times[0,\tau],\tag{9.13}$$

where $C_1>0$ is as in Proposition 9.5. Let $L^*_\lambda:T\hat M\to\mathbb{R}$, $\lambda\in\hat\Lambda$, be given by Proposition 9.5 with $(M,\Lambda)=(\hat M,\hat\Lambda)$. Then the C^2 functional

$$\mathcal{E}^*_{\lambda,\mathbf{N}}:\mathcal{C}_{\tau,\mathbf{N}}(\hat M)\to\mathbb{R},\ \gamma\mapsto\int_0^\tau L^*_\lambda(\gamma(t),\dot\gamma(t))dt\tag{9.14}$$

and the functional $\mathcal{E}_{\lambda,\mathbf{N}}$ in (9.9) coincide in the following open subset of $\mathcal{C}_{\tau,\mathbf{N}}(\hat M)_{\rm reg}$,

$$\mathcal{C}_{\tau,\mathbf{N}}(\hat M,\{F_\lambda\,|\,\lambda\in\hat\Lambda\},c/C_1):=\left\{\alpha\in\mathcal{C}_{\tau,\mathbf{N}}(\hat M)\,\middle|\,\min_{(\lambda,t)\in\hat\Lambda\times[0,\tau]}[F_\lambda(\alpha(t),\dot\alpha(t))]^2>2c/C_1\right\}.\tag{9.15}$$

Since $\{\gamma_\lambda\,|\,\lambda\in\hat\Lambda\}\subset\mathcal{C}_{\tau,\mathbf{N}}(\hat M,\{F_\lambda\,|\,\lambda\in\hat\Lambda\},c/C_1)$ by (9.13), they are critical points of $\mathcal{E}^*_{\lambda,\mathbf{N}}$ and

$$m^-(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda)=m^-(\mathcal{E}^*_{\lambda,\mathbf{N}}\gamma_\lambda)\quad\text{and}\quad m^0(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda)=m^0(\mathcal{E}^*_{\lambda,\mathbf{N}},\gamma_\lambda),\tag{9.16}$$
$$C_m(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda;\mathbf{K})=C_m(\mathcal{E}^*_{\lambda,\mathbf{N}}\gamma_\lambda;\mathbf{K})\quad\forall m\in\mathbb{Z}\tag{9.17}$$

for any Abel group $\mathbf{K}$.

Claim 9.6. *Under Assumption 9.3, if $\gamma:[0,\tau]\to M$ is a constant (non-zero) speed F_λ-geodesic (hence C^ℓ) with boundary condition (9.8) is close to γ_λ in $C^1([0,\tau];M)$ then it is a critical point of*

$$\mathcal{C}_{\tau,\mathbf{N}}(\hat M)\ni\gamma\mapsto\mathcal{E}^*_{\lambda,\mathbf{N}}(\gamma)=\int_0^\tau L^*_\lambda(\gamma(t),\dot\gamma(t))dt.\tag{9.18}$$

Conversely, if $\gamma\in\mathcal{C}_{\tau,\mathbf{N}}(\hat M)$ near γ_λ is a critical point of $\mathcal{E}^*_{\lambda,\mathbf{N}}$ then it is a C^ℓ constant (non-zero) speed F_λ-geodesic with boundary condition (9.8) and is near γ_λ in $C^2([0,\tau];M)$.

Proof. If $d\mathcal{E}_{\lambda,\mathbf{N}}(\gamma)=0$ and γ is close to γ_λ in C^1-topology then $\gamma\in\mathcal{C}_{\tau,\mathbf{N}}(\hat M,\{F_\lambda\,|\,\lambda\in\hat\Lambda\},c/C_1)$ and therefore $d\mathcal{E}^*_{\lambda,\mathbf{N}}(\gamma)=0$. Conversely, we only need to prove that γ is also near γ_λ in $C^2([0,\tau];M)$. This may follow from Lemma 2.6(ii) by localization arguments. □

Geometric characteristics of F-geodesics and their Morse indexes and nullities. Without special statements, from now on we always assume $\ell = 6$, i.e., F is a C^6-Finsler metric on M. In this case the Christoffel symbols of the Chern connection ∇ (on the pulled-back tangent bundle π^*TM) with respect to a coordinate chart (Ω, x^i) on M are C^3 functions $\Gamma^i_{jm} : T\Omega \setminus 0_{T\Omega} \to \mathbb{R}$ such that

$$\nabla_{\partial_{x^i}} \partial_{x^j}(v) = \sum_m \Gamma^i_{jm}(v)\partial_{x^m}, \quad i, j \in \{1, \cdots, n\} \tag{9.19}$$

(cf. [38]); and there exist C^2 functions $R^i_{jkl} : T\Omega \setminus 0_{T\Omega} \to \mathbb{R}$, $1 \le i, j, k, l \le n$, such that the trilinear map R_v from $T_{\pi(v)}M \times T_{\pi(v)}M \times T_{\pi(v)}M$ to $T_{\pi(v)}M$ given by

$$R_v(\xi, \eta)\zeta = \sum_{i,j,k,l} \xi^k \eta^l \zeta^j R_j{}^i{}_{kl}(v)\partial_{x^i}|_{\pi(v)} \tag{9.20}$$

defines the Chern curvature tensor (or [4, (3.3.2) & Exercise 3.9.6]) R_V on $\Omega \subset M$ (cf. [38]).

For a curve $c \in W^{1,2}([a, b], M)$ and $r \in \{0, 1\}$ let $W^{r,2}(c^*TM)$ denote the space of all $W^{r,2}$ vector fields along c. Then $\dot c \in L^2(c^*TM) := W^{0,2}(c^*TM)$. Let (x^i, y^i) be the canonical coordinates around $\dot c(t) \in TM$. Write $\dot c(t) = \dot c^i(t)\partial_{x^i}|_{c(t)}$ and $\zeta(t) = \zeta^i(t)\partial_{x^i}|_{c(t)}$ for $\zeta \in W^{1,2}(c^*TM)$. Call $\xi \in C^0(c^*TM)$ admissible if $\xi(t) \in TM \setminus 0_{TM}$ for all $t \in [a, b]$. The Chern connection induces a covariant derivative of ζ along c (with this admissible ξ as reference vector) is defined by

$$D^\xi_{\dot c}\zeta(t) := \sum_m \big(\dot\zeta^m(t) + \sum_{i,j} \zeta^i(t)\dot c^j(t)\Gamma^m_{ij}(c(t), \xi(t))\big)\partial_{x^m}|_{c(t)}. \tag{9.21}$$

Clearly, $D^\xi_{\dot c}\zeta$ belongs to $L^2(c^*TM)$, and sits in $C^{\min\{1,r\}}(c^*TM)$ provided that c is of class C^{r+1}, $\zeta \in C^{r+1}(c^*TM)$ and $\xi \in C^r(c^*TM)$ for some $0 \le r \le 6$; $D^\xi_{\dot c}\zeta(t)$ depends only on $\xi(t)$, $\dot c(t)$ and behavior of ζ near t. If c, ξ and ζ are C^3, C^1 and C^2, respectively, then $D^\xi_{\dot c}\zeta$ is C^1 and $D^\xi_{\dot c}D^\xi_{\dot c}\zeta$ is well-defined and is C^0. It may be proved that a C^2 admissible curve γ in (M, F) is a F-geodesic of constant speed if and only if $D^{\dot\gamma}_{\dot\gamma}\dot\gamma(t) \equiv 0$. In this case γ must be C^6.

For the above C^7 submanifolds P and Q, define the normal bundle of P in (M, F) by

$$TP^\perp := \{v \in TM \setminus 0_{TM} \mid \pi(v) \in P,\ g^F_v(v, w) = 0\ \forall w \in T_{\pi(v)}P\}$$

(though it is not a vector bundle over P). In fact, it is only an n-dimensional C^6 submanifold of TM and the restriction $\pi : TP^\perp \to P$ is a submersion ([20, Lemma 3.3]). For $v \in TP^\perp$ with $\pi(v) = p$, there exists a splitting $T_pM = T_pP \oplus (T_pP)^\perp_v$, where $(T_pP)^\perp_v$ is the subspace of T_pM consisting of g_v-orthogonal vectors to T_pP. Notice that $v \in (T_pP)^\perp_v$ and that each $u \in T_pM$ has a decomposition $\tan^P_v(u) + \mathrm{nor}^P_v(u)$, where $\tan^P_v(u) \in T_pP$ and $\mathrm{nor}^P_v(u) \in (T_pP)^\perp_v$. Let $\tilde S^P_v : T_pP \to T_pP$ be the normal second fundamental form (or shape operator) of P in the direction v. Then (9.7) implies $\dot\gamma(0) \in TP^\perp$ and $\dot\gamma(\tau) \in TQ^\perp$. In terms of these the Hessian of $\mathcal{E}_{P,Q}$ at γ is given by

$$\begin{aligned} D^2\mathcal{E}_{P,Q}(\gamma)[V, W] \;&=\; \int_0^\tau \Big(g_{\dot\gamma}(R_{\dot\gamma}(\dot\gamma, V)\dot\gamma, W) + g_{\dot\gamma}(D^{\dot\gamma}_{\dot\gamma}V, D^{\dot\gamma}_{\dot\gamma}W)\Big)\,dt \\ &\quad + g_{\dot\gamma(0)}\Big(\tilde S^P_{\dot\gamma(0)}(V(0)), W(0)\Big) - g_{\dot\gamma(\tau)}\Big(\tilde S^Q_{\dot\gamma(\tau)}(V(\tau)), W(\tau)\Big) \end{aligned} \tag{9.22}$$

for $V, W \in C^1_{P\times Q}(\gamma^*M) = T_\gamma C^1([0, \tau]; M, P, Q)$. (Here we use the equality $R^\gamma(\dot\gamma, V)\dot\gamma = R_{\dot\gamma}(\dot\gamma, V)\dot\gamma$ in [20, page 66].) The right side of (9.22) can be extended into a continuous symmetric bilinear form $\mathbf{I}^\gamma_{P,Q}$ on $W^{1,2}_{P\times Q}(\gamma^*TM)$, called as the (P, Q)-index form of γ. Since all R^i_{jkl} are

C^2 it can be proved that $V \in W^{1,2}_{P\times Q}(\gamma^*TM)$ belongs to $\mathrm{Ker}(\mathbf{I}^{\gamma}_{P,Q})$ if and only if it is C^4 and satisfies

$$\left.\begin{aligned} D^{\dot\gamma}_{\dot\gamma}D^{\dot\gamma}_{\dot\gamma}V - R_{\dot\gamma}(\dot\gamma, V)\dot\gamma = 0,\\ \tan^P_{\dot\gamma(0)}\Big((D^{\dot\gamma}_{\dot\gamma}V)(0)\Big) = \tilde S^P_{\dot\gamma(0)}(V(0)), \quad \tan^Q_{\dot\gamma}\Big((D^{\dot\gamma}_{\dot\gamma}V)(\tau)\Big) = \tilde S^Q_{\dot\gamma(\tau)}(V(\tau)). \end{aligned}\right\} \tag{9.23}$$

Let $\gamma : [0, \tau] \to M$ be a F-geodesic of (nonzero) constant speed. (It is C^6.) A C^2 vector field J along γ is said to be a Jacobi field if it satisfies the so-called Jacobi equation

$$D^{\dot\gamma}_{\dot\gamma}D^{\dot\gamma}_{\dot\gamma}J - R_{\dot\gamma}(\dot\gamma, J)\dot\gamma = 0. \tag{9.24}$$

(Jacobi fields along γ must be C^4 because each R^i_{jkl} is C^2.) The set $\mathfrak{J}_\gamma$ of all Jacobi fields along γ is a $2n$-dimensional vector space. For $0 \le t_1 < t_2 \le \tau$ if there exists a nonzero Jacobi field J along $\gamma|_{[t_1,t_2]}$ such that J vanishes at $\gamma(t_1)$ and $\gamma(t_2)$, then $\gamma(t_1)$ and $\gamma(t_2)$ are said to be mutually conjugate along $\gamma|_{[t_1,t_2]}$. Suppose that the geodesic γ orthogonally starts at P. That is, $\gamma(0) \in P$ and $\dot\gamma(0)$ is $g^F_{\dot\gamma(0)}$-orthogonal to P. A Jacobi field J along γ is called a P-Jacobi if

$$J(0) \in T_{\gamma(0)}P \quad \text{and} \quad \tan^P_{\dot\gamma(0)}\Big((D^{\dot\gamma}_{\dot\gamma}J)(0)\Big) = \tilde S^P_{\dot\gamma(0)}(J(0)). \tag{9.25}$$

An instant $t_0 \in (0, \tau]$ is called P-focal if there exists a non-null P-Jacobi field J such that $J(t_0) = 0$; and $\gamma(t_0)$ is said to be a P-focal point along γ. The dimension of the space $\mathfrak{J}^P_\gamma$ of all P-Jacobi fields along γ is equal to $n = \dim M$. The dimension $\mu^P_\gamma(t_0)$ of

$$\mathfrak{J}^P_\gamma(t_0) := \big\{J \in \mathfrak{J}^P_\gamma \,\big|\, J(t_0) = 0\big\}$$

is called the (geometrical) multiplicity of $\gamma(t_0)$. For convenience we understand $\mu^P_\gamma(t_0) = 0$ if $\gamma(t_0)$ is not a P-focal point along γ. Then the claim near (9.23) implies that for any $t \in (0, \tau]$,

$$\mathrm{Ker}(\mathbf{I}^{\gamma_t}_{P,\gamma(t)}) = \mathfrak{J}^P_{\gamma_t}(t) \quad \text{with } \gamma_t = \gamma|_{[0,t]}. \tag{9.26}$$

In particular, $\mathrm{Ker}(\mathbf{I}^{\gamma}_{P,q}) = \mathfrak{J}^P_\gamma(\tau)$ with $q = \gamma(\tau)$. If γ is $g^F_{\dot\gamma}$-orthogonal to Q at $\gamma(\tau)$, elements in $\mathfrak{J}^{P,Q}_\gamma := \mathrm{Ker}(\mathbf{I}^{\gamma}_{P,Q})$ are called (P,Q)-Jacobi fields along γ. Then with $q = \gamma(\tau)$ we have

$$m^0(\mathcal{E}_{P,q}, \gamma) = \dim \mathfrak{J}^P_\gamma(\tau) \quad \text{and} \quad m^0(\mathcal{E}_{P,Q}, \gamma) = \dim \mathfrak{J}^{P,Q}_\gamma. \tag{9.27}$$

For a constant (nonzero) speed F-geodesic $\gamma : [0, \tau] \to M$ orthogonally starting at P, (9.26) shows that an instant $t_0 \in (0, \tau]$ is P-focal if and only if it is a P-focal point along the Euler-Lagrange curve γ of $L = F^2$ and their multiplicities are same, i.e., $\nu^P_\gamma(t_0) = \mu^P_\gamma(t_0)$. Therefore Theorem 3.14 with $q = \gamma(\tau)$ gives:

Corollary 9.7. *Under the above assumptions it holds that*

$$\mathrm{Index}(\mathbf{I}^{\gamma}_{P,q}) = m^-(\mathcal{E}_{P,q}, \gamma) = \sum_{t_0 \in (0,\tau)} \nu^P_\gamma(t_0) = \sum_{t_0 \in (0,\tau)} \mu^P_\gamma(t_0). \tag{9.28}$$

Moreover, if γ is also perpendicular to Q at $q = \gamma(\tau)$, and $\{X(\tau) \,|\, X \in \mathfrak{J}^P_\gamma\} \supseteq T_{\dot\gamma(\tau)}Q$ (the latter may be satisfied if $\gamma(\tau)$ is not a P-focal point), [38, Theorem 1.1(iii)] gives

$$\mathrm{Index}(\mathbf{I}^{\gamma}_{P,Q}) = \mathrm{Index}(\mathbf{I}^{\gamma}_{P,q}) + \mathrm{Index}(\mathcal{A}_\gamma) \tag{9.29}$$

with $q = \gamma(\tau)$, where $\mathcal{A}_\gamma$ is the bilinear symmetric form on $\mathfrak{J}^P_\gamma$ defined by

$$\mathcal{A}_\gamma(J_1, J_2) = g_{\dot\gamma(\tau)}\Big(D^{\dot\gamma}_{\dot\gamma}J_1(\tau) + \tilde S^Q_{\dot\gamma(\tau)}(J_1(\tau)), J_2(\tau)\Big).$$

Remark 9.8. When M, F, P and Q are smooth Ioan Radu Peter [47] proved (9.28) and (9.29) if the Morse index form, P-Jacobi field and the shape operator are introduced by the Cartan connection. Recently, in [38] the author proved the Morse index theorem in the case of two variable endpoints in conic Finsler manifolds by employing the Chern connection to introduce the Morse index form, P-Jacobi field and the shape operator. (9.28) and (9.29) are included in [38, Theorem 1.1(iii)].

Recall that the exponential map of a C^6 Finsler matric F on M is $\exp^F : \mathcal{D} \subset TM \to M$, where $\mathcal{D}$ is the set of vectors v in TM such that the unique geodesic γ_v satisfying $\gamma_v(0) = \pi(v)$ and $\dot{\gamma}_v(0) = v$ is defined at least in $[0, b) \supset [0, 1]$, and $\exp^F(v) = \gamma_v(1)$. $\mathcal{D}$ is a starlike open neighborhood of the zero section 0_{TM} of TM, $\exp^F$ is C^1, C^3 in $TM \setminus 0_{TM}$, and $D(\exp^F_p)(0_p) : T_pM \to T_pM$ is the identity map at the origin $0_p \in T_pM$ for any $p \in M$, where $\exp^F_p$ is the restriction of $\exp^F$ to $\mathcal{D}_p := \mathcal{D} \cap T_pM$. For $v \in \mathcal{D}_p$ and $w \in T_pM$, by [20, Proposition 3.15] or [54, Lemma 11.2.2]) we have $D\exp^F_p(v)[w] = J(1)$, where J is the unique Jacobi field on γ_v such that $J(0) = 0$ and $J'(0) = w$. It follows that $D\exp_p(v) : T_v(T_pM) \equiv T_pM \to T_pM$ is singular if and only if $\gamma_v(1)$ is a conjugate point of p along γ_v (cf. [4, Proposition 7.1.1]). Moreover, the multiplicity (or order) of the conjugate point $\gamma_v(1)$ is $\dim \mathrm{Ker}\big(D\exp^F(v)\big)$.

More generally, let $P \subset M$ and $TP^\perp$ be as above, and let $\exp^{FN}$ be the restriction of $\exp^F$ to $\mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$, where $0_{TM}|_P$ is the restriction of the zero section 0_{TM} of TM to P. We say $\exp^{FN}$ to be the normal exponential map. Note that $v \in \mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$ if and only if $tv \in \mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$ for all $0 \le t \le 1$. A point $q = \exp^{FN}(v)$ with $v \in \mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$ is a P-focal point along $[0, 1] \ni t \mapsto \gamma_v(t) = \exp^{FN}_{\pi(v)}(tv)$ if and only if it is a critical value of $\exp^{FN}$, and in this case the multiplicity (or order) of the focal point q is equal to $\dim \mathrm{Ker}\big(D\exp^{FN}(v)\big)$ (by the definitions above (9.26) and [38, Proposition 3.4]). (See also Lemma 4.8 in [52, page 59] for the case in Riemannian geometry.) Therefore, if $v \in \mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$ is such that $\exp^{FN}(v)$ is not a focal point along $[0, 1] \ni t \mapsto \exp^{FN}(tv)$, and $u \in \mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$ is sufficiently close to v, then $\exp^{FN}(u)$ can not be a focal point along $[0, 1] \ni t \mapsto \exp^{FN}(tu)$ either. Moreover, applying Sard theorem to the C^3 map $\exp^{FN}$ between n-dimensional C^6 manifolds $\mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$ and M we obtain that the focal set of P (i.e., the set of all P-focal points) has measure zero in M.

10 Bifurcations points along a Finsler geodesic

Assumption 10.1. Let M be a n-dimensional, connected C^7 submanifold of $\mathbb{R}^N$, and let P be a C^7 submanifold in M of dimension less than n. For a C^ℓ Finsler metric F on M with $3 \le \ell \le 6$ let $\gamma : [0, \tau] \to M$ be a constant (nonzero) speed F-geodesic which is perpendicular to P at $\gamma(0)$, i.e., $g^F_{\dot{\gamma}(0)}(\dot{\gamma}(0), u) = 0\ \forall u \in T_{\gamma(0)}P$. (Note that γ is C^ℓ.)

Because of Definition 1.8 and [49, Definition 6.1] we introduce:

Definition 10.2. Under Assumption 10.1, $\gamma(\mu)$ with $\mu \in (0, \tau]$ is called a bifurcation point on γ relative to P if there exists a sequence $(t_k) \subset (0, \tau]$ converging to μ and a sequence constant (non-zero) speed F-geodesics $\gamma_k : [0, t_k] \to M$ emanating perpendicularly from P such that

$$\gamma_k(t_k) = \gamma(t_k) \text{ for all } k \in \mathbb{N}, \tag{10.1}$$

$$0 < \|\gamma_k - \gamma|_{[0,t_k]}\|_{C^1([0,t_k],\mathbb{R}^N)} \to 0 \quad \text{as } k \to \infty. \tag{10.2}$$

Remark 10.3. As pointed out below Definition 1.8, using Lemma 2.6(ii) we can prove that the limit in (10.2) is equivalent to one of the conditions: (1) $\gamma_k(0) \to \gamma(0)$ and $\dot{\gamma}_k(0) \to \dot{\gamma}(0)$, (2) $\|\gamma_k - \gamma|_{[0,t_k]}\|_{C^2([0,t_k],\mathbb{R}^N)} \to 0$ as $k \to \infty$.

Theorem 10.4. *Under Assumption 10.1, the following are true.*

(i) *There exists only finitely many P-focal points along γ.*

(ii) *If $\gamma(\mu)$ with $\mu \in (0, \tau]$ is a bifurcation point on γ relatively to P, then it is also a P-focal point along γ.*

(iii) *If $\gamma(\mu)$ with $\mu \in (0, \tau)$ is a P-focal point along γ, then it is a bifurcation point on γ relative to P and one of the following alternatives occurs:*

(iii-1) *There exists a sequence of distinct C^ℓ constant (non-zero) speed F-geodesics emanating perpendicularly from P and ending at $\gamma(\mu)$, $\alpha_k : [0, \mu] \to M$, $\alpha_k \neq \gamma|_{[0,\mu]}$, $k = 1, 2, \cdots$, such that $\alpha_k \to \gamma|_{[0,\mu]}$ in $C^2([0, \mu], \mathbb{R}^N)$ as $k \to \infty$.*

(iii-2) *For every $\lambda \in (0, \tau) \setminus \{\mu\}$ near μ there exists a C^ℓ constant (non-zero) speed F-geodesics emanating perpendicularly from P and ending at $\gamma(\lambda)$, $\alpha_\lambda : [0, \lambda] \to M$, $\alpha_\lambda \neq \gamma|_{[0,\lambda]}$, such that $\|\alpha_\lambda - \gamma|_{[0,\lambda]}\|_{C^2([0,\lambda],\mathbb{R}^N)} \to 0$ as $\lambda \to \mu$.*

(iii-3) *For a given small $\epsilon > 0$ there is an one-sided neighborhood Λ^* of μ such that for any $\lambda \in \Lambda^* \setminus \{\mu\}$, there exist at least two C^ℓ constant (non-zero) speed F-geodesics emanating perpendicularly from P and ending at $\gamma(\lambda)$, $\beta^i_\lambda : [0, \lambda] \to M$, $\beta^i_\lambda \neq \gamma|_{[0,\lambda]}$, $i = 1, 2$, to satisfy the condition that $\|\beta^i_\lambda - \gamma|_{[0,\lambda]}\|_{C^1([0,\lambda],\mathbb{R}^N)} < \epsilon$, $i = 1, 2$. Moreover, the geodesics β^1_λ and β^2_λ can also be chosen to have distinct speeds (or lengths) if the multiplicity of $\gamma(\mu)$ as a P-focal point along γ is greater than one and there exist only finitely many C^ℓ constant (non-zero) speed F-geodesics emanating perpendicularly from P and ending at $\gamma(\lambda)$, $\alpha_1, \cdots, \alpha_m$, such that $\|\alpha_i - \gamma|_{[0,\lambda]}\|_{C^1([0,\lambda],\mathbb{R}^N)} < \epsilon$, $i = 1, \cdots, m$.*

Proof. For any $\lambda \in (0, \tau]$, $\gamma_\lambda := \gamma|_{[0,\lambda]}$ is a critical point of the C^2-functional

$$\mathcal{E}_{P,\gamma(\lambda)} : C^1_{P\times\{\gamma(\lambda)\}}([0, \lambda]; M)_{\text{reg}} \to \mathbb{R},\ \alpha \mapsto \int_0^\lambda [F(\alpha(t), \dot{\alpha}(t))]^2 dt.$$

Since $\gamma([0, \tau])$ is compact in M and the geodesics involved are near this compact subset **we may assume that M is compact**. Therefore there exists an Riemannian metric g on M and an constant $C_1 > 0$ such that $|v|^2_x \leq [F(x, v)]^2 \leq C_1|v|^2_x$ for all $(x, v) \in TM$, where $|v|_x = \sqrt{g_x(v, v)}$. Clearly, there exists a constant $c > 0$ such that $[F(\gamma(t), \dot{\gamma}(t))]^2 > 2c/C_1$ for all $t \in [0, \tau]$. As in Proposition 9.5 we define $L^* : TM \to \mathbb{R},\ (x, v) \mapsto \psi_{\varepsilon,\delta}([F(x, v)]^2) + \phi_{\mu,b}(|v|^2_x) - b$, which is C^ℓ and gives a family of C^2-functionals

$$\mathcal{E}^*_{P,\gamma(\lambda)} : C^1_{P\times\{\gamma(\lambda)\}}([0, \lambda]; M) \to \mathbb{R},\ \alpha \mapsto \int_0^\lambda L^*(\alpha(t), \dot{\alpha}(t))dt, \qquad \lambda \in (0, \tau].$$

By Proposition 9.5(i), $L^*(x, v) = L(x, v)$ if $L(x, v) \geq \frac{2c}{3C_1}$. Hence the functionals $\mathcal{E}_{P,\gamma(\lambda)}$ and $\mathcal{E}^*_{P,\gamma(\lambda)}$ agree on the following open subset of $C^1_{P\times\{\gamma(\lambda)\}}([0, \lambda]; M)_{\text{reg}}$ containing γ_λ,

$$C^1_{P\times\{\gamma(\lambda)\}}([0, \lambda]; M, F, c/C_1) := \left\{\alpha \in C^1_{P\times\{\gamma(\lambda)\}}([0, \lambda]; M) \mid \min_t [F(\alpha(t), \dot{\alpha}(t))]^2 > 2c/C_1\right\}.$$

Then each γ_λ is also a critical point of $\mathcal{E}^*_{P,\gamma(\lambda)}$ and the Hessians

$$D^2\mathcal{E}_{P,\gamma(\lambda)}(\gamma_\lambda) = D^2\mathcal{E}^*_{P,\gamma(\lambda)}(\gamma_\lambda), \quad \forall\lambda \in (0, \tau]. \tag{10.3}$$

By Assumption 10.1 we see that L^* satisfies Assumption 1.7 with $S_0 = P$ and γ is a L^*-curve emanating perpendicularly from P. From Theorem 1.9(i) and (10.3) there only exist finitely many numbers $0 < s_1 < \cdots < s_m \le \tau$ such that

$$\dim \mathrm{Ker}(D^2\mathcal{E}_{P,\gamma(s_i)}(\gamma_{s_i})) = \dim \mathrm{Ker}(D^2\mathcal{E}^*_{P,\gamma(s_i)}(\gamma_{s_i})) > 0, \quad i = 1, \cdots, m.$$

These and (9.27) lead to (i).

Suppose that $\gamma(\mu)$ with $\mu \in (0, \tau]$ is a bifurcation point on γ relatively to P. By Definition 10.2 it is easy to see that μ is a bifurcation instant for (P, γ) where γ is as a L^*-curve. Therefore Theorem 1.9(ii) tells us $\dim \mathrm{Ker}(D^2\mathcal{E}^*_{P,\gamma(\mu)}(\gamma_\mu)) > 0$. Combing the latter with (10.3) and (9.27) we arrive at (ii).

Finally, let us assume that $\gamma(\mu)$ with $\mu \in (0, \tau)$ is a P-focal point along γ. Then by (10.3) we see that μ is a P-focal point along γ (as a L^*-curve relative P). It follows from this result and Theorem 1.9(iii) that μ is a bifurcation instant for (P, γ) and that one of Theorem 1.9(iii-k), $k = 1, 2, 3$, holds after L is replaced by L^*. These and (10.3) easily yield the desired results because

- we can take $\epsilon > 0$ so small that for $\alpha \in C^1_{P\times\{\gamma(\lambda)\}}([0, \lambda]; M)$ the inequality $\|\alpha - \gamma|_{[0,\lambda]}\|_{C^1([0,\lambda],\mathbb{R}^N)} < \epsilon$ implies $\alpha \in C^1_{P\times\{\gamma(\lambda)\}}([0, \lambda]; M, F, c/C_1)$, and
- relations

$$\begin{aligned}\int_0^\lambda L(\beta^1_\lambda(t), \dot\beta^1_\lambda(t))dt &= \int_0^\lambda L^*(\beta^1_\lambda(t), \dot\beta^1_\lambda(t))dt \\ &\ne \int_0^\lambda L^*(\beta^2_\lambda(t), \dot\beta^2_\lambda(t))dt = \int_0^\lambda L(\beta^2_\lambda(t), \dot\beta^2_\lambda(t))dt\end{aligned}$$

  imply that β^1_λ and β^2_λ have different speeds since β^1_λ and β^2_λ have constant (nonzero) speeds.

□

Theorem 10.4 has the following deep geometrical consequence.

Theorem 10.5. *Let M and P be as in Assumption 10.1, and let F be a C^6 Finsler metric on M. Suppose that $v \in \mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P)$ is a critical point of $\exp^{FN}$. Then $\exp^{FN}$ is not injective near v, precisely one of the following alternatives occurs:*

(i) *There exists a sequence (v_k) of distinct points in $\mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P) \setminus \{v\}$ converging to v, such that $\exp^{FN}(v_k) = \exp^{FN}(v)$ for each $k = 1, 2, \cdots$.*

(ii) *For every $\lambda \in \mathbb{R} \setminus \{1\}$ near 1 there exists $v_\lambda \in \mathcal{D} \cap (TP^\perp \cup 0_{TM}|_P) \setminus \{v\}$ such that $\exp^{FN}(\lambda v_\lambda) = \exp^{FN}(\lambda v)$ and $v_\lambda \to v$ as $\lambda \to 1$.*

(iii) *Given a small neighborhood $\mathcal{O}$ of v in $\mathcal{D}\cap(TP^\perp\cup 0_{TM}|_P)$ there is an one-sided neighborhood Λ^* of 1 in $\mathbb{R}$ such that for any $\lambda \in \Lambda^* \setminus \{1\}$, there exist at least two points v^1_λ and v^2_λ in $\mathcal{O} \setminus \{v\}$ such that $\exp^{FN}(\lambda v^k_\lambda) = \exp^{FN}(\lambda v)$ for each $k = 1, 2$. Moreover the points v^1_λ and v^2_λ above can also be chosen to satisfy $F(v^1_\lambda) \ne F(v^2_\lambda)$ if $\dim \mathrm{Ker}(D\exp^{FN}(v)) > 1$ and $\mathcal{O}\setminus\{v\}$ only contains finitely many points, $v_1, \cdots, v_m$, such that $\exp^{FN}(\lambda v_i) = \exp^{FN}(\lambda v)$, $i = 1, \cdots, m$.*

Proof. Let $\gamma_v(t)=\exp^{FN}(tv)$. It is well-defined on $[0,\tau]$ for some $\tau>1$ because $\mathcal{D}$ is a starlike open neighborhood of the zero section 0_{TM} of TM. By the last paragraph of Section 9 the assumption about v implies that $q=\exp^{FN}(v)$ is a P-focal point along γ_v. Therefore $q=\gamma_v(1)$ is a bifurcation point on γ_v relatively to P by Theorem 10.4(iii).

Let (α_k) be as in (iii-1) of Theorem 10.4 with $\mu=1$ and $\gamma=\gamma_v$. Then $v_k:=(\alpha_k(0),\dot\alpha_k(0))\to(\gamma_v(0),\dot\gamma_v(0))$ and all v_k sit in $\mathcal{D}\cap(TP^\perp\cup 0_{TM}|_P)$ by the definition of $\mathcal{D}$. Since $\alpha_k(t)=\exp^{FN}(tv_k)$ for $t\in[0,1]$, we have $\exp^{FN}(v_k)=\alpha_k(1)=\gamma_v(1)=\exp^{FN}(v)$ and $v_k\ne v$ for all k. (i) is proved.

Similarly, let α_λ be as in (iii-2) of Theorem 10.4 with $\mu=1$ and $\gamma=\gamma_v$. Then as $\lambda\to 1$ we have $v_\lambda:=(\alpha_\lambda(0),\dot\alpha_\lambda(0))\to(\gamma_v(0),\dot\gamma_v(0))=v$ because $0<\|\alpha_\lambda-\gamma_v|_{[0,\lambda]}\|_{C^2([0,\lambda],\mathbb{R}^N)}\to 0$ as $\lambda\to 1$. Hence shrinking Λ^* towards 1 (if necessary) we may assume that all geodesics α_λ are well-defined over $[0,1]$ (because γ_v is well-defined on $[0,\tau]$ for some $\tau>1$). Therefore $v_\lambda\in\mathcal{D}\cap(TP^\perp\cup 0_{TM}|_P)\setminus\{v\}$ and $\exp^{FN}(\lambda v_\lambda)=\alpha_\lambda(\lambda)=\gamma_v(\lambda)=\exp^{FN}(\lambda v)$ for all $\lambda\in\Lambda^*$. (ii) is proved.

Finally, let us show how (iii-3) of Theorem 10.4 leads to (iii) of Theorem 10.5. Since γ_v is well-defined on $[0,\tau']$ for some $\tau'>\tau$, we may take a neighborhood $\mathcal{O}$ of v in $\mathcal{D}\cap(TP^\perp\cup 0_{TM}|_P)$ such that for each $u\in\mathcal{O}$ the geodesic $t\mapsto\gamma_u(t):=\exp^{FN}(tu)$ is well-defined on $[0,\tau]$. By Remark 10.3, for a given small number $\delta\in(0,1)$ there exists a small $\epsilon>0$ such that $(\alpha(0),\dot\alpha(0))\in\mathcal{O}$ for any C^ℓ constant (non-zero) speed F-geodesic $\alpha:[0,\lambda]\to M$ emanating perpendicularly from P and ending at $\gamma_v(\lambda)$ with $\lambda\in[1-\delta,1+\delta]$ and satisfying $\|\alpha-\gamma_v|_{[0,\lambda]}\|_{C^1([0,\lambda],\mathbb{R}^N)}<\epsilon$. Let Λ^* and β^i_λ with $\lambda\in\Lambda^*$ be as in (iii-3) of Theorem 10.4 with $\mu=1$ and $\gamma=\gamma_v$. We may shrink Λ^* towards 1 so that $\Lambda^*\subset[1-\delta,1+\delta]$. Then the choice of ϵ implies that

$$v^j_\lambda:=(\beta^j_\lambda(0),\dot\beta^j_\lambda(0))\in\mathcal{O}\setminus\{v\}\text{ and }\exp^{FN}(\lambda v^j_\lambda)=\exp^{FN}(\lambda v)\text{ for }j=1,2,\text{ and }v^1_\lambda\ne v^2_\lambda.$$

Suppose that $\dim\mathrm{Ker}\big(D\exp^{FN}(v)\big)>1$, i.e., the multiplicity of $\gamma_v(1)$ as a P-focal point along γ_v is greater than one by the last paragraph of Section 9, and that $\mathcal{O}\setminus\{v\}$ only contains finitely many points, $v_1,\cdots,v_m$, such that $\exp^{FN}(\lambda v_i)=\exp^{FN}(\lambda v)$, $i=1,\cdots,m$. The second assumption implies that there are no infinitely many C^ℓ constant (non-zero) speed F-geodesics emanating perpendicularly from P and ending at $\gamma_v(\lambda)$, $\alpha_i:[0,\lambda]\to M$, $i=1,2,\cdots$, such that $\|\alpha_i-\gamma_v|_{[0,\lambda]}\|_{C^1([0,\lambda],\mathbb{R}^N)}<\epsilon$, $i=1,2,\cdots$. (Otherwise, by the choice of ϵ we have $v_i:=(\alpha_i(0),\dot\alpha_i(0))\in\mathcal{O}\setminus\{v\}$ and $\exp^{FN}(\lambda v_i)=\exp^{FN}(\lambda v)$ for each $i=1,2,\cdots$.) Therefore there exist only finitely many such α, saying $\alpha_1,\cdots,\alpha_k$. In this case, by (iii-3) of Theorem 10.4 the geodesics β^1_λ and β^2_λ above can also be chosen to have distinct speeds, i.e., $F(v^1_\lambda)\ne F(v^2_\lambda)$. The proof of (iii) of Theorem 10.5 is complete.

□

11 Bifurcations of geodesics with two kinds of special boundary conditions

Let (M,g), $\mathbb{I}_g$ and submanifolds P, Q be as at the beginning of Section 9. In order to use the results in Section 1.1 conveniently, we write P and Q as S_0 and S_1, respectively. Then the boundary condition (9.8) becomes

$$\left\{\begin{array}{ll} g^{F_\lambda}_{\dot\gamma(0)}(u,\dot\gamma(0))=0 & \forall u\in T_{\gamma(0)}S_0,\\ g^{F_\lambda}_{\dot\gamma(1)}(v,\dot\gamma(\tau))=0 & \forall v\in T_{\gamma(\tau)}S_1\end{array}\right.\tag{11.1}$$

if $\mathbf{N}=S_0\times S_1$, and

$$g^{F_\lambda}_{\dot\gamma(0)}(u,\dot\gamma(0))=g^{F_\lambda}_{\dot\gamma(\tau)}(\mathbb{I}_{g*}u,\dot\gamma(\tau))\quad\forall u\in T_{\gamma(0)}M\tag{11.2}$$

if $\mathbf{N} = \mathrm{Graph}(\mathbb{I}_g)$. In these two cases the Morse index and nullity in (9.5) have more precise explanations. See (9.27), (9.28) and (9.29) and [32, §6].

Theorem 11.1. *Under Assumptions 9.2, 9.3 with* $\mathbf{N} = S_0 \times S_1$ *or* $\mathrm{Graph}(\mathbb{I}_g)$, *for* $\mu \in \Lambda$ *such that* $\gamma_\mu(0) \neq \gamma_\mu(\tau)$ *in the case* $\dim S_0 > 0$ *and* $\dim S_1 > 0$, *there holds:*

(I) (Necessary condition): *Suppose that constant (non-zero) speed* $(F_\lambda, \mathbf{N})$*-geodesics with a parameter* $\lambda \in \Lambda$ *bifurcate at some* $\mu \in \Lambda$ *along sequences with respect to the branch* $\{\gamma_\lambda \,|\, \lambda \in \Lambda\}$. *Then* $m^0_\tau(\mathcal{E}_{\mu,\mathbf{N}}, \gamma_\mu) \neq 0$.

(II) (Sufficient condition): *Suppose that* Λ *is first countable and that there exist two sequences in* Λ *converging to* μ, (λ^-_k) *and* (λ^+_k), *such that one of the following conditions is satisfied:*

(II.1) *For each* $k \in \mathbb{N}$, *either* $\gamma_{\lambda^+_k}$ *is not an isolated critical point of* $\mathcal{E}_{\lambda^+_k,\mathbf{N}}$, *or* $\gamma_{\lambda^-_k}$ *is not an isolated critical point of* $\mathcal{E}_{\lambda^-_k,\mathbf{N}}$, *or* $\gamma_{\lambda^+_k}$ *(resp.* $\gamma_{\lambda^-_k}$*) is an isolated critical point of* $\mathcal{E}_{\lambda^+_k,\mathbf{N}}$ *(resp.* $\mathcal{E}_{\lambda^-_k,\mathbf{N}}$*) and* $C_m(\mathcal{E}_{\lambda^+_k,\mathbf{N}}, \gamma_{\lambda^+_k}; \mathbf{K})$ *and* $C_m(\mathcal{E}_{\lambda^-_k,\mathbf{N}}, \gamma_{\lambda^-_k}; \mathbf{K})$ *are not isomorphic for some Abel group* $\mathbf{K}$ *and some* $m \in \mathbb{Z}$.

(II.2) *For each* $k \in \mathbb{N}$, *there exists* $\lambda \in \{\lambda^+_k, \lambda^-_k\}$ *such that* γ_λ *is an either nonisolated or homological visible critical point of* $\mathcal{E}_{\lambda,\mathbf{N}}$, *and*

$$\left.\begin{array}{l}[m^-(\mathcal{E}_{\lambda^-_k,\mathbf{N}}, \gamma_{\lambda^-_k}), m^-(\mathcal{E}_{\lambda^-_k,\mathbf{N}}, \gamma_{\lambda^-_k}) + m^0(\mathcal{E}_{\lambda^-_k,\mathbf{N}}, \gamma_{\lambda^-_k})] \\ \cap[m^-(\mathcal{E}_{\lambda^+_k,\mathbf{N}}, \gamma_{\lambda^+_k}), m^-(\mathcal{E}_{\lambda^+_k,\mathbf{N}}, \gamma_{\lambda^+_k}) + m^0(\mathcal{E}_{\lambda^+_k,\mathbf{N}}, \gamma_{\lambda^+_k})] = \emptyset.\end{array}\right\} \qquad (\spadesuit_k)$$

(II.3) *For each* $k \in \mathbb{N}$, *(*$\spadesuit_k$*) holds true, and either* $m^0(\mathcal{E}_{\lambda^-_k,\mathbf{N}}, \gamma_{\lambda^-_k}) = 0$ *or* $m^0(\mathcal{E}_{\lambda^+_k,\mathbf{N}}, \gamma_{\lambda^+_k}) = 0$.

Then there exists a sequence $(\lambda_k) \subset \hat{\Lambda} := \{\mu, \lambda^+_k, \lambda^-_k \,|\, k \in \mathbb{N}\}$ *converging to* μ *and constant speed* F_{λ_k}*-geodesic* $\gamma^k : [0, \tau] \to M$ *satisfying the boundary condition (9.8) with* $\lambda = \lambda_k$, $k = 1, 2, \cdots$, *such that* $\gamma^k \to \gamma_\mu$ *in* $C^2([0, \tau]; M)$. *In particular, constant (non-zero) speed* $(F_\lambda, \mathbf{N})$*-geodesics with a parameter* $\lambda \in \Lambda$ *bifurcate at* $\mu \in \Lambda$ *along sequences with respect to the branch* $\{\gamma_\lambda \,|\, \lambda \in \Lambda\}$.

Proof. *Step 1* [Prove (I)]. By Definition 9.4 there exists an infinite sequence $\{(\lambda_k, \gamma^k)\}_{k=1}^\infty$ in $\Lambda \times C^1([0, \tau], M) \setminus \{(\mu, \gamma_\mu)\}$ converging to (μ, γ_μ), such that each $\gamma^k \neq \gamma_{\lambda_k}$ is a F_{λ_k}-geodesic satisfying the boundary condition (9.8) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$. Let $\hat{\Lambda} = \{\mu\} \cup \{\lambda_k \,|\, k \in \mathbb{N}\}$. It is compact and sequential compact. (Note that all γ^m and γ_λ are C^ℓ, $4 \leq \ell \leq 6$.) It is easy to find an open subset $\hat{M}$ of M with compact closure such that the closure

$$Cl\left(\cup_{(\lambda,m)\in\hat{\Lambda}\times\mathbb{N}}\gamma_\lambda([0, \tau]) \cup \gamma^m([0, \tau])\right) \subset \hat{M}.$$

Then the conditions in Proposition 9.5 can be satisfied with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$. Therefore for the constant $C_1 > 0$ as in Proposition 9.5 with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$ we have $c > 0$ such that for all $(m, \lambda, t) \in \mathbb{N} \times \hat{\Lambda} \times [0, \tau]$,

$$[F_\lambda(\gamma_\lambda(t), \dot{\gamma}_\lambda(t))]^2 > \frac{2c}{C_1} \quad \text{and} \quad [F_{\lambda_m}(\gamma^m(t), \dot{\gamma}^m(t))]^2 > \frac{2c}{C_1}.$$

Let $L^*_\lambda : T\hat{M} \to \mathbb{R}$, $\lambda \in \hat{\Lambda}$, be given by Proposition 9.5 with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$. Then the corresponding C^2 functional $\mathcal{E}^*_{\lambda,\mathbf{N}}$ given by (9.14) and the C^{2-0} functional $\mathcal{E}_{\lambda,\mathbf{N}}$ in (9.9) coincide in the open subset of $\mathcal{C}_{\tau,\mathbf{N}}(\hat{M})_{\mathrm{reg}}$ as in (9.15),

$$\mathcal{C}_{\tau,\mathbf{N}}(\hat{M}, \{F_\lambda, F_{\lambda_m} \,|\, (\lambda, m) \in \hat{\Lambda} \times \mathbb{N}\}, c/C_1) \qquad (11.3)$$

consisting of all $\alpha \in \mathcal{C}_{\tau,\mathbf{N}}(\hat{M})$ such that

$$\min_{(\lambda,t)\in\hat{\Lambda}\times[0,\tau]}[F_\lambda(\alpha(t),\dot{\alpha}(t))]^2 > 2c/C_1 \quad \text{and} \quad \min_{(m,t)\in\mathbb{N}\times[0,\tau]}[F_{\lambda_m}(\alpha(t),\dot{\alpha}(t))]^2 > 2c/C_1.$$

For any $(m,\lambda)\in\mathbb{N}\times\hat{\Lambda}$, since γ_λ and γ^m belong to $\mathcal{C}_{\tau,\mathbf{N}}(\hat{M},\{F_\lambda,F_{\lambda_m}\,|\,(\lambda,m)\in\hat{\Lambda}\times\mathbb{N}\},c/C_1)$, each γ_λ (resp. γ^m) is a critical point of $\mathcal{E}^*_{\lambda,\mathbf{N}}$ (resp. $\mathcal{E}^*_{\lambda_m,\mathbf{N}}$) and we have also (9.16). Hence when $\mathbf{N}=\mathrm{Graph}(\mathbb{I}_g)$ (resp. $\mathbf{N}=S_0\times S_1$), (μ,γ_μ) is a bifurcation point of the problem (1.13) [resp. (1.5)–(1.6)] with respect to the trivial branch $\{(\lambda,\gamma_\lambda)\,|\,\lambda\in\hat{\Lambda}\}$ in $C^1([0,\tau];M)$. It follows from Theorem 1.13(I) [resp. Theorem 1.4(I)] that $m^0(\mathcal{E}^*_{\mu,\mathbf{N}},\gamma_\mu)>0$ and so $m^0(\mathcal{E}_{\mu,\mathbf{N}},\gamma_\mu)>0$ by (9.16).

Step 2 [Prove (II)]. Since $\hat{\Lambda}=\{\mu,\lambda_k^+,\lambda_k^-\,|\,k\in\mathbb{N}\}$ is compact and sequential compact, as above we can find an open subset $\hat{M}$ of M with compact closure such that the closure

$$Cl\left(\cup_{(\lambda,m)\in\hat{\Lambda}\times\mathbb{N}}\gamma_\lambda([0,\tau])\cup\gamma_{\lambda_m^+}([0,\tau])\cup\gamma_{\lambda_m^-}([0,\tau])\right)\subset\hat{M}.$$

For the constant $C_1>0$ as in Proposition 9.5 with $(M,\Lambda)=(\hat{M},\hat{\Lambda})$ we have $c>0$ such that for all $(m,\lambda,t)\in\mathbb{N}\times\hat{\Lambda}\times[0,\tau]$,

$$[F_\lambda(\gamma_\lambda(t),\dot{\gamma}_\lambda(t))]^2>\frac{2c}{C_1}\quad\text{and}\quad[F_{\lambda_m^\pm}(\gamma_{\lambda_m^\pm}(t),\dot{\gamma}_{\lambda_m^\pm}(t))]^2>\frac{2c}{C_1}.$$

Let $L^*_\lambda:T\hat{M}\to\mathbb{R}$, $\lambda\in\hat{\Lambda}$, be given by Proposition 9.5 with $(M,\Lambda)=(\hat{M},\hat{\Lambda})$. As above, for all $(m,\lambda)\in\mathbb{N}\times\hat{\Lambda}$ we have

$$\gamma_\lambda,\gamma_{\lambda_m^+},\gamma_{\lambda_m^-}\in\mathcal{C}_{\tau,\mathbf{N}}(\hat{M},\{F_\lambda,F_{\lambda_m^+},F_{\lambda_m^-}\,|\,(\lambda,m)\in\hat{\Lambda}\times\mathbb{N}\},c/C_1),$$

and (9.16) and (9.17) lead to

$$\begin{aligned}&m^-(\mathcal{E}_{\lambda_m^\pm,\mathbf{N}},\gamma_{\lambda_m^\pm})=m^-(\mathcal{E}^*_{\lambda_m^\pm,\mathbf{N}},\gamma_{\lambda_m^\pm})\quad\text{and}\quad m^0(\mathcal{E}_{\lambda_m^\pm,\mathbf{N}},\gamma_{\lambda_m^\pm})=m^0(\mathcal{E}^*_{\lambda_m^\pm,\mathbf{N}},\gamma_{\lambda_m^\pm}),\\&C_k(\mathcal{E}_{\lambda_m^\pm,\mathbf{N}},\gamma_{\lambda_m^\pm};\mathbf{K})=C_k(\mathcal{E}^*_{\lambda_m^\pm,\mathbf{N}},\gamma_{\lambda_m^\pm};\mathbf{K})\quad\forall(k,m)\in\mathbb{Z}\times\mathbb{N}\end{aligned}$$

for any Abel group $\mathbf{K}$. By these we see that for $\mathbf{N}=S_0\times S_1$ [resp. $\mathbf{N}=\mathrm{Graph}(\mathbb{I}_g)$] the conditions (II.1), (II.2) and (II.3) in Theorem 11.1, respectively, give rise to the corresponding conditions (II.1), (II.2) and (II.3) in Theorem 1.4 (resp. Theorem 1.13). Hence there exists an infinite sequence $\{(\lambda_k,\gamma^k)\}_{k=1}^\infty$ in $\hat{\Lambda}\times C^2([0,\tau],\hat{M})\setminus\{(\mu,\gamma_\mu)\}$ converging to (μ,γ_μ), such that each $\gamma^k\ne\gamma_{\lambda_k}$ satisfies

$$\frac{d}{dt}\left(\partial_v L^*_\lambda(t,\gamma(t),\dot{\gamma}(t))\right)-\partial_x L^*_\lambda(t,\gamma(t),\dot{\gamma}(t))=0 \tag{11.4}$$

with $\lambda=\lambda_k$ and the boundary condition (1.6) [resp. (1.13)] with $\lambda=\lambda_k$ if $\mathbf{N}=S_0\times S_1$ [resp. $\mathbf{N}=\mathrm{Graph}(\mathbb{I}_g)$], $k=1,2,\cdots$. From these and Claim 9.6 we conclude that for each k large enough, γ^k is a constant (nonzero) speed F_{λ_k}-geodesic satisfying the boundary condition (11.1) [resp. (11.2)] with $\lambda=\lambda_k$ if $\mathbf{N}=S_0\times S_1$ (resp. $\mathbf{N}=\mathrm{Graph}(\mathbb{I}_g)$), $k=1,2,\cdots$. □

Theorem 11.2 (Existence for bifurcations). *Under Assumptions 9.2, 9.3, let* $\mathbf{N}=S_0\times S_1$ *or* $\mathrm{Graph}(\mathbb{I}_g)$. *Suppose that* Λ *is path-connected and there exist two points* $\lambda^+,\lambda^-\in\Lambda$ *such that one of the following conditions is satisfied:*

(i) *Either* γ_{λ^+} *is not an isolated critical point of* $\mathcal{E}_{\lambda^+,\mathbf{N}}$*, or* $\gamma_{\lambda^-,\mathbf{N}}$ *is not an isolated critical point of* $\mathcal{E}_{\lambda^-,\mathbf{N}}$*, or* $\gamma_{\lambda^+,\mathbf{N}}$ *(resp.* $\gamma_{\lambda^-,\mathbf{N}}$*) is an isolated critical point of* $\mathcal{E}_{\lambda^+,\mathbf{N}}$ *(resp.* $\mathcal{E}_{\lambda^-,\mathbf{N}}$*) and* $C_m(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+};\mathbf{K})$ *and* $C_m(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-};\mathbf{K})$ *are not isomorphic for some Abel group* $\mathbf{K}$ *and some* $m\in\mathbb{Z}$*.*

(ii) *The intervals* $[m^-(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-}), m^-(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-})+m^0(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-})]$ *and*

$$[m^-(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+}), m^-(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+})+m^0(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+})]$$

are disjoint, and there exists $\lambda\in\{\lambda^+,\lambda^-\}$ *such that* γ_λ *is an either non-isolated or homological visible critical point of* $\mathcal{E}_{\lambda,\mathbf{N}}$*.*

(iii) *The intervals* $[m^-(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-}), m^-(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-})+m^0(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-})]$ *and*

$$[m^-(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+}), m^-(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+})+m^0(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+})]$$

are disjoint, and either $m^0(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+})=0$ *or* $m^0(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-})=0$*.*

Then for any path $\alpha:[0,1]\to\Lambda$ *connecting* λ^+ *to* λ^- *such that* $\gamma_{\alpha(s)}(0)\neq\gamma_{\alpha(s)}(\tau)$ *for any* $s\in[0,1]$ *in the case* $\mathbf{N}=S_0\times S_1$ *and* $\dim S_0\dim S_1>0$*, there exists a sequence* $(\lambda_k)\subset\alpha([0,1])$ *converging to some* $\mu\in\alpha([0,1])$*, and constant (non-zero) speed* F_{λ_k}*-geodesics* $\gamma^k:[0,\tau]\to M$ *satisfying the boundary condition (9.8),* $k=1,2,\cdots$*, such that* $0<\|\gamma^k-\gamma_{\lambda_k}\|_{C^2([0,\tau];\mathbb{R}^N)}\to 0$ *as* $k\to\infty$*. Moreover,* μ *is not equal to* λ^+ *(resp.* λ^-*) if* $m^0_\tau(\mathcal{E}_{\lambda^+,\mathbf{N}},\gamma_{\lambda^+})=0$ *(resp.* $m^0_\tau(\mathcal{E}_{\lambda^-,\mathbf{N}},\gamma_{\lambda^-})=0$*).*

Proof. As above, since $\hat\Lambda:=\alpha([0,1])$ is a compact and sequential compact subset in Λ we can find an open subset $\hat M$ of M with compact closure such that the closure $Cl\left(\cup_{\lambda\in\hat\Lambda\times\mathbb{N}}\gamma_\lambda([0,\tau])\right)\subset\hat M$. For the constant $C_1>0$ as in Proposition 9.5 with $(M,\Lambda)=(\hat M,\hat\Lambda)$ we have $c>0$ such that

$$[F_\lambda(\gamma_\lambda(t),\dot\gamma_\lambda(t))]^2>\frac{2c}{C_1}\quad\text{and}\quad[F_{\lambda^\pm}(\gamma_{\lambda^\pm}(t),\dot\gamma_{\lambda^\pm}(t))]^2>\frac{2c}{C_1}\quad\text{for all }(\lambda,t)\in\hat\Lambda\times[0,\tau].$$

Let $L^*_\lambda:T\hat M\to\mathbb{R}$, $\lambda\in\hat\Lambda$, be given by Proposition 9.5 with $(M,\Lambda)=(\hat M,\hat\Lambda)$. As above, we have $\gamma_\lambda,\gamma_{\lambda^+},\gamma_{\lambda^-}\in\mathcal{C}_{\tau,\mathbf{N}}(\hat M,\{F_\lambda,F_{\lambda^+},F_{\lambda^-}\,|\,\lambda\in\hat\Lambda\},c/C_1)$ for all $\lambda\in\hat\Lambda$, and (9.16) and (9.17) lead to

$$\begin{aligned}&m^-(\mathcal{E}_{\lambda^\pm,\mathbf{N}},\gamma_{\lambda^\pm})=m^-(\mathcal{E}^*_{\lambda^\pm,\mathbf{N}},\gamma_{\lambda^\pm})\quad\text{and}\quad m^0(\mathcal{E}_{\lambda^\pm,\mathbf{N}},\gamma_{\lambda^\pm})=m^0(\mathcal{E}^*_{\lambda^\pm,\mathbf{N}},\gamma_{\lambda^\pm}),\\&C_k(\mathcal{E}_{\lambda^\pm,\mathbf{N}},\gamma_{\lambda^\pm};\mathbf{K})=C_k(\mathcal{E}^*_{\lambda^\pm,\mathbf{N}},\gamma_{\lambda^\pm};\mathbf{K})\quad\forall k\in\mathbb{Z}\end{aligned}$$

for any Abel group $\mathbf{K}$. As above, for $\mathbf{N}=S_0\times S_1$ [resp. $\mathbf{N}=\mathrm{Graph}(\mathbb{I}_g)$] the corresponding results may follow from these and Theorem 1.5 (resp. Theorem 1.14). □

Theorem 11.3 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumptions 9.2, 9.3 with* Λ *being a real interval, suppose that* $\mathbf{N}=S_0\times S_1$ *or* $\mathrm{Graph}(\mathbb{I}_g)$*, and that* $\mu\in\mathrm{Int}(\Lambda)$ *satisfies conditions:* $\gamma_\mu(0)\neq\gamma_\mu(\tau)$ *(in the case* $\dim S_0>0$ *and* $\dim S_1>0$*),* $m^0(\mathcal{E}_{\mu,\mathbf{N}},\gamma_\mu)>0$*,* $m^0(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda)=0$ *for each* $\lambda\in\Lambda\setminus\{\mu\}$ *near* μ*, and* $m^-(\mathcal{E}_{\lambda,\mathbf{N}},\gamma_\lambda)$ *take, respectively, values* $m^-(\mathcal{E}_{\mu,\mathbf{N}},\gamma_\mu)$ *and* $m^-(\mathcal{E}_{\mu,\mathbf{N}},\gamma_\mu)+m^0(\mathcal{E}_{\mu,\mathbf{N}},\gamma_\mu)$ *as* $\lambda\in\Lambda$ *varies in two deleted half neighborhoods of* μ*. Then one of the following alternatives occurs:*

(i) *There exists a sequence* C^ℓ *constant (non-zero) speed* F_μ*-geodesics* $\gamma^m\neq\gamma_\mu$ *satisfying the boundary condition (9.8) such that* $\gamma^m\to\gamma_\mu$ *in* $C^2([0,\tau],M)$*.*

(ii) *For every $\lambda \in \Lambda \setminus \{\mu\}$ near μ there is a C^ℓ constant (non-zero) speed F_λ-geodesic $\gamma'_\lambda \neq \gamma_\lambda$ satisfying the boundary condition (9.8) such that $\gamma'_\lambda - \gamma_\gamma$ converges to zero in $C^2([0,\tau],\mathbb{R}^N)$ as $\lambda \to \mu$.*

(iii) *For a given neighborhood $\mathcal{W}$ of γ_μ in $C^2([0,\tau],\mathbb{R}^N)$, there is an one-sided neighborhood Λ^0 of μ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, $\mathcal{W}$ contains at least two distinct C^ℓ constant (non-zero) speed F_λ-geodesics satisfying the boundary condition (9.8), $\gamma^1_\lambda \neq \gamma_\lambda$ and $\gamma^2_\lambda \neq \gamma_\lambda$, which can also be chosen to satisfy $F_\lambda(\gamma^1_\lambda(t), \dot\gamma^1_\lambda(t)) \neq F_\lambda(\gamma^2_\lambda(t), \dot\gamma^2_\lambda(t))$ $\forall t$ provided that $m^0_\tau(\mathcal{E}_{\mu,\mathbf{N}}, \gamma_\mu) > 1$ and $\mathcal{W}$ only contains finitely many distinct constant (non-zero) speed F_λ-geodesics satisfying the boundary condition (9.8).*

Proof. Since Λ is a real interval and $\mu \in \mathrm{Int}(\Lambda)$, for some real $\rho > 0$ the compact set $\hat\Lambda := [\mu-\rho, \mu+\rho]$ is contained in Λ. The continuous map $\hat\Lambda \times [0,\tau] \ni (\lambda, t) \to \gamma_\lambda(t) \in M$ has compact image set and therefore the latter is contained in an open subset $\hat M$ of M with compact closure. For the constant $C_1 > 0$ as in Proposition 9.5 with $(M,\Lambda) = (\hat M, \hat\Lambda)$ we have $c > 0$ such that

$$[F_\lambda(\gamma_\lambda(t), \dot\gamma_\lambda(t))]^2 > \frac{2c}{C_1}, \quad \forall(\lambda, t) \in \hat\Lambda \times [0,\tau].$$

Let $L^*_\lambda : T\hat M \to \mathbb{R}$, $\lambda \in \hat\Lambda$, be given by Proposition 9.5 with $(M,\Lambda) = (\hat M, \hat\Lambda)$. Then for all $\lambda \in \hat\Lambda$ we have $\gamma_\lambda \in \mathcal{C}_{\tau,\mathbf{N}}(\hat M, \{F_\lambda \,|\, \lambda \in \hat\Lambda\}, c/C_1)$, and

$$m^-(\mathcal{E}_{\lambda,\mathbf{N}}, \gamma_\lambda) = m^-(\mathcal{E}^*_{\lambda,\mathbf{N}}, \gamma_\lambda) \quad \text{and} \quad m^0(\mathcal{E}_{\lambda,\mathbf{N}}, \gamma_\lambda) = m^0(\mathcal{E}^*_{\lambda,\mathbf{N}}, \gamma_\lambda).$$

By these and the assumptions of Theorem 11.3 we obtain that $m^0(\mathcal{E}^*_{\mu,\mathbf{N}}, \gamma_\mu) > 0$, $m^0(\mathcal{E}^*_{\lambda,\mathbf{N}}, \gamma_\lambda) = 0$ for each $\lambda \in \hat\Lambda \setminus \{\mu\}$ near μ, and $m^-(\mathcal{E}^*_{\lambda,\mathbf{N}}, \gamma_\lambda)$ take, respectively, values $m^-(\mathcal{E}^*_{\mu,\mathbf{N}}, \gamma_\mu)$ and $m^-(\mathcal{E}^*_{\mu,\mathbf{N}}, \gamma_\mu) + m^0(\mathcal{E}^*_{\mu,\mathbf{N}}, \gamma_\mu)$ as $\lambda \in \hat\Lambda$ varies in two deleted half neighborhoods of μ. Hence the desired results may follow from Theorems 1.15, 1.6 and Claim 9.6 as above. □

12 Bifurcations of $\mathbb{I}_g$-invariant geodesics

For an $\mathbb{I}_g$-invariant Finsler metric F on M, a F-geodesic $\gamma : \mathbb{R} \to M$ said to be $\mathbb{I}_g$-invariant if $\gamma(t+1) = \mathbb{I}_g(\gamma(t))$ $\forall t \in \mathbb{R}$. Clearly, $s \cdot \gamma$ is also an $\mathbb{I}_g$-invariant F-geodesics for any $s \in \mathbb{R}$, where $(s\cdot\gamma)(t) = \gamma(t+s)$ for $t \in \mathbb{R}$. Two $\mathbb{I}_g$-invariant F-geodesics γ_1 and γ_2 are said to be $\mathbb{R}$-distinct if there is no $s \in \mathbb{R}$ such that $s \cdot \gamma_1 = \gamma_2$. If a F-geodesic $\gamma : [0,1] \to M$ satisfies $\mathbb{I}_{g*}(\dot\gamma(0)) = \dot\gamma(1)$, then it may be extended into an $\mathbb{I}_g$-invariant F-geodesic $\gamma^\star : \mathbb{R} \to M$ via

$$\gamma^\star(t) = \mathbb{I}_g^{[t]}(\gamma(t - [t]))\ \forall t \in \mathbb{R}, \tag{12.1}$$

where $[s]$ denotes the greatest integer at most s, called the corresponding (maximal) $\mathbb{I}_g$-invariant F-geodesic (determined by γ).

Assumption 12.1. Under Assumption 9.2 with $\ell = 6$, all F_λ are also $\mathbb{I}_g$-invariant, and $\bar\gamma : \mathbb{R} \to M$ is an $\mathbb{I}_g$-invariant constant (non-zero) speed F_λ-geodesic for each $\lambda \in \Lambda$.

Under this assumption, each element in $\mathbb{R} \cdot \bar\gamma := \{\bar\gamma(\theta + \cdot) \,|\, \theta \in \mathbb{R}\}$ ($\mathbb{R}$-orbit) is also an $\mathbb{I}_g$-invariant constant (non-zero) speed F_λ-geodesic for each $\lambda \in \Lambda$. Because of this reason, similar to Definition 1.22 we have:

Definition 12.2. $\mathbb{R}$-orbits of $\mathbb{I}_g$-invariant constant (non-zero) speed F_λ-geodesics with a parameter $\lambda \in \Lambda$ is said sequently bifurcating at μ with respect to the $\mathbb{R}$-orbit $\mathbb{R} \cdot \bar{\gamma}$ if there exists a sequence $(\lambda_k) \subset \Lambda$ converging to μ, and $\mathbb{I}_g$-invariant constant (non-zero) speed F_{λ_k}-geodesics γ^k, $k = 1, 2, \cdots$, such that: (i) $\gamma^k \notin \mathbb{R} \cdot \bar{\gamma}\ \forall k$, (ii) all γ^k are $\mathbb{R}$-distinct, (iii) $\gamma^k|_{[0,1]} \to \bar{\gamma}|_{[0,1]}$ in $C^1([0,1]; M)$. [Passing to a subsequence, (i) is implied in (ii).]

By definition of the fundamental tensor g^{F_λ} in (9.1), $\mathbb{I}_g$-invariance of F_λ implies that

$$g^{F_\lambda}(\mathbb{I}_g(x), \mathbb{I}_{g*}(v))[\mathbb{I}_{g*}(u)), \mathbb{I}_{g*}(w))] = g^{F_\lambda}(x, v)[u, w]$$

for all $(x, v) \in TM \setminus 0_{TM}$ and $u, w \in T_xM$. The following claim easily follows from these and the $\mathbb{I}_g$-invariance of g.

Claim 12.3. *For a given sequential compact subset $\tilde{\Lambda} \subset \Lambda$ and a given open neighborhood $\mathcal{M}$ of $\bar{\gamma}([0,1])$ with compact closure, on the open submanifold $\tilde{M} := \cup_{k\in\mathbb{Z}}(\mathbb{I}_g)^k(\mathcal{M})$ of M the corresponding numbers defined by Proposition 9.5 with $(\Lambda, M) = (\tilde{\Lambda}, \tilde{M})$,*

$$\tilde{\alpha}_g := \inf_{\lambda\in\Lambda} \inf_{(x,v)\in T\tilde{M},\, |v|_x=1} \inf_{u\neq 0} \frac{g_v^{F_\lambda}(u,u)}{g_x(u,u)} = \inf_{\lambda\in\Lambda} \inf_{(x,v)\in T\mathcal{M},\, |v|_x=1} \inf_{u\neq 0} \frac{g_v^{F_\lambda}(u,u)}{g_x(u,u)} \quad \textit{and}$$
$$\tilde{\beta}_g := \sup_{\lambda\in\Lambda} \sup_{(x,v)\in T\tilde{M},\, |v|_x=1} \sup_{u\neq 0} \frac{g_v^{F_\lambda}(u,u)}{g_x(u,u)} = \sup_{\lambda\in\Lambda} \sup_{(x,v)\in T\mathcal{M},\, |v|_x=1} \sup_{u\neq 0} \frac{g_v^{F_\lambda}(u,u)}{g_x(u,u)}$$

are positive, and by scaling down or up g (if necessary) it holds that for some constant $C_1 > 0$,

$$|v|_x^2 \le L_\lambda(x, v) \le C_1|v|_x^2 \quad \forall(\lambda, x, v) \in \tilde{\Lambda} \times T\tilde{M}.$$

Moreover (since $\bar{\gamma}$ is $\mathbb{I}_g$-invariant) there exists $c > 0$ such that

$$[F_\lambda(\bar{\gamma}(t), \dot{\bar{\gamma}}(t))]^2 > \frac{2c}{C_1}, \quad \forall(\lambda, t) \in \tilde{\Lambda} \times \mathbb{R}. \tag{12.2}$$

Claim 12.4. *Under Claim 12.3 let $L^*_\lambda : T\tilde{M} \to \mathbb{R}$, $\lambda \in \tilde{\Lambda}$, be given by Proposition 9.5 with $(\Lambda, M) = (\tilde{\Lambda}, \tilde{M})$. Then there exists a neighborhood $\mathcal{U}$ of $\bar{\gamma}|_{[0,1]}$ in $C^1([0,1], M)$ such that if $\gamma : \mathbb{R} \to M$ is a constant (non-zero) speed $\mathbb{I}_g$-invariant F_λ-geodesic whose restriction to $[0,1]$ sits in $\mathcal{U}$, where $\lambda \in \tilde{\Lambda}$, then it is C^6, sits in $\tilde{M}$ and satisfies*

$$\left.\begin{array}{l} \frac{d}{dt}\Big(\partial_v L^*_\lambda(\gamma(t), \dot{\gamma}(t))\Big) - \partial_x L^*_\lambda(\gamma(t), \dot{\gamma}(t)) = 0\ \forall t \in \mathbb{R}, \\ \mathbb{I}_g(\gamma(t)) = \gamma(t+1) \quad \forall t \in \mathbb{R}. \end{array}\right\} \tag{12.3}$$

Conversely, for a solution γ of (12.3), which must be C^6, if $\gamma|_{[0,1]}$ is in $\mathcal{U}$, then γ is a constant (non-zero) speed $\mathbb{I}_g$-invariant F_λ-geodesic.

Proof. Since $\tilde{\Lambda} \subset \Lambda$ is sequential compact, it follows from (12.2) that there exists a neighborhood $\mathcal{U}$ of $\bar{\gamma}|_{[0,1]}$ in $C^1([0,1], M)$ such that if the restriction of an $\mathbb{I}_g$-invariant C^1 curve $\gamma : \mathbb{R} \to M$ to $[0,1]$ belongs to $\mathcal{U}$ then $[F_\lambda(\gamma(t), \dot{\gamma}(t))]^2 > \frac{2c}{3C_1}$ for all $(\lambda, t) \in \tilde{\Lambda} \times \mathbb{R}$. Therefore for an $\mathbb{I}_g$-invariant C^2 curve $\gamma : \mathbb{R} \to M$, if $\gamma|_{[0,1]}$ is in $\mathcal{U}$ then γ satisfies

$$\frac{d}{dt}\Big(\partial_v L^*_\lambda(\gamma(t), \dot{\gamma}(t))\Big) - \partial_x L^*_\lambda(\gamma(t), \dot{\gamma}(t)) = \frac{d}{dt}\Big(\partial_v L_\lambda(\gamma(t), \dot{\gamma}(t))\Big) - \partial_x L_\lambda(\gamma(t), \dot{\gamma}(t))$$

by Proposition 9.5(i) with $(\Lambda, M) = (\tilde{\Lambda}, \tilde{M})$. These imply the desired conclusions. $\square$

Let $\mathcal{X}^1_\tau(\tilde{M}, \mathbb{I}_g)$ be the C^4 Banach manifold defined in (1.20), which may be identified with $C^1_{\mathbb{I}_g}([0,1]; \tilde{M})$. Define functionals

$$\mathcal{E}_{\lambda,\mathbb{I}_g} : \mathcal{X}^1_\tau(\tilde{M}, \mathbb{I}_g) \to \mathbb{R},\ \gamma \mapsto \int_0^1 [F_\lambda(\gamma(t), \dot\gamma(t))]^2 dt,$$
$$\mathcal{E}^*_{\lambda,\mathbb{I}_g} : \mathcal{X}^1_\tau(\tilde{M}, \mathbb{I}_g) \to \mathbb{R},\ \gamma \mapsto= \int_0^1 L^*_\lambda(\gamma(t), \dot\gamma(t)) dt.$$

Clearly, they agree on the open subset

$$C^1(\mathbb{R}; \tilde{M}, \mathbb{I}_g, \{F_\lambda \,|\, \lambda \in \tilde\Lambda\}, c/C_1) := \left\{\alpha \in \mathcal{X}^1_\tau(\tilde{M}, \mathbb{I}_g) \,\Big|\, \min_{(\lambda,t)\in\tilde\Lambda\times[0,1]} [F_\lambda(\alpha(t), \dot\alpha(t))]^2 > 2c/C_1\right\}$$

of $\mathcal{X}^1_\tau(\tilde{M}, \mathbb{I}_g)$ containing $\bar\gamma$, and their critical points on $\mathcal{X}^1_\tau(\tilde{M}, \mathbb{I}_g)$ near $\bar\gamma$ correspond, respectively, to constant speed $\mathbb{I}_g$-invariant F_λ-geodesics near $\bar\gamma$ in $\tilde{M}$ and solutions of (12.3) near $\bar\gamma$ in $\tilde{M}$. For each $\lambda \in \tilde\Lambda$ we write

$$m^-(\mathcal{E}_{\lambda,\mathbb{I}_g}, \bar\gamma) := m^-(\mathcal{E}_{\lambda,\mathbb{I}_g}, \bar\gamma|_{[0,1]}) \quad\text{and}\quad m^0(\mathcal{E}_{\lambda,\mathbb{I}_g}, \bar\gamma) := m^0(\mathcal{E}_{\lambda,\mathbb{I}_g}\bar\gamma|_{[0,1]}),$$
$$m^-(\mathcal{E}^*_{\lambda,\mathbb{I}_g}, \bar\gamma) := m^-(\mathcal{E}^*_{\lambda,\mathbb{I}_g}, \bar\gamma|_{[0,1]}) \quad\text{and}\quad m^0(\mathcal{E}^*_{\lambda,\mathbb{I}_g}, \bar\gamma) := m^0(\mathcal{E}^*_{\lambda,\mathbb{I}_g}, \bar\gamma|_{[0,1]}).$$

Then $m^-(\mathcal{E}^*_{\lambda,\mathbb{I}_g}, \bar\gamma) = m^-(\mathcal{E}_{\lambda,\mathbb{I}_g}, \bar\gamma)$ and $m^0(\mathcal{E}^*_{\lambda,\mathbb{I}_g}, \bar\gamma) = m^0(\mathcal{E}_{\lambda,\mathbb{I}_g}, \bar\gamma)$.

Theorem 12.5 (Necessary condition). *Under Assumptions 12.1, if $\mathbb{R}$-orbits of $\mathbb{I}_g$-invariant constant (non-zero) speed F_λ-geodesics with a parameter $\lambda \in \Lambda$ sequently bifurcate at μ with respect to the $\mathbb{R}$-orbit $\mathbb{R}\cdot\bar\gamma$, then $m^0(\mathcal{E}_{\mu,\mathbb{I}_g}, \bar\gamma) \geq 2$.*

Proof. By the assumption there exists a sequence $(\lambda_k) \subset \Lambda$ converging to $\mu \in \Lambda$, and constant (non-zero) speed $\mathbb{I}_g$-invariant F_{λ_k}-geodesics $\gamma^k : \mathbb{R} \to M$ (which must be C^6), $k = 1, 2, \cdots$, such that these γ^k are $\mathbb{R}$-distinct each other and satisfy $\gamma^k|_{[0,1]} \to \bar\gamma|_{[0,1]}$ in $C^1([0,1]; M)$. Let $\tilde\Lambda = \{\mu, \lambda_k \,|\, k \in \mathbb{N}\}$, which is sequential compact. Choose $\tilde{M}$ as in Claim 12.3, and $L^*_\lambda : T\tilde{M} \to \mathbb{R}$, $\lambda \in \tilde\Lambda$, as in Claim 12.4. Since $\gamma^k|_{[0,1]} \to \bar\gamma|_{[0,1]}$ in $C^1([0,1]; M)$, for the neighborhood $\mathcal{U}$ of $\bar\gamma|_{[0,1]}$ in $C^1([0,1], M)$ in Claim 12.4, we can assume that all $\gamma^k|_{[0,1]}$ belong to $\mathcal{U}$. By Claim 12.4 each γ^k is C^6, sits in $\tilde{M}$ and satisfies (12.3) with $\lambda = \lambda_k$. Then Theorem 1.23 concludes $m^0(\mathcal{E}^*_{\lambda,\mathbb{I}_g}, \bar\gamma) \geq 2$ and so $m^0(\mathcal{E}_{\lambda,\mathbb{I}_g}, \bar\gamma) \geq 2$. □

Theorem 12.6 (Sufficient condition). *Under Assumption 12.1 suppose the following conditions hold.*

(a) *$\bar\gamma$ is periodic, and $m^0(\mathcal{E}_{\mu,\mathbb{I}_g}, \bar\gamma) \geq 2$.*

(b) *There exist two sequences in Λ converging to μ, (λ^-_k) and (λ^+_k), such that for each $k \in \mathbb{N}$,*

$$[m^-_\tau(\mathcal{E}_{\lambda^-_k,\mathbb{I}_g}, \bar\gamma), m^-_\tau(\mathcal{E}_{\lambda^-_k,\mathbb{I}_g}, \bar\gamma) + m^0_\tau(\mathcal{E}_{\lambda^-_k,\mathbb{I}_g}, \bar\gamma) - 1]$$
$$\cap[m^-_\tau(\mathcal{E}_{\lambda^+_k,\mathbb{I}_g}, \bar\gamma), m^-_\tau(\mathcal{E}_{\lambda^+_k,\mathbb{I}_g}, \bar\gamma) + m^0_\tau(\mathcal{E}_{\lambda^+_k,\mathbb{I}_g}, \bar\gamma) - 1] = \emptyset$$

and either $m^0_\tau(\mathcal{E}_{\lambda^-_k,\mathbb{I}_g}, \bar\gamma) = 1$ or $m^0_\tau(\mathcal{E}_{\lambda^+_k,\mathbb{I}_g}, \bar\gamma) = 1$.

(c) *For any constant (non-zero) speed $\mathbb{I}_g$-invariant F_μ-geodesic $\gamma : \mathbb{R} \to M$, if there exists a sequence (s_k) of reals such that $s_k \cdot \gamma$ converges to $\bar\gamma$ on any compact interval $I \subset \mathbb{R}$ in C^1-topology, then γ is periodic. (Clearly, this holds if $(\mathbb{I}_g)^l = id_M$ for some $l \in \mathbb{N}$.)*

Then there exists a sequence $(\lambda_k)\subset\tilde\Lambda:=\{\mu,\lambda_k^+,\lambda_k^-\,|\,k\in\mathbb{N}\}$ *converging to* μ *and* C^6 *constant (non-zero) speed* $\mathbb{I}_g$*-invariant* F_{λ_k}*-geodesics* $\gamma^k:\mathbb{R}\to M$, $k=1,2,\cdots$, *such that any two of these* γ_k *are* $\mathbb{R}$*-distinct and that* $\gamma^k|_{[0,1]}\to\bar\gamma|_{[0,1]}$ *in* $C^2([0,1];M)$*. In particular,* $\mathbb{R}$*-orbits of* $\mathbb{I}_g$*-invariant constant (non-zero) speed* F_λ*-geodesics with a parameter* $\lambda\in\Lambda$ *sequently bifurcate at* μ *with respect to the* $\mathbb{R}$*-orbit* $\mathbb{R}\cdot\bar\gamma$*.*

Proof. Since $\tilde\Lambda=\{\mu,\lambda_k^+,\lambda_k^-\,|\,k\in\mathbb{N}\}$ is compact and sequential compact, we may choose $\tilde M$ as in Claim 12.3, and $L^*_\lambda:T\tilde M\to\mathbb{R}$, $\lambda\in\tilde\Lambda$, as in Claim 12.4. Clearly, the assumptions (a) and (b) lead to the corresponding conditions (a) and (b) in Theorem 1.24 with $(M,L_\lambda)=(\tilde M,L^*_\lambda)$, respectively.

For any solution γ of (1.23) with $(M,L_\lambda)=(\tilde M,L^*_\lambda)$ and $\lambda=\mu$, that is, a solution of (12.3) with $\lambda=\mu$, suppose that there exists a sequence (s_k) of reals such that $s_k\cdot\gamma$ converges to $\bar\gamma$ on any compact interval $I\subset\mathbb{R}$ in C^1-topology. Then $s_k\cdot\gamma|_{[0,1]}\to\bar\gamma|_{[0,1]}$ in $C^1([0,1];M)$. By Claim 12.4, for each k large enough, $s_k\cdot\gamma$ and hence γ is a constant (non-zero) speed $\mathbb{I}_g$-invariant F_μ-geodesic. The assumption (c) assures that γ is periodic. Hence the condition (c) in Theorem 1.24 with $(M,L_\lambda)=(\tilde M,L^*_\lambda)$ is satisfied.

By Theorem 1.24 there exists a sequence $(\lambda_k)\subset\tilde\Lambda$ converging to μ and C^6 solutions γ^k of (12.3) with $\lambda=\lambda_k$, $k=1,2,\cdots$, such that any two of these γ^k are $\mathbb{R}$-distinct and that (γ^k) converges to $\bar\gamma$ on any compact interval $I\subset\mathbb{R}$ in C^2-topology as $k\to\infty$. By Claim 12.4, for each k large enough γ^k is a constant (non-zero) speed $\mathbb{I}_g$-invariant F_{λ_k}-geodesic. □

Theorem 12.7 (Existence for bifurcations)**.** *Under Assumption 12.1, suppose that* Λ *is path-connected,* $(\mathbb{I}_g)^l=id_M$ *for some* $l\in\mathbb{N}$*, and the following is satisfied:*

(d) *There exist two points* $\lambda^+,\lambda^-\in\Lambda$ *such that*

$$\begin{aligned}&[m^-_\tau(\mathcal{E}_{\lambda^-,\mathbb{I}_g},\bar\gamma),m^-_\tau(\mathcal{E}_{\lambda^-,\mathbb{I}_g},\bar\gamma)+m^0_\tau(\mathcal{E}_{\lambda^-,\mathbb{I}_g},\bar\gamma)-1]\\&\cap[m^-_\tau(\mathcal{E}_{\lambda^+,\mathbb{I}_g},\bar\gamma),m^-_\tau(\mathcal{E}_{\lambda^+,\mathbb{I}_g},\bar\gamma)+m^0_\tau(\mathcal{E}_{\lambda^+,\mathbb{I}_g},\bar\gamma)-1]=\emptyset\end{aligned}$$

and either $m^0_\tau(\mathcal{E}_{\lambda^-,\mathbb{I}_g},\bar\gamma)=1$ *or* $m^0_\tau(\mathcal{E}_{\lambda^+,\mathbb{I}_g},\bar\gamma)=1$*.*

Then for any path $\alpha:[0,1]\to\Lambda$ *connecting* λ^+ *to* λ^- *there exists a sequence* (λ_k) *in* $\alpha([0,1])$ *converging to* $\mu\in\alpha([0,1])\subset\Lambda$*, and* C^6 *constant (non-zero) speed* $\mathbb{I}_g$*-invariant* F_{λ_k}*-geodesics,* $k=1,2,\cdots$*, such that any two of these* γ_k *are* $\mathbb{R}$*-distinct and that* (γ_k) *converges to* $\bar\gamma$ *on any compact interval* $I\subset\mathbb{R}$ *in* C^2*-topology as* $k\to\infty$*. Moreover, this* μ *is not equal to* λ^+ *(resp.* λ^-*) if* $m^0_\tau(\mathcal{E}_{\lambda^+,\mathbb{I}_g},\bar\gamma)=1$ *(resp.* $m^0_\tau(\mathcal{E}_{\lambda^-,\mathbb{I}_g},\bar\gamma)=1$*).*

Proof. Since $\tilde\Lambda:=\alpha([0,1])$ is a compact and sequential compact subset in Λ we may choose $\tilde M$ as in Claim 12.3, and $L^*_\lambda:T\tilde M\to\mathbb{R}$, $\lambda\in\tilde\Lambda$, as in Claim 12.4. Then the assumption (d) yields the corresponding condition (d) in Theorem 1.25 with $(M,L_\lambda)=(\tilde M,L^*_\lambda)$. Hence there exists a sequence (λ_k) in $\alpha([0,1])$ converging to $\mu\in\alpha([0,1])\subset\Lambda$, and C^6 solutions γ_k of the corresponding problem (1.23) on $(M,L_\lambda)=(\tilde M,L^*_\lambda)$ with $\lambda=\lambda_k$, $k=1,2,\cdots$, such that any two of these γ_k are $\mathbb{R}$-distinct and that (γ_k) converges to $\bar\gamma$ on any compact interval $I\subset\mathbb{R}$ in C^2-topology as $k\to\infty$. Moreover, this μ is not equal to λ^+ (resp. λ^-) if $m^0_\tau(\mathcal{E}_{\lambda^+,\mathbb{I}_g},\bar\gamma)=1$ (resp. $m^0_\tau(\mathcal{E}_{\lambda^-,\mathbb{I}_g},\bar\gamma)=1$). The required results easily follow from these as before. □

Theorem 12.8 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumption 12.1 with* Λ *being a real interval, suppose*

(a) $\mu\in\mathrm{Int}(\Lambda)$*,* $\mathbb{I}_g=id_M$ *and* $\bar\gamma$ *have least period* 1*;*

(b) $m^0(\mathcal{E}_{\mu,\mathbb{I}_g},\bar{\gamma}) \ge 2$, $m^0(\mathcal{E}_{\lambda,\mathbb{I}_g},\bar{\gamma}) = 1$ *for each* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ*;*

(c) $m^-(\mathcal{E}_{\lambda,\mathbb{I}_g},\bar{\gamma})$ *take, respectively, values* $m^0(\mathcal{E}_{\mu,\mathbb{I}_g},\bar{\gamma})$ *and* $m^-(\mathcal{E}_{\mu,\mathbb{I}_g},\bar{\gamma}) + m^0(\mathcal{E}_{\mu,\mathbb{I}_g},\bar{\gamma}) - 1$ *as* $\lambda \in \Lambda$ *varies in two deleted half neighborhoods of* μ*.*

Then one of the following alternatives occurs:

(i) *There exists a sequence* C^6 *constant (non-zero) speed* $\mathbb{I}_g$*-invariant* F_μ*-geodesic* $\gamma^k : \mathbb{R} \to M$, $k = 1, 2, \cdots$*, such that these* γ^k *are* $\mathbb{R}$*-distinct each other and converge to* $\bar{\gamma}$ *on any compact interval* $I \subset \mathbb{R}$ *in* C^2*-topology as* $k \to \infty$*.*

(ii) *For every* $\lambda \in \Lambda \setminus \{\mu\}$ *near* μ *there is a* C^6 *constant (non-zero) speed* $\mathbb{I}_g$*-invariant* F_λ*-geodesic* γ^λ*, which is* $\mathbb{R}$*-distinct with* $\bar{\gamma}$ *and converges to* $\bar{\gamma}$ *on any compact interval* $I \subset \mathbb{R}$ *in* C^2*-topology as* $\lambda \to \mu$*.*

(iii) *For a given neighborhood* $\mathcal{W}$ *of* $\bar{\gamma}$ *in* $C^1(\mathbb{R}; M, \mathbb{I}_g)$*, there exists an one-sided neighborhood* Λ^0 *of* μ *such that for any* $\lambda \in \Lambda^0 \setminus \{\mu\}$*, there exist at least two* $\mathbb{R}$*-distinct* C^6 *constant (non-zero) speed* $\mathbb{I}_g$*-invariant* F_λ*-geodesics,* $\gamma_\lambda^1 \notin \mathbb{R} \cdot \bar{\gamma}$ *and* $\gamma_\lambda^2 \notin \mathbb{R} \cdot \bar{\gamma}$*, which can also be chosen to have different* F_λ*-speeds (i.e.,* $F_\lambda(\gamma_\lambda^1(t), \dot{\gamma}_\lambda^1(t)) \ne F_\lambda(\gamma_\lambda^2(t), \dot{\gamma}_\lambda^2(t))\ \forall t$*) provided that* $m_\tau^0(\mathcal{E}_{\mu,\mathbb{I}_g},\bar{\gamma}) \ge 3$ *and there exist only finitely many* $\mathbb{R}$*-distinct* C^6 *constant (non-zero) speed* $\mathbb{I}_g$*-invariant* F_λ*-geodesics in* $\mathcal{W}$ *which are* $\mathbb{R}$*-distinct from* $\bar{\gamma}$*.*

Proof. As in the proof of Theorem 11.3 we have a number $\rho > 0$ such that the compact (and sequential compact) set $\tilde{\Lambda} := [\mu - \rho, \mu + \rho] \subset \Lambda$. Choose $\tilde{M}$ as in Claim 12.3, and $L_\lambda^* : T\tilde{M} \to \mathbb{R}$, $\lambda \in \tilde{\Lambda}$, as in Claim 12.4. Since $m^-(\mathcal{E}^*_{\lambda,\mathbb{I}_g},\bar{\gamma}) = m^-(\mathcal{E}_{\lambda,\mathbb{I}_g},\bar{\gamma})$ and $m^0(\mathcal{E}^*_{\lambda,\mathbb{I}_g},\bar{\gamma}) = m^0(\mathcal{E}_{\lambda,\mathbb{I}_g},\bar{\gamma})$ for each $\lambda \in \tilde{\Lambda}$, the assumptions (b) and (c) imply that $m^0(\mathcal{E}^*_{\mu,\mathbb{I}_g},\bar{\gamma}) \ge 2$, $m^0(\mathcal{E}^*_{\lambda,\mathbb{I}_g},\bar{\gamma}) = 1$ for each $\lambda \in \tilde{\Lambda} \setminus \{\mu\}$ near μ and that $m^-(\mathcal{E}^*_{\lambda,\mathbb{I}_g},\bar{\gamma})$ take, respectively, values $m^0(\mathcal{E}^*_{\mu,\mathbb{I}_g},\bar{\gamma})$ and $m^-(\mathcal{E}^*_{\mu,\mathbb{I}_g},\bar{\gamma}) + m^0(\mathcal{E}^*_{\mu,\mathbb{I}_g},\bar{\gamma}) - 1$ as $\lambda \in \tilde{\Lambda}$ varies in two deleted half neighborhoods of μ. Hence the desired results may follow from Theorem 1.26 and Claim 12.4 as above. □

Remark 12.9. As noted below Theorem 1.26, if M is an open subset U of $\mathbb{R}^n$ and $\mathbb{I}_g$ is an orthogonal matrix E of order n which maintain U invariant, "Assumption 12.1" and all "C^6" in Theorem 12.5, 12.6, 12.8 can be replaced by "Assumption 12.1 with $\ell = 4$" and "C^4" respectively.

13 Bifurcations of reversible geodesics

For a reversible Finsler metric F on M, and any geodesic of F, $\gamma : (-r, r) \to M$, the reverse curve $\gamma^- : (-r, r) \to M$ defined by $\gamma^-(t) = \gamma(-t)$ is a geodesic of F that coincide pointwise with γ. (See [42, Remark 3.1]). The irreversibility of a Finsler metric is a very strong restriction that excludes a lot of interesting examples, for instance Randers metrics, which are Finsler metrices of form $F = \alpha + \beta$, where α is a Riemannian metric and β is a nonzero 1-form on M.

Assumption 13.1. Under Assumption 9.2, for each $\lambda \in \Lambda$ suppose that F_λ is reversible and that $\gamma_\lambda : \mathbb{R} \to M$ is a 1-periodic F_λ-geodesic of constant (non-zero) speed, (which must be C^ℓ and satisfies $\gamma_\lambda(-t) = \gamma_\lambda(t)$ for all $t \in \mathbb{R}$). It is also required that the maps $\Lambda \times \mathbb{R} \ni (\lambda, t) \to \gamma_\lambda(t) \in M$ and $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto \dot{\gamma}_\lambda(t) \in TM$ are continuous.

Definition 13.2. Under Assumption 13.1, 1-periodic F_λ-geodesics of constant (non-zero) speed with a parameter $\lambda \in \Lambda$ is said bifurcating at $\mu \in \Lambda$ along sequences with respect to the branch $\{\gamma_\lambda \,|\, \lambda \in \Lambda\}$ if there exists an infinite sequence $\{(\lambda_k, \gamma^k)\}_{k=1}^\infty$ in $\Lambda \times EC^1(S_1, M) \setminus \{(\mu, \gamma_\mu)\}$ converging to (μ, γ_μ), such that each $\gamma^k \ne \gamma_{\lambda_k}$ is a 1-periodic F_{λ_k}-geodesic of constant (non-zero) speed, $k = 1, 2, \cdots$. (Actually it is not hard to prove that $\gamma^k \to \gamma_\mu$ in $C^2(S_1, M)$.)

Under Assumption 13.1 nonconstant critical points of the C^{2-0}-functional

$$\gamma \mapsto \mathcal{E}^E_\lambda(\gamma) = \int_0^1 [F_\lambda(\gamma(t), \dot{\gamma}(t))]^2 dt. \tag{13.1}$$

on the Banach manifold $EC^1(S_1; M)$ given by (1.27) correspond to 1-periodic F_λ-geodesics of constant (non-zero) speed. Since $\mathcal{E}^E_\lambda$ is C^2 near such a critical point γ, the Morse index and nullity $m^-(\mathcal{E}^E_\lambda, \gamma)$ and $m^0(\mathcal{E}^E_\lambda, \gamma)$ are well-defined as before.

Theorem 13.3. *Let Assumption 13.1 be satisfied, $\mu \in \Lambda$.*

(I) (Necessary condition): *Suppose that 1-periodic F_λ-geodesics of constant (non-zero) speed with a parameter $\lambda \in \Lambda$ bifurcate at $\mu \in \Lambda$ along sequences with respect to the branch $\{\gamma_\lambda \,|\, \lambda \in \Lambda\}$. Then $m^0(\mathcal{E}^E_\mu, \gamma_\mu) \geq 1$.*

(II) (Sufficient condition): *Let Λ be first countable. Suppose that there exist two sequences in Λ converging to μ, (λ^-_k) and (λ^+_k), such that one of the following conditions is satisfied:*

(II.1) *For each $k \in \mathbb{N}$, either $\gamma_{\lambda^+_k}$ is not an isolated critical point of $\mathcal{E}^E_{\lambda^+_k}$, or $\gamma_{\lambda^-_k}$ is not an isolated critical point of $\mathcal{E}^E_{\lambda^-_k}$, or $\gamma_{\lambda^+_k}$ (resp. $\gamma_{\lambda^-_k}$) is an isolated critical point of $\mathcal{E}^E_{\lambda^+_k}$ (resp. $\mathcal{E}^E_{\lambda^-_k}$) and $C_m(\mathcal{E}^E_{\lambda^+_k}, \gamma_{\lambda^+_k}; \mathbf{K})$ and $C_m(\mathcal{E}^E_{\lambda^-_k}, \gamma_{\lambda^-_k}; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$.*

(II.2) *For each $k \in \mathbb{N}$, there exists $\lambda \in \{\lambda^+_k, \lambda^-_k\}$ such that γ_λ is an either nonisolated or homological visible critical point of $\mathcal{E}^E_\lambda$, and*

$$\left.\begin{aligned} &[m^-(\mathcal{E}^E_{\lambda^-_k}, \gamma_{\lambda^-_k}), m^-(\mathcal{E}^E_{\lambda^-_k}, \gamma_{\lambda^-_k}) + m^0(\mathcal{E}^E_{\lambda^-_k}, \gamma_{\lambda^-_k})] \\ &\cap [m^-(\mathcal{E}^E_{\lambda^+_k}, \gamma_{\lambda^+_k}), m^-(\mathcal{E}^E_{\lambda^+_k}, \gamma_{\lambda^+_k}) + m^0(\mathcal{E}^E_{\lambda^+_k}, \gamma_{\lambda^+_k})] = \emptyset. \end{aligned}\right\} \tag{$\clubsuit_k$}$$

(II.3) *For each $k \in \mathbb{N}$, ($\clubsuit_k$) holds true, and either $m^0(\mathcal{E}^E_{\lambda^-_k}, \gamma_{\lambda^-_k}) = 0$ or $m^0(\mathcal{E}^E_{\lambda^+_k}, \gamma_{\lambda^+_k}) = 0$.*

Then there exists a sequence $(\lambda_k) \subset \hat{\Lambda} := \{\mu, \lambda^+_k, \lambda^-_k \,|\, k \in \mathbb{N}\}$ converging to μ, 1-periodic F_{λ_k}-geodesics $\gamma^k \neq \gamma_{\lambda_k}$ of constant (non-zero) speed, $k = 1, 2, \cdots$, such that $\gamma^k \to \gamma_\mu$ in $C^2(S_1; M)$. In particular, 1-periodic F_λ-geodesics of constant (non-zero) speed with a parameter $\lambda \in \Lambda$ bifurcate at $\mu \in \Lambda$ along sequences with respect to the branch $\{\gamma_\lambda \,|\, \lambda \in \Lambda\}$.

Proof. *Step 1* [Prove (I)]. By the assumptions there exists a sequence $(\lambda_k) \subset \Lambda$ converging to $\mu \in \Lambda$ such that for each k there exists a nonconstant reversible and 1-periodic F_{λ_k}-geodesic $\gamma^k \neq \gamma_{\lambda_k}$ to satisfy $\gamma^k \to \gamma_\mu$ in $C^1(S_1; M)$. Let $\hat{\Lambda} = \{\mu\} \cup \{\lambda_k \,|\, k \in \mathbb{N}\}$. It is sequential compact. Then we can choose an open subset $\hat{M}$ of M with compact closure such that $\gamma_\lambda(S_1) \cup \gamma^m(S_1) \subset \hat{M}$ for all $(\lambda, m) \in \hat{\Lambda} \times \mathbb{N}$, and therefore the conditions in Proposition 9.5 can be satisfied with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$. For the constant $C_1 > 0$ as in Proposition 9.5 with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$ we have $c > 0$ such that for all $(m, \lambda, t) \in \mathbb{N} \times \hat{\Lambda} \times \mathbb{R}$,

$$[F_\lambda(\gamma_\lambda(t), \dot{\gamma}_\lambda(t))]^2 > \frac{2c}{C_1} \quad \text{and} \quad [F_{\lambda_m}(\gamma^m(t), \dot{\gamma}^m(t))]^2 > \frac{2c}{C_1}.$$

Let $L^*_\lambda : T\hat{M} \to \mathbb{R}$, $\lambda \in \hat{\Lambda}$, be given by Proposition 9.5 with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$. Then $L^*_\lambda(x, v) = L^*_\lambda(x, -v)$ for all $(x, v) \in T\hat{M}$, and on $EC^1(S_1; M)$ the corresponding C^2 functional $\mathcal{E}^{E*}_\lambda$ given by (9.14) with $\tau = 1$ and the C^{2-0} functional $\mathcal{E}^E_\lambda$ in (9.9) with $\tau = 1$ coincide in the open subset

$$EC^1(S_1, \hat{M}, \{F_\lambda\}, c/C_1) := \{\alpha \in EC^1(S_1, \hat{M}) \,|\, \min_{(\lambda,t) \in \hat{\Lambda} \times S_1} [F_\lambda(\alpha(t), \dot{\alpha}(t))]^2 > 2c/C_1\} \tag{13.2}$$

of $EC^1(S_1; M)$. Moreover, at a critical point γ of them on $EC^1(S_1, \hat{M}, \{F_\lambda\}, c/C_1)$ it holds that their Morse indexes and nullities satisfy $m^-(\mathcal{E}^{E*}_\lambda, \gamma) = m^-(\mathcal{E}^E_\lambda, \gamma)$ and $m^0(\mathcal{E}^{E*}_\lambda, \gamma) = m^0(\mathcal{E}^E_\lambda, \gamma)$. As in Step 1 of proof of Theorem 11.1 the desired conclusion may follow from Theorem 1.29(I).

Step 2 [Prove (II)]. Since $\hat{\Lambda} = \{\mu, \lambda^+_k, \lambda^-_k \mid k \in \mathbb{N}\}$ is sequential compact, as above we can find an open subset $\hat{M}$ of M with compact closure such that $\gamma_\lambda(S_1) \cup \gamma_{\lambda^+_m}(S_1) \cup \gamma_{\lambda^-_m}(S_1) \subset \hat{M}$ for all $(\lambda, m) \in \hat{\Lambda} \times \mathbb{N}$. For the constant $C_1 > 0$ as in Proposition 9.5 with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$ we have $c > 0$ such that for all $(m, \lambda, t) \in \mathbb{N} \times \hat{\Lambda} \times \mathbb{R}$,

$$[F_\lambda(\gamma_\lambda(t), \dot{\gamma}_\lambda(t))]^2 > \frac{2c}{C_1} \quad \text{and} \quad [F_{\lambda_m}(\gamma_{\lambda^\pm_m}(t), \dot{\gamma}_{\lambda^\pm_m}(t))]^2 > \frac{2c}{C_1}.$$

Let $L^*_\lambda : T\hat{M} \to \mathbb{R}$, $\lambda \in \hat{\Lambda}$, be given by Proposition 9.5 with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$. As above, for all $(m, \lambda) \in \mathbb{N} \times \hat{\Lambda}$ we have $\gamma_\lambda, \gamma_{\lambda^+_m}, \gamma_{\lambda^-_m} \in EC^1(S_1, \hat{M}, \{F_\lambda\}, c/C_1)$, and

$$m^-(\mathcal{E}^E_{\lambda^\pm_m}, \gamma_{\lambda^\pm_m}) = m^-(\mathcal{E}^{E*}_{\lambda^\pm_m}, \gamma_{\lambda^\pm_m}) \quad \text{and} \quad m^0(\mathcal{E}^E_{\lambda^\pm_m}, \gamma_{\lambda^\pm_m}) = m^0(\mathcal{E}^{E*}_{\lambda^\pm_m}, \gamma_{\lambda^\pm_m}),$$
$$C_k(\mathcal{E}^E_{\lambda^\pm_m}, \gamma_{\lambda^\pm_m}; \mathbf{K}) = C_k(\mathcal{E}^{E*}_{\lambda^\pm_m}, \gamma_{\lambda^\pm_m}; \mathbf{K}) \quad \forall (k, m) \in \mathbb{Z} \times \mathbb{N}$$

for any Abel group $\mathbf{K}$. The other reasoning may be derived from Theorem 1.29(II) as in Step 2 of proof of Theorem 11.1. □

Theorem 13.4 (Existence for bifurcations)**.** *Under Assumption 13.1, suppose that Λ is path-connected and there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that one of the following conditions is satisfied:*

(i) *Either γ_{λ^+} is not an isolated critical point of $\mathcal{E}^E_{\lambda^+}$, or γ_{λ^-} is not an isolated critical point of $\mathcal{E}^E_{\lambda^-}$, or γ_{λ^+} (resp. γ_{λ^-}) is an isolated critical point of $\mathcal{E}^E_{\lambda^+}$ (resp. $\mathcal{E}^E_{\lambda^-}$) and $C_m(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+}; \mathbf{K})$ and $C_m(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-}; \mathbf{K})$ are not isomorphic for some Abel group $\mathbf{K}$ and some $m \in \mathbb{Z}$.*

(ii) *The intervals $[m^-(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-}), m^-(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-}) + m^0(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-})]$ and*
$$[m^-(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+}), m^-(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+}) + m^0(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+})]$$
are disjoint, and there exists $\lambda \in \{\lambda^+, \lambda^-\}$ such that γ_λ is an either non-isolated or homological visible critical point of $\mathcal{E}^E_\lambda$.

(iii) *The intervals $[m^-(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-}), m^-(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-}) + m^0(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-})]$ and*
$$[m^-(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+}), m^-(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+}) + m^0(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+})]$$
are disjoint, and either $m^0(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+}) = 0$ or $m^0(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-}) = 0$.

Then for any path $\alpha : [0, 1] \to \Lambda$ connecting λ^+ to λ^- there exists a sequence (λ_k) in $\alpha([0, 1])$ converging to some $\mu \in \alpha([0, 1])$, and 1-periodic F_{λ_k}-geodesics γ^k of constant (non-zero) speed, $k = 1, 2, \cdots$, such that $0 < \|\gamma^k - \gamma_{\lambda_k}\|_{C^2(S_1;\mathbb{R}^N)} \to 0$ as $k \to \infty$. Moreover, μ is not equal to λ^+ (resp. λ^-) if $m^0_\tau(\mathcal{E}^E_{\lambda^+}, \gamma_{\lambda^+}) = 0$ (resp. $m^0_\tau(\mathcal{E}^E_{\lambda^-}, \gamma_{\lambda^-}) = 0$).

Proof. Since $\hat{\Lambda} := \alpha([0, 1])$ is compact and sequential compact subset we can find an open subset $\hat{M}$ of M with compact closure such that the closure $Cl\left(\cup_{\lambda \in \hat{\Lambda} \times \mathbb{N}} \gamma_\lambda(S_1)\right) \subset \hat{M}$. For the constant $C_1 > 0$ as in Proposition 9.5 with $(M, \Lambda) = (\hat{M}, \hat{\Lambda})$ we have $c > 0$ such that

$$[F_\lambda(\gamma_\lambda(t), \dot{\gamma}_\lambda(t))]^2 > \frac{2c}{C_1} \quad \text{for all } (\lambda, t) \in \hat{\Lambda} \times S_1.$$

Let $L^*_\lambda : T\hat{M} \to \mathbb{R}$, $\lambda \in \hat{\Lambda}$, be given by Proposition 9.5 with $(M,\Lambda) = (\hat{M},\hat{\Lambda})$. Then $L^*_\lambda(x,v) = L^*_\lambda(x,-v)$ for all $(x,v) \in T\hat{M}$, and on $EC^1(S_1; M)$ the corresponding C^2 functional $\mathcal{E}^{E*}_\lambda$ given by (9.14) with $\tau = 1$ and the C^{2-0} functional $\mathcal{E}^E_\lambda$ in (9.9) with $\tau = 1$ coincide in the open subset $EC^1(S_1, \hat{M}, \{F_\lambda\}, c/C_1)$ in (13.2), which contains $\{\gamma_\lambda \,|\, \lambda \in \hat{\Lambda}\}$. Therefore

$$m^-(\mathcal{E}^E_{\lambda\pm}, \gamma_{\lambda\pm}) = m^-(\mathcal{E}^{E*}_{\lambda\pm}, \gamma_{\lambda\pm}) \quad \text{and} \quad m^0(\mathcal{E}^E_{\lambda\pm}, \gamma_{\lambda\pm}) = m^0(\mathcal{E}^{E*}_{\lambda\pm}, \gamma_{\lambda\pm}),$$
$$C_k(\mathcal{E}^E_{\lambda\pm}, \gamma_{\lambda\pm}; \mathbf{K}) = C_k(\mathcal{E}^{E*}_{\lambda\pm}, \gamma_{\lambda\pm}; \mathbf{K}) \quad \forall k \in \mathbb{Z}$$

for any Abel group $\mathbf{K}$. The desired results may follow from these and Theorem 1.30. □

Theorem 13.5 (Alternative bifurcations of Rabinowitz's type)**.** *Under Assumptions 13.1 with Λ being a real interval, let $\mu \in \mathrm{Int}(\Lambda)$ satisfy $m^0(\mathcal{E}^E_\mu, \gamma_\mu) \neq 0$. Suppose that $m^0(\mathcal{E}^E_\lambda, \gamma_\lambda) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near μ, and that $m^-(\mathcal{E}^E_\lambda, \gamma_\lambda)$ take, respectively, values $m^-(\mathcal{E}^E_\mu, \gamma_\mu)$ and $m^-(\mathcal{E}^E_\mu, \gamma_\mu) + m^0(\mathcal{E}^E_\mu, \gamma_\mu)$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of μ. Then one of the following alternatives occurs:*

- (i) *There exists a sequence of nonconstant 1-periodic F_μ-geodesics $\gamma^k \neq \gamma_\mu$, $k = 1, 2, \cdots$, such that $\gamma^k \to \gamma_\mu$ in $C^2(S_1; M)$.*
- (ii) *For every $\lambda \in \Lambda \setminus \{\mu\}$ near μ there is a nonconstant 1-periodic F_λ-geodesic $\alpha_\lambda \neq \gamma_\lambda$, such that $\alpha_\lambda - \gamma_\lambda$ converges to zero in $C^2(S_1, \mathbb{R}^N)$ as $\lambda \to \mu$. (Recall $M \subset \mathbb{R}^N$.)*
- (iii) *For a given neighborhood $\mathcal{W}$ of γ_μ in $C^2(S_1, M)$, there exists an one-sided neighborhood Λ^0 of μ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, there exist at least two distinct 1-periodic F_λ-geodesics of constant (non-zero) speed in $\mathcal{W}$, $\gamma^1_\lambda \neq \gamma_\lambda$ and $\gamma^2_\lambda \neq \gamma_\lambda$, which can also be chosen to have different F_λ-speeds (i.e., $F_\lambda(\gamma^1_\lambda(t), \dot{\gamma}^1_\lambda(t)) \neq F_\lambda(\gamma^2_\lambda(t), \dot{\gamma}^2_\lambda(t))$ $\forall t$) provided that $m^0(\mathcal{E}^E_\mu, \gamma_\mu) > 1$ and there exist only finitely many 1-periodic F_λ-geodesics of constant (non-zero) speed in $\mathcal{W}$.*

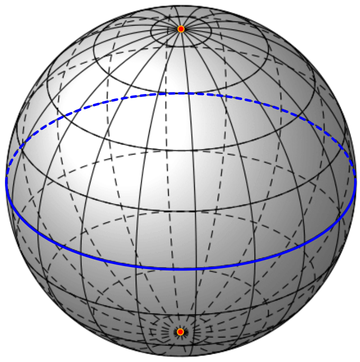
Proof. As in the proof of Theorem 11.3 we have a number $\rho > 0$ such that the compact set $\hat{\Lambda} := [\mu - \rho, \mu + \rho] \subset \Lambda$. Choose an open subset $\hat{M}$ of M with compact closure such that $\gamma_\lambda(S_1) \subset \hat{M}$ for all $\lambda \in \hat{\Lambda}$. Then for the constant $C_1 > 0$ as in Proposition 9.5 with $(M,\Lambda) = (\hat{M},\hat{\Lambda})$ we have $c > 0$ such that $[F_\lambda(\gamma_\lambda(t), \dot{\gamma}_\lambda(t))]^2 > 2c/C_1$ for all $(\lambda, t) \in \hat{\Lambda} \times \mathbb{R}$. Let $L^*_\lambda : T\hat{M} \to \mathbb{R}$, $\lambda \in \hat{\Lambda}$, be given by Proposition 9.5 with $(M,\Lambda) = (\hat{M},\hat{\Lambda})$. The other reasoning may be derived from Theorem 1.31 as in the proof of Theorem 11.3. □

14 The Riemannian case

14.1. Let M and P be as in Assumption 10.1. For a C^ℓ Riemannian metric h on M with $3 \le \ell \le 6$ let $\gamma : [0,\tau] \to M$ be a nonconstant h-geodesic which is perpendicular to P at $\gamma(0)$, i.e., $h(\dot{\gamma}(0), u) = 0$ $\forall u \in T_{\gamma(0)}P$. (Note that γ is C^ℓ.) Definition 1.8 directly yields the notion of bifurcation points along the geodesic γ (cf. Definition 9.4). The notion of P-focal points along γ can be founded in [52]. Applying either Theorem 1.9 to $L(t,q,v) = h_q(v,v)$ and $S_0 = P$ or Theorem 10.4 to $F = h^{1/2}$ we arrive at:

Theorem 14.1. *Under the above assumptions, (i)-(iii) are still true after phrases "constant (non-zero) speed F-geodesics" are changed into "h-geodesics".*

This result directly leads to the following deep geometrical consequence (a special case of Theorem 10.5):



Theorem 14.2. *Theorem 10.5 also holds after "F" and "$\exp^{FN}$" therein are changed into "h" and "$\exp^{hN}$", respectively.*

Example 14.3. Let $M = \mathbb{S}^2$ with the round metric h_0 and $P = \mathbb{S}^1 = \{(x, y, 0) \,|\, x^2 + y^2 = 1\}$. The cut locus of P is $\mathbb{S}^0 = \{\mathbf{e}_3, -\mathbf{e}_3\}$, where $\mathbf{e}_3 = (0, 0, 1)$. For any given $p \in P$, if the norm of $v \in T_p\mathbb{S}^n$ is equal to $\pi/2$, then $\exp_p(v) \in \mathbb{S}^0$ is the north pole $\mathbf{e}_3$. Consider the geodesic $\gamma_v : [0, \infty) \to \mathbb{S}^2$ given by $\gamma_v(t) = \exp_p^{h_0N}(tv)$. $\mathbf{e}_3 = \gamma_v(1)$ is the first P-focal point along γ_v and so a bifurcation point on γ_v relative to P. It is easily seen that only (iii-1) in Theorem 14.1 for $(M, h) = (\mathbb{S}^2, h_0)$ (i.e., Theorem 10.4 for $(M, F) = (\mathbb{S}^2, h_0)$) with $\gamma = \gamma_v$ and $\mu = 1$ occurs. Hence we have only (i) in Theorem 10.5 (i.e., Theorem 14.2 for $\exp^{FN} = \exp^{h_0N}$) with this v).

14.2. The following is a special case of Assumptions 1.1, 1.2 and 1.11.

Assumption 14.4. $\{h_\lambda \,|\, \lambda \in \Lambda\}$ is a family of C^ℓ Riemannian metrics on M with $4 \le \ell \le 6$ parameterized by a topological space Λ, such that $\Lambda \times TM \ni (\lambda, x, v) \to h_\lambda(x, v) \in \mathbb{R}$ is a continuous, and that all partial derivatives of each h_λ of order less than three depend continuously on $(\lambda, x, v) \in \Lambda \times TM$. For each $\lambda \in \Lambda$ let $\gamma_\lambda : [0, \tau] \to M$ be a h_λ-geodesic satisfying the boundary condition

$$h_\lambda(u, \dot{\gamma}_\lambda(0)) = h_\lambda(v, \dot{\gamma}_\lambda(\tau)) \quad \forall (u, v) \in T_{(\gamma_\lambda(0), \gamma_\lambda(\tau))}\mathbf{N}, \tag{14.1}$$

where $\mathbf{N} \subset M \times M$ is a C^7 submanifold. (Therefore γ_λ is C^ℓ by Claim 9.1.) It is also required that the maps $\Lambda \times [0, \tau] \ni (\lambda, t) \to \gamma_\lambda(t) \in M$ and $\Lambda \times [0, \tau] \ni (\lambda, t) \mapsto \dot{\gamma}_\lambda(t) \in TM$ are continuous.

Without occurring of confusions, we use $\mathcal{E}_{\lambda, \mathbf{N}}$ to denote the functional

$$\mathcal{C}_{\tau, \mathbf{N}}(M) \to \mathbb{R}, \; \gamma \mapsto \int_0^\tau h_\lambda(\gamma(t), \dot{\gamma}(t))dt, \tag{14.2}$$

and $m^-(\mathcal{E}_{\lambda, \mathbf{N}}, \gamma_\lambda)$ and $m^0(\mathcal{E}_{\lambda, \mathbf{N}}, \gamma_\lambda)$ to denote the Morse index and nullity at γ_λ of it. For $\mathbf{N} = S_0 \times S_1$ [resp. $\mathrm{Graph}(\mathbb{I}_g)$], the following theorem may directly follow from Theorem 1.4 (resp. Theorem 1.13).

Theorem 14.5. *Under Assumption 14.4 with $\mathbf{N} = S_0 \times S_1$ or $\mathrm{Graph}(\mathbb{I}_g)$, for $\mu \in \Lambda$ there holds:*

(I) *The phrase "constant (non-zero) speed $(F_\lambda, \mathbf{N})$-geodesics" in (I) of Theorem 11.1 is changed into "$(h_\lambda, \mathbf{N})$-geodesics".*

(II) *The phrases "constant speed F_{λ_k}-geodesic $\gamma^k : [0,\tau] \to M$ satisfying the boundary condition (9.8) with $\lambda = \lambda_k$" and "constant (non-zero) speed $(F_\lambda, \mathbf{N})$-geodesics" are changed into "h_{λ_k}-geodesic $\gamma^k : [0,\tau] \to M$ satisfying the boundary condition (14.1) with $\lambda = \lambda_k$" and "$(h_\lambda, \mathbf{N})$-geodesics", respectively.*

For $\mathbf{N} = S_0 \times S_1$ [resp. $\mathbf{N} = \mathrm{Graph}(\mathbb{I}_g)$] Theorem 1.5 (resp. Theorem 1.14) directly leads to:

Theorem 14.6 (Existence for bifurcations)**.** *The phrases "Assumptions 9.2, 9.3" and " constant (non-zero) speed F_{λ_k}-geodesics $\gamma^k : [0,\tau] \to M$ satisfying the boundary condition (9.8)" in Theorem 11.2 are changed into "Assumption 14.4" and "h_{λ_k}-geodesics $\gamma^k : [0,\tau] \to M$ satisfying the boundary condition (14.1)", respectively.*

Similarly, for $\mathbf{N} = S_0 \times S_1$ [resp. $\mathbf{N} = \mathrm{Graph}(\mathbb{I}_g)$] from Theorem 1.6 (resp. Theorem 1.15) we directly derive:

Theorem 14.7 (Alternative bifurcations of Rabinowitz's type)**.** *The phrase "Under Assumptions 9.2, 9.3" in Theorem 11.3 is changed into "Under Assumption 14.4"; and*

- *"constant (non-zero) speed F_μ-geodesics $\gamma^m \neq \gamma_\mu$ satisfying the boundary condition (9.8)" in (i) of Theorem 11.3 is changed into "h_μ-geodesics $\gamma^m \neq \gamma_\mu$ satisfying the boundary condition (14.1)";*
- *"constant (non-zero) speed F_λ-geodesic $\gamma'_\lambda \neq \gamma_\lambda$ satisfying the boundary condition (9.8)" in (ii) of Theorem 11.3 is changed into "h_λ-geodesic $\gamma'_\lambda \neq \gamma_\lambda$ satisfying the boundary condition (14.1)"*
- *"constant (non-zero) speed F_λ-geodesics satisfying the boundary condition (9.8), $\gamma^1_\lambda \neq \gamma_\lambda$ and $\gamma^2_\lambda \neq \gamma_\lambda$, which can also be chosen to satisfy $F_\lambda(\gamma^1_\lambda(t), \dot\gamma^1_\lambda(t)) \neq F_\lambda(\gamma^2_\lambda(t), \dot\gamma^2_\lambda(t))$ $\forall t$" in (iii) of Theorem 11.3 is changed into "h_λ-geodesics satisfying the boundary condition (14.1), $\gamma^1_\lambda \neq \gamma_\lambda$ and $\gamma^2_\lambda \neq \gamma_\lambda$, which can also be chosen to satisfy $h_\lambda(\gamma^1_\lambda(t), \dot\gamma^1_\lambda(t)) \neq h_\lambda(\gamma^2_\lambda(t), \dot\gamma^2_\lambda(t))$ $\forall t$";*
- *"constant (non-zero) speed F_λ-geodesics satisfying the boundary condition (9.8)" in (iii) of Theorem 11.3 is changed into "h_λ-geodesics satisfying the boundary condition (14.1)".*

14.3. Replaceing Assumption 12.1 we make:

Assumption 14.8. $\{h_\lambda \,|\, \lambda \in \Lambda\}$ is a family of C^6 $\mathbb{I}_g$-invariant Riemannian metrics on M parameterized by a topological space Λ, such that $\Lambda \times TM \ni (\lambda, x, v) \to h_\lambda(x, v) \in \mathbb{R}$ is a continuous, and that all partial derivatives of each h_λ of order less than three depend continuously on $(\lambda, x, v) \in \Lambda \times (TM \setminus 0_{TM})$. $\bar\gamma : \mathbb{R} \to M$ is an $\mathbb{I}_g$-invariant nonconstant h_λ-geodesic for each $\lambda \in \Lambda$.

As consequences of results in Section 12 or Theorems 1.23, 1.24, 1.25, 1.26 we obtain:

Theorem 14.9. *The following is true.*

- *In Theorem 12.5, "Assumptions 12.1" and "constant (non-zero) speed F_λ-geodesics" are changed into "Assumptions 14.8" and "nonconstant speed h_λ-geodesics", respectively.*
- *In Theorem 12.6, "Assumption 12.1", "constant (non-zero) speed $\mathbb{I}_g$-invariant F_μ-geodesic", " C^6 constant (non-zero) speed $\mathbb{I}_g$-invariant F_{λ_k}-geodesics" "constant (non-zero) speed F_λ-geodesics" are changed into "Assumption 14.8", "nonconstant $\mathbb{I}_g$-invariant h_μ-geodesic", " C^6 nonconstant $\mathbb{I}_g$-invariant h_{λ_k}-geodesics" "nonconstant h_λ-geodesics", respectively.*

- *In Theorem 12.7, "Assumptions 12.1" and "constant (non-zero) speed $\mathbb{I}_g$-invariant F_{λ_k}-geodesics" are changed into "Assumptions 14.8" and "nonconstant speed h_{λ_k}-geodesics", respectively.*

- *In Theorem 12.8, "Assumption 12.1", "constant (non-zero) speed $\mathbb{I}_g$-invariant F_μ-geodesic", "constant (non-zero) speed $\mathbb{I}_g$-invariant F_λ-geodesic", "constant (non-zero) speed $\mathbb{I}_g$-invariant F_λ-geodesics", "F_λ-speeds (i.e., $F_\lambda(\gamma^1_\lambda(t), \dot\gamma^1_\lambda(t)) \neq F_\lambda(\gamma^2_\lambda(t), \dot\gamma^2_\lambda(t))$ $\forall t$)" are changed into "Assumption 14.8", "nonconstant $\mathbb{I}_g$-invariant h_μ-geodesic", "nonconstant $\mathbb{I}_g$-invariant h_λ-geodesic", "nonconstant $\mathbb{I}_g$-invariant h_λ-geodesics", "h_λ-speeds (i.e., $h_\lambda(\dot\gamma^1_\lambda(t), \dot\gamma^1_\lambda(t)) \neq h_\lambda(\dot\gamma^2_\lambda(t), \dot\gamma^2_\lambda(t))$ $\forall t$)", respectively.*

A Proofs of some lemmas

Exponential map We begin with the standard knowledge from textbooks in Riemannian geometry. Let M be a n-dimensional, C^k-smooth manifold. Its tangent bundle TM is a C^{k-1}-smooth manifold of dimension $2n$, whose points are denoted by (x,v), with $x \in M$ and $v \in T_xM$. Let g be a C^{k-1} Riemannian metric on M. Let $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$, $q \mapsto \varphi(q) = (x^1(q), \cdots, x^n(q))$ be a coordinate chart on M, $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $(g^{ij}) = (g_{ij})^{-1}$. Then the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kj}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

are C^{k-2} and hence the exponential map $\exp : TM \to M$ is a C^{k-2} map. There exists a fibrewise convex neighborhood $\mathcal{U}(0_{TM})$ of the zero section of TM such that the map

$$\mathbb{F} : \mathcal{U}(0_{TM}) \to M \times M,\ (q,v) \mapsto (q, \exp_q v) \tag{A.1}$$

is a C^{k-2} immersion.

For a H^1-curve $\gamma : [a,b] \to M$, a vector field V along γ and $t \in \gamma^{-1}(U)$ we can write

$$V(t) = \sum_j v^j(t) \frac{\partial}{\partial x^j}|_{\gamma(t)} \quad \text{and} \quad \dot\gamma(t) = \sum_j \dot\gamma^j(t) \frac{\partial}{\partial x^j}|_{\gamma(t)},$$

where $\gamma^j(t) = x^j(\gamma(t))$. The covariant derivative of V along γ

$$\frac{DV}{dt} = \sum_i \frac{Dv^i}{dt}\frac{\partial}{\partial x^i}|_{\gamma(t)} = \sum_i \left(\frac{dv^i}{dt} + \Gamma^i_{jk}(\gamma(t))\dot\gamma^j(t) v^k(t)\right)\frac{\partial}{\partial x^i}|_{\gamma(t)}.$$

The vector field V is called parallel if $\frac{DV}{dt} \equiv 0$.

Let $\pi : TM \to M$ be the bundle projection and $\mathfrak{U} = \pi^{-1}(U)$. φ induces a chart on M,

$$\Phi : \mathfrak{U} \to \varphi(U) \times \mathbb{R}^n,\ (q,v) \mapsto (x^1(q,v), \cdots, x^n(q,v), u^1(q,v), \cdots, u^n(q,v)), \tag{A.2}$$

where $x^i(q,v) = x^i(q)$ and $u^i(q,v) = dx^i(q)[v]$ for $i = 1, \cdots, n$, i.e., $v = \sum_{i=1}^n u^1(q,v)\frac{\partial}{\partial x^i}|_q$. It is computed in the standard textbooks in Riemannian geometry that

$$d\mathbb{F}(q,0)\left[\frac{\partial}{\partial x^i}(q,0)\right] = \left(\frac{\partial}{\partial x^i}(q), \frac{\partial}{\partial x^i}(q)\right), \quad d\mathbb{F}(q,0)\left[\frac{\partial}{\partial u^i}(q,0)\right] = \left(0, \frac{\partial}{\partial x^i}(q)\right). \tag{A.3}$$

Let $\{e_1, \cdots, e_n\}$ be a parallel frame field and $v^i : [a,b] \to \mathbb{R}$ be C^1 for each $i = 1, \cdots, n$. Put $V(t) = \sum_{i=1}^n v^i(t) e_i(t)$. Suppose that $(\gamma(t), V(t)) \in \mathcal{U}(0_{TM})$ for all t. Define

$$\Upsilon(V)(t) = \exp_{\gamma(t)} V(t).$$

Claim A.1. *Let $k > 2$ and $A(t) = (\gamma(t), V(t))$. Suppose for some $\bar{t} \in \gamma^{-1}(U)$ that $\dot{\gamma}(\bar{t}) = 0$, i.e., $\dot{\gamma}^j(\bar{t}) = 0$, $j = 1, \cdots, n$, and that $|V(\bar{t})|_g$ is so small that*

$$d\mathbb{F}(A(\bar{t})) : T_{A(\bar{t})}\mathcal{U}(0_{TM}) \to T_{\gamma(\bar{t})}M \times T_{\Upsilon(V)(\bar{t})}M$$

is an isomorphism by (A.3). Then $\frac{d}{dt}\Upsilon(V)(t)|_{t=\bar{t}} = 0$ if and only if $\dot{v}^j(\bar{t}) = 0$, $j = 1, \cdots, n$.

Proof. Then for $t \in \gamma^{-1}(U)$ it holds that

$$\begin{aligned} \dot{A}(t) &= \frac{d}{dt}x^1(A(t))\frac{\partial}{\partial x^1}(A(t)) + \cdots + \frac{d}{dt}x^n(A(t))\frac{\partial}{\partial x^n}(A(t)) \\ &+ \frac{d}{dt}u^1(A(t))\frac{\partial}{\partial u^1}(A(t)) + \cdots + \frac{d}{dt}u^n(A(t))\frac{\partial}{\partial u^n}(A(t)). \end{aligned}$$

For each $i = 1, \cdots, n$, note that $x^i(A(t)) = x^i(\gamma(t)) = \gamma^i(t)$ and

$$u^i(A(t)) = dx^i(\gamma(t))[V(t)] = \sum_{k=1}^n v^k(t)dx^i(\gamma(t))[e_k(t)] = \sum_{k=1}^n v^k(t)e_k^i(t)$$

since $e_k(t) = \sum_{i=1}^n e_k^l(t)\frac{\partial}{\partial x^l}(\gamma(t))$. We obtain $\frac{d}{dt}x^i(A(t)) = \dot{\gamma}^i(t)$ and

$$\frac{d}{dt}u^i(A(t)) = \sum_{k=1}^n \dot{v}^k(t)e_k^i(t) + \sum_{k=1}^n v^k(t)\dot{e}_k^i(t), \quad i = 1, \cdots, n.$$

Because $\{e_1, \cdots, e_n\}$ is a parallel frame field along γ, that is,

$$\frac{d}{dt}e_l^i(t) + \Gamma_{jk}^i(\gamma(t))\dot{\gamma}^j(t)e_l^k(t) = 0, \quad i = 1, \cdots, n,$$

we deduce that $\dot{e}_k^i(\bar{t}) = 0$ for $i, k = 1, \cdots, n$ and hence

$$\begin{aligned} \dot{A}(\bar{t}) &= \dot{\gamma}^1(\bar{t})\frac{\partial}{\partial x^1}(A(\bar{t})) + \cdots + \dot{\gamma}^n(\bar{t})\frac{\partial}{\partial x^n}(A(\bar{t})) \\ &+ \left(\sum_{k=1}^n \dot{v}^k(\bar{t})e_k^1(\bar{t})\right)\frac{\partial}{\partial u^1}(A(\bar{t})) + \cdots + \left(\sum_{k=1}^n \dot{v}^k(\bar{t})e_k^n(\bar{t})\right)\frac{\partial}{\partial u^n}(A(\bar{t})). \end{aligned}$$

Moreover $\mathbb{E}(A(t)) = (\gamma(t), \Upsilon(V)(t))$ implies

$$\left(\dot{\gamma}(\bar{t}), \frac{d}{dt}\Upsilon(V)(t)\Big|_{t=\bar{t}}\right) = d\mathbb{E}(A(\bar{t}))[\dot{A}(\bar{t})].$$

Since $\dot{\gamma}(\bar{t}) = 0$ it follows that

$$\frac{d}{dt}\Upsilon(V)(t)\Big|_{t=\bar{t}} = 0 \quad \Leftrightarrow \quad \dot{A}(\bar{t}) = 0 \quad \Leftrightarrow \quad \dot{v}^j(\bar{t}) = 0, \ j = 1, \cdots, n.$$

□

Claim A.2. *Let $k > 2$. For an even and τ-periodic C^{k-2}-curve $\gamma : \mathbb{R} \to M$, there exist unit orthogonal parallel C^{k-2} frame fields along γ, $\{e_1, \cdots, e_n\}$, such that for all $t \in \mathbb{R}$,*

$$(e_1(-t), \cdots, e_n(-t)) = (e_1(t), \cdots, e_n(t)), \tag{A.4}$$

$$(e_1(t + \tau), \cdots, e_n(t + \tau)) = (e_1(t), \cdots, e_n(t)). \tag{A.5}$$

Proof. Starting with a unit orthogonal frame at $T_{\gamma(0)}M$ and using the parallel transport along $\overline{\gamma}$ with respect to the Levi-Civita connection of the Riemannian metric g we get a unit orthogonal parallel C^5 frame field $\mathbb{R} \to \gamma^*TM,\ t \mapsto (e_1(t), \cdots, e_n(t))$.

Firstly, we prove that (A.4) is satisfied. In fact, let $(U; x^j)$ be a local coordinate system around a point in $\gamma(\mathbb{R})$. Then we can write

$$e_k(t) = \sum_{i=1}^{n} e_k^l(t) \frac{\partial}{\partial x^l}(\gamma(t)) \quad \forall t \in \gamma^{-1}(U), \quad k = 1, \cdots, n.$$

Since $\{e_1, \cdots, e_n\}$ is a parallel frame field,

$$\frac{d}{dt} e_l^i(t) + \Gamma_{jk}^i(\gamma(t))\dot{\gamma}^j(t) e_l^k(t) = 0, \quad \forall t \in \gamma^{-1}(U), \quad i = 1, \cdots, n.$$

Note that $\gamma(-t) = \gamma(t)$ implies $t \in \gamma^{-1}(U)$ if and only if $-t \in \gamma^{-1}(U)$. We have

$$\begin{aligned}
\frac{d}{dt}(e_l^i(-t)) = -\dot{e}_l^i(-t) &= \Gamma_{jk}^i(\gamma(-t))\dot{\gamma}^j(-t) e_l^k(-t) \\
&= -\Gamma_{jk}^i(\gamma(-t)) \frac{d}{dt}(\gamma^j(-t)) e_l^k(-t) \\
&= -\Gamma_{jk}^i(\gamma(t)) \frac{d}{dt}(\gamma^j(t)) e_l^k(-t).
\end{aligned}$$

It follows that $\{e_1(-\cdot), \cdots, e_n(-\cdot)\}$ is also a parallel frame field along γ. Since $e_k(t)$ and $e_k(-t)$ agree at $t = 0$, $k = 1, \cdots, n$, by the theorem of existence and uniqueness of ODE we obtain (A.4).

Next, we prove that for any $k \in \mathbb{N}$ there holds

$$(e_1(k\tau - t), \cdots, e_n(k\tau - t)) = (e_1(t), \cdots, e_n(t)) \quad \forall t \in [0, k\tau]. \tag{A.6}$$

Since for any $t \in [0, k\tau]$ it holds that $t \in \gamma^{-1}(U)$ if and only if $-t \in \gamma^{-1}(U)$, as above we get

$$\begin{aligned}
\frac{d}{dt}(e_l^i(k\tau - t)) = -\dot{e}_l^i(k\tau - t) &= \Gamma_{jk}^i(\gamma(k\tau - t))\dot{\gamma}^j(k\tau - t) e_l^k(k\tau - t) \\
&= -\Gamma_{jk}^i(\gamma(-t)) \frac{d}{dt}(\gamma^j(k\tau - t)) e_l^k(k\tau - t) \\
&= -\Gamma_{jk}^i(\gamma(t)) \frac{d}{dt}(\gamma^j(t)) e_l^k(k\tau - t), \quad l = 1, \cdots, n.
\end{aligned}$$

Hence $[0, k\tau] \to e_l(t)$ and $[0, k\tau] \to e_l(k\tau - t)$ are parallel frame fields along $\gamma|_{[0,k\tau]}$ and have the same value at $t = k\tau/2$, we deduce that $e_l(k\tau - t) = e_l(t)$ for any $t \in [0, k\tau]$ and $l = 1, \cdots, n$.

Finally, we prove (A.5). For any $t > 0$, let us choose $k \in \mathbb{N}$ such that $t < k\tau$. Then

$$e_l(\tau + t) = e_l((k+1)\tau - (\tau + t)) = e_l(k\tau - t) = e_l(t).$$

If $t < -\tau$, this and (A.4) lead to

$$e_l(\tau + t) = e_l(-\tau - t) = e_l(\tau + (-\tau - t)) = e_l(-t) = e_l(t).$$

If $-\tau \le t < 0$ then $0 \le t + \tau < \tau$. By (A.6) and (A.4) we derive

$$e_l(\tau + t) = e_l(\tau - (\tau + t)) = e_l(-t) = e_l(t).$$

Summarizing these (A.5) is proved.

Note: Since γ is even, as a loop $\gamma : S_\tau \to M$ is contractible. Thus $\gamma^*TM \to S_\tau$ has an orthogonal trivialization. This can only lead to the existence of an unit orthogonal frame fields along γ satisfying (A.4). $\square$

Proof of Lemma 3.1. By Lemma 2.6 it suffices to prove the third and fourth assertions. Follow the notations above Lemma 3.1.

Step 1(Prove that $(\lambda,t)\mapsto \mathbf{u}_\lambda(t)$ is continuous, and C^2 with respect to t). By (♠) in the first paragraph in Section 3.1.1 and (3.2), for all $t\in[0,\tau]$ we have

$$\mathbb{F}\left(\overline{\gamma}(t),\sum_{i=1}^n \mathbf{u}^i_\lambda(t)e_i(t)\right)=\left(\overline{\gamma}(t),\exp_{\overline{\gamma}(t)}\left(\sum_{i=1}^n \mathbf{u}^i_\lambda(t)e_i(t)\right)\right)=\left(\overline{\gamma}(t),\gamma_\lambda(t)\right). \tag{A.7}$$

By (3.4), $(\lambda,t)\mapsto\gamma_\lambda(t)$ is a continuous map from $\Lambda\times[0,\tau]$ into the open subset $\mathbf{U}_{3\iota}(\gamma_\mu([0,\tau]))$ of M. These and (♣) in the first paragraph in Section 3.1.1 imply the composition

$$\Lambda\times[0,\tau]\to TM,\ (\lambda,t)\mapsto\left(\mathbb{F}|_{\mathcal{W}(0_{TM})}\right)^{-1}\left(\overline{\gamma}(t),\gamma_\lambda(t)\right)=\left(\overline{\gamma}(t),\sum_{i=1}^n \mathbf{u}^i_\lambda(t)e_i(t)\right)$$

is continuous, and C^2 with respect to t. Since g is a C^6 Riemannian metric on M, we obtain

$$\Lambda\times[0,\tau]\to\mathbb{R},\ (\lambda,t)\mapsto g\left(\left(\overline{\gamma}(t),\sum_{i=1}^n \mathbf{u}^i_\lambda(t)e_i(t)\right),\left(\overline{\gamma}(t),e_j(t)\right)\right)=\mathbf{u}^j_\lambda(t)$$

is continuous, and C^2 with respect to t, for each $j=1,\cdots,n$.

Step 2(Prove that $(\lambda,t)\mapsto\dot{\mathbf{u}}_\lambda(t)$ is continuous). Fix a point $t_0\in[0,\tau]$. Let $\varphi:U\to\varphi(U)\subset\mathbb{R}^n$, $q\mapsto\varphi(q)=(x^1(q),\cdots,x^n(q))$ be a coordinate chart centered at $\overline{\gamma}(t_0)$. For example, we can take $U=\phi_{\overline{\gamma}}(t_0,B^n_{2\iota}(0))$ and $\varphi=(\phi_{\overline{\gamma}}(t_0,\cdot))^{-1}$. It has the induced chart $(\pi^{-1}(U),\Phi)$ on TM as in (A.2). Let $J=(\overline{\gamma})^{-1}(U)$, which is an open neighborhood of t_0 in $[0,\tau]$. For each $t\in J$, we have two basses of $T_{\overline{\gamma}(t)}M$,

$$e_1(t),\cdots,e_n(t)\quad\text{and}\quad\frac{\partial}{\partial x^1}|_{\overline{\gamma}(t)},\cdots,\frac{\partial}{\partial x^n}|_{\overline{\gamma}(t)}.$$

Hence there exists a unique non-degenerate matrix $(A_{ij}(t))$ of order n such that

$$e_i(t)=\sum_{j=1}^n A_{ij}(t)\frac{\partial}{\partial x^j}|_{\overline{\gamma}(t)},\quad i=1,\cdots,n.$$

Then

$$\sum_{i=1}^n \mathbf{u}^i_\lambda(t)e_i(t)=\sum_{i=1}^n \mathbf{u}^i_\lambda(t)\sum_{j=1}^n A_{ij}(t)\frac{\partial}{\partial x^j}|_{\overline{\gamma}(t)}=\sum_{j=1}^n\left(\sum_{i=1}^n \mathbf{u}^i_\lambda(t)A_{ij}(t)\right)\frac{\partial}{\partial x^j}|_{\overline{\gamma}(t)}.$$

Let $\varphi(\overline{\gamma}(t))=(x^1(\overline{\gamma}(t)),\cdots,x^n(\overline{\gamma}(t))$. Then for each $t\in J$ we have

$$\Phi\left(\overline{\gamma}(t),\sum_{i=1}^n \mathbf{u}^i_\lambda(t)e_i(t)\right)=\left(x^1(\overline{\gamma}(t)),\cdots,x^n(\overline{\gamma}(t),\sum_{i=1}^n \mathbf{u}^i_\lambda(t)A_{i1}(t),\cdots,\sum_{i=1}^n \mathbf{u}^i_\lambda(t)A_{in}(t)\right).$$

It is clear that

$$\varphi\times\varphi:U\times U\to B^n_{2\iota}(0)\times B^n_{2\iota}(0),\ (x,y)\mapsto(\varphi(x),\varphi(y)).$$

is a chart on $M\times M$ centered at $(\overline{\gamma}(t_0),\overline{\gamma}(t_0))$. Take a small neighborhood $J_0\subset J$ of t_0 in $[0,\tau]$ such that $d_g(\overline{\gamma}(t),\overline{\gamma}(t_0))<\iota$ for all $t\in J_0$. Then (3.3) implies

$$d_g(\gamma_\lambda(t),\overline{\gamma}(t_0))<2\iota,\quad\forall(\lambda,t)\in\Lambda\times J_0. \tag{A.8}$$

Hence $\{\gamma_\lambda(t)\,|\,(\lambda,t)\in\Lambda\times J_0\}$ is contained in the chart (U,φ). Let

$$(\varphi\times\varphi)\left(\overline{\gamma}(t),\gamma_\lambda(t)\right)=\left(x^1(\overline{\gamma}(t)),\cdots,x^n(\overline{\gamma}(t),x^1(\gamma_\lambda(t)),\cdots,x^n(\gamma_\lambda(t))\right).$$

By this, (A.7) and (A.8) we obtain

$$\begin{aligned}&\Phi\circ(\mathbb{F})^{-1}\circ(\varphi\times\varphi)^{-1}\left(x^1(\overline{\gamma}(t)),\cdots,x^n(\overline{\gamma}(t),x^1(\gamma_\lambda(t)),\cdots,x^n(\gamma_\lambda(t))\right)\\&=\left(x^1(\overline{\gamma}(t)),\cdots,x^n(\overline{\gamma}(t),\sum_{i=1}^n\mathbf{u}^i_\lambda(t)A_{i1}(t),\cdots,\sum_{i=1}^n\mathbf{u}^i_\lambda(t)A_{in}(t)\right)\ \forall(\lambda,t)\in\Lambda\times J_0.\end{aligned}\tag{A.9}$$

Note that $\Psi:=\Phi\circ\left(\mathbb{F}|_{\mathcal{W}(0_{TM})}\right)^{-1}\circ(\varphi\times\varphi)^{-1}$ is a C^5 diffeomorphism onto its image set. Taking the derivative of t for the equation in (A.9) we arrive at

$$\begin{aligned}&D\Psi\left(x^1(\overline{\gamma}(t)),\cdots,x^n(\overline{\gamma}(t),x^1(\gamma_\lambda(t)),\cdots,x^n(\gamma_\lambda(t))\right)[\frac{d}{dt}x^1(\overline{\gamma}(t)),\cdots,\frac{d}{dt}x^n(\overline{\gamma}(t),\frac{d}{dt}x^1(\gamma_\lambda(t)),\\&\qquad\qquad\cdots,\frac{d}{dt}x^n(\gamma_\lambda(t)]\\&=\left(\frac{d}{dt}x^1(\overline{\gamma}(t)),\cdots,\frac{d}{dt}x^n(\overline{\gamma}(t),\sum_{i=1}^n\mathbf{u}^i_\lambda(t)\frac{d}{dt}A_{i1}(t),\cdots,\sum_{i=1}^n\mathbf{u}^i_\lambda(t)\frac{d}{dt}A_{in}(t)\right)\\&+\left(\frac{d}{dt}x^1(\overline{\gamma}(t)),\cdots,\frac{d}{dt}x^n(\overline{\gamma}(t),\sum_{i=1}^n\frac{d}{dt}\mathbf{u}^i_\lambda(t)A_{i1}(t),\cdots,\sum_{i=1}^n\frac{d}{dt}\mathbf{u}^i_\lambda(t)A_{in}(t)\right)\\&\ \forall(\lambda,t)\in\Lambda\times J_0.\end{aligned}\tag{A.10}$$

Since all A_{ij} are C^5, $D\Psi$ is C^4, $(\lambda,t)\mapsto\mathbf{u}_\lambda(t)$ is continuous (by Step 1), and

$$(\lambda,t)\mapsto x^1(\gamma_\lambda(t))\quad\text{and}\quad(\lambda,t)\mapsto\frac{d}{dt}x^1(\gamma_\lambda(t))$$

are continuous in $\Lambda\times J_0$ (by Assumption 1.2), it follows from (A.10) that

$$\Lambda\times J_0\ni(\lambda,t)\mapsto\left(\sum_{i=1}^n\frac{d}{dt}\mathbf{u}^i_\lambda(t)A_{i1}(t),\cdots,\sum_{i=1}^n\frac{d}{dt}\mathbf{u}^i_\lambda(t)A_{in}(t)\right)=\left(\frac{d}{dt}\mathbf{u}^1_\lambda(t),\cdots,\frac{d}{dt}\mathbf{u}^n_\lambda(t)\right)(A_{ij}(t))$$

is continuous. Moreover, all $(A_{ij}(t))$ are invertible. These lead to

$$\Lambda\times J_0\ni(\lambda,t)\mapsto\left(\frac{d}{dt}\mathbf{u}^1_\lambda(t),\cdots,\frac{d}{dt}\mathbf{u}^n_\lambda(t)\right)$$

is continuous. Since $t_0\in[0,\tau]$ is arbitrarily chosen, the required claim is proved. □

Funding

This work was supported by National Natural Science Foundation of China [12371108].

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School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875, The People's Republic of China
E-mail address: gclu@bnu.edu.cn